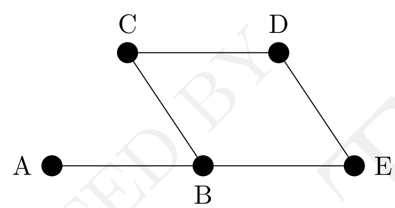
Discrete Mathematics

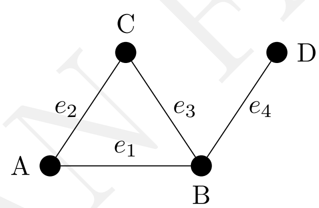
Section 7: Graph Theory

Questions

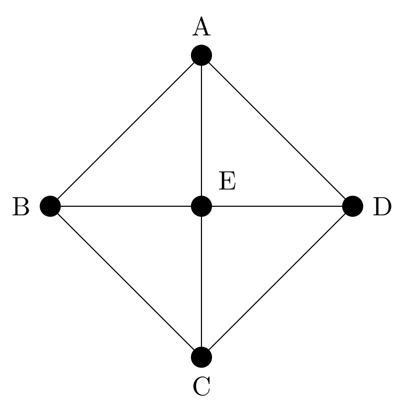
1. Show that in any graph, the sum of the degrees of all vertices is equal to twice the number of edges.
2. Draw a simple graph with 6 vertices where the sum of the degrees of all vertices is 10.
3. Given the graph below, construct its adjacency matrix.



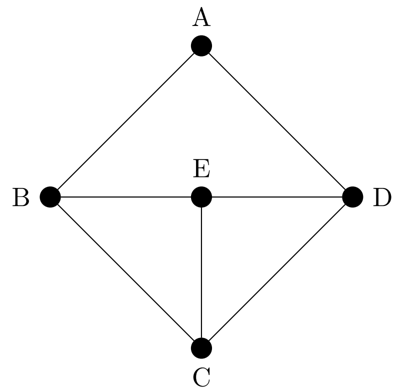
4. Given the graph below, construct its incidence matrix.



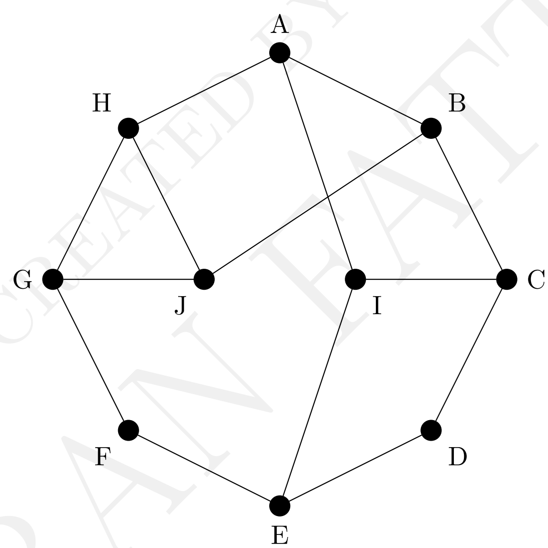
5. Consider the following graph. Determine if there exists an Eulerian circuit, and if so, provide the circuit:



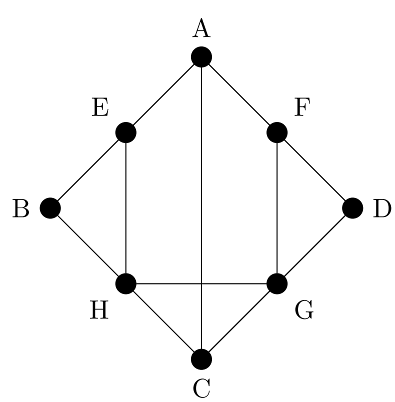
6. Given the following graph, determine if there exists a Hamiltonian path, an Eulerian path, or an Eulerian circuit. If so, provide an example for each:



7. Consider the following graph. Determine if there exists an Eulerian circuit. If so, provide the circuit and justify why it is an Eulerian circuit. If not, explain why an Eulerian circuit does not exist.



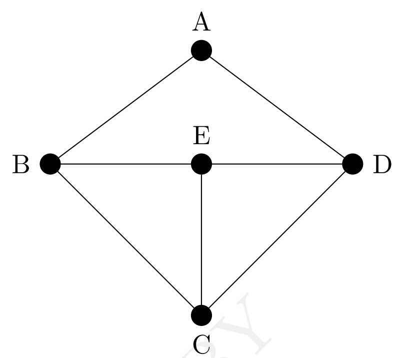
8. Given the following graph, determine if there exists a Hamiltonian circuit or an Eulerian circuit. If they exist, provide the circuit for each:



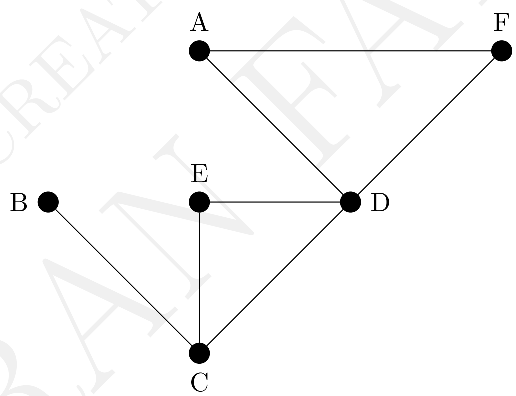
9. Given a connected undirected graph with the following degree sequence: $[3, 3, 3, 3, 2, 2, 1]$, determine if there exists a simple graph that corresponds to this degree sequence. If it does, provide the adjacency matrix of one such graph and prove that the graph is connected and has the given degree sequence.

Degree Sequence: [3, 3, 3, 3, 2, 2, 1]

10. Consider the following graph. Calculate the eccentricity (ecc) of each vertex. Then determine the diameter and radius of the graph.



11. Given the following graph, calculate the eccentricity of each vertex and determine the diameter of the graph.

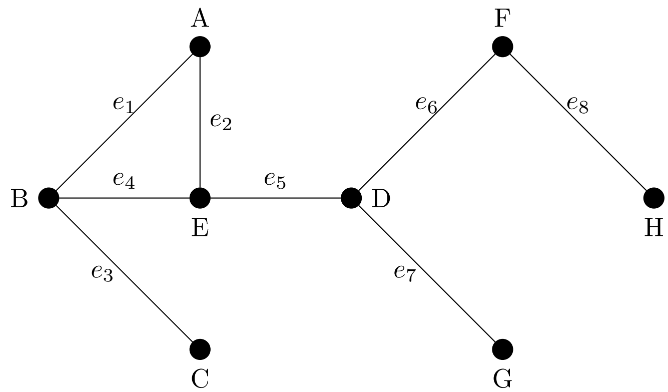


12. Given the following adjacency matrix, draw the graph. Then, calculate the degree of each vertex and find the overall degree of the graph.

Adjacency Matrix

	A	B	C	D	F	G
A	0	1	1	0	0	0
B	1	0	1	1	1	0
C	1	1	0	1	0	0
D	0	1	1	0	1	0
F	0	1	0	1	0	1
G	0	0	0	0	1	0

13. Consider the following graph. Identify all the bridges in the graph.



14. Given the following adjacency and incidence matrices, determine the graph. Then, determine if the graph is connected.

Adjacency Matrix

	A	B	C	D	E	F	G	H	I
A	0	1	0	1	0	0	0	0	0
B	1	0	1	0	1	0	0	0	0
C	0	1	0	1	1	0	0	0	0
D	1	0	1	0	0	1	1	0	0
E	0	1	1	0	0	0	0	0	0
F	0	0	0	1	0	0	0	1	0
G	0	0	0	1	0	0	0	0	1
H	0	0	0	0	0	1	0	0	0
I	0	0	0	0	0	0	1	0	0

Incidence Matrix

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
A	1	1	0	0	0	0	0	0	0	0
B	1	0	1	1	0	0	0	0	0	0
C	0	0	1	0	1	1	0	0	0	0
D	0	1	0	0	1	0	1	1	0	0
E	0	0	0	1	0	1	0	0	0	0
F	0	0	0	0	0	0	1	0	1	0
G	0	0	0	0	0	0	0	1	0	1
H	0	0	0	0	0	0	0	0	1	0
I	0	0	0	0	0	0	0	0	0	1

15. Consider a network of seven cities connected by roads with the following distances:

- The road between City A and City B has a distance of 2 units.
- The road between City A and City D has a distance of 8 units.
- The road between City B and City C has a distance of 1 unit.
- The road between City C and City D has a distance of 1 unit.
- The road between City D and City E has a distance of 3 units.
- The road between City D and City F has a distance of 5 units.
- The road between City D and City G has a distance of 2 units.
- The road between City F and City G has a distance of 1 unit.

Find the shortest path from City A to City F and provide the total path length.

16. Given the following adjacency and incidence matrices, reconstruct the two graphs. Determine if the graphs are isomorphic.

Adjacency Matrix 1

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	1	0
C	1	1	0	1	1
D	0	1	1	0	1
E	0	0	1	1	0

Incidence Matrix 1

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
A	1	1	0	0	0	0	0
B	1	0	1	1	0	0	0
C	0	0	1	1	1	1	0
D	0	0	0	1	1	0	1
E	0	0	0	0	0	1	1

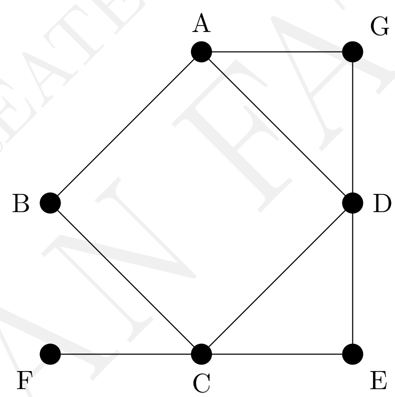
Adjacency Matrix 2

	U	V	W	X	Y
U	0	1	1	0	0
V	1	0	1	1	0
W	1	1	0	1	1
X	0	1	1	0	1
Y	0	0	1	1	0

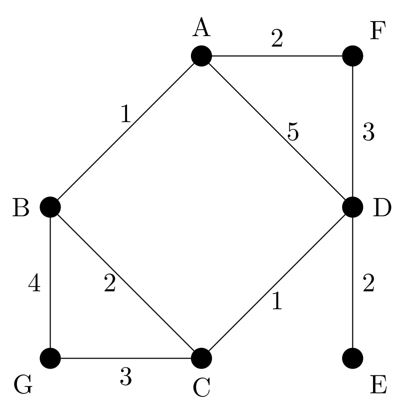
Incidence Matrix 2

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
U	1	1	0	0	0	0	0
V	1	0	1	1	0	0	0
W	0	0	1	1	1	1	0
X	0	0	0	1	1	0	1
Y	0	0	0	0	0	1	1

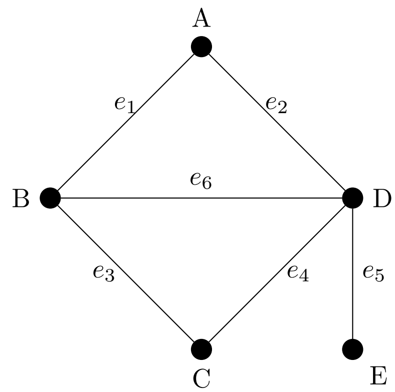
17. Given the graph below, calculate the eccentricity of each vertex, the diameter, and the radius of the graph.



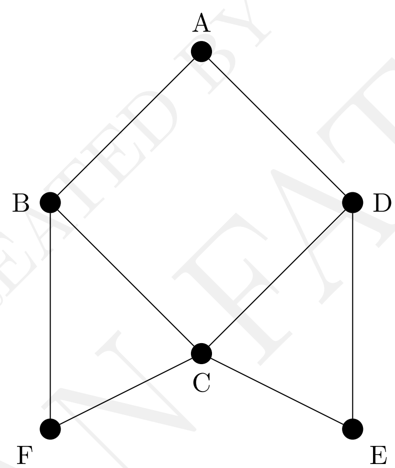
18. Given the graph below, determine the length of the shortest path from vertex A to vertex E.



19. Given the graph below, construct its incidence matrix and determine if there is a Hamiltonian path:



20. Given the graph below, what is the minimum number of edges that must be removed to make the graph disconnected?



21. Given the adjacency matrix of a graph below determine the total degree of the graph.

Adjacency Matrix:

	A	B	C	D	E	F	G	H
A	0	1	1	0	0	0	0	1
B	1	0	1	1	0	0	0	0
C	1	1	0	1	1	0	0	0
D	0	1	1	0	0	1	0	0
E	0	0	1	0	0	1	1	0
F	0	0	0	1	1	0	1	1
G	0	0	0	0	1	1	0	1
H	1	0	0	0	0	1	1	0

22. Given the degrees of each vertex in a graph below, determine if the graph is connected. deg(A) = 3, deg(B) = 2, deg(C) = 2, deg(D) = 2, deg(E) = 1, deg(F) = 2, deg(G) = 0, deg(H) = 2.

23. Given the adjacency matrix of a graph below, determine the shortest path from vertex A to vertex G, where each edge has a weight of 1.

Adjacency Matrix

	A	B	C	D	E	F	G
A	0	1	0	0	1	1	0
B	1	0	0	0	0	1	0
C	0	0	0	1	0	1	1
D	0	0	1	0	1	0	0
E	1	0	0	1	0	1	0
F	1	1	1	0	1	0	0
G	0	0	1	0	0	0	0

24. Given the adjacency matrix of the graph below, determine if there is a Hamiltonian path.

Adjacency Matrix:

	A	B	C	D	E	F
A	0	1	0	1	0	1
B	1	0	1	0	0	0
C	0	1	0	1	0	0
D	1	0	1	0	1	1
E	0	0	0	1	0	0
F	1	0	0	1	0	0

25. Consider the following graph represented by its adjacency matrix.

Adjacency Matrix:

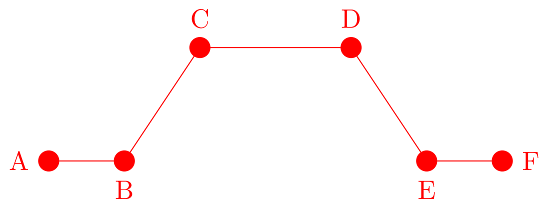
	A	B	C	D	E	F	G
A	0	1	1	0	0	0	0
B	1	0	1	1	0	0	0
C	1	1	0	1	0	1	0
D	0	1	1	0	1	0	0
E	0	0	0	1	0	0	0
F	0	0	1	0	0	0	1
G	0	0	0	0	0	1	0

Using the given adjacency matrix, answer the following questions:

Questions and Answers

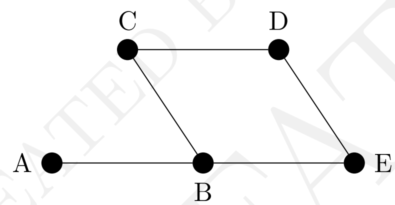
1. Show that in any graph, the sum of the degrees of all vertices is equal to twice the number of edges.
   Each edge contributes exactly 2 to the sum of the degrees of the vertices, as each edge has two endpoints. Thus, if there are m edges, the sum of the degrees of all vertices is $2m$.

2. Draw a simple graph with 6 vertices where the sum of the degrees of all vertices is 10.
   Below is a representation of a graph that satisfies the given conditions:



   The degrees of the vertices (from left to right) are 1, 2, 2, 2, 2, 1. Summing these degrees: 1 + 2 + 2 + 2 + 2 + 1 = 10.

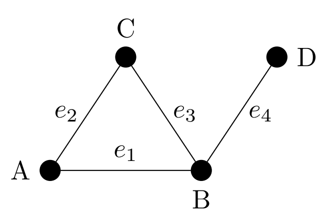
3. Given the graph below, construct its adjacency matrix.



   The adjacency matrix is:

	A	B	C	D	E
A	0	1	0	0	0
B	1	0	1	0	1
C	0	1	0	1	0
D	0	0	1	0	1
E	0	1	0	1	0

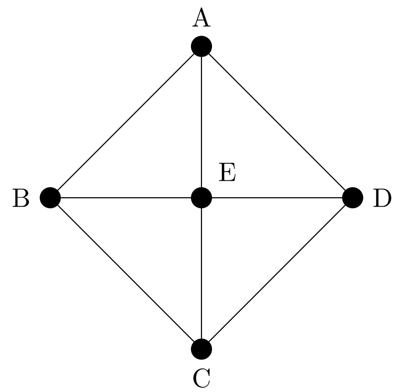
4. Given the graph below, construct its incidence matrix.



   The incidence matrix is:

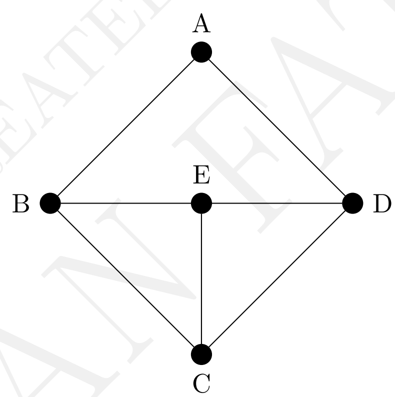
	e_1	e_2	e_3	e_4
A	1	1	0	0
B	1	0	1	1
C	0	1	1	0
D	0	0	0	1

5. Consider the following graph. Determine if there exists an Eulerian circuit, and if so, provide the circuit:



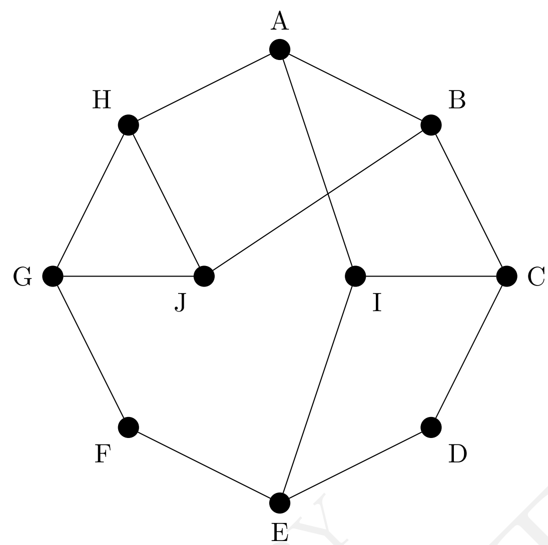
An Eulerian circuit is a circuit that uses every edge of a graph exactly once and starts and ends at the same vertex. For a graph to have an Eulerian circuit, every vertex must have an even degree. In this graph, vertices A, B, C, and D have odd degrees, so an Eulerian circuit does not exist.

6. Given the following graph, determine if there exists a Hamiltonian path, an Eulerian path, or an Eulerian circuit. If so, provide an example for each:



A Hamiltonian path is a path that visits each vertex exactly once. One example of a Hamiltonian path is: A, B, E, C, D.
An Eulerian path is a path that visits each edge exactly once. For a graph to have an Eulerian path, it must have exactly zero or two vertices of odd degree. In this graph, vertices A, B, C, and D have odd degrees, which means there are more than two vertices of odd degree. Therefore, no Eulerian path exists.
An Eulerian circuit is a circuit that visits each edge exactly once and returns to the starting vertex. For a graph to have an Eulerian circuit, all vertices must have even degrees. Since vertices A, B, C, and D have odd degrees, the graph does not have an Eulerian circuit.

7. Consider the following graph. Determine if there exists an Eulerian circuit. If so, provide the circuit and justify why it is an Eulerian circuit. If not, explain why an Eulerian circuit does not exist.

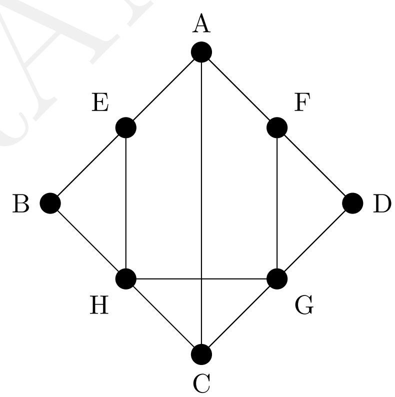


An Eulerian circuit is a circuit that uses every edge of a graph exactly once and starts and ends at the same vertex. For a graph to have an Eulerian circuit, every vertex must have an even degree. In this graph, the degrees of the vertices are as follows:

- Vertices A, B, C, E, G, H, I, and J have degree 3 (odd).
- Vertices D and F have degree 2 (even).

Since there are multiple vertices with odd degrees (A, B, C, E, G, H, I, J), an Eulerian circuit does not exist.

8. Given the following graph, determine if there exists a Hamiltonian circuit or an Eulerian circuit. If they exist, provide the circuit for each:



A Hamiltonian circuit is a circuit that visits each vertex exactly once and returns to the starting vertex. In this graph, one example of a Hamiltonian circuit is: A, E, B, H, C, G, D, F, A.

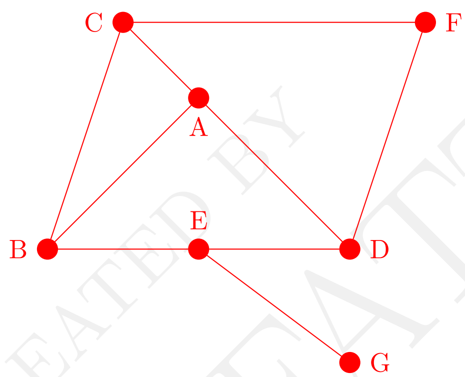
An Eulerian circuit is a circuit that uses every edge of a graph exactly once and returns to the starting vertex. For a graph to have an Eulerian circuit, all vertices must have even degrees. This graph does not have an Eulerian circuit since it has vertices with odd degrees, such as vertex E, which has a degree of 3.

9. Given a connected undirected graph with the following degree sequence: $[3, 3, 3, 3, 2, 2, 1]$, determine if there exists a simple graph that corresponds to this degree sequence. If it does, provide the adjacency matrix of one such graph and prove that the graph is connected and has the given degree sequence.

Degree Sequence: [3, 3, 3, 3, 2, 2, 1]

One possible adjacency matrix for this graph is:

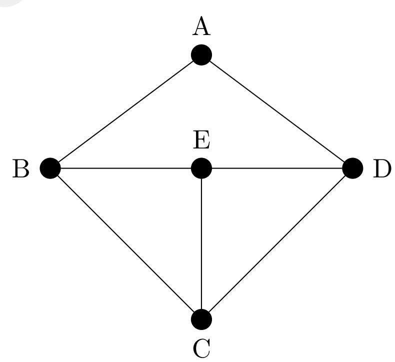
	A	B	C	D	E	F	G
A	0	1	1	1	0	0	0
B	1	0	1	0	1	0	0
C	1	1	0	0	0	1	0
D	1	0	0	0	1	1	0
E	0	1	0	1	0	0	1
F	0	0	1	1	0	0	0
G	0	0	0	0	1	0	0



To prove that the graph is connected and has the given degree sequence, we first confirm that each vertex has the specified degree based on the adjacency matrix. The degree of each vertex matches the provided degree sequence, ensuring that the graph is structured correctly.

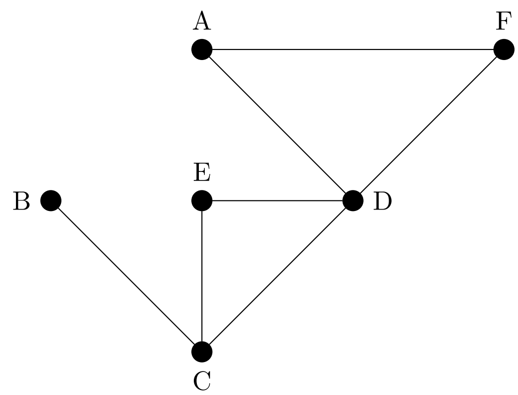
Additionally, the graph is connected because starting from any vertex, it is possible to reach all other vertices either directly or through a sequence of edges. This demonstrates that the graph satisfies the connectedness condition, as there exists a path between any pair of vertices.

10. Consider the following graph. Calculate the eccentricity (ecc) of each vertex. Then determine the diameter and radius of the graph.



The eccentricity (ecc) of a vertex is the distance from that vertex to a vertex farthest from it. The ecc of all the vertices (A, B, C, D, E) equal to 2.
The diameter of the graph is the maximum eccentricity, which is 2. The radius of the graph is the minimum eccentricity, which is also 2.

11. Given the following graph, calculate the eccentricity of each vertex and determine the diameter of the graph.



The eccentricity (ecc) of a vertex is the maximum distance from that vertex to any other vertex in the graph. The eccentricity of vertex A = 3, B = 3, C = 2, D = 2, E = 2 and F = 3. The diameter of the graph is the maximum eccentricity of any vertex in the graph, which is 3.

12. Given the following adjacency matrix, draw the graph. Then, calculate the degree of each vertex and find the overall degree of the graph.

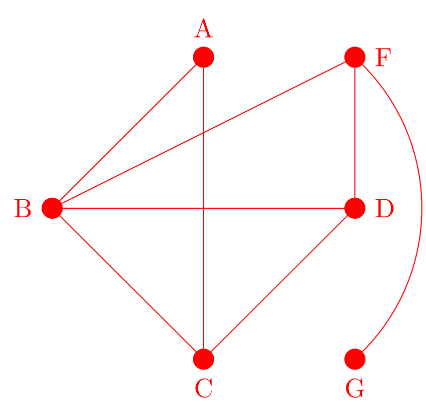
Adjacency Matrix

	A	B	C	D	F	G
A	0	1	1	0	0	0
B	1	0	1	1	1	0
C	1	1	0	1	0	0
D	0	1	1	0	1	0
F	0	1	0	1	0	1
G	0	0	0	0	1	0

The graph corresponding to the given adjacency matrix, named Graph G, is illustrated on the right.

Next, we calculate the degree of each vertex as follows:

- Vertex A: degree 2 (connected to B, C)
- Vertex B: degree 4 (connected to A, C, D, F)
- Vertex C: degree 3 (connected to A, B, D)
- Vertex D: degree 3 (connected to B, C, F)
- Vertex F: degree 3 (connected to B, D, G)
- Vertex G: degree 1 (connected to F)



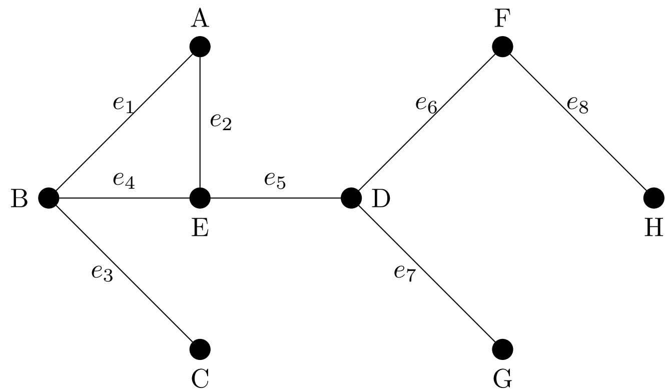
Graph G

The total degree of the graph, which is the sum of the degrees of all vertices, is:

$$2 + 4 + 3 + 3 + 3 + 1 = 16$$

Thus, the overall degree of the graph is 16.

13. Consider the following graph. Identify all the bridges in the graph.



A bridge is an edge whose removal disconnects a graph. The bridges in the graph are: e_3, e_5, e_6, e_7 and e_8.

14. Given the following adjacency and incidence matrices, determine the graph. Then, determine if the graph is connected.

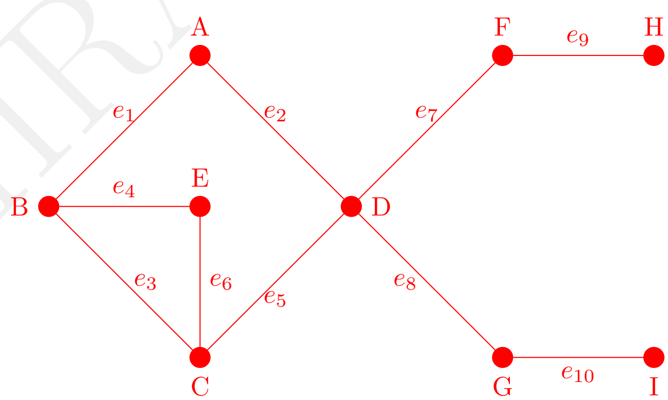
Adjacency Matrix

	A	B	C	D	E	F	G	H	I
A	0	1	0	1	0	0	0	0	0
B	1	0	1	0	1	0	0	0	0
C	0	1	0	1	1	0	0	0	0
D	1	0	1	0	0	1	1	0	0
E	0	1	1	0	0	0	0	0	0
F	0	0	0	1	0	0	0	1	0
G	0	0	0	1	0	0	0	0	1
H	0	0	0	0	0	1	0	0	0
I	0	0	0	0	0	0	1	0	0

Incidence Matrix

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
A	1	1	0	0	0	0	0	0	0	0
B	1	0	1	1	0	0	0	0	0	0
C	0	0	1	0	1	1	0	0	0	0
D	0	1	0	0	1	0	1	1	0	0
E	0	0	0	1	0	1	0	0	0	0
F	0	0	0	0	0	0	1	0	1	0
G	0	0	0	0	0	0	0	1	0	1
H	0	0	0	0	0	0	0	0	1	0
I	0	0	0	0	0	0	0	0	0	1

The corresponding graph is:

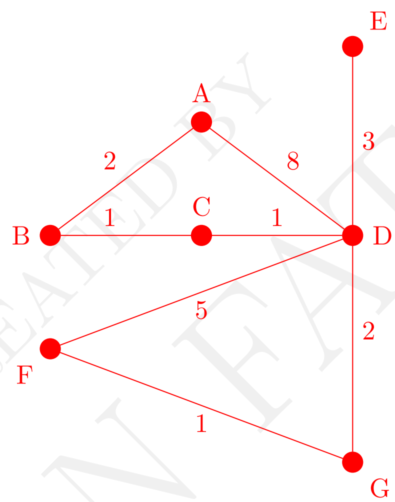


The graph is connected. All vertices are reachable from one another.

15. Consider a network of seven cities connected by roads with the following distances:

   - The road between City A and City B has a distance of 2 units.
   - The road between City A and City D has a distance of 8 units.
   - The road between City B and City C has a distance of 1 unit.
   - The road between City C and City D has a distance of 1 unit.
   - The road between City D and City E has a distance of 3 units.
   - The road between City D and City F has a distance of 5 units.
   - The road between City D and City G has a distance of 2 units.
   - The road between City F and City G has a distance of 1 unit.

   Find the shortest path from City A to City F and provide the total path length.
   The shortest path from City A to City F is: A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ G $\rightarrow$ F, with a total path length of 7 units. Refer to the graph below for further clarification.



16. Given the following adjacency and incidence matrices, reconstruct the two graphs. Determine if the graphs are isomorphic.

Adjacency Matrix 1

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	1	0
C	1	1	0	1	1
D	0	1	1	0	1
E	0	0	1	1	0

Incidence Matrix 1

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
A	1	1	0	0	0	0	0
B	1	0	1	1	0	0	0
C	0	0	1	1	1	1	0
D	0	0	0	1	1	0	1
E	0	0	0	0	0	1	1

Adjacency Matrix 2

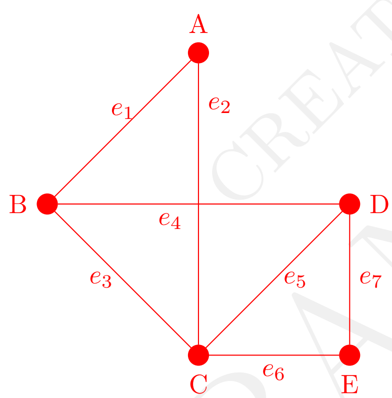
	U	V	W	X	Y
U	0	1	1	0	0
V	1	0	1	1	0
W	1	1	0	1	1
X	0	1	1	0	1
Y	0	0	1	1	0

Incidence Matrix 2

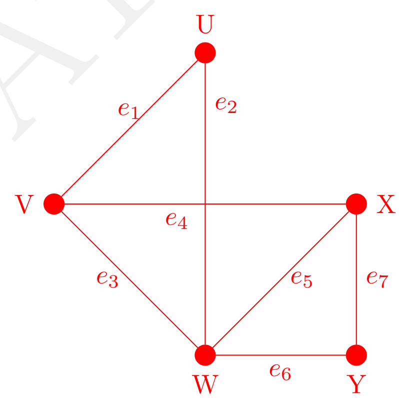
	e_1	e_2	e_3	e_4	e_5	e_6	e_7
U	1	1	0	0	0	0	0
V	1	0	1	1	0	0	0
W	0	0	1	1	1	1	0
X	0	0	0	1	1	0	1
Y	0	0	0	0	0	1	1

The two graphs reconstructed from the matrices are:

Graph 1



Graph 2



To determine if the graphs are isomorphic, we need to verify if there is a one-to-one correspondence between the vertices of the two graphs that preserves adjacency. This means that if two vertices are connected by an edge in one graph, their corresponding vertices in the other graph must also be connected. Examining the reconstructed graphs, we first check the degree sequence of each vertex:

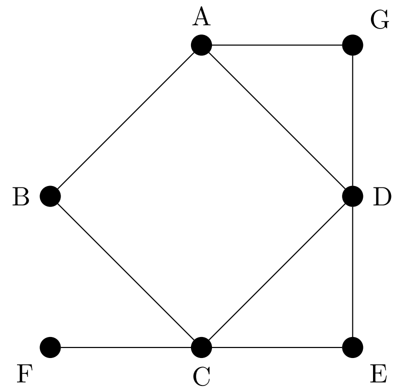
Graph 1	**Graph 2**
A: 2	U: 2
B: 3	V: 3
C: 3	W: 3
D: 3	X: 3
E: 2	Y: 2

Both graphs have the same degree sequence: 2, 3, 3, 3, 2. This suggests that the graphs might be isomorphic. After checking all possible vertex mappings, we conclude that the two graphs are indeed isomorphic, as they have the same structure and can be mapped one-to-one onto each other.

One possible mapping is:

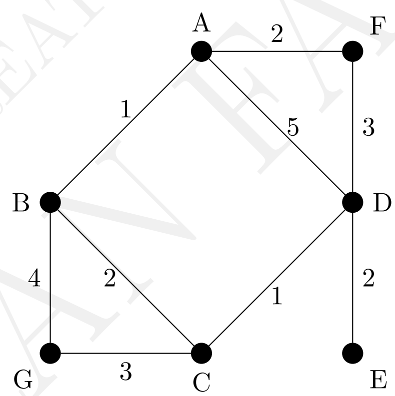
$$A \leftrightarrow U, \quad B \leftrightarrow V, \quad C \leftrightarrow W, \quad D \leftrightarrow X, \quad E \leftrightarrow Y.$$

17. Given the graph below, calculate the eccentricity of each vertex, the diameter, and the radius of the graph.



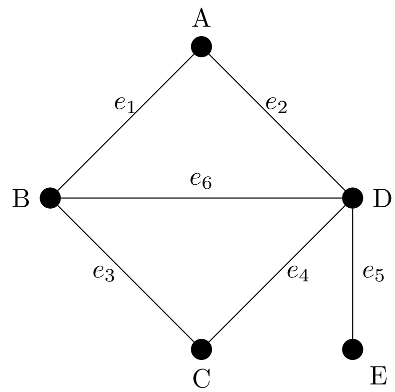
The eccentricity of each vertex is: ecc(A) = 3, ecc(B) = 2, ecc(C) = 2, ecc(D) = 2, ecc(E) = 2, ecc(F) = 3, ecc(G) = 3.
The diameter of the graph is the maximum eccentricity, which is 3.
The radius of the graph is the minimum eccentricity, which is 2.

18. Given the graph below, determine the length of the shortest path from vertex A to vertex E.



Using Dijkstra's Algorithm, the shortest path from A to E is A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ E with a total length of 6.

19. Given the graph below, construct its incidence matrix and determine if there is a Hamiltonian path:

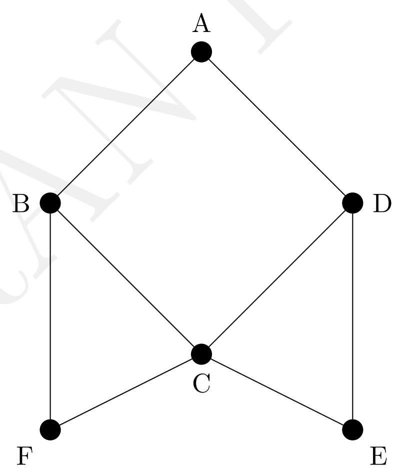


The incidence matrix is shown below:

	e_1	e_2	e_3	e_4	e_5	e_6
A	1	1	0	0	0	0
B	1	0	1	0	0	1
C	0	0	1	1	0	0
D	0	1	0	1	1	1
E	0	0	0	0	1	0

There is a Hamiltonian path: A $\to$ B $\to$ C $\to$ D $\to$ E.

20. Given the graph below, what is the minimum number of edges that must be removed to make the graph disconnected?



The minimum number of edges that must be removed to make the graph disconnected is 2.

21. Given the adjacency matrix of a graph below determine the total degree of the graph.

Adjacency Matrix:

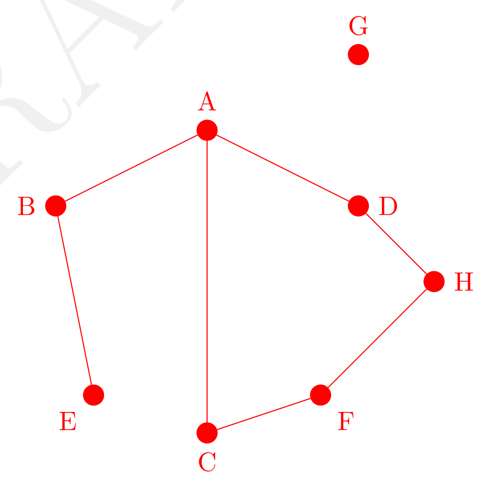
	A	B	C	D	E	F	G	H
A	0	1	1	0	0	0	0	1
B	1	0	1	1	0	0	0	0
C	1	1	0	1	1	0	0	0
D	0	1	1	0	0	1	0	0
E	0	0	1	0	0	1	1	0
F	0	0	0	1	1	0	1	1
G	0	0	0	0	1	1	0	1
H	1	0	0	0	0	1	1	0

The total degree of a graph is the sum of the degrees of all its vertices. The degree of a vertex is the number of edges connected to it. Using the given adjacency matrix, the degrees of the vertices are:

- deg(A) = 3 (connections to B, C, H)
- deg(B) = 3 (connections to A, C, D)
- deg(C) = 4 (connections to A, B, D, E)
- deg(D) = 3 (connections to B, C, F)
- deg(E) = 3 (connections to C, F, G)
- deg(F) = 4 (connections to D, E, G, H)
- deg(G) = 3 (connections to E, F, H)
- deg(H) = 3 (connections to A, F, G)

The total degree of the graph $= 3 + 3 + 4 + 3 + 3 + 4 + 3 + 3 = 26$

22. Given the degrees of each vertex in a graph below, determine if the graph is connected. deg(A) = 3, deg(B) = 2, deg(C) = 2, deg(D) = 2, deg(E) = 1, deg(F) = 2, deg(G) = 0, deg(H) = 2.
To determine if the graph is connected, we need to reconstruct the graph from the given degrees and check if there is a path between every pair of vertices.



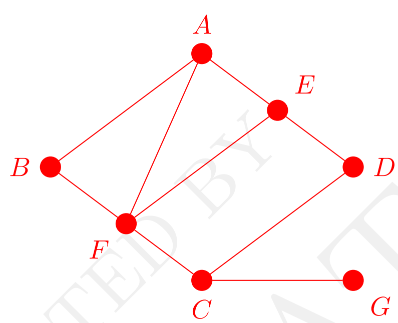
In this graph, vertex G is not connected to any other vertex, meaning there is no path between G and any other vertex. Hence, the graph is not connected.

23. Given the adjacency matrix of a graph below, determine the shortest path from vertex A to vertex G, where each edge has a weight of 1.

Adjacency Matrix

	A	B	C	D	E	F	G
A	0	1	0	0	1	1	0
B	1	0	0	0	0	1	0
C	0	0	0	1	0	1	1
D	0	0	1	0	1	0	0
E	1	0	0	1	0	1	0
F	1	1	1	0	1	0	0
G	0	0	1	0	0	0	0

To determine the shortest path from vertex A to vertex G, we need to reconstruct the graph.



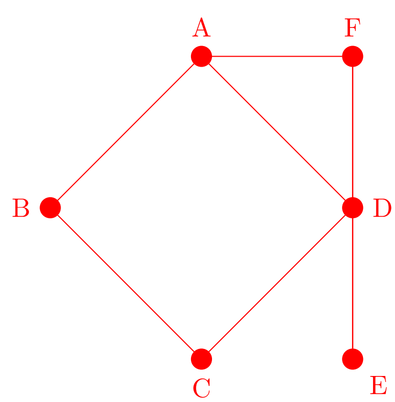
Using the graph, the shortest path from A to G is: $A \to F \to C \to G$, with a total path length of 3.

24. Given the adjacency matrix of the graph below, determine if there is a Hamiltonian path.

Adjacency Matrix:

	A	B	C	D	E	F
A	0	1	0	1	0	1
B	1	0	1	0	0	0
C	0	1	0	1	0	0
D	1	0	1	0	1	1
E	0	0	0	1	0	0
F	1	0	0	1	0	0

To determine the existence of a Hamiltonian path, we begin by reconstructing the graph as below. A Hamiltonian path exists: $F \to A \to B \to C \to D \to E$.



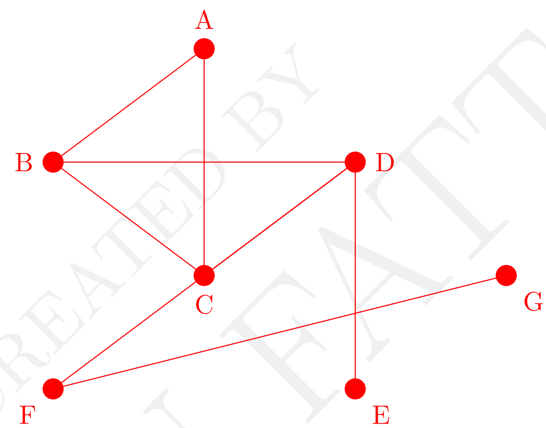
25. Consider the following graph represented by its adjacency matrix.

Adjacency Matrix:

	A	B	C	D	E	F	G
A	0	1	1	0	0	0	0
B	1	0	1	1	0	0	0
C	1	1	0	1	0	1	0
D	0	1	1	0	1	0	0
E	0	0	0	1	0	0	0
F	0	0	1	0	0	0	1
G	0	0	0	0	0	1	0

Using the given adjacency matrix, answer the following questions:

(a) Reconstruct the graph.
The graph reconstructed from the adjacency matrix is:



(b) Calculate the total degree of the graph.
The total degree of the graph is 16.

(c) Determine if the graph is connected.
The graph is connected because there is a path between every pair of vertices.

(d) Find the shortest path from vertex A to vertex G, where each edge has a weight of 1.
The shortest path from A to G is A $\rightarrow$ C $\rightarrow$ F $\rightarrow$ G with a total path length of 3.

(e) Determine if there is a Hamiltonian path in the graph.
There is no Hamiltonian path in the graph because vertices A and G have degree 1, making it impossible to include all vertices exactly once without repetition.

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