Discrete Mathematics

Section 6: Relations

Questions

1. Define a relation R from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mRn \Leftrightarrow m - n$ is divisible by 3.
   (a) For each of the following pairs (m, n), determine whether mRn holds. Explain your reasoning: $(6, 3)$, $(4, 1)$, $(5, 2)$, $(9, 0)$, $(101, 98)$, and $(27, 10)$.
   (b) List five integers that are related by R to 4.
   (c) Prove that if n is any multiple of 3, then $nR0$.

2. Define a relation S from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mSn \Leftrightarrow m + n$ is odd.
   (a) For each of the following pairs (m, n), determine whether mSn holds. Explain your reasoning: $(3, 4)$, $(2, 2)$, $(7, 1)$, $(5, 6)$, $(10, 3)$, and $(8, 8)$.
   (b) List five integers that are related by S to 5.
   (c) Prove that if n is any odd integer, then $nS0$.

3. Define a relation T from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mTn \Leftrightarrow m^2 = n^2$.
   (a) For each of the following pairs (m, n), determine whether mTn holds. Explain your reasoning: $(4, 4)$, $(2, -2)$, $(3, -3)$, $(1, -1)$, $(5, -5)$, and $(6, -6)$.
   (b) List five integers that are related by T to 9.
   (c) Prove that if n is any integer, then $nT(-n)$.

4. Define a relation U from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mUn \Leftrightarrow m \leq n$.
   (a) For each of the following pairs (m, n), determine whether mUn holds. Explain your reasoning: $(2, 3)$, $(4, 4)$, $(5, 1)$, and $(0, 2)$.
   (b) List five integers that are related by U to 2.
   (c) Prove that if n is any integer, then nUn.

5. Define a relation V from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mVn \Leftrightarrow m \cdot n$ is even.
   (a) For each of the following pairs (m, n), determine whether mVn holds. Explain your reasoning: $(2, 4)$, $(3, 5)$, $(6, 1)$, and $(7, 2)$.
   (b) List five integers that are related by V to 3.
   (c) Prove that if n is any even integer, then $nV1$.

6. Define a relation R on $\mathbb{Z} \times \mathbb{Z}$ as follows: For all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$, $(a, b)R(c, d) \Leftrightarrow a + d = b + c$.
   (a) Is $(1, 2)R(3, 0)$? Is $(4, 5)R(6, 7)$? Is $(0, 0)R(0, 0)$? Is $(2, 3)R(5, 1)$?
   (b) List five pairs of integers that are related by R to $(2, 3)$.
   (c) Show that R is an equivalence relation.
   (d) Determine the equivalence class of $(2, 3)$ under R knowing that $(2, 3) = (c, d)$.
   (e) How many distinct equivalence classes does R have on $\mathbb{Z} \times \mathbb{Z}$?

7. Define a relation M on $\mathbb{R}$ as follows: For all $x, y \in \mathbb{R}$, $xMy \Leftrightarrow x - y \in \mathbb{Q}$.

   (a) For each of the following pairs (x, y), determine whether xMy holds. Explain your reasoning: $(2, 2.5)$, $(\sqrt{2}, 1)$, $(1, 0)$, $(\pi, 3.141)$ (where 3.141 approximates π).

   (b) Show that M is an equivalence relation.

8. Define a relation U on $\mathbb{Z}$ as follows: For all $m, n \in \mathbb{Z}$, $mUn \Leftrightarrow$ the sum of the digits of m equals the sum of the digits of n.

   (a) For each of the following pairs (m, n), determine whether mUn holds. Explain your reasoning: $(14, 5)$, $(23, 32)$, $(19, 91)$, $(100, 1)$, $(58, 85)$, and $(123, 321)$.

   (b) List five integers that are related by U to 18.

   (c) Prove that U is an equivalence relation.

   (d) Describe the equivalence class of 28 under U.

   (e) What is the equivalence class of 81 under U?

9. Define a relation R on $\mathbb{N}$ as follows: For all $a, b \in \mathbb{N}$, $aRb \Leftrightarrow a^b > b^a$.

   (a) For each of the following pairs (a, b), determine whether aRb holds. Explain your reasoning: $(2, 4)$, $(3, 2)$, $(4, 5)$, $(2, 5)$, and $(3, 3)$.

   (b) Determine whether the relation R is an equivalence relation. Justify your answer by checking reflexivity, symmetry, and transitivity.

   (c) List four pairs of integers where aRb holds. Explain your reasoning in detail for each pair.

   (d) Explore whether R holds for values where a is a prime number. Justify whether the relation holds or not for these values.

10. Define a relation R on $\mathbb{Z}$ as follows: For all $a, b \in \mathbb{Z}$, $aRb \Leftrightarrow a^2 + b^2$ is a perfect square (i.e., the sum of their squares is a perfect square).

   (a) Is $3R4$? Is $5R12$? Is $7R24$? Is $0R5$?

   (b) List integers that are related by R to 1.

   (c) Determine whether the relation R on $\mathbb{Z}$, defined by $aRb \Leftrightarrow a^2 + b^2$ is a perfect square, is an equivalence relation.

   (d) Find all integers related to 5 under the relation R on $\mathbb{Z}$ defined by $aRb \Leftrightarrow a^2 + b^2$ is a perfect square.

11. Let $A = \{1, 2, 4\}$ and $B = \{2, 4, 8\}$ and let R be the "less than" relation from A to B: For all $(x, y) \in A \times B$, $xRy \Leftrightarrow x < y$.

   (a) State explicitly which ordered pairs are in R and R^{-1}.

   (b) Describe R^{-1} in words.

12. Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ and let R be the relation defined by $xRy \Leftrightarrow (x + y)$ is even and $(x - y)$ is divisible by 3. Note that x and y must be distinct, meaning pairs like (x, x) do not satisfy the relation.

   (a) List all the ordered pairs in the relation R.

   (b) Find the equivalence classes of the elements of A under the relation R.

13. Let $A = \{a, b, c, d, e, f\}$ and let R be the relation defined by

$$R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d), (e, e), (e, a), (a, e), (f, f), (f, a), (a, f)\}.$$

   (a) Find the equivalence classes of the elements of A under the relation R.

   (b) Is R an equivalence relation? Justify your answer.

14. Let $A = \{1, 2, 3, 4, 5, 6\}$ and define a relation R on A by $xRy \Leftrightarrow x$ and y have the same parity (i.e., both are either even or odd). Note that $x = y$ is valid in this relation, as every element has the same parity as itself, so pairs like (x, x) are included in R.
    (a) List all the ordered pairs in the relation R.
    (b) Determine whether the relation R is an equivalence relation. If it is, explain why and list all the equivalence classes. If it is not, explain which properties of an equivalence relation it fails to satisfy.
15. Let $A = \{1, 2, 3, 4\}$ and let R be the relation defined by $R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 4)\}$.
    (a) Find the equivalence classes of the elements of A under the relation R.
    (b) Is R an equivalence relation? Justify your answer.
16. Define a relation S on $\mathbb{Z}$ as follows: For all $a, b \in \mathbb{Z}$, $aSb \Leftrightarrow a - b$ is even.
    (a) Prove that S is reflexive.
    (b) Prove that S is symmetric.
    (c) Prove that S is transitive.
17. Define a relation T on $\mathbb{R}$ as follows: For all $x, y \in \mathbb{R}$, $xTy \Leftrightarrow x^2 \leq y^2$.
    (a) Is T reflexive?
    (b) Is T symmetric?
    (c) Determine if T is transitive and justify your answer.
18. If R and S are reflexive, is $R \cap S$ reflexive? Is $R \cup S$ reflexive? Why or why not?
19. Assume that R is a relation on a set A, and that R is reflexive, symmetric, and transitive. Prove or disprove whether R^{-1} is reflexive, symmetric, and transitive.
    (a) Is R^{-1} reflexive?
    (b) Is R^{-1} symmetric?
    (c) Is R^{-1} transitive?
20. Consider the set $A = \{1, 2, 3, 4, 5\}$ and define a relation R on A as follows:
    $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 4), (4, 5)\}$.
    Determine whether the relation R is reflexive, symmetric, and transitive.
    (a) Is R reflexive?
    (b) Is R symmetric?
    (c) Is R transitive?
21. Consider the set $A = \{1, 2, 3, 4, 5, 6\}$, and define a relation R on A based on the condition that aRb if and only if the absolute difference between a and b is less than or equal to 2.
    (a) List all the ordered pairs (a, b) that satisfy the relation R on A.
    (b) Determine whether the relation R is an equivalence relation. Justify your answer.

22. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A based on the following pairs:

$$R = \left\{ \begin{array}{l} (1,1), (1,2), (2,2), (2,3), (3,1), (3,3), (4,4), (4,5), (5,5), (5,6), \\ (6,6), (6,4), (7,7), (7,8), (8,7), (8,8), (9,9), (1,4), (4,1) \end{array} \right\}$$

   (a) Is the relation R reflexive? Prove your answer.

   (b) Is the relation R symmetric? Prove your answer and provide at least two counterexamples if R is not symmetric.

23. Let $C = \{1, 2, 3, 4, 5\}$. Define a relation T on C as follows: For all $a, b \in C$, $aTb \Leftrightarrow a$ and b are relatively prime (i.e., their greatest common divisor is 1).

   (a) Show that T is symmetric.

   (b) Show that T is not transitive.

24. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A as follows: $aRb \Leftrightarrow$ both a and b are divisible by 2 or neither is divisible by 2. Further, consider a set $B \subseteq A$ such that $B = \{x \in A \mid x \text{ is prime}\}$.

   (a) Prove whether R is an equivalence relation on A. Justify your answer with respect to reflexivity, symmetry, and transitivity.

   (b) Define the set $S = A \cap B$ and find the equivalence classes of the relation R restricted to S.

25. Let $C = \{6, 10, 15, 20, 25, 30, 35\}$ and define a relation T on C as follows: $aTb \Leftrightarrow$ the greatest common divisor (gcd) of a and b is greater than 1. Let $P(x)$ be a predicate that returns true if x is divisible by 5.

   (a) Prove whether T is an equivalence relation on C.

   (b) Consider the predicate $P(x)$, which returns true if x is divisible by 5. Construct the truth table for $P(x)$ for all elements in C, and explain whether $P(x)$ induces a partition on C.

Questions and Answers

1. Define a relation R from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mRn \Leftrightarrow m - n$ is divisible by 3.

   (a) For each of the following pairs (m, n), determine whether mRn holds. Explain your reasoning: $(6, 3)$, $(4, 1)$, $(5, 2)$, $(9, 0)$, $(101, 98)$, and $(27, 10)$.
   - $6R3$: $6 - 3 = 3$, which is divisible by 3. Thus, $6R3$ is true.
   - $4R1$: $4 - 1 = 3$, which is divisible by 3. Thus, $4R1$ is true.
   - $5R2$: $5 - 2 = 3$, which is divisible by 3. Thus, $5R2$ is true.
   - $9R0$: $9 - 0 = 9$, which is divisible by 3. Thus, $9R0$ is true.
   - $101R98$: $101 - 98 = 3$, which is divisible by 3. Thus, $101R98$ is true.
   - $27R10$: $27 - 10 = 17$, which is not divisible by 3. Thus, $27R10$ is false.

   (b) List five integers that are related by R to 4.
   Five integers related by R to 4 are: $1, 7, 10, -2, 13$, because $4 - 1 = 3$, $7 - 4 = 3$, $10 - 4 = 6$, $4 - (-2) = 6$, and $13 - 4 = 9$, all of which are divisible by 3.

   (c) Prove that if n is any multiple of 3, then $nR0$.
   If n is a multiple of 3, we can write $n = 3k$ for some integer k. Then $n - 0 = 3k - 0 = 3k$, which is divisible by 3. Therefore, $nR0$ for any multiple of 3.

2. Define a relation S from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mSn \Leftrightarrow m + n$ is odd.

   (a) For each of the following pairs (m, n), determine whether mSn holds. Explain your reasoning: $(3, 4)$, $(2, 2)$, $(7, 1)$, $(5, 6)$, $(10, 3)$, and $(8, 8)$.
   - $3S4$: $3 + 4 = 7$, which is odd. Thus, $3S4$ is true.
   - $2S2$: $2 + 2 = 4$, which is even. Thus, $2S2$ is false.
   - $7S1$: $7 + 1 = 8$, which is even. Thus, $7S1$ is false.
   - $5S6$: $5 + 6 = 11$, which is odd. Thus, $5S6$ is true.
   - $10S3$: $10 + 3 = 13$, which is odd. Thus, $10S3$ is true.
   - $8S8$: $8 + 8 = 16$, which is even. Thus, $8S8$ is false.

   (b) List five integers that are related by S to 5.
   Five integers related by S to 5 are: $6, 4, 8, 2, 10$, because $5 + 6 = 11$, $5 + 4 = 9$, $5 + 8 = 13$, $5 + 2 = 7$, and $5 + 10 = 15$ are all odd sums.

   (c) Prove that if n is any odd integer, then $nS0$.
   If n is odd, we can write $n = 2k + 1$ for some integer k. Then $n + 0 = 2k + 1 + 0 = 2k + 1$, which is odd. Therefore, $nS0$ for any odd integer n.

3. Define a relation T from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mTn \Leftrightarrow m^2 = n^2$.

   (a) For each of the following pairs (m, n), determine whether mTn holds. Explain your reasoning: $(4, 4)$, $(2, -2)$, $(3, -3)$, $(1, -1)$, $(5, -5)$, and $(6, -6)$.
   - $4T4$: $4^2 = 16$ and $4^2 = 16$. Thus, $4T4$ is true.
   - $2T-2$: $2^2 = 4$ and $(-2)^2 = 4$. Thus, $2T-2$ is true.
   - $3T-3$: $3^2 = 9$ and $(-3)^2 = 9$. Thus, $3T-3$ is true.
   - $1T-1$: $1^2 = 1$ and $(-1)^2 = 1$. Thus, $1T-1$ is true.
   - $5T-5$: $5^2 = 25$ and $(-5)^2 = 25$. Thus, $5T-5$ is true.
   - $6T-6$: $6^2 = 36$ and $(-6)^2 = 36$. Thus, $6T-6$ is true.

   (b) List five integers that are related by T to 9.
   Five integers related by T to 9 are: $9, -9, 3, -3, 0$, because $9^2 = 81$, $(-9)^2 = 81$, $3^2 = 9$, $(-3)^2 = 9$, and $0^2 = 0$.

   (c) Prove that if n is any integer, then $nT(-n)$.
   If n is any integer, then $n^2 = (-n)^2$, since squaring both n and $-n$ gives the same result. Therefore, $nT(-n)$ for any integer n.

4. Define a relation U from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mUn \Leftrightarrow m \leq n$.

   (a) For each of the following pairs (m, n), determine whether mUn holds. Explain your reasoning: $(2, 3)$, $(4, 4)$, $(5, 1)$, and $(0, 2)$.
   - $2U3$: $2 \leq 3$. Thus, $2U3$ is true.
   - $4U4$: $4 \leq 4$. Thus, $4U4$ is true.
   - $5U1$: $5 \leq 1$ is false. Thus, $5U1$ is false.
   - $0U2$: $0 \leq 2$. Thus, $0U2$ is true.

   (b) List five integers that are related by U to 2.
   Five integers related by U to 2 are: $0, 1, 2, -1, -2$, because $0 \leq 2$, $1 \leq 2$, $2 \leq 2$, $-1 \leq 2$, and $-2 \leq 2$.

   (c) Prove that if n is any integer, then nUn.
   For any integer n, we have $n \leq n$, which is always true. Therefore, nUn for any integer n.

5. Define a relation V from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $mVn \Leftrightarrow m \cdot n$ is even.

   (a) For each of the following pairs (m, n), determine whether mVn holds. Explain your reasoning: $(2, 4)$, $(3, 5)$, $(6, 1)$, and $(7, 2)$.
   - $2V4$: $2 \cdot 4 = 8$, which is even. Thus, $2V4$ is true.
   - $3V5$: $3 \cdot 5 = 15$, which is odd. Thus, $3V5$ is false.
   - $6V1$: $6 \cdot 1 = 6$, which is even. Thus, $6V1$ is true.
   - $7V2$: $7 \cdot 2 = 14$, which is even. Thus, $7V2$ is true.

   (b) List five integers that are related by V to 3.
   Five integers related by V to 3 are: $2, 4, 6, 8, 10$ because $3 \cdot 2 = 6$, $3 \cdot 4 = 12$, $3 \cdot 6 = 18$, $3 \cdot 8 = 24$, and $3 \cdot 10 = 30$ are all even.

   (c) Prove that if n is any even integer, then $nV1$.
   If n is any even integer, we can write $n = 2k$ for some integer k. Then, $n \cdot 1 = 2k \cdot 1 = 2k$, which is even. Thus, $nV1$ for any even integer n.

6. Define a relation R on $\mathbb{Z} \times \mathbb{Z}$ as follows: For all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$, $(a, b)R(c, d) \Leftrightarrow a + d = b + c$.

   (a) Is $(1, 2)R(3, 0)$? Is $(4, 5)R(6, 7)$? Is $(0, 0)R(0, 0)$? Is $(2, 3)R(5, 1)$?
   - $(1, 2)R(3, 0)$: $1 + 0 = 1$ and $2 + 3 = 5$. Since $1 \neq 5$, $(1, 2)R(3, 0)$ is false.
   - $(4, 5)R(6, 7)$: $4 + 7 = 11$ and $5 + 6 = 11$. Since $11 = 11$, $(4, 5)R(6, 7)$ is true.
   - $(0, 0)R(0, 0)$: $0 + 0 = 0$ and $0 + 0 = 0$. Since $0 = 0$, $(0, 0)R(0, 0)$ is true.
   - $(2, 3)R(5, 1)$: $2 + 1 = 3$ and $3 + 5 = 8$. Since $3 \neq 8$, $(2, 3)R(5, 1)$ is false.

   (b) List five pairs of integers that are related by R to $(2, 3)$.
   Five pairs related by R to $(2, 3)$ are $(0, 1)$, $(1, 2)$, $(3, 4)$, $(4, 5)$, and $(5, 6)$.

   (c) Show that R is an equivalence relation.
   - **Reflexive:** For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $(a, b)R(a, b)$ because $a + b = a + b$, which is true.
   - **Symmetric:** If $(a, b)R(c, d)$, then $a + d = b + c$. This implies $c + b = d + a$, so $(c, d)R(a, b)$. Therefore, R is symmetric.
   - **Transitive:** If $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then $a+d = b+c$ and $c+f = d+e$. From these, we can conclude $a + f = b + e$. Therefore, $(a, b)R(e, f)$, which shows that R is transitive.

   (d) Determine the equivalence class of $(2, 3)$ under R knowing that $(2, 3) = (c, d)$.
   To determine the equivalence class of $(2, 3)$ under R, we find all pairs (a, b) such that $(a, b)R(2, 3)$. This means that $a + 3 = b + 2$, which simplifies to $a - b = -1$. Therefore, the equivalence class of $(2, 3)$ under R is the set of all pairs (a, b) such that $a - b = -1$. Examples include $(4, 5)$, $(2, 3)$, $(1, 2)$, $(0, 1)$, and $(-1, 0)$.

   (e) How many distinct equivalence classes does R have on $\mathbb{Z} \times \mathbb{Z}$?
   R has infinitely many distinct equivalence classes on $\mathbb{Z} \times \mathbb{Z}$, one for each possible sum $k \in \mathbb{Z}$, such as $a + b = k$.

7. Define a relation M on $\mathbb{R}$ as follows: For all $x, y \in \mathbb{R}$, $xMy \Leftrightarrow x - y \in \mathbb{Q}$.

   (a) For each of the following pairs (x, y), determine whether xMy holds. Explain your reasoning: $(2, 2.5)$, $(\sqrt{2}, 1)$, $(1, 0)$, $(\pi, 3.141)$ (where 3.141 approximates π).

   - $2M2.5$: $2 - 2.5 = -0.5$, which is rational. So, $2M2.5$ is true.
   - $\sqrt{2}M1$: $\sqrt{2} - 1$ is irrational. So, $\sqrt{2}M1$ is false.
   - $1M0$: $1 - 0 = 1$, which is rational. So, $1M0$ is true.
   - $\pi M3.141$: $\pi - 3.141$ is irrational. So, $\pi M3.141$ is false.

   (b) Show that M is an equivalence relation.

   - **Reflexive:** For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Q}$, so xMx.
   - **Symmetric:** If xMy, then $x - y \in \mathbb{Q}$. This implies $-(x - y) = y - x \in \mathbb{Q}$, so yMx. Therefore, M is symmetric.
   - **Transitive:** If xMy and yMz, then $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$. To determine if xMz holds, we need to find the difference $x - z$. Notice that $x - z$ can be expressed as the sum of $(x - y)$ and $(y - z)$, that is:
   $$x - z = (x - y) + (y - z)$$
   Since both $x - y$ and $y - z$ are in $\mathbb{Q}$ (they are rational numbers), their sum $(x - y) + (y - z)$ is also in $\mathbb{Q}$. Therefore, $x - z \in \mathbb{Q}$, which implies xMz. This shows that M is transitive because if x is related to y, and y is related to z, then x is also related to z.

8. Define a relation U on $\mathbb{Z}$ as follows: For all $m, n \in \mathbb{Z}$, $mUn \Leftrightarrow$ the sum of the digits of m equals the sum of the digits of n.

   (a) For each of the following pairs (m, n), determine whether mUn holds. Explain your reasoning: $(14, 5)$, $(23, 32)$, $(19, 91)$, $(100, 1)$, $(58, 85)$, and $(123, 321)$.

   - $14U5$: Sum of digits of 14 is $1 + 4 = 5$, and of 5 is 5. So, $14U5$ is true.
   - $23U32$: Sum of digits of 23 is $2 + 3 = 5$, and of 32 is $3 + 2 = 5$. So, $23U32$ is true.
   - $19U91$: Sum of digits of 19 is $1 + 9 = 10$, and of 91 is $9 + 1 = 10$. So, $19U91$ is true.
   - $100U1$: Sum of digits of 100 is $1 + 0 + 0 = 1$, and of 1 is 1. So, $100U1$ is true.
   - $58U85$: Sum of digits of 58 is $5 + 8 = 13$, and of 85 is $8 + 5 = 13$. So, $58U85$ is true.
   - $123U321$: Sum of digits of 123 is $1 + 2 + 3 = 6$, and of 321 is $3 + 2 + 1 = 6$. So, $123U321$ is true.

   (b) List five integers that are related by U to 18.
   Five integers related by U to 18 are 9, 27, 36, 45, and 81 (all of which have a digit sum of 9).

   (c) Prove that U is an equivalence relation.

   - **Reflexive:** For any $m \in \mathbb{Z}$, the sum of the digits of m equals itself, so mUm.
   - **Symmetric:** If mUn, then the sum of the digits of m equals the sum of the digits of n. Since equality is symmetric, nUm. Therefore, U is symmetric.
   - **Transitive:** If mUn and nUp, then the sum of the digits of m equals the sum of the digits of n, and the sum of the digits of n equals the sum of the digits of p. Therefore, the sum of the digits of m equals the sum of the digits of p, so mUp. Thus, U is transitive.

   (d) Describe the equivalence class of 28 under U.
   The equivalence class of 28 under U is the set $[28] = \{x \in \mathbb{Z}_{\geq 0} \mid \text{the sum of the digits of } x \text{ is } 10\}$, such as 19, 37, 46, 55, and 82

   (e) What is the equivalence class of 81 under U?
   The equivalence class of 81 under U is the set $[81] = \{x \in \mathbb{Z} \mid \text{the sum of the digits of } x \text{ is } 9\}$, such as 18, 27, 36, 45, and 90. This is because the sum of the digits of 81 is 9.

9. Define a relation R on $\mathbb{N}$ as follows: For all $a, b \in \mathbb{N}$, $aRb \Leftrightarrow a^b > b^a$.

(a) For each of the following pairs (a, b), determine whether aRb holds. Explain your reasoning: $(2, 4)$, $(3, 2)$, $(4, 5)$, $(2, 5)$, and $(3, 3)$.

- $(2, 4)$: $2^4 = 16$ and $4^2 = 16$, so $2^4 = 4^2$, and therefore $(2, 4)R$ does not hold.
- $(3, 2)$: $3^2 = 9$ and $2^3 = 8$, so $3^2 > 2^3$, and therefore $(3, 2)R$ holds.
- $(4, 5)$: $4^5 = 1024$ and $5^4 = 625$, so $4^5 > 5^4$, and therefore $(4, 5)R$ holds.
- $(2, 5)$: $2^5 = 32$ and $5^2 = 25$, so $2^5 > 5^2$, and therefore $(2, 5)R$ holds.
- $(3, 3)$: $3^3 = 27$ and $3^3 = 27$, so $3^3 = 3^3$, and therefore $(3, 3)R$ does not hold.

(b) Determine whether the relation R is an equivalence relation. Justify your answer by checking reflexivity, symmetry, and transitivity.

- **Reflexive:** R is reflexive if for all $a \in \mathbb{N}$, aRa. This requires $a^a > a^a$, which is clearly false for all a because no number is greater than itself. Therefore, R is **not reflexive**.
- **Symmetric:** R is symmetric if, for all $a, b \in \mathbb{N}$, aRb implies bRa. This requires $a^b > b^a$ to imply $b^a > a^b$. Consider the pair $(3, 2)$. We know that:
  - $3^2 = 9$ and $2^3 = 8$, so $3^2 > 2^3$, which means $3R2$ holds.
  - However, for symmetry, we would also need $2R3$, but $2^3 = 8$ and $3^2 = 9$, so $2^3 < 3^2$, meaning $2R3$ does not hold.

  Since aRb does not imply bRa in this case, R is **not symmetric**.
- **Transitive:** R is transitive if for all $a, b, c \in \mathbb{N}$, if aRb and bRc, then aRc. This requires that if $a^b > b^a$ and $b^c > c^b$, then $a^c > c^a$. Consider the numbers $a = 1$, $b = 2$, and $c = 4$. We have:
  - $1R2$: $1^2 = 1$ and $2^1 = 2$, but 1^2 is not greater than 2^1. Therefore, $1R2$ does not hold.
  - $2R4$: $2^4 = 16$ and $4^2 = 16$, so $2^4 = 4^2$, meaning $2R4$ does not hold.
  - $1R4$: $1^4 = 1$ and $4^1 = 4$, but 1^4 is not greater than 4^1. Therefore, $1R4$ does not hold.

  Since $1R2$, $2R4$, and $1R4$ do not hold, R is **not transitive**.
  Since R is not reflexive, symmetric, or transitive, it cannot be an equivalence relation.

(c) List four pairs of integers where aRb holds. Explain your reasoning in detail for each pair.
Four pairs where aRb holds are:

- $(2, 5)$: $2^5 = 32$ and $5^2 = 25$. Since $2^5 > 5^2$, aRb holds for $(2, 5)$.
- $(3, 4)$: $3^4 = 81$ and $4^3 = 64$. Since $3^4 > 4^3$, aRb holds for $(3, 4)$.
- $(4, 5)$: $4^5 = 1024$ and $5^4 = 625$. Since $4^5 > 5^4$, aRb holds for $(4, 5)$.
- $(5, 6)$: $5^6 = 15625$ and $6^5 = 7776$. Since $5^6 > 6^5$, aRb holds for $(5, 6)$.

In each case, we compute a^b and b^a and compare the results to determine whether aRb holds. We choose pairs such that $a^b > b^a$ is true, ensuring that the relation holds for these pairs.

(d) Explore whether R holds for values where a is a prime number. Justify whether the relation holds or not for these values.
Let's explore some prime numbers for a:

- $(3, 2)$: $3^2 = 9$ and $2^3 = 8$. Since $3^2 > 2^3$, R holds for $a = 3$.
- $(5, 3)$: $5^3 = 125$ and $3^5 = 243$. Since $3^5 > 5^3$, R does not hold for $a = 5$.
- $(7, 2)$: $7^2 = 49$ and $2^7 = 128$. Since $2^7 > 7^2$, R does not hold for $a = 7$.
- $(11, 4)$: $11^4 = 14,641$ and $4^{11} = 4,194,304$. Since $4^{11} > 11^4$, R does not hold for $a = 11$.

As seen in the examples, the relation R holds for some prime values of a, such as $a = 3$, but does not hold for others like $a = 5$, $a = 7$, and $a = 11$. The outcome depends on how quickly the powers grow relative to each other.

10. Define a relation R on $\mathbb{Z}$ as follows: For all $a, b \in \mathbb{Z}$, $aRb \Leftrightarrow a^2 + b^2$ is a perfect square (i.e., the sum of their squares is a perfect square).

(a) Is $3R4$? Is $5R12$? Is $7R24$? Is $0R5$?

- $3^2 + 4^2 = 9 + 16 = 25$, which is a perfect square (5^2), so $3R4$ is true.
- $5^2 + 12^2 = 25 + 144 = 169$, which is a perfect square (13^2), so $5R12$ is true.
- $7^2 + 24^2 = 49 + 576 = 625$, which is a perfect square (25^2), so $7R24$ is true.
- $0^2 + 5^2 = 0 + 25 = 25$, which is a perfect square (5^2), so $0R5$ is true.

(b) List integers that are related by R to 1.
Only the integer 0 is related by R to 1, since $1^2 + 0^2 = 1$, which is a perfect square.

(c) Determine whether the relation R on $\mathbb{Z}$, defined by $aRb \Leftrightarrow a^2 + b^2$ is a perfect square, is an equivalence relation.

- **Reflexive:** For R to be reflexive, every integer a must satisfy aRa. This means $a^2 + a^2 = 2a^2$ must be a perfect square. We observe that $2a^2$ is a perfect square only when $a = 0$. To illustrate that R cannot be reflexive for all a, consider any non-zero integer, say $a = 1$. In this case, $2 \times 1^2 = 2$, which is not a perfect square. Hence, R is **not reflexive**.
- **Symmetric:** The relation R is symmetric if for all $a, b \in \mathbb{Z}$, whenever aRb is true, bRa is also true. Since $a^2 + b^2 = b^2 + a^2$ (addition is commutative), if aRb, then bRa. Therefore, R is **symmetric**.
- **Transitive:** The relation R is transitive if whenever aRb and bRc, then aRc. Consider the integers $a = 0$, $b = 1$, and $c = 0$. We have:
  - $0R1$: $0^2 + 1^2 = 1$, which is a perfect square, so $0R1$ is true.
  - $1R0$: $1^2 + 0^2 = 1$, which is a perfect square, so $1R0$ is true.
  - $0R0$: $0^2 + 0^2 = 0$, which is a perfect square, so $0R0$ is true.

  While the example above holds, consider the integers $a = 1$, $b = 0$, and $c = 2$. We have:
  - $1R0$ is true because $1^2 + 0^2 = 1$.
  - $0R2$ is true because $0^2 + 2^2 = 4$.
  - $1R2$ is false because $1^2 + 2^2 = 5$, which is not a perfect square.

  Therefore, R is **not transitive**.
  Conclusion: R is not an equivalence relation since R is symmetric but not reflexive and not transitive.

(d) Find all integers related to 5 under the relation R on $\mathbb{Z}$ defined by $aRb \Leftrightarrow a^2 + b^2$ is a perfect square.
To find the integers related to 5 under R, we need to solve the equation $5^2 + a^2 = k^2$ for some integer k. This becomes:

$$25 + a^2 = k^2 \quad \Rightarrow \quad k^2 - a^2 = 25 \quad \Rightarrow \quad (k - a)(k + a) = 25.$$

The expression $k^2 - a^2 = (k - a)(k + a)$ comes from factoring the difference of squares. Now, we need to consider the integer pairs whose product equals 25. The possible factorizations of 25 are:

$$(1, 25), (-1, -25), (5, 5), (-5, -5), (25, 1), (-25, -1).$$

We now solve for k and a in each case:

- $k - a = 1$ and $k + a = 25$: Adding these two equations, we get $2k = 26$, so $k = 13$. Substituting $k = 13$ into $k - a = 1$ gives $13 - a = 1$, so $a = 12$.
- $k - a = 25$ and $k + a = 1$: Adding these, we get $2k = 26$, so $k = 13$. Substituting $k = 13$ into $k - a = 25$ gives $13 - a = 25$, so $a = -12$.
- $k - a = 5$ and $k + a = 5$: Adding these, we get $2k = 10$, so $k = 5$. Substituting $k = 5$ into $k - a = 5$ gives $5 - a = 5$, so $a = 0$.

Thus, the integers related to 5 under R are: {0, 12, -12}.

11. Let $A = \{1, 2, 4\}$ and $B = \{2, 4, 8\}$ and let R be the "less than" relation from A to B: For all $(x, y) \in A \times B$, $xRy \Leftrightarrow x < y$.

 (a) State explicitly which ordered pairs are in R and R^{-1}.
 In R: $\{(1, 2), (1, 4), (1, 8), (2, 4), (2, 8), (4, 8)\}$.
 In R^{-1}: $\{(2, 1), (4, 1), (4, 2), (8, 1), (8, 2), (8, 4)\}$.

 (b) Describe R^{-1} in words.
 The inverse relation R^{-1} is formed by swapping the elements of each pair in R. Specifically, R^{-1} contains the pairs: $(2, 1)$, $(4, 1)$, $(4, 2)$, $(8, 1)$, $(8, 2)$, $(8, 4)$.

12. Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ and let R be the relation defined by $xRy \Leftrightarrow (x + y)$ is even and $(x - y)$ is divisible by 3. Note that x and y must be distinct, meaning pairs like (x, x) do not satisfy the relation.

 (a) List all the ordered pairs in the relation R.
 The relation R contains the following ordered pairs:

$$\{(0, 6), (6, 0)\}.$$

 These pairs satisfy both conditions: the sum is even, and the difference is divisible by 3.

 (b) Find the equivalence classes of the elements of A under the relation R.
 The equivalence classes under the relation R are:

$$[0] = \{0, 6\}, \quad [6] = \{0, 6\}.$$

 The equivalence classes are formed by grouping elements that are related under R. For example, 0 and 6 are related because $0 + 6$ is even, and $0 - 6$ is divisible by 3.

13. Let $A = \{a, b, c, d, e, f\}$ and let R be the relation defined by

$$R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d), (e, e), (e, a), (a, e), (f, f), (f, a), (a, f)\}.$$

 (a) Find the equivalence classes of the elements of A under the relation R.
 The equivalence classes under the relation R are:
 $[a] = \{a, e, f\}$ $\quad [b] = [d] = \{b, d\}$ $\quad [c] = \{c\}$ $\quad [e] = \{a, e\}$ $\quad [f] = \{a, f\}$.

 (b) Is R an equivalence relation? Justify your answer.
 Yes, R is an equivalence relation because it satisfies the properties of reflexivity, symmetry, and transitivity.

 - **Reflexive:** A relation is reflexive if every element in A is related to itself. In R, we have the pairs (a, a), (b, b), (c, c), (d, d), (e, e), and (f, f), meaning that each distinct element of $A = a, b, c, d, e, f$ is related to itself. Therefore, we conclude that R is reflexive.
 - **Symmetric:** A relation is symmetric if whenever $(x, y) \in R$, then $(y, x) \in R$ as well. In R, we have the pairs (b, d) and (d, b), (a, e) and (e, a), and (a, f) and (f, a), showing that the relation is symmetric between these elements. Other pairs like (a, a), (b, b), (c, c), (d, d), (e, e), and (f, f) are symmetric by definition. Therefore, R is symmetric.
 - **Transitive:** A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. In R, we can see several examples of transitivity:
   - Since $(a, e) \in R$ and $(e, a) \in R$, we must have (a, a) in R, which is true.
   - Similarly, since $(b, d) \in R$ and $(d, b) \in R$, we must have (b, b), which is present.
   - Also, (f, a) and (a, f) imply that (f, f) must be in R, and indeed it is.

 Thus, since R satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation.

14. Let $A = \{1, 2, 3, 4, 5, 6\}$ and define a relation R on A by $xRy \Leftrightarrow x$ and y have the same parity (i.e., both are either even or odd). Note that $x = y$ is valid in this relation, as every element has the same parity as itself, so pairs like (x, x) are included in R.

   (a) List all the ordered pairs in the relation R.
   The relation R is given by:
   $$R = \left\{ \begin{array}{l} (1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5), \\ (4,2),(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6) \end{array} \right\}.$$
   These pairs represent all elements of A where x and y have the same parity (both even or both odd).

   (b) Determine whether the relation R is an equivalence relation. If it is, explain why and list all the equivalence classes. If it is not, explain which properties of an equivalence relation it fails to satisfy.
   The relation R is an equivalence relation because it satisfies all three properties required for an equivalence relation:
   - **Reflexive:** Every element in A is related to itself, since $(x, x) \in R$ for all $x \in A$. This holds because each element has the same parity as itself.
   - **Symmetric:** If x is related to y (i.e., they have the same parity), then y is also related to x. In other words, if $(x, y) \in R$, then $(y, x) \in R$.
   - **Transitive:** If x is related to y and y is related to z (i.e., they all have the same parity), then x is also related to z. So if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

   Since R is an equivalence relation, the equivalence classes are as follows:
   $$\begin{aligned} [1] &= \{1, 3, 5\} && \text{(all odd numbers)} \\ [2] &= \{2, 4, 6\} && \text{(all even numbers)} \\ [3] &= \{1, 3, 5\} && \text{(since 3 is odd)} \\ [4] &= \{2, 4, 6\} && \text{(since 4 is even)} \\ [5] &= \{1, 3, 5\} && \text{(since 5 is odd)} \\ [6] &= \{2, 4, 6\} && \text{(since 6 is even)} \end{aligned}$$
   These equivalence classes partition the set A into two disjoint subsets: one class containing all the odd numbers and the other containing all the even numbers. This demonstrates how the relation based on parity naturally divides the set into these two subsets.

15. Let $A = \{1, 2, 3, 4\}$ and let R be the relation defined by $R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 4)\}$.

   (a) Find the equivalence classes of the elements of A under the relation R.
   The equivalence classes are: $[1] = \{1, 3\}$ $\quad [2] = \{2\}$ $\quad [3] = \{1, 3\}$ $\quad [4] = \{4\}$.

   (b) Is R an equivalence relation? Justify your answer.
   Yes, R is an equivalence relation because it satisfies the properties of reflexivity, symmetry, and transitivity:
   - **Reflexive:** A relation is reflexive if every element in the set is related to itself. In R, we have the pairs $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$, which shows that every element of $A = \{1, 2, 3, 4\}$ is related to itself. Therefore, R is reflexive.
   - **Symmetric:** A relation is symmetric if whenever $(x, y) \in R$, then $(y, x) \in R$. In R, we have the pairs $(1, 3)$ and $(3, 1)$, which shows that the relation is symmetric between 1 and 3. For all other pairs, such as $(1, 1)$, $(2, 2)$, and $(4, 4)$, symmetry is trivially satisfied because these elements are related to themselves. Therefore, R is symmetric.
   - **Transitive:** A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. In R, we have $(1, 3)$ and $(3, 1)$, so transitivity holds because $(1, 1) \in R$. For the other elements, transitivity is trivially satisfied because there are no further pairs to check. Therefore, R is transitive.

   Since R satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation.

16. Define a relation S on $\mathbb{Z}$ as follows: For all $a, b \in \mathbb{Z}$, $aSb \Leftrightarrow a - b$ is even.

   (a) Prove that S is reflexive.
   To prove S is reflexive, we need to show that aSa for all $a \in \mathbb{Z}$. Since $a - a = 0$ and 0 is even, aSa is true for all $a \in \mathbb{Z}$. Therefore, S is reflexive.

   (b) Prove that S is symmetric.
   To prove S is symmetric, we need to show that if aSb, then bSa. If aSb, then $a - b$ is even. This means $a - b = 2k$ for some integer k. Then $b - a = -2k$, which is also even. Therefore, bSa is true, and S is symmetric.

   (c) Prove that S is transitive.
   To prove that S is transitive, we need to show that if aSb and bSc, then aSc.
   Assume aSb and bSc. This means, by the definition of the relation S, that $a-b$ is even and $b-c$ is even. In mathematical terms, there exist integers k and m such that $a-b = 2k$ and $b-c = 2m$. Adding the equations:
   $$a - b + b - c = 2k + 2m$$
   Simplifying, we get:
   $$a - c = 2(k + m)$$
   Since $k+m$ is an integer, this shows that $a-c$ is an even number, and therefore, by the definition of the relation S, we have aSc. Since we have shown that if aSb and bSc, then aSc, this proves that the relation S is transitive.

17. Define a relation T on $\mathbb{R}$ as follows: For all $x, y \in \mathbb{R}$, $xTy \Leftrightarrow x^2 \leq y^2$.

   (a) Is T reflexive?
   To prove that T is reflexive, we must show that xTx for all $x \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have $x^2 \leq x^2$, which is true for all real numbers. Therefore, xTx holds for all $x \in \mathbb{R}$. Hence, T is reflexive.

   (b) Is T symmetric?
   To determine if T is symmetric, we must check whether xTy implies yTx for all $x, y \in \mathbb{R}$. Assume xTy, which means $x^2 \leq y^2$. Symmetry would require that $y^2 \leq x^2$ as well. However, this is not necessarily true. For example, if $x = 1$ and $y = 2$, then $x^2 = 1^2 = 1 \leq 2^2 = 4$, but $2^2 \not\leq 1^2$. Therefore, T is not symmetric.

   (c) Determine if T is transitive and justify your answer.
   To determine whether T is transitive, we must check if xTy and yTz imply xTz for all $x, y, z \in \mathbb{R}$. Assume xTy and yTz. By the definition of T, this means $x^2 \leq y^2$ and $y^2 \leq z^2$. Since inequality is transitive, we can conclude that $x^2 \leq z^2$. Therefore, xTz holds, which shows that T is transitive.

18. If R and S are reflexive, is $R \cap S$ reflexive? Is $R \cup S$ reflexive? Why or why not?
   To determine that $R \cap S$ is reflexive, we must show that for all $x \in A$, $(x, x) \in R \cap S$.

   - Since R is reflexive, for every element $x \in A$, $(x, x) \in R$.
   - Similarly, since S is reflexive, for every element $x \in A$, $(x, x) \in S$.
   - The intersection $R \cap S$ contains all elements that are in both R and S. Therefore, for every $x \in A$, since $(x, x) \in R$ and $(x, x) \in S$, it follows that $(x, x) \in R \cap S$.

   Thus, for every element $x \in A$, $(x, x) \in R \cap S$, which proves that $R \cap S$ is reflexive.

   To determine whether $R \cup S$ is reflexive, we must check if for all $x \in A$, $(x, x) \in R \cup S$.

   - Since R is reflexive, $(x, x) \in R$ for all $x \in A$.
   - Similarly, since S is reflexive, $(x, x) \in S$ for all $x \in A$.
   - The union $R \cup S$ contains all elements that are in either R or S. Therefore, for every $x \in A$, since $(x, x) \in R$ and $(x, x) \in S$, it follows that $(x, x) \in R \cup S$.

   Thus, $R \cup S$ is reflexive because for all $x \in A$, $(x, x) \in R \cup S$.

19. Assume that R is a relation on a set A, and that R is reflexive, symmetric, and transitive. Prove or disprove whether R^{-1} is reflexive, symmetric, and transitive.

    (a) Is R^{-1} reflexive?
    Since R is reflexive, for all $a \in A$, $(a,a) \in R$. By the definition of R^{-1}, if $(a,a) \in R$, then $(a,a) \in R^{-1}$, since reversing the pair gives (a,a) again. Thus, if R is reflexive, R^{-1} is also reflexive.

    (b) Is R^{-1} symmetric?
    To determine if R^{-1} is symmetric, assume $(b,a) \in R^{-1}$. By the definition of R^{-1}, this means that $(a,b) \in R$. Since R is symmetric, $(a,b) \in R$ implies that $(b,a) \in R$. Therefore, by the definition of R^{-1}, this means $(a,b) \in R^{-1}$. Thus, R^{-1} is symmetric.

    (c) Is R^{-1} transitive?
    To determine if R^{-1} is transitive, assume $(b,a) \in R^{-1}$ and $(a,c) \in R^{-1}$. This means, by the definition of R^{-1}, that $(a,b) \in R$ and $(c,a) \in R$. Since R is transitive, $(c,a) \in R$ and $(a,b) \in R$ imply that $(c,b) \in R$. By the definition of R^{-1}, this means that $(b,c) \in R^{-1}$. Therefore, R^{-1} is transitive.

20. Consider the set $A = \{1,2,3,4,5\}$ and define a relation R on A as follows:
    $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,4),(4,5)\}$.
    Determine whether the relation R is reflexive, symmetric, and transitive.

    (a) Is R reflexive?
    A relation R is reflexive if for all $a \in A$, $(a,a) \in R$. In this case, R includes $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$, but it does not include $(5,5)$. Therefore, R is not reflexive because $(5,5) \notin R$.

    (b) Is R symmetric?
    A relation R is symmetric if for all $a,b \in A$, $(a,b) \in R$ implies $(b,a) \in R$. In this case, R includes $(1,2)$ and $(2,1)$, so symmetry holds for this pair. However, R includes $(2,3)$ but does not include $(3,2)$. Therefore, R is not symmetric.

    (c) Is R transitive?
    A relation R is transitive if for all $a,b,c \in A$, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$. In this case, $(2,3) \in R$ and $(3,4) \in R$, but $(2,4) \notin R$, violating transitivity. Therefore, R is not transitive.

21. Consider the set $A = \{1,2,3,4,5,6\}$, and define a relation R on A based on the condition that aRb if and only if the absolute difference between a and b is less than or equal to 2.

    (a) List all the ordered pairs (a,b) that satisfy the relation R on A.
    To satisfy aRb, we need $|a-b| \leq 2$. The ordered pairs that satisfy this condition are:
    $$\mathrm{R} = \left\{ \begin{array}{c} (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(3,5), \\ (4,2),(4,3),(4,4),(4,5),(4,6),(5,3),(5,4),(5,5),(5,6),(6,4),(6,5),(6,6) \end{array} \right\}.$$

    (b) Determine whether the relation R is an equivalence relation. Justify your answer.
    To determine if R is an equivalence relation, we need to check if it is reflexive, symmetric, and transitive.

    - **Reflexive**: The relation is reflexive because for all $a \in A$, we have $(a,a) \in R$. This is true for $1,2,3,4,5$, and 6.
    - **Symmetric**: The relation is symmetric because if $(a,b) \in R$, then $(b,a) \in R$. For example, if $(1,2) \in R$, then $(2,1) \in R$ as well. This holds for all pairs in the set.
    - **Transitive**: The relation is **not transitive**. For example, we have $(1,2) \in R$ and $(2,4) \in R$, but $(1,4) \notin R$, which violates transitivity. There are other examples as well, such as $(1,3) \in R$ and $(3,5) \in R$, but $(1,5) \notin R$.

    Since R is reflexive and symmetric but not transitive, it is **not** an equivalence relation.

22. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A based on the following pairs:

$$R = \left\{ \begin{array}{l} (1,1),(1,2),(2,2),(2,3),(3,1),(3,3),(4,4),(4,5),(5,5),(5,6), \\ (6,6),(6,4),(7,7),(7,8),(8,7),(8,8),(9,9),(1,4),(4,1) \end{array} \right\}$$

   (a) Is the relation R reflexive? Prove your answer.
   A relation R is reflexive if for all $a \in A$, $(a, a) \in R$. In this case, we observe that R contains $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8)$, and $(9,9)$. Therefore, the relation is reflexive because for every $a \in A$, we have $(a, a) \in R$.

   (b) Is the relation R symmetric? Prove your answer and provide at least two counterexamples if R is not symmetric.
   A relation R is symmetric if for all $a, b \in A$, $(a, b) \in R$ implies $(b, a) \in R$. In this case: We observe that $(1, 2) \in R$, but $(2, 1) \notin R$, which violates symmetry. We also observe that $(5, 6) \in R$, but $(6, 5) \in R$, so symmetry holds for this pair. Therefore, R is not symmetric because there exists at least one pair where $(a, b) \in R$ but $(b, a) \notin R$.

23. Let $C = \{1, 2, 3, 4, 5\}$. Define a relation T on C as follows: For all $a, b \in C$, $aTb \Leftrightarrow a$ and b are relatively prime (i.e., their greatest common divisor is 1).

   (a) Show that T is symmetric.
   To show that T is symmetric, we need to demonstrate that if aTb, then bTa. If aTb, then $\gcd(a, b) = 1$. Since $\gcd(b, a) = \gcd(a, b) = 1$, bTa is also true. Therefore, T is symmetric.

   (b) Show that T is not transitive.
   To show that T is not transitive, we need to find a counterexample where aTb and bTc are true, but aTc is false. Consider $a = 2$, $b = 3$, and $c = 4$. We have:

$$\gcd(2, 3) = 1 \quad \text{and} \quad \gcd(3, 4) = 1,$$

   but

$$\gcd(2, 4) = 2 \neq 1.$$

   Therefore, aTb and bTc do not imply aTc, so T is not transitive.

24. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A as follows: $aRb \Leftrightarrow$ both a and b are divisible by 2 or neither is divisible by 2. Further, consider a set $B \subseteq A$ such that $B = \{x \in A \mid x \text{ is prime}\}$.

   (a) Prove whether R is an equivalence relation on A. Justify your answer with respect to reflexivity, symmetry, and transitivity.
   Reflexivity: For all $a \in A$, either a is divisible by 2 or it is not. In either case, we have aRa. Therefore, R is reflexive.
   Symmetry: If aRb, then both a and b are either divisible by 2 or neither is divisible by 2. This implies that bRa. Therefore, R is symmetric.
   Transitivity: If aRb and bRc, then both a and b must either both be divisible by 2 or neither is divisible by 2, and the same applies to b and c. Hence, a and c must also either both be divisible by 2 or neither, implying that aRc. Thus, R is transitive.
   Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

   (b) Define the set $S = A \cap B$ and find the equivalence classes of the relation R restricted to S.
   The set $B = \{2, 3, 5, 7\}$ consists of all prime numbers in A. The intersection $S = A \cap B = \{2, 3, 5, 7\}$ contains these elements. Under the relation R, we need to check for pairs in S:

   - $[2]$ forms its own equivalence class because it is divisible by 2.
   - $[3, 5, 7]$ form another equivalence class because none of them are divisible by 2.

   Therefore, the equivalence classes are $[2]$ and $[3, 5, 7]$.

25. Let $C = \{6, 10, 15, 20, 25, 30, 35\}$ and define a relation T on C as follows: $aTb \Leftrightarrow$ the greatest common divisor (gcd) of a and b is greater than 1. Let $P(x)$ be a predicate that returns true if x is divisible by 5.

   (a) Prove whether T is an equivalence relation on C.
   Reflexivity: For all $a \in C$, $\gcd(a, a) = a > 1$, so aTa is true for all $a \in C$. Thus, T is reflexive.
   Symmetry: If aTb, then $\gcd(a, b) > 1$, and since gcd is symmetric, we also have $\gcd(b, a) > 1$, so bTa. Thus, T is symmetric.
   Transitivity: If aTb and bTc, then $\gcd(a, b) > 1$ and $\gcd(b, c) > 1$. However, this does not necessarily imply $\gcd(a, c) > 1$. For example, $\gcd(6, 10) = 2$, $\gcd(10, 35) = 5$, but $\gcd(6, 35) = 1$. Therefore, T is not transitive.
   Since T is reflexive and symmetric but not transitive, T is not an equivalence relation.

   (b) Consider the predicate $P(x)$, which returns true if x is divisible by 5. Construct the truth table for $P(x)$ for all elements in C, and explain whether $P(x)$ induces a partition on C.
   The truth table for $P(x)$ is as follows:

x	$P(x)$
6	false
10	true
15	true
20	true
25	true
30	true
35	true

   $P(x)$ induces a partition on C by dividing the set into elements that are divisible by 5 and those that are not. The partition is:

   - $\{10, 15, 20, 25, 30, 35\}$ (divisible by 5)
   - $\{6\}$ (not divisible by 5)

   Thus, $P(x)$ creates a valid partition on C.

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