I/IV-B.Tech-(ODDSem), AcademicYear:2025-2026

B.Tech. (AIDS,CSE, CSIT, CE, EEE &IOT),2025 BatchI/IV,ODDSemester

Curse code: 25MT1002 **Course: Discrete Mathematics**

Session-1(Sets: Sets and Subsets. Power Set, Cartesian Product, Set Operations, Venn Diagram)

Instructional Objective:

1. To learn definition Sets: Sets and Subsets. Power Set, Cartesian Product, Set Operations,
2. To learn how to construct Venn Diagram

Learning Outcomes:

To learn Sets and Subsets. Power Set, Cartesian Product, Set Operations, Venn Diagram

Introduction:

SETS

Set Definition

Set can be defined as a collection of elements enclosed within curly brackets. In other words, we can describe the Set as a Collection of Distinct Objects or Elements.

Example of Sets

Some examples of sets include:

- *A set of fruits: {apple, banana, orange}*
- *A set of numbers: {1, 2, 3, 4}*
- *A set of even numbers: {2, 4, 6, 8, 10, ...}*

Representation of a Set

Set can be denoted using two common forms:

- Roaster Form or Tabular Form
- Set Builder Form

Roaster Form or Tabular Form: In Roaster Form, elements of the set are enclosed within a pair of brackets and separated by commas. Example:

N is a set of Natural Numbers less than 7 { 1, 2, 3, 4, 5, 6}

Set of Vowels in Alphabet = { a, e, i, o, u}

Set Builder Form: In this representation, Set is given by a Property that the members need to satisfy.

{x: x is an odd number divisible by 3 and less than 10}

{x: x is a whole number less than 5}

Size of a Set

The number of elements in the set is called cardinality or size of the set. In general, the Cardinality of the Set A is given by |A| and can be either finite or infinite.

Types of Sets

Finite Set: A Set containing a finite number of elements is called Finite Set. Empty Sets come under the Category of Finite Sets. If at all the Finite Set is Non-Empty then they are called Non- Empty Finite Sets.

Example: A = {x: x is the first month in a year}; Set A will have 31 elements.

Infinite Set: In Contrast to the finite set if the set has infinite elements then it is called Infinite Set.

Example: A = {x : x is an integer}; There are infinite integers. Hence, A is an infinite set.

Sub Set: If Set A contains the elements that are in Set B as well then Set A is said to be the Subset of Set B. A subset is indicated by the symbol '⊆' and read as 'is a subset of' in set theory.

It can be expressed using this symbol as follows:

"A ⊆ B" this signifies that Set A is a subset of Set B.

Example:

- A = {2, 3, 10} is a subset of B = {1, 2, 3, 4, 10},
- **P** = Set of All Prime Numbers is a subset of **N** = Set of All Natural Numbers, and
- X = {a, e, i, o ,u} are collection of vowels and is a subset of Y = Set of all Alphabets .

Types of Subsets :

There are two types of subsets that are:

- Proper Subset
- Improper Subset

Proper Subset

A proper subset only comprises a few members of the original set. Proper subset can never be equal to the original set. The number of elements of a proper subset is always less than the parent set. A proper subset is denoted by ⊂,

We can express a proper subset for set A and set B as; A ⊂ B

Example: Let set A = {1, 3, 5}, then proper subsets of A are {}, {1}, {3}, {5}, {1, 3} {3, 5} {1, 5}.

Improper Subset

An improper subset contains includes both the null set and each member of the initial set. Improper subset is to be equal to the original set. This is represented by the symbol **⊆**.

Example: What will be the improper subset of set A = {1, 3, 5}?

Answer:

Improper subset: {}, {1}, {3}, {5}, {1,3}, {1,5}, {3,5} and {1,3,5}

Power Set: Power Set of A is the set that contains all the subsets of Set A. It is represented as P(A).

Example: If set A = {-5,7,6}, then power set of A will be:

P(A)={ϕ, {-5}, {7}, {6}, {-5,7}, {7,6}, {6,-5}, {-5,7,6}}

Universal Set:

This is the base for all the other sets formed. Based on the Context universal set is decided and it can be either finite or infinite. All the other Sets are Subsets of Universal Set and is given by U.

Example: Set of Real Numbers is a Universal Set of Integers .

Empty Set:

There will be no elements in the set and is represented by the symbol ϕ or {}. The other names of Empty Set are Null Set or Void Set.

Example: S = { x | x ∈ N and 9 < x < 10 } = ∅

Singleton Set:

If a Set contains only one element then it is called a Singleton Set.

Example: A = {x : x is an odd prime number}

Venn diagram:

A Venn diagram is a diagram that shows the relationship between and among a finite collection of sets. If we have two or more sets, we can use a Venn diagram to show the logical relationship among these sets as well as the cardinality of those sets.

Set Operations:

Set Operations can be defined as the operations performed on two or more sets to obtain a single set containing a combination of elements from all the sets being operated upon.

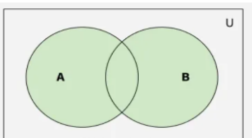
Consider Two different sets A and B.

Union: Union Operation is given by the symbol U. Set **A U B** denotes the union between Sets A and B. It is read as A union B or Union of A and B. It is defined as the Set that contains all the elements belonging to either of the Sets.

$A \cup B = \{x: x \in A \text{ or } x \in B\}$

Venn Diagram For Union of Sets

The area shaded in green represents $A \cup B$ or the union of sets A and B.



A U B is shaded

Example: Find the union of A = {2, 3, 4} and B = {3, 4, 5}.
Solution:

$A \cup B = \{2, 3, 4, 5\}$.

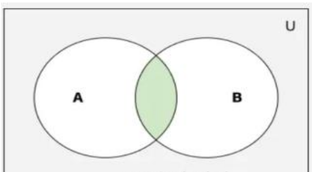
Intersection

The intersection of the sets A and B, denoted by **A ∩ B,** is the set of elements that belong to both A and B, i.e. set of the common elements in A and B. This operation is represented as:

$A \cap B = \{x: x \in A \text{ and } x \in B\}$

Venn Diagram For Intersection of Sets

The area shaded in green represents A∩B or the intersection of sets A and B, which includes the elements common to both sets A and B.



A ∩ B is shaded

Example: Find the intersection of A = {2, 3, 4} and B = {3, 4, 5}

Solution:

The elements that are **common to both** sets A and B.

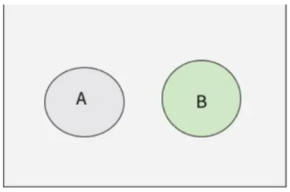
$A \cap B = \{3, 4\}$.

Disjoint Set

Two sets are said to be disjoint if their intersection is the empty set. i.e., sets have no common elements. In simpler terms, they don't "overlap" at all.
So if you try to find their intersection, you'll get the empty set, which we denote by the symbol ϕ or {}.

Venn Diagram For Disjoint Sets

The sets A and B are disjoint, meaning they have no common elements (no overlap).



For Example: Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8}

Solution:

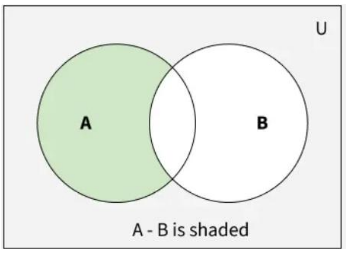
A and B are disjoint sets since both of them have no common elements.

Set Difference

The difference between sets is denoted by '**A - B**', which is the set containing elements that are in A but not in B i.e., all elements of A except the element of B.

Venn Diagram For Set Difference

In the below diagram, the set difference A−B contains all the elements that are in A but not in B.



Example: If A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8}, find A - B.
Solution:

A - B = {1, 3, 5}

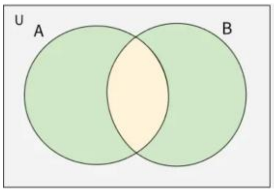
Symmetric Difference

The symmetric difference of A and B includes elements in **A or B** but **not both**.

- It is denoted by: **A △ B or A ⊕ B.**
- The symmetric difference is like saying, "Give me everything that's not shared.
- It is defined as: A △ B = (A−B) ∪ (B−A)

Venn Diagram For Symmetric Difference

The symmetric difference AΔB includes elements that are in either A or B but not in both.



Example: Let set A = {1, 2, 3}, and set B = {3, 4, 5}, then Find the symmetric difference

Solution:

A △ B = {1, 2, 4, 5}

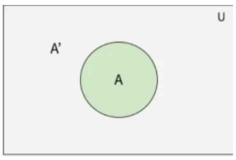
Complement of a Set

If U is a universal set and X is any subset of U, then the complement of A consists of all the elements in U that are not in A.

A' = {x : x∈ U and x ∉ A}

Venn Diagram For Complement of a Set

In the diagram below, set A' includes all elements not in A, relative to the universal set.



Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8} And A = {1, 2, 5, 6}

Solution:

Then, the complement of A, denoted as A′, will be: A′ = {3, 4, 7, 8}

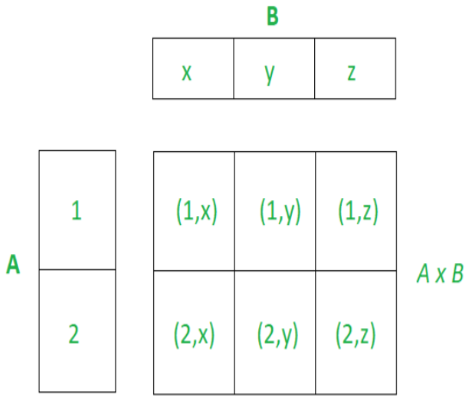
Cartesian Product: Consider A and B to be Two Sets. The Cartesian product of the two sets is given by AxB i.e. the set containing all the ordered pairs (a, b) where a belong to Set A, b belongs to Set B.

Representation of Cartesian product A × B = {(a, b) | a ∈ A and b ∈ B}.

Example:

Let $A = \{1, 2\}$ and $B = \{x, y, z\}$

$A \times B = \{(1, x), (1, y), (1, z), (2,x), (2, y), (2, z)\}$



Properties of Cartesian Product:

Various properties of cartesian product includes,

1. Cartesian Product is non-commutative: $A \times B \neq B \times A$

Example:

$A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

$B \times A = \{(a, 1), (b, 1), (b, 1), (b, 2)\}$

Therefore as $A \neq B$ we have $A \times B \neq B \times A$

2. $A \times B = B \times A$, only if $A = B$

3. Cardinality of Cartesian Product is defined as number of elements in $A \times B$ and is equal to the product of cardinality of both sets i.e.,

$|A \times B| = |A| \times |B|$

Then the bit string representation of A is: 01010

(1 for presence, 0 for absence; indexed by U)

Review Questions:

1. **Explain** the Sets and Subsets.
2. What is Power Set?
3. What is Cartesian Product of a Set?

Summary:

A **set** is a collection of distinct elements, and a **subset** contains elements all belonging to another set. The **power set** is the set of all subsets of a given set, having 2n2^n2n elements if the original set has n elements. The **Cartesian product** of two sets is the set of all ordered pairs formed by taking an element from each set. **Set operations** include union (all elements in either set), intersection (common elements), difference (elements in one set but not the other), and complement (elements not in the set with respect to a universal set). **Venn diagrams** visually represent these relationships and operations using overlapping circles.

Self-assessment questions:

Terminal Questions:

Classroom Delivery Problems

. Convert the following Sets given in Roster form into Set-Builder form.

(i) A = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}
(ii) B = {2, 4, 6, 8, 10}
(iii)C = {5, 7, 9, 11, 13, 15, 17, 19}

2. Express the given Sets in Roster form.
   (i) A = {a: a = n/2, n ∈ N, n < 10}
   (ii) B = {b: b = n2, n ∈ N, n ≤ 5}

3. If U={1,2,3,4,5,6,7,8,9}, A={1,2,4,6,8}, and B={2,4,5,9} compute the following:

(i) $\bar{A}$ (ii) $\bar{B}$ (iii) $\bar{A} \cup \bar{B}$ (iv) $\overline{A \cup B}$ (v) $\bar{A} \cap \bar{B}$ (vi) $\overline{A \cap B}$ (vii) B-A (viii) A-B (xi) AΔB

4 . Determine the sets A and B , given that A-B={1,3,7,11}, B-A={2,6,8}, and $A \cap B$={4,9}.

5. Draw the venn diagrams for the following:

(i) $A \cup B \cup C$ (ii) $(A\Delta B) \cup C$ (iii) $\overline{A \cap B \cap C}$

6. Using venn diagrams, prove that , for any sets A,B,C,

(i) $\overline{(A \cap B) \cup C}$ = $(\bar{A} \cup \bar{B}) \cap \bar{C}$

(ii) $\{\overline{(A \cup B) \cap C}\} \cup \bar{B}$= $\overline{B \cap C}$

7. Find the elements of the power set P(A) in the following cases

(i) A={3,7} (ii) A={ 2,3,7} (iii) A={ a,b,c,d}

8. **How many elements are in set A if set A has a power set with 64 subsets?**

9. What will be the cardinality of Power Set of the set containing first 7 multiples of 3.

10. Let A={1,3,5}, B={2,3}, and C={4,6}. Write down the following:

(i) $(A \cup B)XC$ **(ii)** $A \cup (BXC)$ **(iii)** $(AXB) \cup (BXA)$

11. Given A = {2, 3, 4 , 5} and B = {4 , 16 , 23}, a ∈ A, b ∈ B, find the set of ordered pairs such that $a^2 < b$?

12. Given A × B has 15 ordered pairs and A has 5 elements, find the number of elements in B?

Tutorial Problems

1. **For sets A = {x | x is an integer, 1 ≤ x ≤ 6} and B = {x | x is an even integer, 2 ≤ x ≤ 8}, find the set A - B.**

2. In a college, 200 students are randomly selected. 140 like tea, 120 like coffee and 80 like both tea and coffee. Solve the following problems by using venn diagram.
   (i) How many students like only tea?
   (ii) How many students like only coffee?
   (iii) How many students like neither tea nor coffee?
   (iv) How many students like only one of tea or coffee?
   (v) How many students like at least one of the beverages?
3. a company has three promotional tactics: {Email, Social Media, TV Ads}. Write the power set to explore all possible combinations of marketing strategies.
4. **Let A = {1, 2} and B = {a, b}. Find |P(A × B)|, where P denotes the power set and × denotes the Cartesian product.**
5. If A = {5, 6}, B = {s, t}, and C = {g, h}, to find the Cartesian product of A × B × C.

Home Assignment Problems

1. **If set M = {x | x is a prime number less than 20} and set N = {x | x is an odd number less than 10}, what is M ∩ N?**
2. Consider two shopping lists: List A contains fruits (apples, oranges, bananas), and List B contains vegetables (carrots, tomatoes, spinach, potatoes). Create a new list representing items that are either fruits or vegetables, and another list representing items that are both fruits and vegetables.
3. **What are the number of elements in the union of two sets P and Q with number of elements 10, 8 respectively? Also the number of common elements in both sets is 3.**
4. Given A={1,2,3}, $B = \{2,3,5\}$, $C = \{3,4\}$ and $D = \{1,3,5\}$, check if $(A \cap C)\times(B \cap D) = (A \times B) \cap (C \times D)$ is true?

Session-3(Inclusion-Exclusion Principle, Computer Representation of Sets)

Instructional Objective:

3. To learn definition and examples Inclusion-Exclusion Principle
4. To learn Computer Representation of Sets

Learning Outcomes:

To learn applications of two or more sets and ensures each element is counted only once. Uses bits to indicate presence or absence of elements from a universal set. Efficient for performing operations like union and intersection.

Introduction:

The Principle of Inclusion and Exclusion (PIE) is a fundamental combinatorial method used to calculate the cardinality (size) of the union of multiple sets, especially when they overlap. It ensures that elements belonging to more than one set are not overcounted.

Explanation:

Formula for Two and Three Sets

For two sets A and B:

$|A \cup B| = |A| + |B| - |A \cap B|$

For three sets A, B, and C:

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

General Formula for n Sets

Let $A_1, A_2, \ldots, A_n$ be n sets. The inclusion-exclusion principle generalizes as:

$|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$

Example

In a survey of 100 students:
- 40 like Mathematics
- 50 like Physics
- 30 like Chemistry
- 20 like both Mathematics and Physics
- 15 like both Mathematics and Chemistry
- 10 like both Physics and Chemistry
- 5 like all three subjects

Find how many students like at least one of the subjects.

Solution:
Let M = Mathematics, P = Physics, C = Chemistry

Using PIE:
$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|$
$= 40 + 50 + 30 - 20 - 15 - 10 + 5 = 80$

Computer Representation of Sets

In computer science and discrete mathematics, sets can be represented in various ways for efficient storage, retrieval, and manipulation. The choice of representation depends on the operations to be performed and the nature of the data.

Bit String Representation

A bit string (or bit vector) is a sequence of bits (0s and 1s) used to represent a set. Each bit represents whether an element from a universal set U is present in the subset.

Example:

Let U = {1, 2, 3, 4, 5} and A = {2, 4}.

Then the bit string representation of A is: 01010

(1 for presence, 0 for absence; indexed by U)

Review Questions:

4. **Explain** the principle of inclusion and exclusion for three sets
5. What is a bit string?

Summary: Student will be able to learn The **Inclusion and Exclusion Principle It** is a combinatorial technique used to compute the cardinality of the union of overlapping sets by correcting for overcounting. It applies to two or more sets and ensures each element is counted only once. Uses bits to indicate presence or absence of elements from a universal set. Efficient for performing operations like union and intersection.

Self-assessment questions:

Terminal Questions:

Classroom Delivery Problems

1. A computer company requires 30 programmers to handle system programming jobs and 40 programmers for application programming. If the company appoints 55 programmers to carry out these jobs, how many of these perform jobs of both types? How many handle only system programming jobs and how many handle only application programming?

2. In a school, all pupils play either Hockey or Football or both. 400 play Football, 150 play Hockey, and 130 play both the games. Find
    i. The number of pupils who play Football only,
    ii. The number of pupils who play Hockey only,
    iii. The total number of pupils in the school.

3. In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?

4. In a sample of 100 logic chips, 23 have a defect D1, 26 have a defect D2, 30 have a defect D3, 7 have defects D1 and D2, 8 have defect D1 and D3, 10 have defect D2 and D3 and 3 have all three defects. Find the number of chips having
    a) At least one defect   b) No defect.

5. Among a group of students, 50 played cricket, 50 played hockey and 40 played volley ball. 15 played both cricket and hockey, 20 played both hockey and volley ball, 15 played cricket and volley ball and 10 played all three. If every student played at least one game, find the number of students and how many played only cricket, only hockey and only volleyball?

6. Let U ={ 1,2,3,4,5,6,7,8,9,10, and the ordering of elements of U has the elements in increasing order; that is, $a_i = 1$.

a) What bit strings represent the subset of all odd integers in U and the subset of all even integers in U.
b) What bit strings represent the subset of integers not exceeding 5 in U?
c) What is the bit string for the complement of the {1,3,5,7,9}
d) The bit strings for the sets {1,2,3,4,5} and {1,3,5,7,9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Tutorial Problems

6. In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French?
7. In a college, 200 students are randomly selected. 140 like tea, 120 like coffee and 80 like both tea and coffee. Solve the following problems by using venn diagram.
    (vi) How many students like only tea?
    (vii) How many students like only coffee?
    (viii) How many students like neither tea nor coffee?
    (ix) How many students like only one of tea or coffee?
    (x) How many students like at least one of the beverages?

8. In a survey of 80 people, it was found that 35 people read newspaper H, 20 read newspaper T, 15 read the newspaper I, 5 read both H and I, 10 read both H and T, 7 read both T and I, 4 read all three newspapers. Find the number of people who read at least one of the newspapers?
9. Suppose that the universal set is U ={ 1,2,3,4, 5,6,7,8,9,10}. Express each of these sets with bit strings where the i^{th} bit in the string is 1 if i is in the set and 0 otherwise.

a) {3,4,5} b) {1,3,6,10} c) {2,3,4,7,8,9}

10. Suppose that the universal set is U ={ 1,2,3,4, 5,6,7,8,9,10}, find the set specified by each of these bit strings.

a) 11 1100 1111 b) 01 0111 1000 c) 10 0000 0001.

Home Assignment Problems

5. In a survey of 200 people, 80 people drink coffee, 70 people drink tea, and 60 people drink juice. It was found that 30 people drink both coffee and tea, 25 drink both coffee and juice, 20 drink both tea and juice, and 15 drink all three beverages. Draw the venn diagram for the above information. Make use of Inclusion- Exclusion principle Obtain the number of people drink at least one of the beverages?

6. In a group of students, 65 play foot ball, 45 play hockey, 42 play cricket, 20 play foot ball and hockey, 25 play foot ball and cricket, 15 play hockey and cricket and 8 play all the three games. Find the total number of students in the group (Assume that each student in the group plays at least one game).

Session-4 & 5
Relations and their properties

Instructional Objective:

1. To learn relations
2. To learn properties of relations

Learning Outcomes:

1. Able to understand the relations
2. Able to understand the properties of relations

Introduction:

A relation can be thought of as a structure that represents the relationship of elements of a set to the elements of another set. We come across many situations where relationship between elements of sets, such as those between roll numbers of students in a class and their names, industries and their telephone numbers, employees in an organization and their salaries occur. Relations can be used to solve problems such as producing a useful way to store information in computers.

Explanation:

If A and B be two sets. The Cartesian product of A and B, denoted by AXB, is the set of all ordered pairs (a, b), where a∈A and b∈B.

Hence

$$AXB = \{(a,b) \ / \ a \in A \text{ and } b \in B\}.$$

The simplest way to express a relationship between elements of two sets is to use ordered pairs consisting of two related elements. Due to this reason, of ordered pairs is called binary relations. In this section, we introduced the basic terminology used to described binary relations, discuss the mathematics of relations defined on sets and explore the various properties of relations,

Definition:-If A and B are sets, a subset R of the Cartesian product A X B is called a relation from A to B.

If R is a relation from A to B, R is a set of ordered pair (a, b), where $a \in A$ and $b \in B$. When $(a,b) \in R$, we use the notation a R b and read it as "a is related to b by R ". If $(a,b) \notin R$, it is denoted as aRb.

Note:- If R is a relation from a set A to itself, viz., if R is a subset of AXA ,

then R is called a relation on the set A.

Examples:-

1. Let A be the set of students in the school, and let B be the set of courses.

Let R be the relation that consists of those pairs (a, b), where a is a student

enrolled in course b.

For instant, if Jason Good friend and Deborah Sheman are enrolled in CS518, the pair (Jason Good friend, CS518) and (Deborah Sherman,CS518) belong to R.

Further if Jason Good friend is also enrolled CS510, then the pair (Jason Good friend, CS510) is also in R.

However, if Deborah Sherman is not enrolled in CS510, then the pair (Deborah Sheman, CS510) is not in R.

Types of Relations:-

a. A relation R on a set A is called a **universal relation**, if R=AXA.

For example if A={1,2,3}, then R = AXA = {(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1)(3,2)(3,3)} is the universal relation on A.

b. A relation R on a set A is called a **void relation**, if R is the null set φ. For example if A={3,4,5} and r is define as a R b if and only if a+b>10,

Then R is a null set, since no element in AXA satisfies the given condition. Note:- The entire Cartesian product AXA and the empty set are subsets of AXA.

c. A relation R on a set A is called a **binary relation**, if R={(a, a)/aϵA}and is
denoted by IA .

For example , If A={1,2,3}, then R={(1,1),(2,2),(3,3)} is the identity relation on A.

d. When R is any relation from set A to a set B , the **inverse** of R, denoted by
R^{-1}, is relation from B to A which consists of those ordered pairs got by
interchanging the elements of the ordered pairs
in R. i.e., R^{-1}={(b,a)/(a,b)∈R}.,
i,e., if a R b, then b R^{-1} a.

For example , if A=[,2,3,5}, B={6,8,10} and a R b if and only if a∈A divides b∈B, then R={(2,6),(2,8),(2,10)(3,6),(5,10)}
Now R^{-1}={(6,2),(8,2),(10,2),(6,3),(10,5)}
We note that b R^{-1}a if and only if b∈B is a multiple of a∈A. Also we note that D(R)=R(R^{-1})={2,3,5} and (R)= D(R^{-1})={6,8,10}.

Composition of Relations:-

If R is a relation from set A to set B and S is a relation from set B to set C, viz., R is a subset of A×B and S is a subset of B×C, then the composition of R and S is denoted by R·S, [some authors use the notation S·R instead of R·S] is defined by a (R·S) c, if for some b∈B, we have a R b and b R c.

viz., R·S={(a, c)/there exists some b∈B for which (a, b)∈R and (b, c) ∈S}.

Note:-

1. For the relation R·S, the domain is a subset of A and the range is a subset of C.
2. R·S is empty, if the intersection of the range of R and the domain of S is empty.
3. If R is relation on a set A, then R·R, the composition of R with itself is always defined and sometimes denoted as R^2.

   For example, let R={(1,1), (1,3), (3,2), (3,4), (4,2)} and S={(2,1),(3,3)(3,4),(4,1)}.

   Any member (ordered pair) of R·S can be obtained only if the second element in the ordered pair of R agrees with the first element in the ordered pair of S. Thus (1,1) cannot combine with any member of S.

(1,3) of R can combine with(3,3) and (3,4) of S producing the members (1,3) and (1,4) respectively of R·S. Similarly the other members of R·S are obtained.

Thus R·S= {(1,3), (1,4), (3,1), (4,1)}

Similarly S·R= {(2,1), (2,3), (3,2), (3,4), (4,1), (4,3)}

R·R= {(1,1),(1,3),(1,2),(1,4),(3,2)}

S·S = {(3,3),(3,4),(3,1)}

R·(S·R)={(1,2),(1,4),(3,1),(3,3),(4,1)(

4,3)} R^3=R·R·R=(R·R)·R=R·(R·R)

={(1,1),(1,3),(1,2),(1,4)}.

4. The compliment of R denoted by R' defined by $a\not{R}b$ = aRb.
For example let A={x,y,z}, B = {1,2,3}. Let R be a relation from A to B
defined by R={(x,1),(x,2),(y, 3)}, then R'={(x,3),
(y,1),(y,2),(z,1),(z,2),(z,3)}.

Definition:- Let R be a relation on the set A . The power R^n, n=1, 2, 3,
...., are defined by

Recursively by R^1=R and $R^{n+1}=R^n \circ R$.

The definition shows that $R^2=R \circ R$, $R^3=R^2 \circ R=(R \circ R) \circ R$, and so on.

Properties of Relations:-

1. A relation R on a set A is said to be **reflexive** , if a R a for every a∈A, viz.,

if (a, a) ∈R for every a∈A.

For example, if R is the relation on A={1,2,3} defined by (a, b)∈R if a≤b, where a, b ∈A , then R={(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)}. Now R is reflexive, since each of the elements of A is related to itself, as (1,1),(2,2) and (3,3) are members in R.

Note: A relation R on a set A is irreflexive, if, for every a∈A, (a, a)∉R viz., if
there is no a∈A such that a R a.

For example, R, {(1,2),(2,3),(1,3)}in the above example is irreflexive.

2. A relation R on a set A is said to be **symmetric**, if whenever a R b then b
R a, viz., if whenever (a, b) ∈R then (b, a) also ∈R

Thus a relation R on A is not symmetric if there exists a, b∈A such that (a, b)∈R, but (b, a)∉R.

3. A relation R on a set A is said to be **anti-symmetric**, whenever (a, b) and
(b, a) $\in$R then a = b. If there exists a, b$\in$A such that (a, b) and
(b, a) $\in$R, but a$\neq$b ,then R is not anti symmetric.

For example, the relation of perpendicularity on a set of line in the plane is symmetric, since if a line a is perpendicular to the line b, then b is perpendicular to a.

The relation$\leq$ on the set Z of integers is not symmetric, since for example,4$\leq$5, but 5 $\nleq$ 4.

The relation of divisibility on N is anti symmetric, since whenever m is divisible by n and n is divisible by m then m=n.

NOTE: Symmetry and anti-symmetry are not negative of each other. For example, the relation R={(1,3),(3,1),(2,3)} is neither symmetric nor anti symmetric, where as the relation S={(1,1),(2,2)} is both symmetric and anti-symmetric.

A relation R on a set A is said to be **transitive**, if whenever a R b and bRc then a R c. viz., if whenever (a, b) $\in$ R and (b, c) $\in$R but (a, c)$\notin$R, then R is not transitive.

For example, the relation of set inclusion on a collection of sets is transitive, since if $A \subseteq B$ and B$\subseteq$C, then $A \subseteq C$.

n-Ary Relations

By an n-ary relation, we mean a set of ordered n-tuples. For any set S, a subset of the product set S^n is called an n-ary relation on S.

In particular, a subset of S^3 is called a ternary relation on S.

Ex: The 3-tuples in a 3-ary relation represent the following attributes of a student database:
student ID number, name, phone number.

Note: Is student ID number likely to be a primary key?
Yes because a student ID number is unique in a system.
Is name likely to be a primary key?
No because multiple students can have the same name

Review questions:

1. What is relation?
2. What are the types of relations?

Summary:

In this session, it was discussed that the definition of relation with suitable examples and their notations. Further, the types of relations are also discussed in this session with suitable examples.

Self-assessment questions:

1. The Cartesian product of A and B, denoted by------- [a]

   (a) A X B (b) B X A (c) A U B (d) Above all

2. A relation R on a set A is said to be -------, if a R a for every $a \in A$, viz.,
   if $(a, a) \in R$ for every $a \in A$
   [a]
   (a) Reflexive (b) Irreflexive (c) Symmetric (d) Above all
3. A relation R on a set A is said to be------, if whenever a R b then b [c]
   (a) Reflexive (b) Irreflexive (c) Symmetric (d) Above all
4. A relation R on a set A is called a ------ **relation**, if R=AXA.
   [b]
   (a) Reflexive (b) universal (c) Symmetric (d) Above all
5. A relation R on a set A is called a ----- **relation**, if $R=\{(a, a)/a \epsilon A\}$ [c]
   (a) Reflexive (b) universal (c) binary (d) Above all

Problems to be discussed:

Question:(2 marks)

1. Let A={1,2}. Define a relation R={(1,1),(2,2)}on set A.Is R reflexive?

2. Is the relation R={(1,2),(2,3),(1,3)} on A={1,2,3} is transitive.

3. Provide an example of relation R which is Reflexive and Symmetric but not Transitive.

4. Provide an example of relation R which is Reflexive and Transitive but not Symmetric

5. Provide an example of relation R which is Transitive and Symmetric but not reflexive.

Questions.(3 or 4 marks)

1. Let A={1,2}, B={3,4}and relation R={(1,3),(2,4)}.Find the **inverse** of relation R^{-1}
.

2. Let set A={a,b}, B={x,y} and relation R={(a,x),(b,y)}.Is R^{-1} a relation from B to A? Justify your answer.

3. Given, Relation R={(1,2),(2,3)} from set A={1,2,3} to set B={2,3}.Relation S={(2,4),(3,5)} from set B to set C={4,5}.Find the **composition** $S \circ R$.

4. Given that Relation R={(1,2),(2,3)} from set A={1,2,3} to set B={2,3}.Relation S={(2,4),(3,5)} from set B to set C={4,5).Find the **composition** $S \circ R$.

Questions (5 marks)

1. **L**et R and S be the following relations on A = {1, 2, 3};
   R = {(1,1),(1,2),(2,3),(3,1),(3,3)}, S = {(1,2),(1,3),(2,1),(3,3)}.
   Determine i) $R \cap S$, $R \cup S$, R^c ii) RoS iii)S^2=SoS

2. Let me be a positive integer. A relation R is defined on the set Z by "aRb if and only if a – b is divisible by m" for a, b ∈ Z. Show that R is an equivalence relation on set Z.
3. Examine that the relation R is an equivalence relation in the set

   A = { 1, 2, 3, 4, 5 } given by the relation R = { (a, b)/ |a-b| is even }.
4. R={$(1,b_2),(1,b_3),(3,b_2)(4,b_1),(4,b_3)$}. Determine the matrix of the relation **(GATE-22)**
5. Find the number of equivalence relations of the set {1,2,3,4} **(GATE-97).**

Tutorials

Problems

1. Let R and S be the following relations on B={a,b,c,d}. R={(a,a), (a,c), (c,b), (c,d), (d,b)}and S={(b,a), (c,c), (c,d), (d,a)}. Determine the following composition relations i) RoS ii)SoR iii)RoR iv)S^2=SoS.
2. A relation R defined from a set A={2,3,4,5} to the set B={3,6,7,10} as follows If (x,y) belongs to R then "x divides y". Write R as a set of ordered pairs and also find R^{-1}.
3. Let S be the set of all lines in 3 dimensional space. A relation 'R' is defined on S by "*l R m if and only if l lies on the plane of m" for l, m ∈ S.* Examine if R is (i) reflexive, (ii) symmetric,(iii) transitive. Give your conclusion.
4. Let R_1 be the relation defined on the set of real numbers are such that as (a, b) belongs to R_1 if and only if 1+ab > 0 for all (a, b) belongs to R_1. Illustrate that R_1 is reflexive, symmetric but not transitive.

Home Assignments:

- Let A={1,2,3,4,5,6}. Define a relation R on A such that (a,b)∈R if and only if a≡b mod 3.Show that Ris an equivalence relation.
- Define a relation R on the set of integers Z by aRb ⟺ a-b is even. Prove that R is an equivalence relation

Session-6 & 7 (Partial order relation and Hasse diagrams)

Instructional Objective:

1. To understand Partial order set
2. To understand the structure of the Hasse diagram

Learning Outcomes:

1. Identify the Partial order sets.
2. Construct the Hasse diagrams.

Introduction: First, we recall the definition of a partially ordered set or poset.

Definition: A partially ordered set is a system consisting of a non-empty set P and a binary relation ≤ on P such that the following conditions are satisfied for all x, y, z∈P

(i) x≤x (reflexive)

(ii) x≤y and y≤x ⇒x=y (anti symmetric)

(iii) x≤y and y≤z ⇒x≤z (transitive)

Note 1: We call the relation ≤(less than or equal to) a partial order on the set P and P is said to be a partly ordered set or a Partially Ordered set or simply a poset by the relation ≤

Note 2:It is easy to observe that if ≤ is a partial order on P then≥ is also a partial order on P and we call the partly ordered set (P, ≥) the dual of the partly ordered set (P,≤).

Comparable and incomparable elements: The elements 'a' and 'b' of a poset (S, ≤) are called 'comparable' if either a≤b or b≤a. Otherwise they are called 'incomparable' elements.

Chain: A poset in which any two elements are comparable is called a chain.

Example 1: Is the relation "greater than or equal to" is a partial ordering on the set of integers? Justify.

Solution: Let I= {--------3, -2, -1, 0, 1, 2, 3, ------} be the set of integers.

Given relation is greater than or equal to (i.e.≥)

Now we have to prove that (I, ≥) is a Poset.

(i) **Reflexive**: x ≥ x for all x in I

(ii) **Anti-symmetric**: Let x≥ y and y≥ x then x=y, for all x, y in I

(iii) **Transitive**: Let x ≥ y and y≥ z then x ≥ z, for all x, y, z in I

Hence (I, ≥) is a partially ordered set.

Example 2: Is the divisibility relation '/' is a partial ordering on the set of positive integers? Justify.

Solution: Given set is the set of positive integers, N= {1, 2, 3, ----------}

Given relation is divisibility (i.e.'|')

(i) **Reflexive**: we have a|a, since a=1a, for all a in N.

(ii) **Anti-symmetric**: Let a|b and b|a then there exists j, k in N such that b=ja and a=kb.

Now b=ja=j (kb) = (jk) b then 1=jk ie. i=j=1

Therefore, a=b, and hence divisibility is anti-symmetric in N.

(iii) **Transitive**: Let a|b and b|c for a, b, c in N.

a|b means there exists k such that b=ka and b|c means there exists j such that c=jb.

And c=jb=j (ka) = (jk) a, therefore a|c and hence divisibility is transitive in N.

Hence (N,|) is a poset.

Example 3: Check whether the inclusion ⊆ is a partial ordering on the power set of a set S?

Solution: Let P(S) be the power set of S= The set of all sub sets of S.

The given relation is inclusion ⊆ .

Now we have to prove that (P(S), ⊆) is a poset.

(i) **Reflexive**: we have A⊆ A for all A in P(S)

(ii) **Anti-Symmetric**: Let A⊆ B *and* $B \subseteq A$ then A=B, for all A, B in P(S)

(iii) **Transitive**: Let A⊆ B *and* $B \subseteq C$ then A⊆ C for all A, B, C in P(S).

Hence (P(S), ⊆) is a poset.

Example 4: Provide an example of partial ordering on R is the set of real numbers.

Solution: Let R be the set of real numbers and less than or equal to (i.e.≤) be a relation on R.

Now we have to prove that (R, ≤) is a partially ordered set.

(i) **Reflexive**: we have x ≤ x for all x in R.

(ii) **Anti-symmetric**: Let x≤y and y≤x then x=y for all x, y in R.

(iii) **Transitivity**: Let x≤ y *and* $y \leq z$ then x≤z for all x, y, z in R.

Example 5: Determine the relation ⊆ is a partial ordering on the power set of a set A, where X= {a, b, c}.

Solution: Given set X= {a, b, c}

Then P (X) = P ({a, b, c}) = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

Given relation is ⊆

Now we have to prove that (P(X), ⊆) is a partially ordered set.

(i) Reflexive: since every set is sub set to itself, A⊆ A, for all A in P(X)

(ii) **Anti-Symmetric**: Let A⊆ B *and* $B \subseteq A$ then A=B, for all A, B in P(X)

(iii) **Transitive**: Let A$\subseteq B$ *and* $B \subseteq C$ then A$\subseteq C$ for all A, B, C in P(X).

Hence (P(X), ⊆) is a poset.

Hasse diagrams

Definition: A finite partly ordered set P can be represented by means of a diagram in the following manner. Represent each element a in P by a small circle k_a in such a way that whenever a<b, then k_b is higher than k_a in the diagram. Further join k_a and k_b by a straight line segment whenever b covers a.

If b>a in a finite partly ordered set if and only if there is a sequence $a=a_1, \ldots\ldots a_n=b$ such that each a_{i+1} covers a_i.

Thus a<b iff there is an ascending Brocken line connecting a to b.

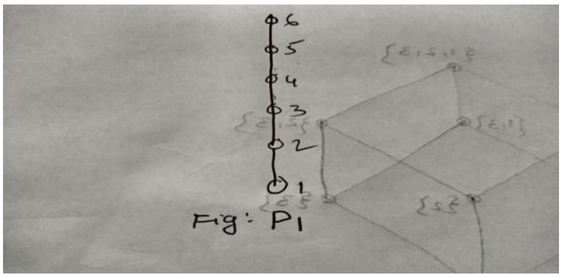
If no line connects a and b$\neq a$ then a and b are in comparable.

The resulting figure is the diagram of the partly ordered set P with respect to the given ordering and this is called the Hasse diagram of P.

Examples:

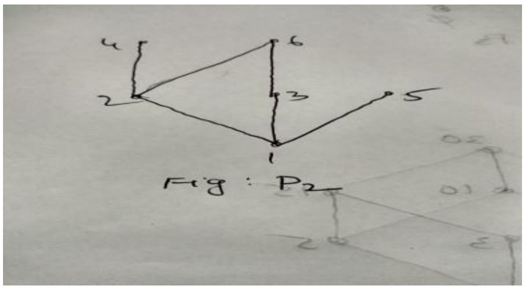
1. Let P_1= {1, 2, 3, 4, 5, 6} ordered by the usual relation "less than or equal to". Draw the Hasse diagram.

Sol:



2. Draw the Hasse diagram of P_2=={1, 2, 3, 4, 5, 6}ordered by divisibility i.e., a≤b $\Leftrightarrow a/b$

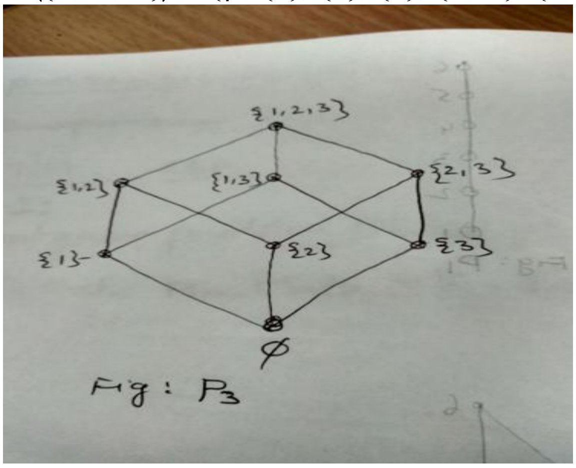
Sol:



3. Draw the Hasse diagram of the partial ordered set P_3=P ({1, 2, 3}) ordered by set inclusion.

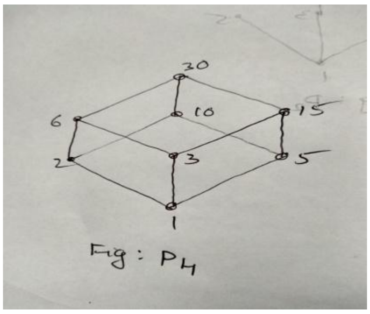
Sol: Given that P_3=P ({1, 2, 3})

P ({1, 2, 3}) = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}



4. Draw the Hasse diagram for the set of all divisors of 30 with respect to the usual divisibility.

Sol:



Review questions:

1. What is relation?
2. What is POSET?

Summary:

In this session, it was discussed that the definition of equivalence relation and then partial ordering relation and followed by partial order set in this session with suitable examples.

Self-assessment questions:

1. If A= {1, 2, 3}, B= {1, 4, 6, 9} and R is a relation from A to B defined by 'x is greater than y'. The range of R is -----------
   [c]

(a) {1, 4, 6, 9} (b) {4, 6, 9} (c) {1} (d) None of the above

2. Let L denote the set of all straight lines in a plane. Let a relation R be defined by lRm

iff l is parallel to m for all l, m $\in$L. Then R is --------
[d]

(a) Reflexive but not symmetric (b) only symmetric (c) only reflexive and transitive (d) equivalence relation

3. Given set A= {1, 2, 3} and relation R= {(1, 2), (2, 1)} the relation R Will be ------

[c]

(a) Reflexive (b) Irreflexive (c) Symmetric (d) above all

4. A relation R is said to be an equivalence relation----- [d]

(a) Reflexive (b) symmetric (c) transitive (d) above all

5. Which of the following is true about partial ordering relation [d]

(a) Reflexive (b) anti-symmetric (c) transitive (d) above all

Terminal questions

1. Let us consider that F is a relation on the set R real numbers that are defined by xFy on a condition if x-y is an integer. Prove F as an equivalence relation on R.
2. Examine whether the relation (x, y) ∈ R, if, x ≥ y defined on the set of +ve integers is a partial order relation.

Some multiple choice questions on Posets and Hasse diagrams:

1. Which of the following properties are satisfied by a partially ordered set?

   (a) Reflexive (b) Symmetric (c) anti-symmetric (d) Transitive

(A) a, b only (B) a, b, c only (C) all a, b, c, d (D) None of these.

2. Which of the following is not a partially ordered set?

   (A) The real numbers ordered by the standard *less-than-or-equal* relation ≤

   (B) The set of subsets of a given set (its power set) ordered by inclusion.

   (C) The set of natural numbers equipped with the relation of divisibility.

   (D) None of the above

3. Divisors of 30 are

(A) 1, 2, 3, 5, 6, 10, 15, 30 (B) 1, 2, 3, 5 (C) 6, 10, 15, 30 (D) None

4. In a Hasse diagram elements in posets are denoted by the following

   (A) a small circle (B) a small dot (c) a small sqare (D) None

5. If the set S containing 4 elements then the number of elements in the power set of S are

(A) 4 (B) 8 (C) 16 (D) 3

Answers: 1-D, 2-D, 3-A, 4-A, 5-C

S-6 Classroom Delivery Problems

1. Is the relation “greater than or equal to” is a partial ordering on the set of integers? Justify.
2. Is the divisibility relation ‘/’ is a partial ordering on the set of positive integers? Justify.
3. Verify the set inclusion ⊆ is a partial ordering on the power set of a set S.
4. Let m be a positive integer. A relation R is defined on the set Z by “aRb if and only if a – b is divisible by m” for a, b ∈ Z. Show that R is an equivalence relation on set Z.
5. Prove that the relation of similarity with respect to a set of triangles T, is an equivalence relation.
6. Examine/Verify that the relation “<” is a partial ordering on Natural number set N.
7. Examine whether the relation (x, y) ∈ R, if, x ≥ y defined on the set of +ve integers is a partial order relation.
8. Let P= {1, 2, 3, 4, 5, 6} be ordered by the usual relation “less than or equal to”. Draw the Hasse diagram.
9. Draw the Hasse diagram for divisibility on the set {1,2,3,6,12,24,36,48}
10. Draw the Hasse diagram for divisibility on the set {2, 3, 5, 9, 12, 15, 18}.

S-6 Tutorial problems:

1. Provide an example of partial ordering on R is the set of real numbers.
2. Determine the relation ⊆ is a partial ordering on the power set of a set A, where

A= {a, b, c}.

3. A set of integers, a relation R is defined by xRy if and only if x-y is divisible by 4, then verify R is an equivalence relation.

4. Verify the relation R is an equivalence relation on the set A = {1, 2, 3, 4, 5} given by the relation R = {(a, b) / |a-b| is even}.
5. Determine whether the relation 'divides' defined on N is a partial order relation.
6. Draw the Hasse diagram of P= {1, 2, 3, 4, 5, 6} ordered by divisibility i.e.,$a \le b \Leftrightarrow a/b$
7. Draw the Hasse diagram of Partial ordered se P=P ({1, 2, 3}) ordered by set inclusion
8. Draw the Hasse diagram for the set of all divisors of 30 with respect to the usual divisibility.

Home Assignment Problems

1. Let R be the relation on the set of integers such that aRb if and only if a = b or a = -b. Verify that the relation R is an equivalence relation.
2. Prove that the relation R is an equivalence type in the set S= { 3, 4, 5,6 } given by the relation R = { (p, q):|p-q| is even }.
3. Consider A = {2, 3, 4, 5} and R = {(5, 5), (5, 3), (2, 2), (2, 4), (3, 5), (3, 3), (4, 2), (4, 4)}. Examine that R is equivalence relation or not.
4. Let us consider that F is a relation on the set R real numbers that are defined by xFy on a condition if x-y is an integer. Prove F as an equivalence relation on R.
5. Examine "a is an ancestor of b" is a partial order relation on the set of all people (provided each person is an ancestor of himself/herself).
6. Let S is set of all real numbers, and "=" is the relation.

Is the relation a) reflexive, b) symmetric, c) anti-symmetric, d) transitive, e) an equivalence relation, f) a partial order.

7. Draw the Hasse diagram of poset ((P, ≤)) where (P = {1, 2, 3, 4, 6, 12}) and the relation (≤) is divisibility.
8. Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, /)
9. Draw Hasse diagram for (D_{12}, /)
10. Draw Hasse diagram for (D45, /)

Session-8 (Lattices)

Instructional Objective:

5. To learn definition and examples of lattices
6. To learn how to construct Hasse diagrams for lattices

Learning Outcomes:

To learn lattices and construction of Hasse diagrams

Introduction:

In this topic we give some basic definitions of lower bound and upper bound, lub, glb, lattice definition and construction of hasse diagrams and examples for each definition.

Explanation:

Lower and Upper bounds:

Definition: Let R be a nonempty subset of a partly ordered se P. An element a$\in P$ is called an upper bound of R, if x$\leq a \forall x \in R$. If R has at least one upper bound then we say that R is bounded above in P.

Definition: An upper bound a of R is said to be a least upper bound of R if, for any upper bound b of R a$\leq b$ (i.e. least among the upper bounds). The least upper bound of R in P is denoted by $lub_P R$ or $Sup_P R$ (supremum of R).

Note: Any sub set R of P has at most one least upper bound in P.

Definition: Let R be a nonempty sub set of a partly ordered set P. An element a$\in P$ is called a lower bound of R if a≤x $\forall x \in R$. If R has at least one lower bound then we say that R is bounded below in P.

Definition: A lower bound a of R is said to be a greatest lower bound of R, for any lower bound b of R b≤a (i.e. greatest among the lower bounds). The greatest lower bound of R is denoted by $g.l.b._P R$ or $Inf_P R$ (infimum of R) or simply g.l.b.R or inf R.

Note: R can have almost one g.l.b in P.

Definition: An element a$\in P$ is said to be a least element of P if a≤x for all x in P.

Clearly P has at most one least element and when it exists, it is denoted by 0 (zero). Clearly 0=inf P.

An element a$\in P$ is said to be a greatest element of P if x≤a for all x in P.

Clearly P has at most one greatest element and when it exists, it is denoted by 1 (one) and clearly 1 SupP.

Definition: Let (P, ≤) be a partially ordered set and a, b∈P such that a<b. We say that a is covered by b or b covers a if there is no element x∈P such that a<x<b.

Definition of Lattice: A lattice is a Partially ordered set (L, ≤) in which every pair of elements has a least upper bound (l.u.b) and a greatest lower bound (g.l.b).

The least upper bound (or supremum, l.u.b) of a,b∈L is denoted by a⊕b, a$\vee b, a \cup bor a + b$ and is called the join or sum of a and b.

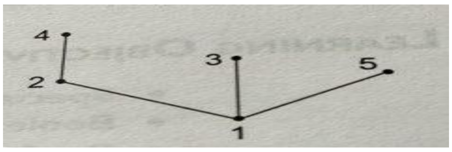
The greatest lower bound (or infimum, g.l.b) of a, b∈L is denoted by a*b, a$\wedge b, a \cap bor a.b$ and is called the meet or product of a and b.

Note 3: $\vee$ *and* $\wedge$ are binary operations on a lattice, since the least upper bound and the greatest lower bound of any subset of a poset are unique.

Note 4: All Partially ordered sets are not lattices.

Example (1): Determine whether the following posets are lattices (i) ({1, 2, 3, 4, 5}, /) and (ii) ({1, 2, 4, 8, 16}, /).

Solution: Consider the poset ({1, 2, 3, 4, 5}, /).

The corresponding Hasse diagram is shown in Fig.1 

There is no upper bound for the pairs(2, 3) and (3, 5). Hence, the least upper bound does not exist.

This implies that ({1, 2, 3, 4, 5}, /) is not a lattice.

So we can say that all the partially ordered sets are not lattices.

(ii) Consider the poset ({1, 2, 4, 8, 16}, /). The corresponding Hasse diagram is shown in Fig.2

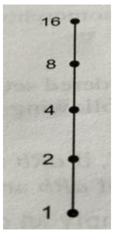


Fig.2

l.u.b.of {1,2}=2 g.l.b.{1,2}=1

l.u.b. of {2, 4} =4 g.l.b. {2, 4}=2

l.u.b. of {4, 8} =8 g.l.b. {4, 8} =4

l.u.b. of {2, 8} =8 g.l.b. {2, 8} =2

l.u.b. of {8, 16} =16 g.l.b. {8, 16}=8

l.u.b. of {4, 16}=16 g.l.b. {4,16}=4

l.u.b. of {2, 16} =16 g.l.b. {2,16}=2

l.u.b. of {1, 4} =4 g.l.b. {1,4}=1

l.u.b. of {1, 8} =8 g.l.b. {1, 8}=1

l.u.b. of {1, 16} =16 g.l.b. {1, 16}=1

Every pair of elements of the poset has a least upper bound and a greatest lower bound.

Hence, the poset ({1, 2, 4, 8, 16}, /) is a lattice.

Example (2): Determine whether the posets represented by each of the Hasse diagrams in Fig.3 (a) to (c) are lattices.

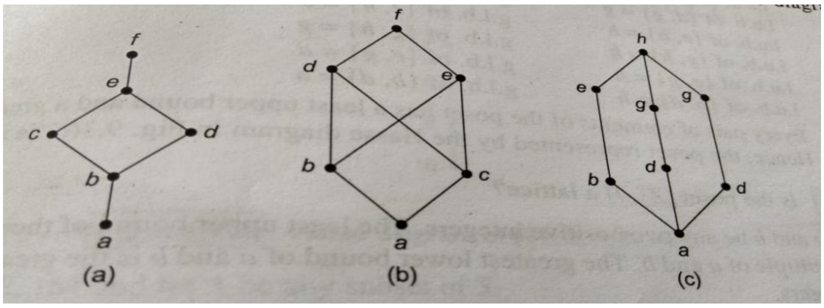


Fig.3

Solution: (i) In Fig.3 (a):

l.u.b.of {a, b} =b	g.l.b. of {a, b} =a
l.u.b.of {b, c} =c	g.l.b. of {b, c} =b
l.u.b.of {b, d} =d	g.l.b. of {b, d} =b
l.u.b.of {c, e} =e	g.l.b. of {c, e} =c
l.u.b.of {d, e} =e	g.l.b. of {d, e} =d
l.u.b.of {b, e} =e	g.l.b. of {b, e} =b
l.u.b.of {e, f} =f	g.l.b. of {e, f} =e
l.u.b.of {c,d} =e	g.l.b. of {c, d} =b

Every pair of elements of this poset has a least upper bound and a greatest lower bound.

Thus, the poset represented by the Hasse diagram in Fig.3(a) is a lattice.

(ii) Consider Fig.3 (b):

l.u.b. of {a, b}=b

l.u.b. of {a, c}=c

l.u.b. of {b, d}=d

l.u.b. of {c, e}=e

l.u.b. of {b, e}=e

l.u.b. of {c, d}=d

l.u.b. of {d, f}=f

l.u.b. of {e, f}=f

l.u.b. of {b, c}=d, e

l.u.b. of {d, e}=f

l.u.b. of {a, f}=f

Since the pair of elements {b, c}does not have a least upper bound, the poset given in Fig.3(b) is not a lattice

(iii)Consider the poset given in Fig.3(c):

l.u.b. of {a, b}=b	g.l.b.of {a, b}=a
l.u.b. of {a, d}=d	g.l.b.of {a, d}=a
l.u.b. of {b, e}=e	g.l.b.of {b, e}=b
l.u.b. of {d, g}=g	g.l.b.of {d, g}=d
l.u.b. of {e, h}=h	g.l.b.of {e, h}=e
l.u.b. of {g, h}=h	g.l.b.of {g, h}=g
l.u.b. of {e, g}=h	g.l.b.of {e, g}=a
l.u.b. of {b, d}=h	g.l.b.of {b, d}=a

Every pair of elements of the poset has a least upper bound and a greatest lower bound.

Hence, the poset represented by the Hasse diagram in Fig.3(c) is a lattice.

Example (3): Is the poset (Z^+, /) a lattice?

Solution: Let a and b be any two positive integers. The least upper bound of these two integers is the least common multiple of a and b. The greatest lower bound of a and b is the greatest common divisor of these two integers.

For example, consider the set {2, 4, 6}.An integer is a lower bound of {2, 4, 6} if 2, 4 and 6 are divisible by this integer. The only such integers are 1 and 2. The greatest lower bound of {2, 4, 6} is 2. An integer is an upper bound of {2, 4, 6} if it is divisible by 2, 4 and 6.

Integers with this property are those divisible by the least common multiple of 2, 4 and 6 is 12

Thus, 12 is the least upper bound of {2, 4, 6}

Hence, in Z^+, every pair of elements has a least upper bound and a greatest lower bound.

Therefore, (Z^+, /) is a lattice.

Example (4): If P(S) is the power set of a set S and $\cup$ *and* $\cap$ are taken as the join and meet, then prove that {P(S), $\subseteq$} is a lattice.

Solution: Let A and B be any two elements of P(S), that is, A and B are any two subsets of S.

Then $A\subseteq A \cup B and B \subseteq A\cup B. \Longrightarrow A\cup B$is an upper bound of {A, B}.

We assume that $A\subseteq C and B \subseteq C, then A\cup B \subseteq C.$

Thus, the least upper bound of {A, B}=$A\cup B$.

Similarly $A\cap B \subseteq A and A\cap B \subseteq B \Rightarrow A \cap B$ is a lower bound of {A, B}

If we assume that $C \subseteq A and C \subseteq B, then C \subseteq A \cap B$.

Thus the greatest lower bound of {A, B}= A$\cap B$.

Since every pair of elements of P(S) has a least upper bound and a greatest lower bound under the relation $\subseteq$, {P(S), $\subseteq$} is a lattice.

Principle of Duality:

Any statement about lattices involving the operations $\wedge$ *and* $\vee$ and the relations $\leq$ *and* $\geq$ remains true if $\wedge$ is replaced by $\vee$ and $\vee$ is replaced by $\wedge$, $\leq$ *by* $\geq$ *and* $\geq$ *by* $\leq$.

The lattices (L, $\leq$) *and* $(L, \geq)$are called duals of each other. Similarly, the operations $\vee$ *and* $\wedge$ are duals of each other and the relations $\leq$ *and* $\geq$ are duals of each other.

Example (5): Determine whether the principle of duality is followed in the poset ({1, 2, 4, 8, 16},·).

Solution: Consider the Hasse diagram given in Fig.4 for the poset ({1, 2, 4, 8, 16}, ·).

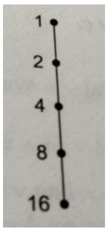


Fig.4

Let S= {1, 2, 4, 8, 16} and let A be any subset of S.

Then, the least upper bound of A with respect to $\leq$ is equal to the greatest lower bound of A with respect to $\geq$ and vice versa.

Thus, when $\leq$ *and* $\geq$ are interchanged, the least upper bound and the greatest lower bound are interchanged.

If (L, $\leq$) is a lattice, then (L, $\geq$) is also a lattice. Also, the operations of join and meet on (L, $\leq$) become the operations of meet and join respectively, on (L, $\geq$).

Thus, the principle of duality is followed in the poset ({1, 2, 4, 8, 16},·).

Properties of Lattices:

In this section we will discuss some important properties of lattices.

1. **Idempotent Law**: Let (L,$\leq$) be a lattice and a$\in$L. Then a$\vee a = a and a \wedge a = a$
2. **Commutative Law**: Let (L,$\leq$) be a lattice and a,b $\in$L. Then a$\vee a = b \vee a and a \wedge b = b \wedge a$.
3. **Associative Law**: Let (L,$\leq$) be a lattice and a,b, c $\in$L. Then, a$\vee (b \vee c) = (a \vee b) \vee c and a \wedge (b \wedge c) = (a \wedge b) \wedge c$
4. **Absorption Law**: Let (L,$\leq$) be a lattice and a,b, c $\in$L. Then, a$\vee (a \wedge b) = a and a \wedge (a \vee b) = a$.
5. Let (L,$\leq$) be a lattice in which $\vee$ *and* $\wedge$denotethe operations of join and meet, respectively. Then for any a,b $\in$L, a$\leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$.
6. **Isotonic Property**: If (L,$\leq$) is a lattice, then for any a,b,c $\in$ L, the following properties hold good:

b≤c ⟹ (i) a∨b≤a ∨ c and (ii) a ∧ b≤ **a ∧ c**

7. **Distributive Inequalities: Let (L,** ≤) be a lattice. Then for any a, b, c ∈ L, the following inequalities hold good:
   (i) **a ∨ (b∧c) ≤ (a ∨b) ∧(a ∨c)**
   (ii) **a ∧ (b∨c) ≥ (a ∧ b) ∨ (a ∧ c)**
8. **Modular Inequality**: If (L, ≤) is a lattice then for any a, b, c∈ $L, a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Example (6): Let (L, ≤) be a lattice and a, b, c∈ L. If a≤c, b≤c, then prove that a∨ $b \leq c and a \wedge b \leq c$

Solution: we assume that a≤c and b≤c. Therefore, c is an upper bound of {a, b}

(i) Now**,** a∨ b = l.u.b.of {a, b}, hence a∨ $b \leq c$
(ii) Also, a∧ b = g.l.b. of {a, b}, therefore, a∧ $b \leq a and a \wedge b \leq b$

⟹a∧ $b \leq c and a \wedge b \leq c$ (by our assumption a≤c and b≤c). Therefore, $a \wedge b \leq c$

Lattice as algebraic systems: A lattice may be defined as an algebraic system (L, ∨, ∧)with two binary operations ∨ *and* ∧ on L, which satisfy the commutative, associative and absorption laws.

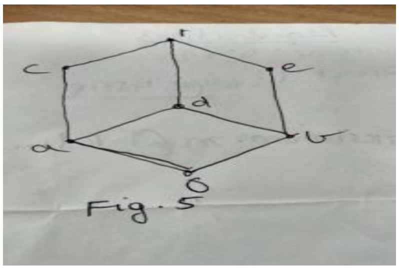
Diagrams of Lattices:

Example (7): Draw the diagram of the lattice for the following table Let L= {0, a, b, c, d, e, 1}, where 0∧ $x = 0, 0 \vee x = x, 1 \wedge x = x, 1 \vee x = 1$ *and*

∧	a	b	c	d	e
a	a	0	a	a	0
b	0	b	0	b	b
c	a	0	c	a	0
d	a	b	a	d	b
e	0	b	0	b	e

∨	a	b	c	d	e
a	a	d	c	d	1
b	d	b	1	d	e
c	c	1	c	1	1
d	d	d	1	d	1
e	1	e	1	1	e

Sol: Then this lattice can be represented by the diagram

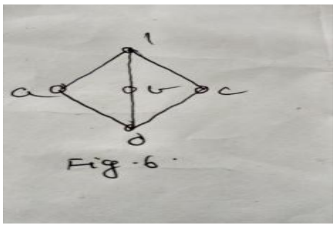


Example (8): Draw the diagram of the lattice for the following table, L= {0, a, b, c, 1} 0<a, b, c<1

∧	a	b	c
a	a	0	0
b	0	b	0
c	0	0	c

∨	a	b	c
a	a	1	1
b	1	b	1
c	1	1	c

Sol: Then this lattice can be represented by the diagram



LATTICES (Self learning)

Definition: A lattice is a special type of poset in which every pair of elements has:

- A least upper bound (join): The join of a and b, denoted by a ∨ b is the least element greater than or equal to both a and b.
- A greatest lower bound (meet): The meet of a and b, denoted by a ∧ b, is the greatest element less than or equal to both a and b.

This means you can always find a unique join and meet for any two elements in the set.

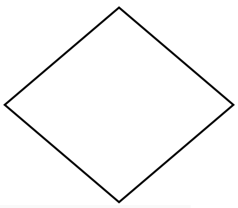
Definition: A Poset (P, ≤ *) is called a lattice if every pair of elements a & b in P has both least upper bound (lub) and greatest lower bound (glb).*

i.e., a lattice is a poset in which any two elements have both lub and glb.

lub{a, b}=a∨ b (join of a and b)

glb{a, b}= a∧ b (meet of a and b)

a∨ b



b

a∧ b

Note: Every chain is a lattice

Example of Lattices:

***Example**: 1*

The set of integers with the divisibility relation (where a ≤ b if a divides b) is a lattice.

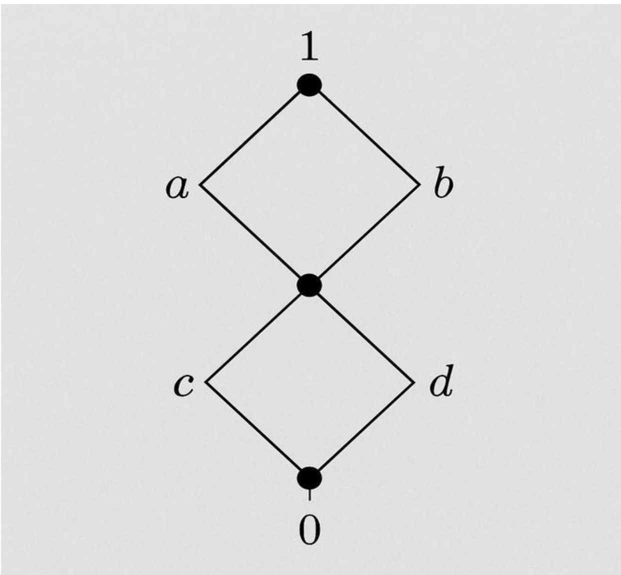
Example: 2
Consider the set L = {1,2,3,6} with the divisibility relation:

- Join: The join of 2 and 3 is 6 since 6 is the smallest number that is divisible by both 2 and 3.
- Meet: The meet of 2 and 6 is 2 since 2 is the largest number that divides both 2 and 6.

The set L with this order is a lattice.

Concept	Key Properties	Example
Partial Order	Reflexive, antisymmetric, transitive	Subset relation on sets
Lattice	Partial order + each pair has join and meet	Power set ordered by inclusion

The concept of lattice and poset are explained in detail with the help of example below:
This is a Hasse diagram of a lattice, a type of partially ordered set (poset) where every pair of elements has a least upper bound (join) and a greatest lower bound (meet).



Hasse Diagram

Elements

- The nodes labeled 0, c, d, middle point, a, b, and 1 represent elements in the poset.
- The bottom element (0) is the least element everything else is above it.
- The top element (1) is the greatest element it is above all others.

Explanation

- An upward path from one node to another means the lower one is less than or equal to the higher one (according to the partial order).

- For example:
    - c and d are both above 0, so 0 ≤ c and 0 ≤ d.
    - The middle node (unlabeled in the diagram but representing the join of c and d or meet of a and b) is above both c and d and below both a and b.
    - 1 is above a and b, so a ≤ 1 and b ≤ 1.

Meet and Join Examples

- Join (least upper bound) of c and d is the middle node.
- Meet (greatest lower bound) of a and b is the same middle node.

This structure shows how any two elements (e.g., c and d, or a and b) have both a meet and a join, which is what makes the set a lattice.

Applications in Engineering

Task Scheduling

- Partial orders are used to model dependencies among tasks.
- Tasks that must occur in a specific sequence are represented as partially ordered sets, enabling efficient scheduling and parallel execution.

Data Structures

- Lattices play a role in the design and optimization of data structures such as:
    - Search trees
    - Heaps
- They help maintain order and support efficient data retrieval and insertion.

Database Theory

- Partial orders and lattices are foundational in:
    - Query optimization determining the most efficient way to execute a query.
    - Schema design modeling hierarchies and constraints within relational databases.

Formal Verification

- Used in formal methods to ensure system correctness.
- Especially useful in concurrent systems, where events may not have a total order.
- Partial orders represent event causality and enable reasoning about different execution paths.

Network Design

- Communication networks benefit from partial order and lattice theory for:
    - Routing optimization
    - Dependency analysis
    - Resource allocation

Review Questions:

6. What is the definition of lattice?
7. How to construct the hasse diagrams for lattices?

Summary: Student will be able to learn lattice definition and construction of hasse diagrams for lattices.

Self-assessment questions:

1. How many least elements in a Poset?
   (A) 0 (B) 1 (C) 2 (D) 3
2. How many greatest elements in a Poset?
   (A) 0 (B) 1 (C) 2 (D) 3
3. Which is the least element in a partially ordered set?
   (A) 0 (B) 1 (C) 2 (D) 3
4. Which is the greatest element in a partially ordered set?
   (A) 0 (B) 1 (C) 2 (D) 3
5. Which of the following is not true?
   (A) Every Lattice is a Partially ordered set
   (B) Every Poset is a Lattice
   (C) The least upper bound and the greatest lower bound of any subset of a poset are unique.
   (D) A lattice is a Partially ordered set $(L, \leq)$ in which every pair of elements has a least upper bound (l.u.b) and a greatest lower bound (g.l.b).

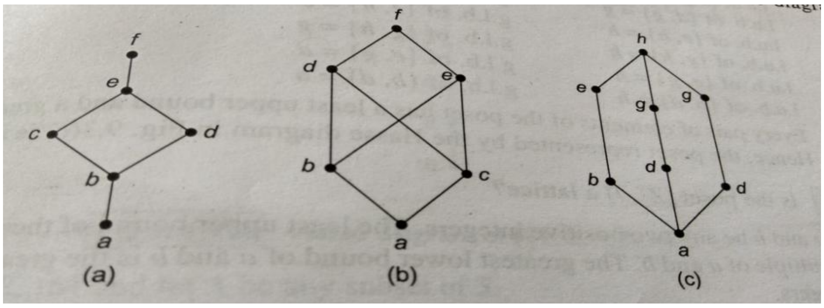
Answers: 1-B, 2-B, 3-A, 4-B, 5-B

Terminal Questions:

Classroom Delivery Problems

1. Determine whether the following posets are lattices (i) ({1, 2, 3, 4, 5}, /) and (ii) ({1, 2, 4, 8, 16}, /).

2. Determine whether the posets represented by each of the Hasse diagrams in (a) to (c) are lattices.



3. Is the poset $(Z^+, /)$ a lattice?

4. If P(S) is the power set of a set S and ∪ *and* ∩ are taken as the join and meet, then prove that {P(S), ⊆} is a lattice.

5. Draw the diagram of the lattice for the following table, L= {0, a, b, c, 1} 0<a, b, c<1

∧	a	b	c
a	a	0	0
b	0	b	0
c	0	0	c

∨	a	b	c
a	a	1	1
b	1	b	1
c	1	1	c

Tutorial Problems

1. Determine whether the following Posets are lattices (i) ({1, 2, 3, 4, 5}, /) and

   (ii) ({1, 2, 4, 8, 16}, /).

2. Is the poset (Z^+, /) a lattice?
3. If P(S) is the power set of a set S and ∪ *and* ∩ are taken as the join and meet, then prove that {P(S), ⊆} is a lattice.

4. Let L= {0, a, b, c, d, e, 1}, where $0 \wedge x = 0, 0 \vee x = x, 1 \wedge x = x, 1 \vee x = 1$ *and* draw the diagram from the following tables.

∧	a	b	c	d	e
a	a	0	a	a	0
b	0	b	0	b	b
c	a	0	c	a	0
d	a	b	a	d	b
e	0	b	0	b	e

∨	a	b	c	d	e
a	a	d	c	d	1
b	d	b	1	d	e
c	c	1	c	1	1
d	d	d	1	d	1
e	1	e	1	1	e

Home Assignment Problems

1. In the poset (Z^+, /), are the integers 3 and 9 comparable? Are 5 and 7 comparable?
2. Provide an example of a sub lattice.
3. Determine whether the poset {(1, 5, 25, 125), /} is a lattice.
4. Draw the diagram of the lattice for the following table L={0, a, b, c, 1} 0<a, b, c<1

∧	a	b	c
a	a	0	0
b	0	b	0
c	0	0	c

∨	a	b	c
a	a	1	1
b	1	b	1
c	1	1	c

5. Let (L, ≤) be a lattice and a, b, c∈ L. If a≤c, b≤c, then prove that a$\vee b \leq c$ and $a \wedge b \leq c$.