Probability & Statistics Workbook Solutions

Hypothesis testing

krista king MATH

INFERENTIAL STATISTICS AND HYPOTHESES

■ 1. A current pain reliever has an 85% success rate of treating pain. A company develops a new pain reliever and wants to show that its success rate of treating pain is better than the current option. Decide if the hypothesis statement would require a population proportion or a population mean, then set up the statistical hypothesis statements for the situation.

Solution:

We're interested in finding out if the new pain reliever has a better success rate than the current one. Since we're given a percentage of success, we'll be using a population proportion p, instead of a population mean μ. And since we're looking at how much better the pain reliever will perform, we use the $>$ symbol in our alternative hypothesis, which means the null hypothesis has to have the $\leq$ symbol.

$$H_0 : p \leq 0.85$$

$$H_a : p > 0.85$$

■ 2. A research study on people who quit smoking wants to show that the average number of attempts to quit before a smoker is successful is less than 3.5 attempts. How should they set up their hypothesis statements?

Solution:

We're interested in finding out if the mean number of attempts is less than 3.5, so we'll be using a population mean μ. And since we're looking at whether the mean is less than 3.5, we use the $<$ symbol in our alternative hypothesis, which means the null hypothesis has to have the $\geq$ symbol.

$$H_0 : \mu \geq 3.5$$

$$H_a : \mu < 3.5$$

■ 3. A factory creates a small metal cylindrical part that later becomes part of a car engine. Because of variations in the process of manufacturing, the diameters are not always identical. The machine was calibrated to create cylinders with an average diameter of $1/16$ of an inch. During a periodic inspection, it became clear that further investigation was needed to determine whether or not the machine responsible for making the part needed recalibration. Write statistical hypothesis statements.

Solution:

The factory wants the mean diameter of the parts it produces to match the diameter that they need, $1/16$ of an inch. That means this is an example of a statistical hypothesis statement that uses the population mean.

Both parts that are too small or too large could create problems, which means the alternative hypothesis needs to have a $\neq$ sign. Which means the null hypothesis will include an $=$ sign.

$$H_0: \mu = \frac{1}{16}$$

$$H_a: \mu \neq \frac{1}{16}$$

■ 4. A marketing study for a clothing company concluded that the mean percentage increase in sales could potentially be over $17\,\%$ for creating a clothing line that focused on lime green and polka dots. Which hypothesis statements do they need to write in order to test their theory?

Solution:

The claim of the marketing study is that creating the clothing line that focuses on lime green and polka dots will increase sales by over $17\,\%$. Which means the alternative hypothesis would need to include the $>$ sign, and therefore that the null hypothesis has to include a $\leq$ sign.

$$H_0: \mu \leq 0.17$$

$$H_a: \mu > 0.17$$

5. A food company wants to ensure that less than $0.0001\ \%$ of its product is contaminated. Which hypothesis statements will it write if it wants to test for this?

Solution:

The food company wants the proportion of contaminated product to be less than $0.0001\ \%$, so they'll be using a population proportion p. And since they're looking at whether the proportion is less than $0.0001\ \%$, they'll use the $<$ symbol in the alternative hypothesis, which means the null hypothesis has to have the $\geq$ symbol.

$$H_0 : p \geq 0.0001\ \%$$

$$H_a : p < 0.0001\ \%$$

6. A new medication is being developed to prevent heart worms in dogs, and the developer wants it to work better than the current medication. The current medication prevents heart worms at a rate of $75\ \%$. What hypothesis statements should they write if they want to test whether or not the new medication works better than the existing one?

Solution:

The developer wants the proportion of dogs in which heart worm is prevented by their medication to be greater than 0.75, so they'll be using a population proportion p. And since we're looking at whether the proportion is greater than 0.75, we use the $>$ symbol in our alternative hypothesis, which means the null hypothesis has to have the $\leq$ symbol.

$$H_0 : p \leq 0.75$$

$$H_a : p > 0.75$$

SIGNIFICANCE LEVEL AND TYPE I AND II ERRORS

1. We're running a statistical test on a new pharmaceutical drug. The stakes are high, because the side effects of the drug could potentially be serious, or even fatal. If we want to reduce the Type I and Type II error rates as low as possible to avoid rejecting the null when it's true or accepting the null when it's false, what should we do when we take the sample?

Solution:

The only way to reduce both the Type I error rate and Type II error rate simultaneously is to increase the sample size. Therefore, if it's important that we reduce error rate as low as possible, we should take the largest possible sample.

2. If the probability of making a Type II error in a statistical test is 5%, what is the power of the test?

Solution:

The power of a statistical test is the probability that we'll reject the null hypothesis when it's false (make that particular correct choice).

	H_0 is true	H_0 is false
Reject H_0	Type I error P(Type I error)=alpha	**CORRECT** **Power**
Accept H_0	CORRECT	Type II error P(Type II error)=beta

Power is always equivalent to $1 - \beta$, and β is another name for Type II error rate. So

Power $= 1 - \beta$

Power $= 1 -$ Type II error rate

Power $= 1 - 0.05$

Power $= 0.95$

The power of the statistical test, given that the probability of making a Type II error is 5%, is Power $= 95\%$.

3. On average, professional golfers make 75% of putts within 5 feet. One golfer believes he does better than this, and wants to use a statistical test to see whether or not he's correct. Unbeknownst to him, in actuality this golfer makes 7 out of 10 of these kinds of putts. When he takes a sample of his putts, he finds $\hat{p} = 0.92$. What kind of error might he be in danger of making?

Solution:

The golfer's null and alternative hypotheses are

$$H_0 : p \leq 0.75$$

$$H_a : p > 0.75$$

In reality, his null hypothesis is true, but based on the sample proportion $\hat{p} = 0.92$, he may be in danger of rejecting the null when he shouldn't.

	H_0 is true	H_0 is false
Reject H_0	**Type I error** **P(Type I error)=alpha**	CORRECT
Accept H_0	CORRECT	Type II error P(Type II error)=beta

Which means the golfer is in danger of making a Type I error.

4. The average age of a guest at an amusement park is 15 years old. One amusement park believes the average age of their guests is younger than this, and wants to use a statistical test to see whether or not they're correct. Unbeknownst to them, in actuality the average guest age at this particular amusement park is 12 years old. When they take a sample of his guests, they find $\bar{x} = 16$ years. What kind of error might they be in danger of making?

Solution:

The park's null and alternative hypotheses are

$$H_0 : \mu \geq 15$$

$$H_a : \mu < 15$$

In reality, they're null hypothesis is false, but based on the sample mean $\bar{x} = 16$, they may be in danger of accepting the null when they shouldn't.

	H_0 is true	H_0 is false
Reject H_0	Type I error P(Type I error)=alpha	CORRECT
Accept H_0	CORRECT	**Type II error** **P(Type II error)=beta**

Which means the amusement park is in danger of making a Type II error.

5. Of all political donations, 70% come from corporations and lobbies, not from individual citizens. One politician believes he receives less than 70% of his own donations from corporations and lobbies, and wants to use a statistical test to see whether or not he's correct. Unbeknownst to him, in actuality the proportion of his donations that come from corporations and lobbies is 65%. When he takes a sample of his donations that come from corporations and lobbies, he finds $\hat{p} = 0.72$. What kind of error might he be in danger of making?

Solution:

The politician's null and alternative hypotheses are

$$H_0 : p \geq 0.7$$

$$H_a : p < 0.7$$

In reality, his null hypothesis is false, but based on the sample proportion $\hat{p} = 0.72$, he may be in danger of accepting the null when he shouldn't.

	H_0 is true	H_0 is false
Reject H_0	Type I error P(Type I error)=alpha	CORRECT
Accept H_0	CORRECT	**Type II error P(Type II error)=beta**

Which means the politician is in danger of making a Type II error.

6. A coffee shop owner believes that he sells 500 cups of coffee each day, on average, and he wants to test this assumption. The truth is, he actually sells fewer than 500 cups each day. He takes a random sample of 10 days and records the number of cups he sells each of those days. What kind of error is the coffee shop owner in danger of making?

Day	1	2	3	4	5	6	7	8	9	10
Cups sold	488	502	496	506	492	489	510	511	506	500

Solution:

The owner's null and alternative hypotheses are

$$H_0 : \mu = 500$$

$$H_a : \mu \neq 500$$

If we look at the data, we can see that the sample mean is $\bar{x} = 500$. In reality, his null hypothesis is false, but based on the sample mean $\bar{x} = 500$, he may be in danger of accepting the null when he shouldn't.

	H_0 is true	H_0 is false
Reject H_0	Type I error P(Type I error)=alpha	CORRECT
Accept H_0	CORRECT	**Type II error** **P(Type II error)=beta**

Which means the coffee shop owner is in danger of making a Type II error.

TEST STATISTICS FOR ONE- AND TWO-TAILED TESTS

■ 1. A local high school states that its students perform much better than average on a state exam. The average score for all high school students in the state is 106 points. A sample of 256 students at this particular school had an average test score of 129 points with a sample standard deviation of 26.8. Choose and calculate the appropriate test statistic.

Solution:

The sample is comparing average scores, which means the population parameter is a population mean (not a proportion) with an unknown standard deviation (since we have the sample standard deviation and not the population standard deviation).

The sample size is large enough at 256 high schoolers that we can assume the distribution is approximately normal. In this case, we use a t-test statistic.

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{129 - 106}{\frac{26.8}{\sqrt{256}}} \approx 13.73$$

■ 2. A dietician is looking into the claim at a local restaurant that the number of calories in its portion sizes is lower than the national average. The national average is $1{,}500$ calories per meal. She samples 35 meals at the restaurant and finds they contain an average of $1{,}250$ calories per meal with a sample standard deviation of 350.2. Choose and calculate the appropriate test statistic.

Solution:

The sample is comparing average number of calories, which means the population parameter is a population mean (not a proportion) with an unknown standard deviation (since we have the sample standard deviation and not the population standard deviation).

The sample size is large enough at 35 meals that we can assume the distribution is approximately normal. In this case, we use a t-test statistic.

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{1{,}250 - 1{,}500}{\frac{350.2}{\sqrt{35}}} \approx -4.22$$

3. In a recent survey, 567 out of a 768 randomly selected dog owners said they used a kennel that was run by their veterinary office to board their dogs while they were away on vacation. The study would like to make a conclusion that the majority (more than 50%) of dog owners use a kennel run by their veterinary office when the owners go on vacation. Choose and calculate the appropriate test statistic.

Solution:

The sample size is large enough at 768 randomly selected individuals that we can state the distribution is approximately normal. We can show this by using the checks for the population proportion, $np \geq 10$ and $n(1-p) \geq 10$. The sample size is $n = 768$ and the sample proportion is

$$\hat{p} = \frac{567}{768} \approx 0.738$$

Therefore,

$$n\hat{p} = (768)(0.738) \approx 567 \geq 10$$

$$n(1-\hat{p}) = (768)(1-0.738) \approx 201 \geq 10$$

So we can say that the test statistic will be the z-test statistic for a population proportion.

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.738 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{768}}} \approx 13.21$$

■ 4. We want to open a day care center, so we take a random sample of 500 households in our town with children under preschool age, and find that 243 of them were using a family member to care for those children. We want to determine if, at a statistically significant level, fewer than half of households in our town are using a family member to care for the kids.

1. Set up the hypothesis statements.

2. Check that the conditions for normality are met.

3. State the type of test: upper-tailed, lower-tailed, or two-tailed.

4. Calculate the test statistic using the appropriate formula.

Solution:

The hypothesis statements would be

$$H_0 : p \geq 0.5$$

$$H_a : p < 0.5$$

We need to see if we have an approximately normal distribution by using the checks for a population proportion. The sample size is from a simple random sample of $n = 500$ households. The proportion is the 243 out of the 500 households, so $\hat{p} = 243/500 = 0.486$.

$$n\hat{p} = (500)(0.486) = 243 \geq 10$$

$$n(1 - \hat{p}) = (500)(1 - 0.486) = 257 \geq 10$$

Because both values are greater than 10, the distribution is approximately normal. This is a lower-tailed test because the alternative hypothesis uses the $<$ sign.

This is a population proportion, so we'll calculate a z-test statistic for a population proportion.

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{0.486 - 0.50}{\sqrt{\frac{0.5(1 - 0.5)}{500}}} \approx -0.6261$$

5. The highest allowable amount of bromate in drinking water is 0.0100 mg/L^2. A survey of a city's water quality took 50 water samples in random locations around the city and found an average of 0.0102 mg/L^2 of bromate with a sample standard deviation of 0.0025 mg/L. The survey committee is interested in testing if the amount of bromate found in the water samples is higher than the allowable amount at a statistically significant level.

1. Set up the hypothesis statements.

2. Check that the conditions for normality are met.

3. State the type of test: upper-tailed, lower-tailed, or two-tailed.

4. Calculate the test statistic using the appropriate formula.

Solution:

The hypothesis statements would be

$$H_0 : \mu \leq 0.0100$$

$$H_a : \mu > 0.0100$$

The sample size is a simple random sample of 50 samples, so the distribution is approximately normal. This is an upper-tailed test because the alternative hypothesis uses the greater than sign.

This is a population mean with an unknown population standard deviation, so we'll calculate a t-test statistic with the population mean formula.

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{0.0102 - 0.0100}{\frac{0.0025}{\sqrt{50}}} \approx 0.5657$$

6. A farmer reads a study that states: The average weight of a day-old chick upon hatching is $\mu_0 = 38.60$ grams with a population standard deviation of $\sigma = 5.7$ grams. The farmer wants to see if her day-old chicks have the same average. She takes a simple random sample of 60 of her day-old chicks and finds their average weight is $\bar{x} = 39.1$ grams.

1. Set up the hypothesis statements.
2. Check that the conditions for normality are met.
3. State the type of test: upper-tailed, lower-tailed, or two-tailed.
4. Calculate the test statistic using the appropriate formula.

Solution:

The hypothesis statements would be

$$H_0 : \mu = 38.60$$

$$H_a : \mu \neq 38.60$$

The sample size is a simple random sample of 60 of her day-old chicks so we can say the distribution is approximately normal. This is a two-tailed test because the alternative hypothesis uses the $\neq$ sign.

This is a population mean with a known population standard deviation, so we'll calculate a z-test statistic with the population mean formula.

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{39.10 - 38.60}{\frac{5.7}{\sqrt{60}}} \approx 0.6795$$

THE P-VALUE AND REJECTING THE NULL

1. A medical trial is conducted to test whether or not a new medicine reduces total cholesterol, when the national average is 230 mg/dL with a standard deviation of 16 mg/dL. The trial takes a simple random sample of 223 adults who take the new medicine, and finds $\bar{x} = 227$ mg/dL. What can the trial conclude at a significance level of $\alpha = 0.01$?

Solution:

The hypothesis statements will be

$$H_0 : \mu \geq 230$$

$$H_a : \mu < 230$$

The test statistic will be

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{227 - 230}{\frac{16}{\sqrt{223}}} = -\frac{3\sqrt{223}}{16} \approx -2.80$$

From the z-table we get 0.0026.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026

Because this is a lower-tail test, the p-value is just this value we found, $p = 0.0026$. Comparing this to $\alpha = 0.01$, we can see that $p \leq \alpha$, which means we'll reject the null hypothesis.

Therefore, at a significance level of $\alpha = 0.01$, the trial can conclude that the new medicine reduces cholesterol. Because the p-value we found is even more significant, the trial could go even further, stating that the new medicine reduces cholesterol at a significance level of $p = 0.0026$.

■ 2. The national average length of pregnancy is 283.6 days with a population standard deviation of 10.5 days. A hospital wants to know if the average length of a pregnancy at their hospital deviates from the national average. They use a sample of $9{,}411$ births at the hospital to calculate a test statistic of $z = -1.60$. Set up the hypothesis statements and find the p-value.

Solution:

The hospital wants to know if mean length of pregnancy at their hospital is different than the national average in a significant way.

$$H_0 : \mu = 283.6$$

$$H_a : \mu \neq 283.6$$

Because the alternative hypothesis uses a $\neq$ sign, this is a two-tailed test. We were told in the problem that the test statistic is $z = -1.60$, so we'll look that up in the z-table.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559

For the lower tail, $z = -1.60$ gives an area of 0.0548. Now to calculate our p-value, we multiply this by 2.

$$p = 2(0.0548)$$

$$p = 0.1096$$

3. The highest allowable amount of bromate in drinking water is 0.0100 (mg/L)^2. A survey of a city's water quality took 31 water samples in random locations around the city and used the data to calculate a test statistic of $t = 2.04$. The city wants to know if the amount of bromate in their drinking water is too high. Set up the hypothesis statements and determine the type of test, then find the p-value.

Solution:

The city wants to know if the amount of bromate in their drinking water is higher than the allowable amount in a significant way.

$$H_0 : \mu \leq 0.0100$$

$$H_a : \mu > 0.0100$$

Because the alternative hypothesis uses a $>$ sign, this is an upper-tailed test. We were told in the problem that the test statistic is $t = 2.04$, so we'll look that up in the t-table, but we'll also need to know the degrees of freedom. We know the study included 31 samples, so degrees of freedom is $n - 1 = 31 - 1 = 30$.

	Upper-tail probability p									
df	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	**2.042**	2.457	2.750	3.385	3.646
	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence level C									

Because the test statistic and degrees of freedom gives a p-value just under $p = 0.025$, we'll round the p-value to $p = 0.025$.

■ 4. A paint company produces glow in the dark paint with an advertised glow time of 15 min. A painter is interested in finding out if the product behaves worse than advertised. She sets up her hypothesis statements as $H_0 : \mu \geq 15$ and $H_a : \mu < 15$, then calculates a test statistic of $z = -2.30$. What would be the conclusions of her hypothesis test at significance levels of $\alpha = 0.05$, $\alpha = 0.01$, and $\alpha = 0.001$?

Solution:

We need to look up $z = -2.30$ in the z-table.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	**.0107**	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110

For a lower-tail test, the p-value is given by this value we found in the z-table, so $p = 0.0107$.

We know that

If $p \leq \alpha$, reject the null hypothesis

If $p > \alpha$, do not reject the null hypothesis

Therefore,

- For $p = 0.0107$ and $\alpha = 0.05$, $p \leq \alpha$, so she'd reject the null
- For $p = 0.0107$ and $\alpha = 0.01$, $p > \alpha$, so she'd fail to reject the null
- For $p = 0.0107$ and $\alpha = 0.001$, $p > \alpha$, so she'd fail to reject the null

■ 5. An article reports that the average wasted time by an employee is 125 minutes every day. A manager takes a small random sample of 16 employees and monitors their wasted time, calculating that average wasted time for her employees is 118 minutes with a standard deviation of 28.7 minutes. She wants to know if 118 minutes is below average at a significance level of $\alpha = 0.05$. She assumes the population is normally distributed.

1. State the population parameter and whether a t-test or z-test should be used.

2. Check that the conditions for performing the statistical test are met.

3. Set up the hypothesis statements.

4. State the type of test: upper-tailed, lower-tailed, or two-tailed.

5. Calculate the test statistic using the appropriate formula.

6. Calculate the p-value.

7. Compare the p-value to the significance level and draw a conclusion.

Solution:

This is a population mean with an unknown population standard deviation because the manager is going to do her analysis based on the sample standard deviation. She also has a small sample size of 16 employees. This means we should use the t-test statistic because we have a small sample size and also an unknown population standard deviation.

The conditions for performing a t-test with a population mean are an approximately normal distribution and a simple random sample, and we've been told in the problem that both of those conditions are met.

The manager wants to know if 118 minutes is below average. We're comparing 118 minutes to the stated average of 125 minutes. Since she wants to know if her measurement is below average, we should use the less than symbol in our alternative hypothesis.

$$H_0 : \mu \geq 125$$

$$H_a : \mu < 125$$

The test statistic will be

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{118 - 125}{\frac{28.7}{\sqrt{16}}} \approx -0.9756$$

The next step is to find the p-value by looking up the test statistic in the t-table. To look up a t-value, we'll also need to know the degrees of freedom from the problem. We know the study included 16 samples, so the degrees of freedom are $16 - 1 = 15$.

We calculated the test statistic as $t \approx -0.9756$. We're looking for the area in the lower tail, but the table will give us the area in the upper tail when $t = 0.9756$. Remember these values are equal because the t-curve is symmetric. Now we look up where our test statistic and degrees of freedom intersect. The value we read from the t-table is somewhere between $p = 0.20$ and $p = 0.15$.

	Upper-tail probability p									
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.691	**0.866**	**1.074**	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
	50%	**60%**	**70%**	**80%**	**90%**	**95%**	**98%**	**99%**	**99.8%**	**99.9%**
	Confidence level C									

Regardless of the exact value of p between $p = 0.20$ and $p = 0.15$, at a significance level of $\alpha = 0.05$, we can say $p > \alpha$, so the manager will fail to reject the null hypothesis, and conclude that there's not enough evidence to conclude that her employees waste less time than the average rate of 125 minutes per day at the significance level of $\alpha = 0.05$.

■ 6. We want to test if college students take fewer than than 5 years to graduate, on average, so we take a simple random sample of 30 students and record their years to graduate. For the sample, $\bar{x} = 4.9$ and $s = 0.5$. What can we conclude at 90% confidence?

Solution:

The hypothesis statements will be

$$H_0 : \mu \geq 5$$

$$H_a : \mu < 5$$

Find the test statistic.

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{4.9 - 5}{\frac{0.5}{\sqrt{30}}} = \frac{-0.1}{\frac{0.5}{\sqrt{30}}} \approx -1.0954$$

The next step is to look up the test statistic in the t-table. To look up a t-value, we'll also need to know the degrees of freedom from the problem. We know the study included 30 students, so the degrees of freedom are $30 - 1 = 29$.

We calculated the test statistic as $t \approx -1.0954$. We're looking for the area in the lower tail, but the table will give us the area in the upper tail when $t = 1.0954$. Remember these values are equal because the t-curve is symmetric. Now we look up where our test statistic and degrees of freedom intersect. The value we read from the t-table is somewhere between $p = 0.15$ and $p = 0.10$.

	Upper-tail probability p									
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.683	0.854	**1.055**	**1.311**	1.699	2.045	2.462	2.756	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
	50%	**60%**	**70%**	**80%**	**90%**	**95%**	**98%**	**99%**	**99.8%**	**99.9%**
	Confidence level C									

Regardless of the exact value of p between $p = 0.15$ and $p = 0.10$, at a significance level of $\alpha = 0.10$, we can say $p > \alpha$, so we'll fail to reject the null hypothesis, because there's not enough evidence that college students take less than 5 years to graduate at a significance level of $\alpha = 0.1$.

HYPOTHESIS TESTING FOR THE POPULATION PROPORTION

1. A large electric company claims that at least 80% of the company's $1{,}000{,}000$ customers are very satisfied. Using a simple random sample, 100 customers were surveyed and 73% of the participants were very satisfied. Based on these results, should we use a one- or two- tailed test, and should we accept or reject the company's hypothesis? Assume a significance level of 0.05.

Solution:

The first step is to state the null and alternative hypotheses for the survey.

$$H_0: \ p \geq 0.80$$

$$H_a: \ p < 0.80$$

These hypotheses require a one-tailed test, specifically a lower-tail test. The null hypothesis will be rejected only if the sample proportion is significantly less than 80%.

We calculate standard error based on the sample,

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.8(1-0.8)}{100}} = 0.04$$

and then compute the z-score test statistic.

$$z = \frac{\hat{p} - p}{\sigma_{\hat{p}}} = \frac{0.73 - 0.80}{0.04} = -1.75$$

The z-table gives 0.0401 for a z-score of $z = -1.75$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	**.0401**	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455

Since we have a one-tailed test, the p-value is $p = 0.0401$, and we were told in the problem that $\alpha = 0.05$. Because the p-value is less than the α-level, $p < \alpha$, we'll reject the null hypothesis.

2. A university is conducting a statistical test to determine whether the percentage of its students who live on its campus is above the national average of $64\,\%$. They've calculated the test statistic to be $z = 1.40$. Set up hypothesis statements and find the p-value.

Solution:

The university wants to know if the proportion of students who live on campus is above the national average in a statistically significant way.

$$H_0 : p \leq 64\,\%$$

$$H_a : p > 64\,\%$$

Because the alternative hypothesis uses a $>$ sign, this is a one-tail, upper-tailed test. We were told that the test statistic is $z = 1.40$, so we'll look that up in the z-table.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	**.9192**	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441

This is an upper-tail test, which means the p-value is the area outside of 0.9192, or

$$p = 1 - 0.9192$$

$$p = 0.0808$$

■ 3. A report claims that 60% of American families take fewer than 6 months to purchase a home, from the time they start looking to the time they make their first offer. A realtor wants to know if her clients purchase at the same rate, so she takes a simple random sample of 50 of her clients and finds $\hat{p} = 0.64$ and $\sigma_{\hat{p}} = \sqrt{0.0048}$ from the sample. What can she conclude with 90% confidence?

Solution:

The first step is to state the null and alternative hypotheses for the survey.

$$H_0: \ p = 0.60$$

$$H_a: \ p \neq 0.60$$

These hypotheses require a two-tailed test. The null hypothesis will be rejected only if the sample proportion is significantly different than $60\,\%$.

The z-score will be

$$z = \frac{\hat{p} - p}{\sigma_{\hat{p}}} = \frac{0.64 - 0.60}{\sqrt{0.0048}} = \frac{0.04}{\sqrt{0.0048}} \approx 0.58$$

The z-table gives 0.7190 for a z-score of $z = 0.58$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549

The p-value is the area outside of 0.7190, or

$$p = 1 - 0.7190$$

$$p = 0.2810$$

Since we have a two-tailed test, the p-value is double this, or

$$p = 2(0.2810)$$

$$p = 0.5620$$

We were told in the problem that $\alpha = 0.10$, so $p \geq \alpha$, which means the realtor will fail to reject the null hypothesis. So she can't say that her clients purchase at a different rate than the report claims.

■ 4. A gambler wins 48% of the hands he plays, but he feels like he's on a losing streak recently, winning fewer hands than normal. He takes a random sample of 40 of his recent hands, and finds the proportion of winning hands in the sample to be $\hat{p} = 0.45$ with $\sigma_{\hat{p}} = \sqrt{0.00624}$. What can he conclude with 90% confidence?

Solution:

The first step is to state the null and alternative hypotheses for the survey.

$$H_0: \ p \geq 0.48$$

$$H_a: \ p < 0.48$$

These hypotheses require a one-tailed test, specifically a lower-tail test. The null hypothesis will be rejected only if the sample proportion is significantly lower than 48%.

The z-score test statistic will be

$$z = \frac{\hat{p} - p}{\sigma_{\hat{p}}} = \frac{0.45 - 0.48}{\sqrt{0.00624}} = \frac{-0.03}{\sqrt{0.00624}} \approx -0.38$$

The z-table gives 0.3520 for a z-score of $z = -0.38$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	**.3520**	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859

This is a lower-tail test, which means this is also the p-value, $p = 0.3520$.

We were told in the problem that $\alpha = 0.10$. Because the p-value is greater than the α-level, $p < \alpha$, the gambler will fall to reject the null hypothesis, and conclude that he hasn't been on a losing streak at a statistically significant level.

5. A study claims that the proportion of new homeowners who purchase an internet subscription plan is 0.92. We take a random sample of 140 new homeowners to test this claim, and find $\hat{p} = 0.9$ with $\sigma_{\hat{p}} \approx 0.0229$. What can we conclude at a significance level of $\alpha = 0.05$?

Solution:

The first step is to state the null and alternative hypotheses for the survey.

$$H_0: \ p = 0.92$$

$$H_a: \ p \neq 0.92$$

These hypotheses require a two-tailed test. The null hypothesis will be rejected only if the sample proportion is significantly different than 92%.

The z-score test statistic will be

$$z = \frac{\hat{p} - p}{\sigma_{\hat{p}}} = \frac{0.90 - 0.92}{0.0229} = -\frac{0.02}{0.0229} \approx -0.87$$

The z-table gives 0.1922 for a z-score of $z \approx -0.87$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148

Since this is a two-tailed test, we double this to find the p-value, and we get

$$p = 2(0.1922)$$

$$p = 0.3844$$

We were told in the problem that $\alpha = 0.05$. Because the p-value is greater than the α-level, $p > \alpha$, we'll fail to reject the null hypothesis, which means that we can't conclude that the number of homeowners who purchase an internet subscription plan is different than $92\,\%$.

■ 6. A recent study reported that the $15.3\,\%$ of patients who are admitted to the hospital with a heart attack die within 30 days of admission. The same study reported that $16.7\,\%$ of the $3{,}153$ patients who went to the hospital with a heart attack died within 30 days of admission when the lead cardiologist was away.

Is there enough evidence to conclude that the percentage of patients who die when the lead cardiologist is away is any different than when they're present? Make conclusions at significance levels of $\alpha = 0.05$ and $\alpha = 0.01$.

1. State the population parameter and whether a t-test or z-test should be used.

2. Check that the conditions for performing the statistical test are met.

3. Set up the hypothesis statements.

4. State the type of test: upper-tailed, lower-tailed, or two-tailed.

5. Calculate the test statistic using the appropriate formula.

6. Calculate the p-value.

7. Compare the p-value to the significance level and draw a conclusion.

Solution:

This is a population proportion because the data is looking at the proportion of heart attack patients admitted to the hospital who die within 30 days of admittance.

The sample size is large at $3{,}153$ with a population proportion of $16.7\,\%$, but to continue with the test we need to assume that the sample was a simple random sample (since it's not stated in the problem).

This sample size is large enough to meet the conditions:

$$np = (3{,}153)(0.167) \approx 527 \geq 10$$

$$n(1-p) = (3{,}153)(1-0.167) \approx 2{,}626 \geq 10$$

When these two conditions are met, then the distribution is approximately normal. Then we can continue with the hypothesis test.

According to the problem, we want to know if the percentage of patients who went to the hospital with a heart attack and died within 30 days of admission when the leading cardiologist was away differs from when they were not away. This means we need to use the $\neq$ symbol in our hypothesis statement.

$$H_0 : p = 0.153$$

$$H_a : p \neq 0.153$$

Since we're dealing with a population proportion, the z-test statistic will be

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.167 - 0.153}{\sqrt{\frac{0.153(1-0.153)}{3{,}153}}} \approx 2.1837$$

The next step is to find the p-value by looking up the test statistic in the z-table. Since this is a two-tailed test, we'll need to double the area we find in either the upper or lower tail. From the z-table, we find a value of 0.9854.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890

But this is the area below the upper tail. Before we can do anything else we need to find the area in the upper tail. The total area under the curve is 1, so we'll subtract this value from 1.

$$1 - 0.9854 = 0.0146$$

Now to calculate the p-value, we multiply the upper tail by 2.

$$p = 2(0.0146)$$

$$p = 0.0292$$

We know that

If $p \leq \alpha$, reject the null hypothesis

If $p > \alpha$, do not reject the null hypothesis

Therefore,

- For $p = 0.0292$ and $\alpha = 0.05$, $p \leq \alpha$, so we'd reject the null
- For $p = 0.0292$ and $\alpha = 0.01$, $p > \alpha$, so we'd fail to reject the null

Which means there's enough evidence to conclude that the percentage of patients who went to the hospital with a heart attack and died within 30 days of admission when the leading cardiologist was away is different than when the leading cardiologist is present, at a statistically significant level of $\alpha = 0.05$, but not at $\alpha = 0.01$.

CONFIDENCE INTERVAL FOR THE DIFFERENCE OF MEANS

1. A researcher wants to compare the effectiveness of new blood pressure medication for males and females. He takes a simple random sample of 25 males and 25 females and finds an average drop in blood pressure of 4.5 with a standard deviation of 0.35 for males, and an average drop in blood pressure of 4.85 with a standard deviation of 0.22 for females. Can he use pooled standard deviation to find the confidence interval?

Solution:

The sample variances are $s_1^2 = 0.35^2 = 0.1225$ and $s_2^2 = 0.22^2 = 0.0484$. The variance s_1^2 is more than twice the other variance, so the researcher will assume unequal population variances.

2. A grocery store wants to know whether families of 3 spend more on groceries than families of 2. They randomly survey ten 3-person families and find a mean weekly grocery spend of $\$258$ with a standard deviation of $\$22$, then randomly survey ten 2-person families and find a mean weekly grocery spend of $\$252$ with a standard deviation of $\$26$. Calculate the number of degrees of freedom.

Solution:

The sample variances are $s_1^2 = 22^2 = 484$ and $s_2^2 = 26^2 = 676$, and neither sample variance is more than twice the other, so we can assume equal population variances.

Which means that the degree so freedom will be given by

$$\text{df} = n_1 + n_2 - 2$$

$$\text{df} = 10 + 10 - 2$$

$$\text{df} = 18$$

3. For the last question, calculate a 95% confidence interval around the difference in mean weekly grocery spending for 3-and 2-person families.

Solution:

Because we already determined in the previous solution that we're working with equal population variances, we'll calculate pooled standard deviation.

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

$$s_p = \sqrt{\frac{(10 - 1)22^2 + (10 - 1)26^2}{10 + 10 - 2}}$$

$$s_p = \sqrt{\frac{9(22^2) + 9(26^2)}{18}}$$

$$s_p = \sqrt{\frac{22^2 + 26^2}{2}}$$

$$s_p \approx 24.083$$

At 95% confidence and df $= 18$, the t-table gives 2.101.

	Upper-tail probability p									
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	**2.101**	2.552	2.878	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
	50%	**60%**	**70%**	**80%**	**90%**	**95%**	**98%**	**99%**	**99.8%**	**99.9%**
	Confidence level C									

Then the confidence interval is

$$(a, b) = (\bar{x}_1 - \bar{x}_2) \pm t_{\frac{\alpha}{2}} \times s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$(a, b) \approx (258 - 252) \pm 2.101 \times 24.083 \sqrt{\frac{1}{10} + \frac{1}{10}}$$

$$(a, b) \approx 6 \pm 22.63$$

Therefore, we can say that the confidence interval is

$$(a, b) \approx (6 - 22.63, 6 + 22.63)$$

$$(a,b) \approx (-16.63, 28.63)$$

We can be 95% confident that the true difference of mean weekly grocery spending for 3- and 2-person families will fall between $-\$16.63$ and $\$28.63$. But because the confidence interval contains 0, it means there's likely no difference between the population means.

4. A researcher is interested in whether a new fitness program lowers systolic blood pressure. He enrolls 50 participants into the study and randomly splits them into two groups of 25 each. The first group kept their same physical activity habits, while the second group followed the new fitness program. After a month, the mean systolic blood pressure in the group of exercisers was 123 with standard deviation of 4, and the mean systolic pressure in the group of non-exercisers was 131 with a standard deviation of 5.5. Calculate the margin of error at 99% confidence.

Solution:

The sample variances are $s_1^2 = 4^2 = 16$ and $s_2^2 = 5.5^2 = 30.25$, and neither sample variance is more than twice the other, so we'll assume equal population variances, which means the degrees of freedom will be

$$\text{df} = n_1 + n_2 - 2$$

$$\text{df} = 25 + 25 - 2$$

$$\text{df} = 48$$

The pooled standard deviation will be

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

$$s_p = \sqrt{\frac{(25 - 1)(16) + (25 - 1)(30.25)}{25 + 25 - 2}}$$

$$s_p = \sqrt{\frac{384 + 726}{48}}$$

$$s_p = \sqrt{23.125}$$

At df $= 48$ and 99% confidence, the t-table gives $t = 2.682$. Now we can calculate the margin of error as

$$ME = t_{\alpha/2} \times s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$ME = 2.682 \times \sqrt{23.125} \sqrt{\frac{1}{25} + \frac{1}{25}}$$

$$ME \approx 2.682\sqrt{1.85}$$

$$ME \approx 3.6479$$

■ 5. Given population standard deviations $\sigma_1 = 2.25$ and $\sigma_2 = 2.02$, with sample means $\bar{x}_1 = 14.5$ and $\bar{x}_2 = 13.6$ and sample sizes $n_1 = 250$ and $n_2 = 250$, calculate a 90% confidence interval around the difference of means.

Solution:

A 90% confidence level is associated with z-scores of $z = \pm 1.65$, so the confidence interval will be

$$(a, b) = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$(a, b) = (14.5 - 13.6) \pm 1.65\sqrt{\frac{2.25^2}{250} + \frac{2.02^2}{250}}$$

$$(a, b) \approx 0.9 \pm 0.316$$

Therefore, we can say that the confidence interval is

$$(a, b) \approx (0.9 - 0.316, 0.9 + 0.316)$$

$$(a, b) \approx (0.584, 1.216)$$

6. Owners of a large shopping center want to determine whether or not there's a difference in the amount of time that men and women spend per visit to the shopping center. Previous studies showed a standard deviation of 0.4 hours for men and 0.2 hours for women. The owners sample 500 men and 500 women and find that the mean time spent per visit was 1.6 hours for men and 2.5 hours for women. Find a 98% confidence interval around the difference of means.

Solution:

The z-values associated with a 98% confidence level are $z \pm 2.33$, so because the population standard deviations are known, the confidence interval will be given by

$$(a, b) = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$(a, b) = (1.6 - 2.5) \pm 2.33\sqrt{\frac{0.4^2}{500} + \frac{0.2^2}{500}}$$

$$(a, b) = -0.9 \pm 0.0466$$

Therefore, we can say that the confidence interval is

$$(a, b) = (-0.9 - 0.0466, -0.9 + 0.0466)$$

$$(a, b) = (-0.9466, -0.8534)$$

We can be 98% confidence that the true difference between mean time spent in the shopping center per visit by men and women will fall between -0.95 and -0.85 hours. Therefore, we've provided support for the hypothesis that men spend less time in the shopping center per visit than women.

HYPOTHESIS TESTING FOR THE DIFFERENCE OF MEANS

1. An ice cream shop owner believes his average daily revenue is higher in August than it is in September. He calculated average daily revenue of $\$496$ in August and $\$456$ in September, with standard deviations of $\$14$ and $\$21.5$, respectively. What can he conclude at a 0.05 significance level using a p-value approach.

Solution:

The sample variances are $s_1^2 = 14^2 = 196$ and $s_2^2 = 21.5^2 = 462.25$, so the second sample variance is more than twice the first. Therefore, because the sample variances are unequal, we can assume unequal population variances. Additionally, we have large samples $n_A = 31$ and $n_S = 30$, since there are 31 days in August and 30 days in September, so we'll use a z-test.

We'll run an upper-tailed test because the shop owner believes the average daily revenue in August was higher than in September.

$H_0 : \mu_A - \mu_S \leq 0$; the average daily revenue in August is not higher than in September.

$H_a : \mu_A - \mu_S > 0$; the average daily revenue in August is higher than in September.

Then the test statistic is

$$z = \frac{\bar{x}_A - \bar{x}_S}{\sqrt{\frac{s_A^2}{n_A} + \frac{s_S^2}{n_S}}}$$

$$z = \frac{496 - 456}{\sqrt{\frac{14^2}{31} + \frac{21.5^2}{30}}}$$

$$z = \frac{40}{\sqrt{\frac{196}{31} + \frac{462.25}{30}}}$$

$$z \approx 8.581$$

The test statistic is much larger than the largest z-value in the z-table, so we could say that the probability of finding $z \approx 8.581$ is almost 0. Therefore, $p \leq \alpha$ and we can reject the null hypothesis, concluding that the average daily revenue in August was higher than in September.

2. A fitness coach wants to determine whether his new weight loss program is more effective than his old program. He randomly samples 50 of his clients following each program, and finds a mean weight loss of 5.5 pounds with a standard deviation of 1.05 pounds for those following the old program, and a mean weight loss of 6.12 pounds with a standard deviation of 0.95 pounds for those following the new program. Using a critical value approach, what can the coach conclude at a 0.01 level of significance?

Solution:

The sample variances are $s_1^2 = 1.05^2 = 1.1025$ and $s_2^2 = 0.95^2 = 0.9025$. Neither sample variance is more than twice the other, which means we can assume equal sample variances, and therefore equal population variances.

The null and alternative hypotheses for the upper-tailed test are

$$H_0 : \mu_1 - \mu_2 \leq 0$$

$$H_a : \mu_1 - \mu_2 > 0$$

Since both samples are larger than 30 and the population variances are equal, we can calculate pooled standard deviation.

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

$$s_p = \sqrt{\frac{(50 - 1)1.05^2 + (50 - 1)0.95^2}{50 + 50 - 2}}$$

$$s_p = \sqrt{\frac{49(1.05^2) + 49(0.95^2)}{98}}$$

$$s_p = \sqrt{\frac{1.05^2 + 0.95^2}{2}}$$

$$s_p \approx 1.00125$$

Then the z-statistic is

$$z = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$z = \frac{6.12 - 5.5}{1.00125\sqrt{\frac{1}{50} + \frac{1}{50}}}$$

$$z = \frac{0.62}{1.00125\sqrt{\frac{1}{25}}}$$

$$z \approx 3.10$$

For an upper-tailed test and $\alpha = 0.01$, the critical z-value is $z = 2.33$. Since $3.10 > 2.33$, the coach can reject the null hypothesis and conclude that the new weight loss program is more effective than the old program.

■ 3. Test the claim that, in 2006, the mean weight of men in the US was not significantly different from the mean weight of women. Previous research showed population standard deviations were 10.25 pounds for men and 8.58 pounds for women. A random sample of 1,500 men has a mean weight of 193.5 pounds and a random sample of 1,500 women has a mean weight of 185.3 pounds. Assuming the population variances are unequal, use a p-value approach to formulate a decision at the 0.05 significance level.

Solution:

Given the sample mean $\mu_m = 193.5$ and population standard deviation $\sigma_m = 10.25$ for men, and the sample mean $\mu_w = 185.3$ and population standard deviation $\sigma_w = 8.58$ for women, the null and alternative hypotheses for the two-tailed test will be

$$H_0 : \mu_m - \mu_w = 0$$

$$H_a : \mu_m - \mu_w \neq 0$$

With unequal population variances and large samples, the test statistic will be

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$z = \frac{193.5 - 185.3}{\sqrt{\frac{10.25^2}{1,500} + \frac{8.58^2}{1,500}}}$$

$$z = \frac{8.2}{\sqrt{\frac{105.0625 + 73.6164}{1,500}}}$$

$$z = 8.2\sqrt{\frac{1,500}{105.0625 + 73.6164}}$$

$$z \approx 23.76$$

The test statistic is much larger than the largest z-value in the z-table, so we could say that the probability of finding $z \approx 23.76$ is almost 0. Therefore,

$p \leq \alpha$ and we can reject the null hypothesis, concluding that the mean weight of men and women was significantly different.

4. A research team wants to determine whether men and women drink a different amount of water each day. They randomly sample 25 men and 25 women and find that the men consumed 1.48 liters of water with a standard deviation of 0.13 liters, and that the women consumed 1.62 liters of water with a standard deviation of 0.20 liters. Using a critical value approach, what can the research team conclude at a 0.10 level of significance?

Solution:

The null and alternative hypotheses for the two-tailed test will be

$$H_0 : \mu_m - \mu_w = 0$$

$$H_a : \mu_m - \mu_w \neq 0$$

With small samples and unequal population variances ($s_2^2 = 0.2^2 = 0.04$ is more than twice $s_1^2 = 0.13^2 = 0.0169$), the t-statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$t = \frac{1.48 - 1.62}{\sqrt{\frac{0.0169}{25} + \frac{0.04}{25}}}$$

$$t = \frac{-0.14}{\sqrt{\frac{0.0569}{25}}}$$

$$t \approx -2.9346$$

The number of degrees of freedom will be

$$\text{df} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2}$$

$$\text{df} = \frac{\left(\frac{0.0169}{25} + \frac{0.04}{25}\right)^2}{\frac{1}{25 - 1}\left(\frac{0.0169}{25}\right)^2 + \frac{1}{25 - 1}\left(\frac{0.04}{25}\right)^2}$$

$$\text{df} \approx 27.9966$$

With df $= 27$ and $\alpha = 0.10$, we find critical t-values of $t \pm 1.703$. Since $-2.9346 < -1.703$, the research team can reject the null hypothesis and conclude that there's a difference in the mean amount of water that men and women drink each day.

5. Given $\bar{x}_1 = 23.55$ and $\bar{x}_2 = 20.12$ with $s_1 = 2.3$, $s_2 = 2.9$, $n_1 = 10$, and $n_2 = 15$, determine whether the two population means differ significantly. Using a critical value approach, and assuming population standard deviations are unequal, what can we conclude at a 0.01 level of significance?

Solution:

We want to determine whether there's a difference in population means, so we need to use a two-tailed test, and our hypothesis statements will be

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 \neq 0$$

With small samples and unequal population variances, we should calculate a t-statistic.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$t = \frac{23.55 - 20.12}{\sqrt{\frac{2.3^2}{10} + \frac{2.9^2}{15}}}$$

$$t = \frac{3.43}{\sqrt{\frac{5.29}{10} + \frac{8.41}{15}}}$$

$$t \approx 3.286$$

Calculate the number of degrees of freedom.

$$\text{df} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2}$$

$$\text{df} = \frac{\left(\frac{2.3^2}{10} + \frac{2.9^2}{15}\right)^2}{\frac{2.3^4}{10^2(10-1)} + \frac{2.9^4}{15^2(15-1)}}$$

$$\text{df} \approx 22.17$$

Rounding down to the nearest whole number gives df $= 22$. From the t-table, we find that the critical t-value for the two-tailed test with $\alpha = 0.01$ and df $= 22$ is $t = 2.819$.

Because $3.286 > 2.819$, we can reject the null hypothesis and conclude that there's a significant difference in population means.

6. John claims that the temperature in July is higher than the temperature in August. He recorded the temperature daily at $12:00$ p.m. throughout July and August. He found a mean temperature of 28.4° C with a standard deviation of 2.1° C in July, and a mean temperature of 27.3° C with a standard deviation of 1.7° C in in August. Using a critical value approach and assuming the population variances are unequal, what can John conclude at a 0.05 level of significance?

Solution:

Using μ_1, $\bar{x}_1$, and s_1 for July and μ_2, $\bar{x}_2$, and s_2 for August, the hypothesis statements for the John's upper-tailed test will be

$$H_0: \mu_1 - \mu_2 \leq 0$$

$$H_a: \mu_1 - \mu_2 > 0$$

With large samples and unequal population variances, John's test statistic will be

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$z = \frac{28.4 - 27.3}{\sqrt{\frac{2.1^2}{31} + \frac{1.7^2}{31}}}$$

$$z = 1.1\sqrt{\frac{31}{4.41 + 2.89}}$$

$$z \approx 2.27$$

The critical z-value for $\alpha = 0.05$ with an upper-tailed test is $z = 1.65$. Since $2.27 > 1.65$, John can reject the null hypothesis and conclude that the mean temperature in July is higher than in August.

MATCHED-PAIR HYPOTHESIS TESTING

1. A golf club manufacturer claims that their new driver delivers 15 yards of extra driving distance. They record the before and after driving distances of 10 top professional players.

Player	1	2	3	4	5	6	7	8	9	10
Before x_1	303	308	295	305	301	312	287	294	300	301
After x_2	307	320	297	315	305	316	299	302	307	315
Difference, d	4	12	2	10	4	4	12	8	7	14
d^2	16	144	4	100	16	16	144	64	49	196

Can the manufacturer conclude at a 5% significance level that their driver delivers 15 yards of extra driving distance?

Solution:

The manufacturer will define the "before" responses as Population 1, and the "after" responses as Population 2, and their null and alternative hypotheses will be

$$H_0: \mu_2 - \mu_1 \leq 15$$

$$H_a: \mu_2 - \mu_1 > 15$$

where μ_1 is the mean driving distance with the players' current drivers, and μ_2 is the mean driving distance with the manufacturer's new driver. And because $\mu_2 - \mu_1$ is the difference in distance, the hypothesis statements could also be written as

$$H_0 : \mu_d \le 15$$

$$H_a : \mu_d > 15$$

where μ_d is the mean difference between the two populations.

To find the mean difference, we'll sum the differences and divide by the number of matched-pairs in our sample, $n = 10$.

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{4 + 12 + 2 + 10 + 4 + 4 + 12 + 8 + 7 + 14}{10} = \frac{77}{10} = 7.7$$

So the sample mean tells us that mean distance gained is 7.7 yards. Then the sample standard deviation is

$$s_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n - 1}}$$

To calculate this, we'll first find

$$\sum_{i=1}^{n} (d_i - \bar{d})^2$$

$$(4 - 7.7)^2 + (12 - 7.7)^2 + (2 - 7.7)^2 + (10 - 7.7)^2 + (4 - 7.7)^2$$

$$+(4 - 7.7)^2 + (12 - 7.7)^2 + (8 - 7.7)^2 + (7 - 7.7)^2 + (14 - 7.7)^2$$

$$(-3.7)^2 + 4.3^2 + (-5.7)^2 + 2.3^2 + (-3.7)^2 + (-3.7)^2 + 4.3^2 + 0.3^2 + (-0.7)^2 + 6.3^2$$

$$13.69 + 18.49 + 32.49 + 5.29 + 13.69 + 13.69 + 18.49 + 0.09 + 0.49 + 39.69$$

$$156.1$$

Then the sample standard deviation is

$$s_d = \sqrt{\frac{156.1}{9}}$$

$$s_d \approx \sqrt{17.34}$$

$$s_d \approx 4.165$$

Because the population standard deviations are unknown, and/or because both sample sizes are small, $n_1, n_2 < 30$, the test statistic will be

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$$

$$t \approx \frac{7.7 - 15}{\frac{4.165}{\sqrt{10}}}$$

$$t \approx -7.3 \cdot \frac{\sqrt{10}}{4.165}$$

$$t \approx -5.543$$

and the degrees of freedom are

$$\text{df} = n - 1 = 10 - 1 = 9$$

At a significance level of 5% (a confidence level of 95% for an upper-tailed test), and $\text{df} = 9$, the t-table gives 1.833.

	Upper-tail probability p									
df	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence level C									

The manufacturer's t-test statistic $t \approx -5.543$ doesn't meet the threshold $t = 1.833$, so the critical value approach tells them that they can't reject the null hypothesis, and therefore can't conclude that their new driver adds 15 yards of extra distance for the professional players.

2. A car company believes that the changes they've made to their hybrid engine will increase miles per gallon by 4. They send out one car with the old engine and one car with the new engine to drive the same route, and record the miles per gallon of each pair of cars.

Route	1	2	3	4	5	6	7	8	9	10
Old engine	39	39	38	42	44	43	42	47	47	47
New engine	50	49	45	46	46	41	42	43	43	49
Difference, d	11	10	7	4	2	-2	0	-4	-4	2
d^2	121	100	49	16	4	4	0	16	16	4

Can the car company conclude at a 1% significance level that the changes they've made to the hybrid engine deliver 4 extra miles per gallon?

Solution:

The car company will define the values for the old engine as Population 1, and the values for the new engine as Population 2, and their null and alternative hypotheses will be

$$H_0 : \mu_2 - \mu_1 \leq 4$$

$$H_a : \mu_2 - \mu_1 > 4$$

where μ_1 is the miles per gallon obtained by the old engine, and μ_2 is the miles per gallon obtained by the old engine. And because $\mu_2 - \mu_1$ is the difference in miles per gallon, the hypothesis statements could also be written as

$$H_0 : \mu_d \leq 4$$

$$H_a : \mu_d > 4$$

where μ_d is the mean difference between the two populations.

To find the mean difference, we'll sum the differences and divide by the number of matched-pairs in our sample, $n = 10$.

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{11 + 10 + 7 + 4 + 2 + (-2) + 0 + (-4) + (-4) + 2}{10} = \frac{26}{10} = 2.6$$

So the sample mean tells us that mean difference is 2.6 miles per gallon. Then the sample standard deviation is

$$s_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n - 1}}$$

To calculate this, we'll first find

$$\sum_{i=1}^{n} (d_i - \bar{d})^2$$

$$(11 - 2.6)^2 + (10 - 2.6)^2 + (7 - 2.6)^2 + (4 - 2.6)^2 + (2 - 2.6)^2$$

$$+(-2 - 2.6)^2 + (0 - 2.6)^2 + (-4 - 2.6)^2 + (-4 - 2.6)^2 + (2 - 2.6)^2$$

$$8.4^2 + 7.4^2 + 4.4^2 + 1.4^2 + (-0.6)^2$$

$$+(-4.6)^2 + (-2.6)^2 + (-6.6)^2 + (-6.6)^2 + (-0.6)^2$$

$$70.56 + 54.76 + 19.36 + 1.96 + 0.36 + 21.16 + 6.67 + 43.56 + 43.56 + 0.36$$

$$262.31$$

Then the sample standard deviation is

$$s_d = \sqrt{\frac{262.31}{9}}$$

$$s_d \approx \sqrt{29.15}$$

$$s_d \approx 5.399$$

Because the population standard deviations are unknown, and/or because both sample sizes are small, $n_1, n_2 < 30$, the test statistic will be

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$$

$$t \approx \frac{2.6 - 4}{\frac{5.399}{\sqrt{10}}}$$

$$t \approx -1.4 \cdot \frac{\sqrt{10}}{5.399}$$

$$t \approx -0.82$$

and the degrees of freedom are

$$\text{df} = n - 1 = 10 - 1 = 9$$

At a significance level of 1% (a confidence level of 99%) for an upper-tailed test, and df $= 9$, the t-table gives 2.821.

	Upper-tail probability p									
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	**2.821**	3.250	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
	50%	**60%**	**70%**	**80%**	**90%**	**95%**	**98%**	**99%**	**99.8%**	**99.9%**
	Confidence level C									

The car company's t-test statistic $t \approx -0.82$ doesn't meet the threshold $t = 2.821$, so the critical value approach tells them that they can't reject the null hypothesis, and therefore can't conclude that their new engine adds 4 miles per gallon.

3. We want to test the claim that listening to classical music while studying makes students complete their homework faster. We ask 10

students to study in silence for the first semester, and study with classical music for the second semester, then we record the mean number of hours spent on homework per week in each semester.

Student	1	2	3	4	5	6	7	8	9	10
In silence	14	13	16	21	15	19	11	20	19	16
With music	12	13	15	22	16	19	8	17	18	17
Difference, d	2	0	1	-1	-1	0	3	3	1	-1
d^2	4	0	1	1	1	0	9	9	1	1

Can we conclude at a 10% significance level that studying with classical music reduces the number of hours spent per week on homework?

Solution:

We'll define the hours spent studying in silence as Population 1, and the hours spent studying with classical music as Population 2, and our null and alternative hypotheses will be

$$H_0 : \mu_1 - \mu_2 \leq 0$$

$$H_a : \mu_1 - \mu_2 > 0$$

where μ_1 is the mean number hours spent studying in silence, and μ_2 is the mean number of hours spent studying with classical music. And because $\mu_1 - \mu_2$ is the difference in study time, the hypothesis statements could also be written as

$$H_0 : \mu_d \leq 0$$

$$H_a : \mu_d > 0$$

where μ_d is the mean difference between the two populations.

To find the mean difference, we'll sum the differences and divide by the number of matched-pairs in our sample, $n = 10$.

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{2 + 0 + 1 + (-1) + (-1) + 0 + 3 + 3 + 1 + (-1)}{10} = \frac{7}{10} = 0.7$$

So the sample mean tells us that mean difference is 0.7 studying hours. Then the sample standard deviation is

$$s_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n - 1}}$$

To calculate this, we'll first find

$$\sum_{i=1}^{n} (d_i - \bar{d})^2$$

$$(2 - 0.7)^2 + (0 - 0.7)^2 + (1 - 0.7)^2 + (-1 - 0.7)^2$$

$$+(-1 - 0.7)^2 + (0 - 0.7)^2 + (3 - 0.7)^2 + (3 - 0.7)^2$$

$$+(1 - 0.7)^2 + (-1 - 0.7)^2$$

$$1.3^2 + (-0.7)^2 + 0.3^2 + (-1.7)^2 + (-1.7)^2 + (-0.7)^2 + 2.3^2 + 2.3^2 + 0.3^2 + (-1.7)^2$$

$$1.69 + 0.49 + 0.09 + 2.89 + 2.89 + 0.49 + 5.29 + 5.29 + 0.09 + 2.89$$

$$22.1$$

Then the sample standard deviation is

$$s_d = \sqrt{\frac{22.1}{9}}$$

$$s_d \approx \sqrt{2.46}$$

$$s_d \approx 1.567$$

Because the population standard deviations are unknown, and/or because both sample sizes are small, $n_1, n_2 < 30$, the test statistic will be

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$$

$$t = \frac{0.7 - 0}{\frac{1.567}{\sqrt{10}}}$$

$$t = 0.7 \cdot \frac{\sqrt{10}}{1.567}$$

$$t \approx 1.413$$

and the degrees of freedom are

$$\text{df} = n - 1 = 10 - 1 = 9$$

At a significance level of $10\,\%$ (a confidence level of $90\,\%$) for an upper-tailed test, and $\text{df} = 9$, the t-table gives 1.383.

	Upper-tail probability p									
df	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.703	0.883	1.100	**1.383**	1.833	2.262	2.821	3.250	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence level C									

Our t-test statistic $t \approx 1.413$ meets the threshold $t = 1.383$, so the critical value approach tells us that we can reject the null hypothesis, and therefore conclude that studying with classical music reduces time spent on homework.

4. A clothing store wants to test the claim that customers who join their VIP program return less merchandise. They track the mean monthly merchandise returns of 10 customers for one year before and after joining the VIP program, then record the mean returns per month.

Customer	1	2	3	4	5	6	7	8	9	10
Before VIP	12	55	48	23	97	103	33	44	17	29
After VIP	15	44	35	20	100	97	30	41	24	40
Difference, d	-3	11	13	3	-3	6	3	3	-7	-11
d^2	9	121	169	9	9	36	9	9	49	121

Can they conclude at a 5% significance level that joining the VIP program reduces the amount of merchandise returns?

Solution:

The clothing store will define the values for returns before enrolling in the VIP program as Population 1, and the values for returns after enrolling in the VIP program as Population 2, and their null and alternative hypotheses will be

$$H_0 : \mu_1 - \mu_2 \le 0$$

$$H_a : \mu_1 - \mu_2 > 0$$

where μ_1 is the mean monthly merchandise returns before the VIP program, and μ_2 is the mean monthly merchandise returns after the VIP program. And because $\mu_1 - \mu_2$ is the difference in monthly merchandise returns, the hypothesis statements could also be written as

$$H_0 : \mu_d \le 0$$

$$H_a : \mu_d > 0$$

where μ_d is the mean difference between the two populations.

To find the mean difference, we'll sum the differences and divide by the number of matched-pairs in our sample, $n = 10$.

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = \frac{(-3) + 11 + 13 + 3 + (-3) + 6 + 3 + 3 + (-7) + (-11)}{10} = \frac{15}{10} = 1.5$$

So the sample mean tells us that mean difference is 1.5 in merchandise returns. Then the sample standard deviation is

$$s_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}}$$

To calculate this, we'll first find

$$\sum_{i=1}^{n} (d_i - \bar{d})^2$$

$$(-3 - 1.5)^2 + (11 - 1.5)^2 + (13 - 1.5)^2 + (3 - 1.5)^2$$

$$+(-3 - 1.5)^2 + (6 - 1.5)^2 + (3 - 1.5)^2 + (3 - 1.5)^2$$

$$+(-7 - 1.5)^2 + (-11 - 1.5)^2$$

$$(-4.5)^2 + 9.5^2 + 11.5^2 + 1.5^2 + (-4.5)^2$$

$$+4.5^2 + 1.5^2 + 1.5^2 + (-8.5)^2 + (-12.5)^2$$

$$20.25 + 90.25 + 132.25 + 2.25 + 20.25 + 20.25 + 2.25 + 2.25 + 72.25 + 156.25$$

$$518.5$$

Then the sample standard deviation is

$$s_d = \sqrt{\frac{518.5}{9}}$$

$$s_d \approx \sqrt{57.61}$$

$$s_d \approx 7.59$$

Because the population standard deviations are unknown, and/or because both sample sizes are small, n_1, $n_2 < 30$, the test statistic will be

$$t=\frac{\bar{d}-\mu_d}{\frac{s_d}{\sqrt{n}}}$$

$$t\approx\frac{1.5-0}{\frac{7.59}{\sqrt{10}}}$$

$$t\approx 1.5\cdot\frac{\sqrt{10}}{7.59}$$

$$t\approx 0.625$$

and the degrees of freedom are

$$\text{df}=n-1=10-1=9$$

At a significance level of 5% (a confidence level of 95%) for an upper-tailed test, and df $=9$, the t-table gives 1.833.

	Upper-tail probability p									
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.703	0.883	1.100	1.383	**1.833**	2.262	2.821	3.250	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
	50%	**60%**	**70%**	**80%**	**90%**	**95%**	**98%**	**99%**	**99.8%**	**99.9%**
	Confidence level C									

The clothing store's t-test statistic $t\approx 0.625$ doesn't meet the threshold $t=1.833$, so the critical value approach tells them that they can't reject the null hypothesis, and therefore can't conclude that their VIP program causes customers to return less merchandise.

5. If the mean difference is $\bar{d} = 10$ on a sample of $n = 25$ with sample standard deviation $s_d = 2.5$, calculate the 95% confidence interval around $\bar{d}$.

Solution:

For 95% confidence with df $= n - 1 = 25 - 1 = 24$, and because we have a small sample $n < 30$, the confidence interval will be

$$(a, b) = \bar{d} \pm t_{\alpha/2}\frac{s_d}{\sqrt{n}}$$

$$(a, b) = 10 \pm 2.064\left(\frac{2.5}{\sqrt{25}}\right)$$

$$(a, b) = 10 \pm 2.064(0.5)$$

$$(a, b) = 10 \pm 1.032$$

Then the confidence interval will be

$$(a, b) = (10 - 1.032, 10 + 1.032)$$

$$(a, b) = (8.968, 11.032)$$

So we're 95% confident that the mean difference falls between 8.968 and 11.032.

6. If the mean difference is $\bar{d} = 24$ on a sample of $n = 49$ with population standard deviation $\sigma_d = 3.2$, calculate the 99% confidence interval around $\bar{d}$.

Solution:

For 99% confidence with critical values $z = \pm 2.58$ and a large sample $n \geq 30$, the confidence interval will be

$$(a, b) = \bar{d} \pm z_{\alpha/2}\frac{\sigma_d}{\sqrt{n}}$$

$$(a, b) = 24 \pm 2.58\left(\frac{3.2}{\sqrt{49}}\right)$$

$$(a, b) \approx 24 \pm 2.58(0.457)$$

$$(a, b) \approx 24 \pm 1.179$$

Then the confidence interval will be

$$(a, b) \approx (24 - 1.179, 24 + 1.179)$$

$$(a, b) \approx (22.821, 25.179)$$

So we're 99% confident that the mean difference falls between 22.821 and 25.179.

CONFIDENCE INTERVAL FOR THE DIFFERENCE OF PROPORTIONS

1. Given $x_1 = 54$ successes in the first sample $n_1 = 150$, and $x_2 = 47$ successes in the second sample $n_2 = 160$, calculate a 95% confidence interval.

Solution:

The sample proportions are

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{54}{150} = 0.36$$

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{47}{160} \approx 0.294$$

At 95% confidence, the critical z-values are $z = \pm 1.96$, so the confidence interval will be

$$(a, b) = (\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$(a, b) \approx (0.36 - 0.294) \pm 1.96\sqrt{\frac{0.36(1-0.36)}{150} + \frac{0.294(1-0.294)}{160}}$$

$$(a, b) \approx 0.066 \pm 1.96\sqrt{\frac{0.36(0.64)}{150} + \frac{0.294(0.706)}{160}}$$

$$(a, b) \approx 0.066 \pm 0.104$$

Then the 95% confidence interval is

$$(a,b) \approx (0.066 - 0.104, 0.066 + 0.104)$$

$$(a,b) \approx (-0.038, 0.170)$$

Because the confidence interval includes 0, we can't conclude that there's a difference between the population proportions.

■ 2. A light bulb manufacturer wants to know whether their own bulbs last longer than a competitor's bulb. They randomly sampled 150 people who bought their bulb, and 72 of them reported that it lasted longer than 250 days. They randomly sampled 150 people who bought the competitor's bulb, and 69 of them reported that it lasted for more than 250 days. Find a 90% confidence interval around the difference of proportions.

Solution:

The sample proportions are

$$\hat{p}_1 = \frac{72}{150} = 0.48$$

$$\hat{p}_2 = \frac{69}{150} = 0.46$$

At 90% confidence, the critical z-values are $z = \pm 1.65$, so the confidence interval will be

$$(a,b) = (\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$(a,b) = (0.48 - 0.46) \pm 1.65\sqrt{\frac{0.48(1-0.48)}{150} + \frac{0.46(1-0.46)}{150}}$$

$$(a,b) = 0.02 \pm 1.65\sqrt{\frac{0.48(0.52)}{150} + \frac{0.46(0.54)}{150}}$$

$$(a,b) \approx 0.02 \pm 0.095$$

Then the 90% confidence interval is

$$(a,b) \approx (0.02 - 0.095, 0.02 - 0.095)$$

$$(a,b) \approx (-0.075, 0.115)$$

Because the confidence interval includes 0, we can't conclude that there's a difference between the proportion of bulbs from each company that last longer than 250 days.

■ 3. A research team wants to know whether Vitamin C shortens recovery time from the common cold. They chose 100 patients with the common cold and randomly assigned 50 of them to the Vitamin C treatment group and 50 of them to the placebo group. In the Vitamin C group, 38 patients recovered in less than 7 days, while 24 patients in the placebo group recovered in less than 7 days. Find a 99% confidence interval around the difference in population proportions.

Solution:

The sample proportions are

$$\hat{p}_1 = \frac{38}{50} = 0.76$$

$$\hat{p}_2 = \frac{24}{50} = 0.48$$

At 99% confidence, the critical z-values are $z = \pm 2.58$, so the confidence interval will be

$$(a, b) = (\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

$$(a, b) = (0.76 - 0.48) \pm 2.58\sqrt{\frac{0.76(1 - 0.76)}{50} + \frac{0.48(1 - 0.48)}{50}}$$

$$(a, b) = 0.28 \pm 2.58\sqrt{\frac{0.76(0.24)}{50} + \frac{0.48(0.52)}{50}}$$

$$(a, b) \approx 0.28 \pm 0.24$$

Then the 99% confidence interval is

$$(a, b) \approx (0.28 - 0.24, 0.28 + 0.24)$$

$$(a, b) \approx (0.04, 0.52)$$

We can be 99% confident that the true difference between population proportions is between 0.04 and 0.52. Which means we can be 99%

confident that the Vitamin C treatment shortens recovery time from the common cold.

4. A researcher randomly chose 900 smokers, 450 men and 450 women. He found that 357 of the male smokers have been diagnosed with coronary artery disease, while 295 of the female smokers have been diagnosed with coronary artery disease. Construct a 95% confidence interval to estimate the difference between the proportions of male and female smokers who have been diagnosed with coronary artery disease.

Solution:

The sample proportions are

$$\hat{p}_1 = \frac{357}{450} \approx 0.793$$

$$\hat{p}_2 = \frac{295}{450} \approx 0.656$$

At 95% confidence, the critical z-values are $z = \pm 1.96$, so the confidence interval will be

$$(a, b) = (\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$(a, b) = (0.793 - 0.656) \pm 1.96\sqrt{\frac{0.793(1-0.793)}{450} + \frac{0.656(1-0.656)}{450}}$$

$$(a,b) = 0.137 \pm 1.96\sqrt{\frac{0.793(0.207)}{450} + \frac{0.656(0.344)}{450}}$$

$$(a,b) \approx 0.137 \pm 0.058$$

Then the 95% confidence interval is

$$(a,b) \approx (0.137 - 0.058, 0.137 + 0.058)$$

$$(a,b) \approx (0.079, 0.195)$$

We can be 95% confident that the true difference between the proportion of male smokers with coronary artery disease and the proportion of female smokers with coronary artery disease is between 0.079 and 0.195.

■ 5. In a simple random sample of $1{,}000$ people aged $20-24$, 7% said they ran at least one marathon in the last year. In a simple random sample of $1{,}200$ people aged $25-29$, 12% said they ran at least one marathon in the last year. Find a 99% confidence interval around the difference of population proportions.

Solution:

With $\hat{p}_1 = 0.07$ for $n_1 = 1{,}000$ and $\hat{p}_2 = 0.12$ for $n_2 = 1{,}200$, and critical values of $z = \pm 2.58$ for a 99% confidence level, the confidence interval will be

$$(a,b) = (\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$(a,b) = (0.07 - 0.12) \pm 2.58\sqrt{\frac{0.07(1-0.07)}{1{,}000} + \frac{0.12(1-0.12)}{1{,}200}}$$

$$(a,b) = -0.05 \pm 2.58\sqrt{\frac{0.07(0.93)}{1{,}000} + \frac{0.12(0.88)}{1{,}200}}$$

$$(a,b) \approx -0.05 \pm 0.032$$

Then the 99% confidence interval is

$$(a,b) \approx (-0.05 - 0.032, -0.05 + 0.032)$$

$$(a,b) \approx (-0.082, -0.018)$$

We can be 99% confident that the true of proportions is between -0.082 and -0.018. Which means it's likely that more people aged $25-29$ ran at least one marathon in the last year, compared to people aged $20-24$.

■ 6. In a simple random sample of 280 Masters students from one university, 24 said they planned to pursue a PhD. In a simple random sample of 350 Masters students at a second university, 34 said they planned to pursue a PhD. Build a 98% confidence interval around the difference of proportions.

Solution:

The sample proportions are

$$\hat{p}_1 = \frac{24}{280} \approx 0.086$$

$$\hat{p}_2 = \frac{34}{350} \approx 0.097$$

At 98% confidence, the critical z-values are $z = \pm 2.33$, so the confidence interval will be

$$(a, b) = (\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

$$(a, b) = (0.086 - 0.097) \pm 2.33\sqrt{\frac{0.086(1 - 0.086)}{280} + \frac{0.097(1 - 0.097)}{350}}$$

$$(a, b) = -0.011 \pm 2.33\sqrt{\frac{0.086(0.914)}{280} + \frac{0.097(0.903)}{350}}$$

$$(a, b) \approx -0.011 \pm 0.054$$

Then the 98% confidence interval is

$$(a, b) \approx (-0.011 - 0.054, -0.011 + 0.054)$$

$$(a, b) \approx (-0.065, 0.043)$$

Because the confidence interval includes 0, we can't conclude that there's a difference between the proportion of Masters students at each university who want to pursue a PhD.

HYPOTHESIS TESTING FOR THE DIFFERENCE OF PROPORTIONS

1. We defined the hypothesis statements below, and then found sample proportions of $\hat{p}_1 = 0.456$ for $n_1 = 278$ and $\hat{p}_2 = 0.384$ for $n_2 = 310$. Using a critical value approach, can we reject the null hypothesis at a confidence level of 95%?

$$H_0 : p_1 - p_2 \leq 0$$

$$H_a : p_1 - p_2 > 0$$

Solution:

The pooled proportion is

$$\hat{p} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$$

$$\hat{p} = \frac{0.456(278) + 0.384(310)}{278 + 310}$$

$$\hat{p} \approx 0.418$$

Then the z-statistic is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z \approx \frac{0.456 - 0.384}{\sqrt{0.418(1 - 0.418)\left(\frac{1}{278} + \frac{1}{310}\right)}}$$

$$z \approx \frac{0.072}{\sqrt{0.418(0.582)\left(\frac{1}{278} + \frac{1}{310}\right)}}$$

$$z \approx 1.767$$

For an upper-tailed test at a confidence level of 95%, the critical value is $z = 1.65$. Since $1.767 > 1.65$, we can reject the null hypothesis and conclude that $p_1 > p_2$ at a 95% confidence level.

2. Given the hypothesis statements below, $x_1 = 234$ with $n_1 = 1{,}150$ and $x_2 = 327$ with $n_2 = 1{,}320$, calculate the test statistic.

$$H_0 : p_1 - p_2 = 0$$

$$H_a : p_1 - p_2 \neq 0$$

Solution:

First calculate the sample proportions.

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{234}{1{,}150} \approx 0.203$$

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{327}{1,320} \approx 0.248$$

The pooled proportion is

$$\hat{p} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$$

$$\hat{p} = \frac{0.203(1,150) + 0.248(1,320)}{1,150 + 1,320}$$

$$\hat{p} \approx 0.227$$

Then the z-statistic is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z \approx \frac{0.203 - 0.248}{\sqrt{0.227(1-0.227)\left(\frac{1}{1,150} + \frac{1}{1,320}\right)}}$$

$$z \approx \frac{-0.045}{\sqrt{0.227(0.773)\left(\frac{1}{1,150} + \frac{1}{1,320}\right)}}$$

$$z \approx -2.663$$

3. A cinema owner wants to know whether there's a difference in the number of boys and girls who watched a new movie last week. She randomly sampled 76 boys and 75 girls and found that 45 boys and 58 girls watched the movie. What can she conclude about the difference of proportions at a 99% confidence level?

Solution:

The owner is running a two-tailed test, so her hypothesis statements will be

$$H_0 : p_1 - p_2 = 0$$

$$H_a : p_1 - p_2 \neq 0$$

The sample proportions are

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{45}{76} \approx 0.592$$

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{58}{75} \approx 0.773$$

Then the pooled proportion is

$$\hat{p} = \frac{45 + 58}{76 + 75}$$

$$\hat{p} \approx 0.682$$

and the test statistic is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z = \frac{0.592 - 0.773}{\sqrt{0.682(1-0.682)\left(\frac{1}{76} + \frac{1}{75}\right)}}$$

$$z = \frac{-0.181}{\sqrt{0.682(0.318)\left(\frac{1}{76} + \frac{1}{75}\right)}}$$

$$z \approx -2.39$$

For a two-tailed test at 99% confidence, the critical value will be $z = -2.58$. Because $-2.58 < -2.39$, the cinema owner fails to reject the null hypothesis, and can't conclude that more boys than girls watched the new movie last week.

4. A store owner believes that women spend at least 22% more in his store than men. He randomly chooses 64 visitors, 32 men and 32 women, and finds that 14 men spent more than $\$100$, while 23 women spent more than $\$100$. Using a p-value approach, what can he conclude at a 90% confidence level?

Solution:

Assuming p_1 is the population proportion of women who spend more than $\$100$ and p_2 is the population proportion of men who spend more than $\$100$, the store owner's hypothesis statements for the upper-tailed test will be

$$H_0: p_1 - p_2 \le 0.22$$

$$H_a: p_1 - p_2 > 0.22$$

The sample proportions are

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{23}{32} \approx 0.7188$$

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{14}{32} = 0.4375$$

Then the test statistic is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

$$z \approx \frac{(0.7188 - 0.4375) - 0.22}{\sqrt{\frac{0.7188(1-0.7188)}{32} + \frac{0.4375(1-0.4375)}{32}}}$$

$$z \approx \frac{0.0613}{\sqrt{\frac{0.20212656 + 0.24609375}{32}}}$$

$$z \approx 0.52$$

This z-value gives

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549

We can see that the area to the left of $z \approx 0.52$ is 0.6985. Because this is an upper-tailed test, we're interested in the area to the left of $z \approx 0.52$, so $1 - 0.6985 = 0.3015$. Therefore, $p = 0.3015$. Since $0.3015 > 0.1$, the store owner fails to reject the null hypothesis. There's not enough evidence to conclude that women spend 22% more than men.

■ 5. In a random sample of 60 people under the age of 30, 14% said they're planning to go hiking next month. In a random sample of 75 people older than 50, 23% said they're planning to go hiking next month. Using a critical value approach at a 95% confidence level, is there enough evidence to conclude that a higher proportion of people over age 50 plan to go hiking next month than the proportion of people under 30 who plan to go hiking?

Solution:

If p_1 is the proportion of people under 30 who plan to hike, and p_2 is the proportion of people over 50 who plan to hike, then the hypothesis statements are

$$H_0 : p_1 - p_2 \geq 0$$

$$H_a : p_1 - p_2 < 0$$

The sample proportions are

$$\hat{p}_1 = 0.14$$

$$\hat{p}_2 = 0.23$$

The pooled proportion is

$$\hat{p} = \frac{0.14(60) + 0.23(75)}{60 + 75}$$

$$\hat{p} = 0.19$$

Then the test statistic is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z = \frac{0.14 - 0.23}{\sqrt{0.19(1 - 0.19)\left(\frac{1}{60} + \frac{1}{75}\right)}}$$

$$z = \frac{-0.09}{\sqrt{0.19(0.81)\left(\frac{1}{60} + \frac{1}{75}\right)}}$$

$$z = -1.325$$

For a 95% confidence interval and a lower-tailed test, the critical z-value is $z = -1.65$. Since $-1.325 > -1.65$, we fail to reject the null hypothesis. There's not enough evidence to conclude that the proportion of people over 50

who plan to hike is higher than the proportion of people under 30 who plan to hike.

6. John and Steven are two fitness trainers who want to compare their client satisfaction rate. John chose a random sample of 85 clients and Steven chose a random sample of 72 clients. John found that 89% of his clients were satisfied and Steve found that 91% of his clients were satisfied. Using a critical value approach at a 95% confidence level, is there a significant difference between proportions?

Solution:

John and Steven are running a two-tailed test, so their hypothesis statements are

$$H_0 : p_1 - p_2 = 0$$

$$H_a : p_1 - p_2 \neq 0$$

The sample proportions are

$$\hat{p}_1 = 0.89$$

$$\hat{p}_2 = 0.91$$

The pooled proportion is

$$\hat{p} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$$

$$\hat{p} = \frac{0.89(85) + 0.91(72)}{85 + 72}$$

$$\hat{p} \approx 0.899$$

Then the test statistic is

$$z = \frac{0.89 - 0.91}{\sqrt{0.899(1 - 0.899)\left(\frac{1}{85} + \frac{1}{72}\right)}}$$

$$z = \frac{-0.02}{\sqrt{0.899(0.101)\left(\frac{1}{85} + \frac{1}{72}\right)}}$$

$$z \approx -0.414$$

For a 95% confidence level and a two-tailed test, the critical z-values are $z = \pm 1.96$. Since -0.414 falls between -1.96 and 1.96, we fail to reject the null hypothesis. There's not enough evidence to conclude that there's a significant difference between John and Steven's client satisfaction rate.

W W W . K R I S T A K I N G M A T H . C O M