KONERU LAKSHMAIAH EDUCATION FOUNDATION
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Department of Mathematics

DISCRTE MATHEMATICS-25MT1002

CO-3 Material

I/IV-B.Tech-(ODD Sem), Academic Year: 2025-2026

Session-15 (Understand Direct proofs and their Applications)

Instructional Objective:

1. To understand the concept of direct proofs.
2. To solve the problems on direct proofs

Learning Outcomes:

1. Understand the concept and structure of a direct proof.
2. Apply direct proof techniques to prove basic statements in mathematics.

Introduction:

In mathematics, a **proof** is a logical argument that establishes the truth of a statement based on previously accepted facts such as definitions, axioms, and previously proven theorems. Among various types of proof techniques, the **Direct Proof** is one of the simplest and most commonly used methods, especially in number theory and algebra.

A direct proof starts from known facts (definitions, axioms, previously proven theorems) and uses logical steps to arrive at the statement to be proven. It is used primarily for implications of the form: If P, then Q.

Structure of a Direct Proof:

1. Assume the hypothesis (P) is true.
2. Use definitions, algebra, or logical steps.
3. Show the conclusion (Q) logically follows.

Key Characteristics:

- Begins with the assumption that the hypothesis is true.
- Follows a logical sequence of steps.
- Ends by establishing the conclusion directly.
- Uses known definitions and theorems (e.g., definition of even/odd, divisibility rules, properties of real numbers).

Example:

Statement: If n is an even integer, then $n^2$is also even.

This will be proven by assuming that n is even (i.e., n=2k) and then showing n^2 also satisfies the definition of even (i.e., divisible by 2).

When to Use Direct Proof:

- When proving universal statements ("for all x, if P(x) then Q(x)").
- When working with algebraic identities or number properties.
- When definitions and known properties are sufficient to derive the conclusion.

Model Problem and Solution

Example:1 Prove: If n is an even integer, then n^2 is also even.

Solution (Direct Proof):

Let n be an even integer.

By definition of even, n = 2k for some integer k.

Then, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

Since $2k^2$ is an integer, n^2 is divisible by 2.

Hence, n^2 is even.

Example 2: The sum of two even integers equals an even integer

Consider two even integers x and y. Since they are even, they can be written as

$$x = 2a, y = 2b$$

respectively for integers a and b. Then the sum can be written as

$x + y = 2a + 2b$ where   , a and b are all integers.

It follows that $x + y$ has 2 as a factor and therefore is even, so the sum of any two even integers is even.

Example 3: The square of an odd number is also odd

By definition, if n is an odd integer, it can be expressed as

n=2k+1

for some integer k. Thus

$$
\begin{aligned}
n^2 &= (2k+1)^2 \\
&= (2k+1)(2k+1) \\
&= 4k^2 + 2k + 2k + 1 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

Since $2k^2 + 2k$ is an integer, n^2 is also odd.

Review questions:

- Prove: If a and b are odd integers, then a + b is even.
- Prove: If n is divisible by 4, then n is even.
- Prove: If $x > 3$, then $x^2 > 9$.
- If n is an even integer then 2n+1 is odd/even
- If m is an even integer and n is an odd integer then m + n is an odd /even integer.

Summary and Recap

- Review definition and structure of a direct proof.
- Compare direct proof to other methods (e.g., contradiction).
- Emphasize practice and clarity in logic and presentation.

Self-assessment questions:

1. Which of the following is not a step in a direct proof?

a) Start with the hypothesis or premises that are given.

b) Use logical reasoning to deduce new information or properties from the premises.

c) Use counterexamples to disprove the conclusion.

d) Keep going until you arrive at the conclusion you want to prove.

2. What is the purpose of a direct proof?

a) To disprove a statement or proposition.

b) To prove a statement or proposition by starting with the premises and using logical steps to arrive at the conclusion.

c) To prove a statement or proposition by contradiction.

d) To prove a statement or proposition using induction.

3. Which of the following is an example of a direct proof?

a) Assume the conclusion is false and derive a contradiction.

b) Assume the conclusion is true and derive a contradiction.

c) Start with the given statement or proposition and apply logical reasoning to deduce new information or properties from the premises until you arrive at the conclusion you want to prove.
d) None of the above.

What is the role of logical reasoning in a direct proof?

a) To start with the conclusion and work backward to the premises.

b) To start with the premises and derive new information or properties until you arrive at the conclusion you want to prove.

c) To assume the conclusion is false and derive a contradiction.

d) To assume the conclusion is true and derive a contradiction.

Which of the following statements can be proven by a direct proof?

a) The sum of two odd numbers is an even number.

b) The square root of 2 is an irrational number.

c) The product of two irrational numbers is an irrational number.

d) All of the above.

Answers:

[1] c
[2] b
[3] c
[4] b
[5] d

CLASSROOM DELIVARY PROBLEMS

1. Prove: If a and b are odd integers, then a + b is even.
2. Prove: If n is divisible by 4, then n is even.
3. Prove: If $x > 3$, then $x^2 > 9$.
4. If n is an even integer then 7n + 4 is an even integer.
5. If m is an even integer and n is an odd integer then m + n is an odd integer.
6. If m is an even integer and n is an odd integer then mn is an even integer.
7. If a, b and c are integers such that a divides b and b divides c then a divides c.

PRACTICE PROBLEMS

1. Prove that the square of an odd integer is odd.
2. Prove that the square of any even integer is divisible by 4.
3. Prove the following statements:

   (a) If n and m are both odd, then n + m is even.

   (b) If n is odd and m is even, then n + m is odd.

   (c) If n and m are both even, then n + m is even. **GATE2023**

   (d) If n and m are both odd, then nm is odd; otherwise, nm is even.

TUTORIAL PROBLEMS

1. Prove that the sum of two odd integers is an even integer.
2. Prove that the product of two even integers is an even integer.
3. Prove that the sum of two rational numbers is a rational number.

HOME ASSIGNMENT PROBLEMS

1. Prove that the square of an odd integer is odd.
2. Prove: If n is an even integer, then n^3 is even.
3. Prove: If a and b are even integers, then ab is even.
4. Prove: If x is a real number and $x > 2$, then $x^2 > 4$.
5. Prove: If n is a multiple of 6, then n is a multiple of 3.
6. Prove: If $a \equiv 0 \pmod 2$ and $b \equiv 0 \pmod 2$, then $a + b \equiv 0 \pmod 2$.
7. Prove that the square of any even integer is divisible by 4.

Session-16 (Understand Indirect proofs and their Applications)

Instructional Objective: 1. To enable students to understand and apply indirect proof techniques, particularly proof by a. contradiction b. contra positive, in solving mathematical propositions and statements.

Learning Outcomes:

1. Define and distinguish between direct and indirect proof methods.
2. Explain the logic behind proof by contradiction and contra positive.
3. Construct logical arguments using indirect proof strategies.
4. Solve mathematical problems using proof by contradiction.
5. Identify statements where indirect methods are preferable or necessary

Introduction:

Indirect proofs involve assuming the **negation of what you want to prove** and then showing this leads to a **contradiction**, thereby validating the original statement.

There are two main types:

- **Proof by Contradiction**: Assume the negation of the conclusion and derive a contradiction.
- **Proof by Contra positive**: To prove "If P, then Q", prove "If not Q, then not P" instead.

Indirect proofs are especially useful when direct proofs are difficult or when dealing with negations or non-constructive existence statements.

Example Problems

Problem1: Prove that $\sqrt{2}$ is irrational.

Solution (By Contradiction):

4. Assume the opposite: $\sqrt{2}$ is rational.
5. Then $\sqrt{2} = a/b$ where a and b are integers with no common factor.
6. Squaring both sides: $2 = a^2/b^2 \Rightarrow a^2 = 2b^2$
7. So a^2 is even $\Rightarrow$ a is even $\Rightarrow a = 2k$
8. Then $a^2 = 4k^2 \Rightarrow 2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b^2$ is even $\Rightarrow$ b is even.
9. So both a and b are even $\Rightarrow$ they have a common factor of 2, contradicting our assumption.
10. Hence, $\sqrt{2}$ is irrational.

Problem2: Prove that if n^2 is even, then n is even.

Contra positive Form: Instead of proving "If n^2 is even, then n is even", we prove the contrapositive: "If n is odd, then n^2 is odd".

Proof:

1. Assume n is odd.
2. Then, by definition, $n = 2k + 1$ for some integer k.
3. Now compute n^2:
4. $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
5. This is of the form $2m + 1$, which is odd.
6. Therefore, if n is odd, then n^2 is odd.

Review questions:

1. What procedure we follow for contradiction?
2. What are the mathematical implications we use for contrapositive and contradiction.

Summary:

In this session, we discussed about indirect proofs.

Indirect proof is a method of mathematical reasoning that involves assuming the negation of what needs to be proved and then showing that this assumption leads to a contradiction. It is a powerful tool for proving many mathematical theorems and propositions, and it can also be applied to real-world problems in various fields.

Self-assessment questions:

1. What is the purpose of an indirect proof by contradiction?

- a) To prove a statement or proposition by starting with the premises and using logical steps to arrive at the conclusion.
- b) To prove a statement or proposition by assuming the opposite of what needs to be proved and then showing that this assumption leads to a contradiction.
- c) To disprove a statement or proposition by assuming the opposite of what needs to be proved and then showing that this assumption leads to a contradiction.
- d) To prove a statement or proposition using mathematical induction.

Correct Answer: b

2. Which of the following is an example of an indirect proof by contradiction?

- a) Assume the conclusion is true and derive a contradiction.
- b) Assume the conclusion is false and derive a contradiction.
- c) Start with the given statement or proposition and apply logical reasoning to deduce new information or properties from the premises until you arrive at the conclusion you want to prove.
- d) None of the above.

Correct Answer: b

3. What is the role of assuming the opposite of what needs to be proved in an indirect proof by contradiction?

- a) To prove the statement or proposition directly.
- b) To disprove the statement or proposition directly.
- c) To show that if the opposite of what needs to be proved is false, then the conclusion must be true.
- d) To show that if the opposite of what needs to be proved is true, then the conclusion must be false.

Correct Answer: c

4. Which of the following statements can be proven by an indirect proof by contradiction?

- a) The sum of two odd numbers is an even number.
- b) The square root of 2 is an irrational number.
- c) The product of two irrational numbers is an irrational number.
- d) All of the above.

Correct Answer: b

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5. What is the final step in an indirect proof by contradiction?

- a) To assume the opposite of what needs to be proved and then derive a contradiction.
- b) To assume the conclusion is true and then derive a contradiction.
- c) To assume the conclusion is false and then derive a contradiction.
- d) To assume the opposite of what needs to be proved and then prove the statement directly.

Correct Answer: a

CLASSROOM DELIVARY PROBLEMS

1. Prove that there are infinitely many prime numbers. (Hint: Assume finite and derive contradiction.)
2. If n^2 is even, prove that n is even. (Use contrapositive.)
3. Prove that if a^2 is divisible by 3, then a is divisible by 3.
4. Prove that 3 is irrational using contradiction.
5. Explain how you can use an indirect proof by contradiction to prove that there are infinitely many prime numbers.
6. Use an indirect proof by contradiction to prove that the statement "If n is an odd integer, then n^2 is odd" is true.
7. Verify the statement "There is a positive integer n such that $n^3 - 5n - 6 = 0$" is true using an indirect proof by contradiction.
8. Use an indirect proof by contradiction to prove that the statement "If n is an even integer, then n^2 is even" is true.

PRACTICE PROBLEMS

1. Prove that the statement "For any two positive integers a and b, if a + b is odd, then a and b cannot both be odd" using an indirect proof by contradiction.
2. Use an indirect proof by contradiction to prove that the statement "If a and b are integers, then a + b is even if and only if a and b have the same parity" is true.

TUTORIAL PROBLEMS

1. Prove that there is no smallest positive rational number.
2. If $x \in \mathbb{R}$ and $x \neq 0$, prove that $1/x \neq 0$.
3. Prove that if n is an integer and n^2 is odd, then n is odd. (Use contrapositive.)
4. Prove that between any two rational numbers, there is another rational number. (Indirectly)
5. Explain the process of an indirect proof by contradiction. Give an example to support your explanation.
6. Use an indirect proof by contradiction to prove that the square root of 2 is irrational.
7. Prove that there is no largest prime number using an indirect proof by contradiction.
8. Explain how you can use an indirect proof by contradiction to prove that there are infinitely many prime numbers.

HOME ASSIGNMENT PROBLEMS

1. Prove that there is no rational number whose square is 5.
2. Prove that if $x^2 - 6x + 5 < 0$, then $1 < x < 5$. (Use indirect reasoning.)
3. Prove that the sum of a rational number and an irrational number is irrational.
4. Use proof by contradiction to prove that there is no integer solution to the equation $x^2 = 2x + 1$.
5. Explain how you can use an indirect proof by contradiction to prove the statement "There is a real number x such that $x^3 + 3x + 1 = 0$."
6. Prove that there is no smallest positive rational number.
7. Prove that the sum of any two irrational numbers is an irrational number.

Session-18: The Pigeonhole Principle

Instructional Objective:

1. **Understand the concept of the pigeonhole principle.**
2. **Be able to understand that how these principles are useful in real world problems.**

Learning Outcomes: Students will learn to understand the concept and application of the pigeonhole principle.

[I] The Pigeonhole Principle:

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called ***the pigeonhole principle***, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 2). Of course, this principle applies to other objects besides pigeons and pigeonholes.

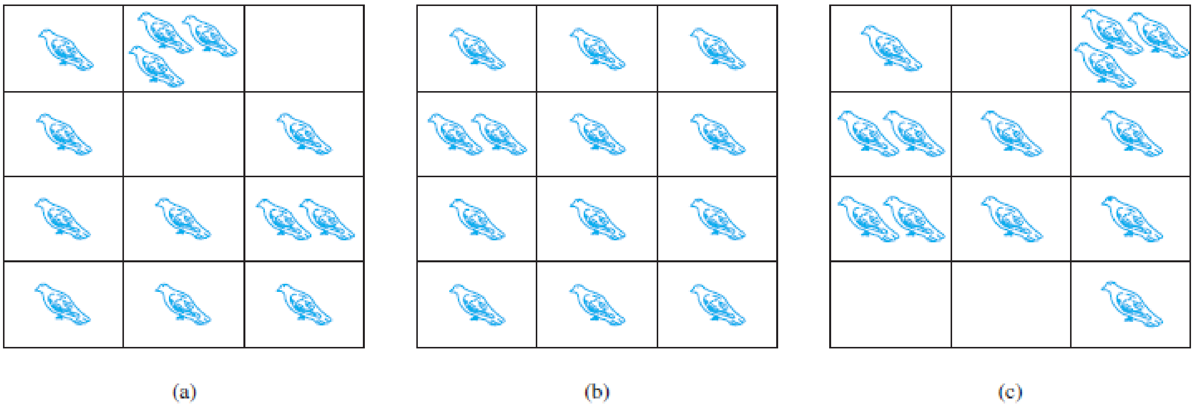


Figure 2: There Are More Pigeons Than Pigeonholes.

Result 1 (THE PIGEONHOLE PRINCIPLE): If **k** is a positive integer and **k + 1** or more objects are placed into **k** boxes, then there is at least one box containing two or more of the objects.

Examples:

1. Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
2. In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

Result 2 (THE GENERALIZED PIGEONHOLE PRINCIPLE):

If **N** objects are placed into **k** boxes, then there is at least one box containing at least $\lceil \mathbf{N/k} \rceil$ objects.

Note: $\lceil \mathbf{x} \rceil$ represents the greatest integer function.

Example: Among 100 people there are at least $\lceil \mathbf{100/12} \rceil = \mathbf{9}$ who were born in the same month.

Review Questions:

1. Give some examples of the pigeonhole principle.

Self-assessment questions:

Classroom Delivery Problems

Session-18

1. Discuss the following with suitable examples
   a. Pigeonhole Principle
2. If 5 colors are used to paint 26 doors, show that at least 6 doors will have the same color.
3. If 7 cars carry 26 passengers prove that at least one car must have four or more passengers.
4. Show that if 50 books in a library contain a total of 27551 pages, one of the books must have at least 552 pages.

Home Assignment Problems

1. A drawer has 10 black socks and 12 white socks. If you pick socks at random, how many must you pull out to ensure you get at least one matching pair?
2. Show that in a group of 13 people, at least two must have birthdays in the same month.
3. If 26 students are assigned to 5 classrooms, prove that at least one room has 6 or more students.
4. A teacher has 31 chocolates to give to 6 students. Show that at least one student gets 6 or more chocolates.
5. A box has 4 compartments. If you put 9 coins into the box, show that one compartment has at least 3 coins.

Tutorial Problems

1. You have 6 black shoes and 6 white shoes mixed in a dark room. How many shoes must you pick to ensure you get a matching pair?
2. In a class of 13 students, prove that at least two students have birthdays in the same month.

Session-19: Counting (The basics of Counting, Permutations)

Instructional Objective:

3. **Understand the concept of basic counting principles in permutations.**
4. **Be able to understand that how these principles are useful in real world problems.**

Learning Outcomes: Students will learn to understand the concept and application of the basics of counting, permutations rules.

[I] The Basics of Counting:

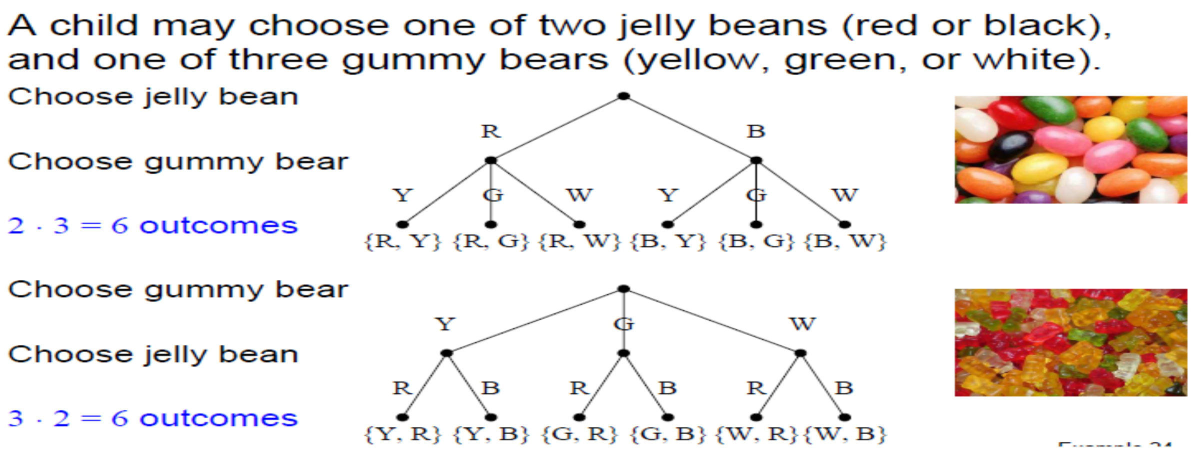
Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics. Enumeration, the counting of objects with certain properties, is an important part of combinatorics. We must count objects to solve many different types of problems.

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there? The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this session. These methods serve as the foundation for almost all counting techniques.

Basic Counting Principles:

We first present two basic counting principles, the product rule and the sum rule. Then we will show how they can be used to solve many different counting problems.

1. The Product Rule:



Statement: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are n_1n_2 ways to do the procedure.

Example 1: Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with?

Solution: There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt.

Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the following figure 1.

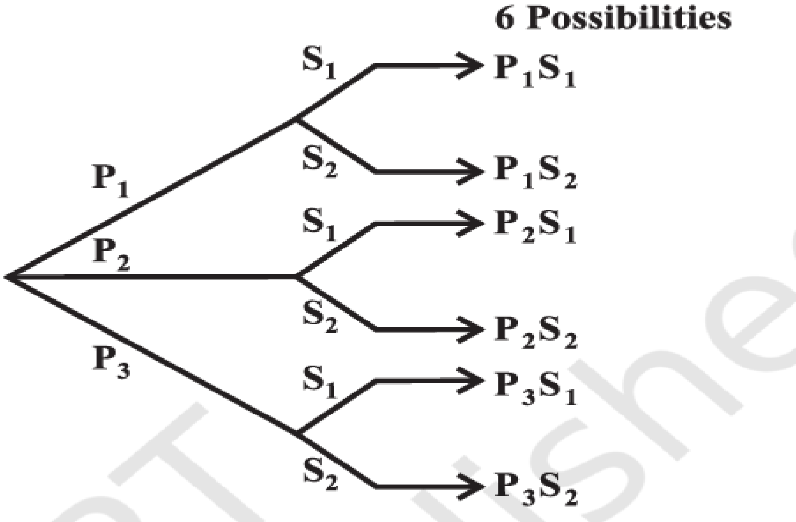


Figure 1

Example 2: A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are 12×11 = 132 ways to assign offices to these two employees.

Example 3: There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution: The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are 32×24 = 768 ports.

Result (*Size of Cartesian Product*): The product rule is often phrased in terms of sets in this way: If A_1, A_2, …, A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \ldots \times |A_m|$$

Example: Let $P = \{R, B\}$; $Q = \{Y, G, W\}$

Then, $P \times Q = \{(R, Y), (R, G), (R, W), (B, Y), (B, G), (B, W)\}$

Here, we have $|P| = 2$; $|Q| = 3$; $|P \times Q| = 6$

We can clearly see that,

$$|P \times Q| = |P| \times |Q|$$

2. The Sum Rule:

A diner may select 1 dessert from 3 pies and 4 cakes.
How many ways can this be done?
Two events (pie and cake) but only one will occur.

$3 + 4 = 7$


Statement: If a task can be done either in one of $\mathbf{n_1}$ ways or in one of $\mathbf{n_2}$ ways, where none of the set of $\mathbf{n_1}$ ways is the same as any of the set of $\mathbf{n_2}$ ways, then there are $\mathbf{n_1 + n_2}$ ways to do the task.

Example 1: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Result (Size of the Set Union): The sum rule can be phrased in terms of sets as: If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \cdots \ldots + |A_m|$$

Example:

Let A and B be disjoint sets.
By the addition principle,
$|A \cup B| = |A| + |B|$

Example

$A = \{\text{Bach, Handel}\}$ $\qquad\qquad$ $|A| = 2$

$B = \{\text{Kraftwerk, Epica, Primal Fear}\}$ $\qquad\qquad$ $|B| = 3$

$A \cup B = \{\text{Bach, Handel, Kraftwerk, Epica, Primal Fear}\}$
$|A \cup B| = |A| + |B| = 2 + 3 = 5$

[II] **Permutation:**

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this section we will develop methods to answer questions such as these.

Let us consider one example.

Example: In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Solution: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $\mathbf{5 \times 4 \times 3 = 60}$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $\mathbf{5 \times 4 \times 3 \times 2 \times 1 = 120}$ ways to arrange all five students in a line for a picture.

The above example illustrates how ordered arrangements of distinct objects can be counted. This leads to some terminology.

Definition (Permutation): A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects, taken some or all at a time. An ordered arrangement of **r** elements of a set is called an ***r-permutation***.

1. **Permutations when all the objects are distinct**

Result 1: The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by $^{n}P_{r}$ or $P(n,r)$.

In other words, if **n** is a positive integer and **r** is an integer with $1 \leq r \leq n$, then there are

$$^{n}P_{r} = P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with **n** distinct elements.

Definition (Factorial Notation): The notation **n!** represents the product of first **n** natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as **n!**. We read this symbol as **'n factorial'**. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

Result: If **n** and **r** are integers with $0 \leq r \leq n$, then

$$^{n}P_{r} = \frac{n!}{(n-r)!}, \; 0 \leq r \leq n$$

Example 1: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$^{100}P_{3} = 100 \times 99 \times 98 = 970{,}200.$$

Example 2: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ ways for the saleswoman to choose her tour. If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths.

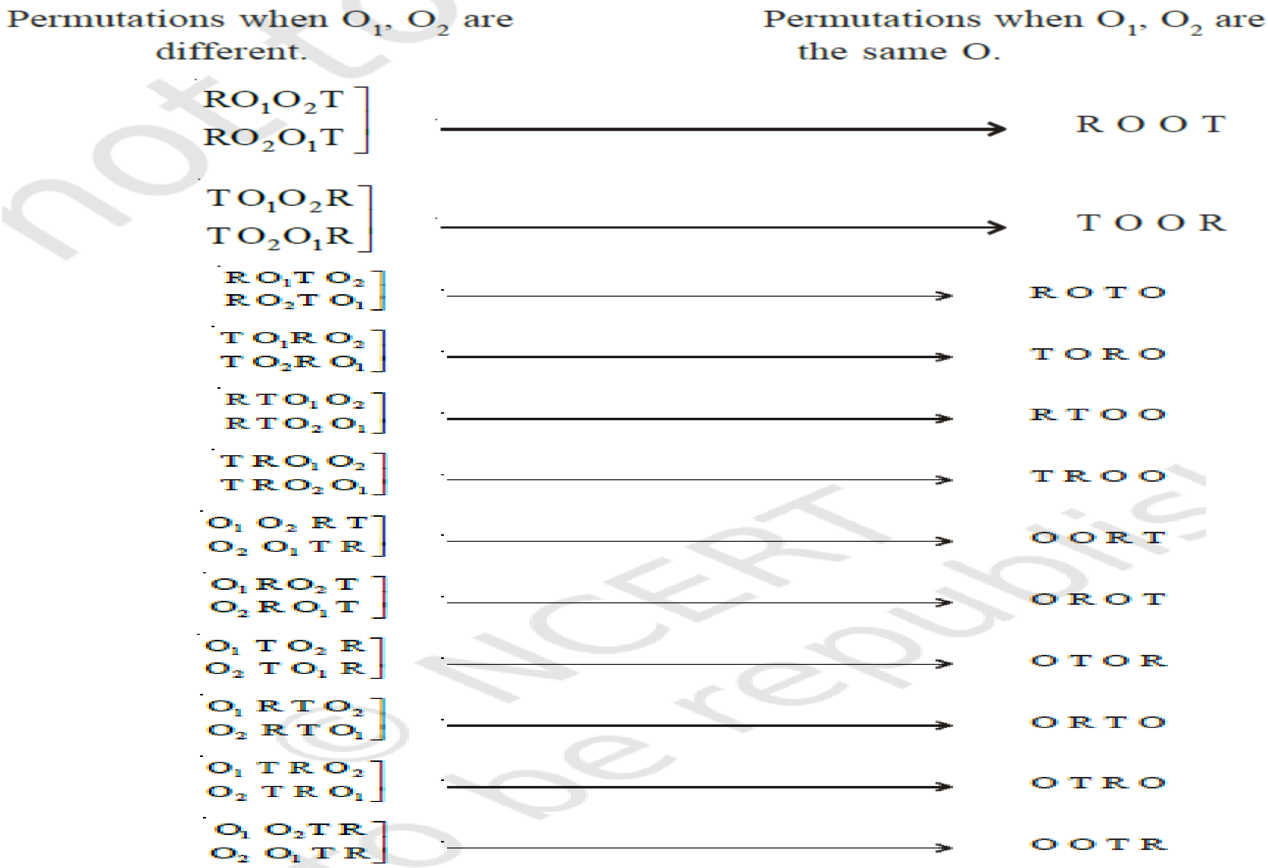
Example 3: Ten athletes compete in an Olympic event; gold, silver, and bronze medals are awarded. How many ways can the awards be made?

Solution: $P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8 = 720.$

2. Permutations when all the objects are not distinct objects

Suppose we have to find the number of ways of rearranging the letters of the word **ROOT**. In this case, the letters of the word are not all different. There are **2 Os**, which are of the same kind. Let us treat, temporarily, the **2 Os** as different, say, $\mathbf{O_1}$ and $\mathbf{O_2}$. The number of permutations of **4-different letters**, in this case, taken all at a time is **4!**. Consider one of these permutations say, $\mathbf{RO_1O_2T}$. Corresponding to this permutation, we have **2!** permutations $\mathbf{RO_1O_2T}$ and $\mathbf{R\,O_2O_1T}$ which will be exactly the same permutation if $\mathbf{O_1}$ and $\mathbf{O_2}$ are not treated as different, i.e., if $\mathbf{O_1}$ and $\mathbf{O_2}$ are the same O at both places.

Therefore, the required number of permutations $= \frac{4!}{2!} = 3 \times 4 = 12$.



Result: The number of permutations of **n** objects, where **p** objects are of the same kind and rest are all different $\frac{\mathbf{n!}}{\mathbf{p!}}$.

In fact, we have a more general result as follows:

Result: The number of permutations of **n** objects, where $\mathbf{p_1}$ objects are of one kind, $\mathbf{p_2}$ are of second kind, ..., $\mathbf{p_k}$ are of k^{th} kind and the rest, if any, are of different kind is

$$\frac{\mathbf{n}!}{\mathbf{p_1}!\,\mathbf{p_2}!\ldots\mathbf{p_k}!}$$

Example: Find the number of permutations of the letters of the word **ALLAHABAD**.

Solution: Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements = $\frac{9!}{4!2!} = \frac{9\times8\times7\times6\times5}{2} = \mathbf{7560}$ **.**

Review Questions:

2. Define Product Rule and Sum Rule of Basic Counting Principle?
3. Define permutations with examples

Self-assessment questions:

1. The value of $C(n,r)$ is:
   a. $n!$
   b. $r!\,P(n,r)$.
   c. $\frac{P(n,r)}{r!}$.
   d. $r!$.

2. From the 9 elements of the set 'A' we have to select 5 elements to form a subset. This can be done in ….. ways.
   a. $^9C_5$.
   b. $^9P_5$.
   c. 9!
   d. 5!

Ans: 1-d; 2-a; 3-b; 4-a

Classroom Delivery Problems

Session-19

5. Discuss the following with suitable examples
   b. Counting techniques
   c. Permutations

Session:19

1. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
2. Suppose that we want to buy a computer from one of two makes A1 and A2. Suppose also that those makes have 12 and 18 different models, respectively. Then how many models are there altogether to choose from?
3. How many ways can the letters in the word "CAT" be arranged?
4. How many permutations of the letters ABCDEFGH contain the string ABC?
5. Find the number of permutations of the letters of the word **MISSISSIPPI**.
6. Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Home Assignment Problems

6. Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

7. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

8. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable?

9. Eight students should be accommodated in two 3-bed and one 2-bed rooms. In how many ways can they be accommodated?

10. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, and 7 if no digit is repeated?

Tutorial Problems

3. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

4. A password of 6 digits is made of digits 9,2,6,0,0,2. How many possible passwords are there? How long would it take to try all the possible passwords if trying one password takes 5 seconds?
5. Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,
   a. do the words start with P
   b. do all the vowels always occur together
   c. do the vowels never occur together
   d. do the words begin with I and end in P?

6. There are 3 elements a, b, c. Use them to make:
   a. permutations (without repetition)
   b. permutations with repetition, where, “a” occurs 2 times, “b” one time and “c” one time.

Session 20 & 21 : COMBINATIONS

AIM OF THE SESSION
To familiarize students with the basic concept of “Permutations and Combinations, With and without repetition of the objects, distinguisahable and indistinguishable objects.

INSTRUCTIONAL OBJECTIVES
This Session is designed to:
To understand the selection of the objects
To understand the combination of the items

LEARNING OUTCOMES
At the end of this session, you should be able to:
1. Identify the number of combinations of the given items.
2. Justify the objects are distinguishable or not..

Introduction:

Permutations and combinations are fundamental concepts in combinatorics that deal with the arrangement and selection of objects from a set. Permutations consider the order of elements, while combinations do not. A simple way to introduce them is by using real-world examples that illustrate the difference between order mattering and not mattering.

Permutations (Order Matters):
Start with a scenario where order is important, like arranging letters in a word or the order of runners finishing a race. For instance, "ABC" is different from "CAB" in a permutation.

Combinations (Order Doesn't Matter):
Then, introduce a scenario where order doesn't matter, like selecting a group of friends or choosing lottery numbers. For example, "apple, banana, grape" is the same combination as "banana, grape, apple".

Use simple language to explain that permutations are about arrangements where order is important, and combinations are about selections where order is not.

Illustration-1: Let us assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

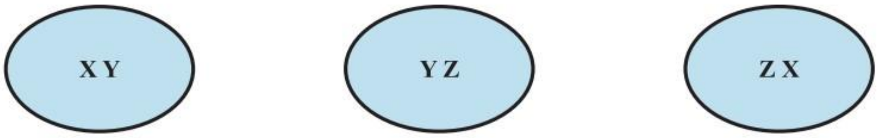


Fig-1

These are XY, YZ and ZX. Here, each selection is called a combination of 3 different objects taken 2 at a time.

Remark: In a combination, the order is not important.

Illustration-2 Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of handshakes. X shaking hands with Y and Y with X will not be two different handshakes. Here, order is not important. There will be as many handshakes as there are combinations of 12 different things taken 2 at a time. Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Definition: An unordered selection of r elements of a set containing n distinct elements is called an r-Combination of n elements and is denoted by $C(n, r)$or n_{C_r}.

we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by $C(n, r)$.

Result: The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$\boldsymbol{C(n,r) = \frac{n!}{r!(n-r)!}}.$$

Example-1 How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Solution: Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are

$$C(52,5) = \frac{52!}{5!\,47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 26 \times 17 \times 10 \times 49 \times 12$$

$$= 2{,}598{,}960$$

Note that there are

$$C(52,47) = \frac{52!}{47!\,5!}.$$

different ways to select 47 cards from a standard deck of 52 cards. We do not need to compute this value because C(52, 47) = C(52, 5).

Corollary: Let n and r be nonnegative integers with $r \leq n$. Then $\mathrm{C(n, r)} = \mathrm{C(n, n - r)}$.

Example-2 How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Solution: The answer is given by the number of 5-combinations of a set with 10 elements. By Theorem 2, the number of such combinations is

$$C(10,5) = \frac{10!}{5!\,5!} = 252.$$

Example-3: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution: The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 1, the number of such combinations is

$$C(30,6) = \frac{30!}{6!\,24!} = 593{,}775.$$

Example-4: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with 9 elements and the number of 4-combinations of a set with 11 elements. Thus, we have,

$$\mathrm{C}(9,3) \cdot \mathrm{C}(11,4) = \frac{9!}{3!\,6!} \cdot \frac{11!}{4!\,7!} = 84 \times 330 = 27{,}720.$$

Review Questions:

4. Define Product Rule and Sum Rule of Basic Counting Principle?
5. Give some examples of the pigeonhole principle.
6. Define permutations and combinations with examples

Self-assessment questions:

3. How many permutations of the letters of the word APPLE are there?
   a. 600
   b. 120
   c. 240
   d. 60

4. In how many ways can 8 Indians and, 4 American and 4 Englishman can be seated in a row so that all persons of same nationality sit together?
   a. 3! 4! 8! 4!
   b. 3! 8!
   c. 4! 4!
   d. 8! 4! 4!

5. The value of $C(n, r)$ is:
   a. $n!$
   b. $r!\,P(n, r)$.
   c. $\frac{P(n,r)}{r!}$.
   d. $r!$.

6. From the 9 elements of the set 'A' we have to select 5 elements to form a subset. This can be done in ….. ways.
   e. ${}^{9}C_{5}$.
   f. ${}^{9}P_{5}$.
   g. 9!
   h. 5!

Ans: 1-d; 2-a; 3-b; 4-a

Classroom Delivery Problems

Session:17

1. A committee of 8 students is to be selected from a class consisting of 19 freshman and 34 sophomores.

   a. In how many ways can 3 freshman and 5 sophomores be selected?
   b. In how many ways can a committee with exactly 1 freshman be selected?
   c. In how many ways can a committee with at most 1freshman be selected?

2. From a club consisting of 6 men and 7 women, in how many ways can we select a committee of
   a. 3 men and 4 women?
   b. 4 persons which has at least one woman?
   c. 4 persons that has at most one man?
   d. 4 persons that has at most one man?

3. In how many ways can 20 students out of a class of 30 be selected for an extra-curricular activity, if
   a. Rama refuses to be selected.
   b. Raja insists on being selected.
   c. Gopal and Govind insist on being selected?
4. What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these
   a. four cards are of the same suit
   b. four cards belong to four different suits
   c. two are red cards and two are black cards.
5. How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE?

Home Assignment Problems

1. A question paper contains 10 questions. A student has to answer 7 of them, but question 1 is compulsory. In how many ways can the student choose the questions?
2. A bookshelf has 10 different books. In how many ways can you select 4 books such that two particular books are either both included or both excluded?
3. From a group of 15 people, in how many ways can a team of 6 be formed such that at least 2 members are women, if the group has 5 women and 10 men?
4. A company wants to select a team of 5 members from 8 engineers and 6 designers. In how many ways can this be done if at least 2 designers must be included?

Tutorial Problems

7. **A student has to choose 3 subjects out of 5 optional ones. In how many ways can the student choose the subjects?**
8. **In a club of 8 men and 6 women, a committee of 4 people is to be formed. How many ways can this be done if the committee must include exactly 2 women?**
9. **Out of 12 people, 5 are to be chosen to form a team. How many ways can this be done if two specific people must be included in the team?**
10. A man has 4 red ties, 3 blue ties, and 2 green ties. If he wants to choose 3 ties, how many different combinations can he select?
11. A school wants to form a committee of 5 students from a group of 10 boys and 8 girls. How many ways can they form a committee that includes at least 2 girls?