Discrete Mathematics

Section 5: Functions

Questions

1. Given the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2x+1}{x-1}$:
   (a) Write the domain of f.
   (b) Find $f(0)$, $f(2)$, and $f(-1)$.
   (c) The function $f(x) = \frac{2x+1}{x-1}$ has an inverse. Find the inverse function, specifying any necessary restrictions.
   (d) Find $f^{-1}(0)$ and $f^{-1}(5)$.
   (e) If f is restricted to the domain $[2, \infty)$, determine whether the function f has an inverse on this domain and, if so, find $f^{-1}(3)$ and $f^{-1}(0)$.
   (f) Find the ordered pairs for $f(x) = \frac{2x+1}{x-1}$ at $x \in \{-2, -1, 0, 1, 2\}$.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$:
   (a) Write the domain of f.
   (b) Find $f(1)$, $f(-1)$, and $f(2)$.
   (c) What is the range of f?
   (d) Does the function f have an inverse? If so, find $f^{-1}(5)$ and $f^{-1}(7)$.
   (e) Find $f^{-1}(1)$ and $f^{-1}(0)$.
   (f) Represent f as a set of ordered pairs for $x \in \{1, -1, 2, -2, 0\}$.
3. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \frac{1}{x}$:
   (a) Write the domain of h.
   (b) Find $h(1)$, $h(-1)$, and $h(2)$.
   (c) What is the range of h?
   (d) Does the function h have an inverse? If so, find $h^{-1}(2)$ and $h^{-1}(0.25)$.
   (e) Find $h^{-1}(-2)$ and $h^{-1}(-0.25)$.
   (f) Represent h as a set of ordered pairs for $x \in \{1, -1, 2, -2, 0.5\}$.
4. Consider the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = \sin(x)$:
   (a) Write the domain of p.
   (b) Find $p(0)$, $p(\pi/2)$, and $p(\pi)$.
   (c) What is the range of p for $x \in [0, 2\pi]$?
   (d) Determine whether the function p has an inverse in the interval $[0, \pi]$. If it does, find $p^{-1}(1)$ and $p^{-1}(0)$.
   (e) For the interval $[0, \pi]$, find $p^{-1}(0.5)$ and $p^{-1}(1)$.
   (f) Represent p as a set of ordered pairs for $x \in \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$.

5. Consider the function $r : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ defined by $r(x) = \frac{3x-4}{x-2}$:
   (a) Write the domain of r.
   (b) Find $r(0)$, $r(1)$, and $r(-1)$.
   (c) What is the range of r?
   (d) Determine whether the function r has an inverse. If so, find $r^{-1}(0)$ and $r^{-1}(2)$.
   (e) Find $r^{-1}(5)$ and $r^{-1}(3)$.
   (f) Represent r as a set of ordered pairs for $x \in \{0, 1, 2, 3, -1\}$.
6. Given the functions $f(x) = 2x + 3$ and $g(x) = x^2 - 1$:
   (a) Find the expression for $(f \circ g)(x)$.
   (b) Find the expression for $(g \circ f)(x)$.
   (c) Evaluate $(f \circ g)(2)$.
   (d) Evaluate $(g \circ f)(1)$.
   (e) Determine the domain of $(f \circ g)(x)$.
7. Given the functions $h(x) = 3x - 4$ and $k(x) = \frac{x+2}{3}$:
   (a) Find the expression for $(h \circ k)(x)$.
   (b) Find the expression for $(k \circ h)(x)$.
   (c) Evaluate $(h \circ k)(1)$.
   (d) Evaluate $(k \circ h)(2)$.
   (e) Determine the domain of $(k \circ h)(x)$.
8. Consider the functions $p(x) = x^2 + 1$ and $q(x) = \sqrt{x-1}$ (for $x \geq 1$):
   (a) Find the expression for $(p \circ q)(x)$.
   (b) Find the expression for $(q \circ p)(x)$.
   (c) Evaluate $(p \circ q)(3)$.
   (d) Evaluate $(q \circ p)(2)$.
   (e) Evaluate $(p \circ q \circ p)(2)$ and $(q \circ p \circ q)(5)$.
9. Given the functions $r(x) = \frac{1}{x}$ and $s(x) = x - 3$:
   (a) Find the expression for $(r \circ s)(x)$.
   (b) Find the expression for $(s \circ r)(x)$.
   (c) Evaluate $(r \circ s)(4)$.
   (d) Evaluate $(s \circ r)(1)$.
   (e) Evaluate $(r \circ s \circ r)(2)$ and $(s \circ r \circ s)(4)$.
10. Given the functions $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$:
   (a) Find the expression for $(f \circ g)(x)$.
   (b) Find the expression for $(g \circ f)(x)$.
   (c) Evaluate $(f \circ g)(4)$.
   (d) Evaluate $(g \circ f)(3)$.
   (e) Evaluate $(f \circ g \circ f)(2)$ and $(g \circ f \circ g)(9)$.

11. Given the functions $f(x) = 2x + 3$ and $g(x) = x^2 - 1$:
    (a) Find the expression for $(f + g)(x)$.
    (b) Find the expression for $(f - g)(x)$.
    (c) Find the expression for $(f \cdot g)(x)$.
    (d) Determine the domain of $\left(\frac{f}{g}\right)(x)$.
12. Given the functions $h(x) = x^3$ and $k(x) = 4x - 5$:
    (a) Find the expression for $(h + k)(x)$.
    (b) Find the expression for $(h - k)(x)$.
    (c) Find the expression for $(h \cdot k)(x)$.
    (d) Find the expression for $\left(\frac{h}{k}\right)(x)$.
    (e) Determine the domain of $\left(\frac{h}{k}\right)(x)$.
13. Given the functions $p(x) = \sqrt{x}$ and $q(x) = x^2 + 1$:
    (a) Find the expression for $(p + q)(x)$.
    (b) Find the expression for $(p - q)(x)$.
    (c) Find the expression for $(p \cdot q)(x)$.
    (d) Find the expression for $\left(\frac{p}{q}\right)(x)$.
    (e) Determine the domain of $\left(\frac{p}{q}\right)(x)$.
14. Given the functions $r(x) = \frac{1}{x}$ and $s(x) = x + 2$:
    (a) Find the expression for $(r + s)(x)$.
    (b) Find the expression for $(r - s)(x)$.
    (c) Find the expression for $(r \cdot s)(x)$.
    (d) Find the expression for $\left(\frac{r}{s}\right)(x)$.
    (e) Determine the domain of $\left(\frac{r}{s}\right)(x)$.
15. Given the functions $f(x) = x^2 + 2$ and $g(x) = x - 3$:
    (a) Find the expression for $(f + g)(x)$.
    (b) Find the expression for $(f - g)(x)$.
    (c) Find the expression for $(f \cdot g)(x)$.
    (d) Find the expression for $\left(\frac{f}{g}\right)(x)$.
    (e) Determine the domain of $\left(\frac{f}{g}\right)(x)$.
16. Determine whether the function $f(x) = x^3 - x$ is even, odd, or neither.
17. Determine whether the function $g(x) = x^4 - x^2 + 1$ is even, odd, or neither.
18. Determine whether the function $h(x) = x^3 + x^2$ is even, odd, or neither.
19. Given the functions $f(x) = \frac{2x-1}{x+2}$ and $g(x) = \frac{1}{x^2-4}$, find the expression for $(f \circ g)(x)$ and determine its domain.
20. Determine whether the function $h(x) = \frac{x^4-4x^2+1}{x^2+1}$ is even, odd, or neither.

21. Given the functions $f(x) = 3x^2 - 1$ and $g(x) = \sqrt{x-1}$ (for $x \geq 1$), find the inverse of the composite function $(g \circ f)(x)$.
22. Find the domain of the function $f(x) = \frac{\sqrt{3x-2}}{x^2-9}$.
23. Perform the long division of the functions $f(x) = x^3 - 6x^2 + 11x - 6$ by $g(x) = x - 2$.
24. Divide the function $f(x) = 3x^5 - 2x^4 + x^3 - 4x^2 + x - 6$ by $g(x) = x^2 - 2x + 1$ using long division.
25. Use long division to divide $f(x) = 4x^3 - 5x^2 + 2x - 3$ by $g(x) = x - 1$.

Questions and Answers

1. Given the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2x+1}{x-1}$:

   (a) Write the domain of f.
   The domain of f is $\mathbb{R} \setminus \{1\}$ because the function has a denominator that is zero when $x = 1$.

   (b) Find $f(0)$, $f(2)$, and $f(-1)$.
   $f(0) = \frac{2(0)+1}{0-1} = -1, \quad f(2) = \frac{2(2)+1}{2-1} = 5, \quad \text{and} \quad f(-1) = \frac{2(-1)+1}{-1-1} = \frac{-2+1}{-2} = \frac{-1}{-2} = \frac{1}{2}.$

   (c) The function $f(x) = \frac{2x+1}{x-1}$ has an inverse. Find the inverse function, specifying any necessary restrictions.
   The function $f(x) = \frac{2x+1}{x-1}$ has an inverse, but we must first note that $f(x)$ is undefined at $x = 1$. Therefore, the domain of the original function is $\mathbb{R} \setminus \{1\}$. Now, let's find the inverse function.
   Set $y = f(x)$, so:
   $$y = \frac{2x+1}{x-1}.$$
   Multiply both sides by $(x-1)$:
   $$y(x-1) = 2x+1.$$
   Expand the left-hand side:
   $$yx - y = 2x + 1.$$
   Now, isolate x on one side:
   $$yx - 2x = y + 1.$$
   Factor out x:
   $$x(y-2) = y+1.$$
   Finally, solve for x:
   $$x = \frac{y+1}{y-2}.$$
   Thus, the inverse function is $f^{-1}(y) = \frac{y+1}{y-2}$ for $y \neq 2$. The domain of the inverse function is $y \neq 2$, so the range of the original function is $\mathbb{R} \setminus \{2\}$.

   (d) Find $f^{-1}(0)$ and $f^{-1}(5)$.
   Using the inverse function $f^{-1}(y) = \frac{y+1}{y-2}$, we get:
   $$f^{-1}(0) = \frac{0+1}{0-2} = \frac{1}{-2} = -\frac{1}{2}, \quad f^{-1}(5) = \frac{5+1}{5-2} = \frac{6}{3} = 2.$$

   (e) If f is restricted to the domain $[2, \infty)$, determine whether the function f has an inverse on this domain and, if so, find $f^{-1}(3)$ and $f^{-1}(0)$.
   On the restricted domain $[2, \infty)$, the function remains injective and thus has an inverse. Using the inverse function $f^{-1}(y) = \frac{y+1}{y-2}$, we get:
   $$f^{-1}(3) = \frac{3+1}{3-2} = \frac{4}{1} = 4, \quad f^{-1}(0) = \frac{0+1}{0-2} = -\frac{1}{2}.$$

   (f) Find the ordered pairs for $f(x) = \frac{2x+1}{x-1}$ at $x \in \{-2, -1, 0, 1, 2\}$.
   $f = \{(-2, 1), (-1, \frac{1}{2}), (0, -1), (1, \text{undefined}), (2, 5)\}.$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$:

   (a) Write the domain of f.
   The domain of f is $\mathbb{R}$.

   (b) Find $f(1)$, $f(-1)$, and $f(2)$.
   $f(1) = 5$, $f(-1) = 1$, $f(2) = 7$.

   (c) What is the range of f?
   The range of f is $\mathbb{R}$ because it is a linear function with no restrictions on x.

   (d) Does the function f have an inverse? If so, find $f^{-1}(5)$ and $f^{-1}(7)$.
   The function f does have an inverse because it is a bijective linear function. The inverse is $f^{-1}(y) = \frac{y-3}{2}$. Therefore, $f^{-1}(5) = 1$ and $f^{-1}(7) = 2$.

   (e) Find $f^{-1}(1)$ and $f^{-1}(0)$.
   $f^{-1}(1) = -1$ and $f^{-1}(0) = \frac{-3}{2}$.

   (f) Represent f as a set of ordered pairs for $x \in \{1, -1, 2, -2, 0\}$.
   $f = \{(1, 5), (-1, 1), (2, 7), (-2, -1), (0, 3)\}$.

3. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \frac{1}{x}$:

   (a) Write the domain of h.
   The domain of h is $\mathbb{R} \setminus \{0\}$, as $h(x)$ is undefined when $x = 0$.

   (b) Find $h(1)$, $h(-1)$, and $h(2)$.
   $h(1) = 1$, $h(-1) = -1$, $h(2) = \frac{1}{2}$.

   (c) What is the range of h?
   The range of h is $\mathbb{R} \setminus \{0\}$, since $h(x)$ can take any real value except 0.

   (d) Does the function h have an inverse? If so, find $h^{-1}(2)$ and $h^{-1}(0.25)$.
   The function $h(x) = \frac{1}{x}$ does have an inverse. In fact, the inverse of h is the function itself, meaning $h^{-1}(y) = \frac{1}{y}$. Therefore, $h^{-1}(2) = \frac{1}{2}$ and $h^{-1}(0.25) = \frac{1}{0.25} = 4$.

   (e) Find $h^{-1}(-2)$ and $h^{-1}(-0.25)$.
   $h^{-1}(-2) = \frac{1}{-2}$ and $h^{-1}(-0.5) = \frac{1}{-0.5} = -2$.

   (f) Represent h as a set of ordered pairs for $x \in \{1, -1, 2, -2, 0.5\}$.
   $h = \{(1, 1), (-1, -1), (2, 0.5), (-2, -0.5), (0.5, 2)\}$.

4. Consider the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = \sin(x)$:

   (a) Write the domain of p.
   The domain of p is $\mathbb{R}$.

   (b) Find $p(0)$, $p(\pi/2)$, and $p(\pi)$.
   $p(0) = 0$, $p(\pi/2) = 1$, $p(\pi) = 0$.

   (c) What is the range of p for $x \in [0, 2\pi]$?
   The range of p for $x \in [0, 2\pi]$ is $[-1, 1]$.

   (d) Determine whether the function p has an inverse in the interval $[0, \pi]$. If it does, find $p^{-1}(1)$ and $p^{-1}(0)$.
   The function p has an inverse in the interval $[0, \pi]$. The inverse function is $p^{-1}(y) = \arcsin(y)$. Therefore, $p^{-1}(1) = \pi/2$ and $p^{-1}(0) = 0$.

   (e) For the interval $[0, \pi]$, find $p^{-1}(0.5)$ and $p^{-1}(1)$.
   For $p^{-1}(0.5)$, we have $\arcsin(0.5) = \frac{\pi}{6}$. For $p^{-1}(1)$, we have $\arcsin(1) = \frac{\pi}{2}$. Both values lie within the interval $[0, \pi]$.

   (f) Represent p as a set of ordered pairs for $x \in \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$.
   $p = \{(0, 0), (\pi/2, 1), (\pi, 0), (3\pi/2, -1), (2\pi, 0)\}$.

5. Consider the function $r : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ defined by $r(x) = \frac{3x-4}{x-2}$:

   (a) Write the domain of r.
   The domain of r is $\mathbb{R} \setminus \{2\}$ because the denominator cannot be zero.

   (b) Find $r(0)$, $r(1)$, and $r(-1)$.
   $r(0) = 2$, $r(1) = 1$, and $r(-1) = \frac{7}{3}$.

   (c) What is the range of r?
   The range of r is $\mathbb{R} \setminus \{3\}$ because the function can take all real values except 3.

   (d) Determine whether the function r has an inverse. If so, find $r^{-1}(0)$ and $r^{-1}(2)$.
   The function r has an inverse because it is one-to-one. The inverse is $r^{-1}(y) = \frac{2y-4}{y-3}$.
   Therefore, $r^{-1}(0) = \frac{4}{3}$ and $r^{-1}(2) = 0$.

   (e) Find $r^{-1}(5)$ and $r^{-1}(3)$.
   For $r^{-1}(5)$, we have $r^{-1}(5) = 3$. For $r^{-1}(3)$, the inverse is undefined because the denominator becomes zero.

   (f) Represent r as a set of ordered pairs for $x \in \{0, 1, 2, 3, -1\}$.
   $r = \{(0, 2), (1, 1), (2, \text{undefined}), (3, 5), (-1, \frac{7}{3})\}$.

6. Given the functions $f(x) = 2x + 3$ and $g(x) = x^2 - 1$:

   (a) Find the expression for $(f \circ g)(x)$.
   $(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = 2(x^2 - 1) + 3 = 2x^2 + 1$.

   (b) Find the expression for $(g \circ f)(x)$.
   $(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 - 1 = 4x^2 + 12x + 8$.

   (c) Evaluate $(f \circ g)(2)$.
   $(f \circ g)(2) = f(g(2)) = f(2^2 - 1) = f(3) = 2(3) + 3 = 9$.

   (d) Evaluate $(g \circ f)(1)$.
   $(g \circ f)(1) = g(f(1)) = g(2(1) + 3) = g(5) = 5^2 - 1 = 24$.

   (e) Determine the domain of $(f \circ g)(x)$.
   The domain of $(f \circ g)(x)$ is $\mathbb{R}$.

7. Given the functions $h(x) = 3x - 4$ and $k(x) = \frac{x+2}{3}$:

   (a) Find the expression for $(h \circ k)(x)$.
   $(h \circ k)(x) = h(k(x)) = h\left(\frac{x+2}{3}\right) = x - 2$.

   (b) Find the expression for $(k \circ h)(x)$.
   $(k \circ h)(x) = k(h(x)) = k(3x - 4) = x - \frac{2}{3}$.

   (c) Evaluate $(h \circ k)(1)$.
   $(h \circ k)(1) = h(k(1)) = h\left(\frac{1+2}{3}\right) = h(1) = -1$.

   (d) Evaluate $(k \circ h)(2)$.
   $(k \circ h)(2) = k(h(2)) = k(3(2) - 4) = k(2) = \frac{4}{3}$.

   (e) Determine the domain of $(k \circ h)(x)$.
   The domain of $(k \circ h)(x)$ is $\mathbb{R}$.

8. Consider the functions $p(x) = x^2 + 1$ and $q(x) = \sqrt{x-1}$ (for $x \geq 1$):

   (a) Find the expression for $(p \circ q)(x)$.
   $(p \circ q)(x) = p(q(x)) = p(\sqrt{x-1}) = (\sqrt{x-1})^2 + 1 = x - 1 + 1 = x.$
   Since the function $p(x) = x^2 + 1$ and the function $q(x) = \sqrt{x-1}$, the square root undoes the squaring from $p(x)$, leaving x as the result. Therefore, $(p \circ q)(x) = x$.

   (b) Find the expression for $(q \circ p)(x)$.
   $(q \circ p)(x) = q(p(x)) = q(x^2 + 1) = x.$

   (c) Evaluate $(p \circ q)(3)$.
   $(p \circ q)(3) = p(q(3)) = p(\sqrt{2}) = 3.$

   (d) Evaluate $(q \circ p)(2)$.
   $(q \circ p)(2) = q(p(2)) = q(5) = 2.$

   (e) Evaluate $(p \circ q \circ p)(2)$ and $(q \circ p \circ q)(5)$.
   $(p \circ q \circ p)(2) = p(q(p(2))) = p(q(2^2 + 1)) = p(q(5)) = p(\sqrt{5-1}) = p(2) = 2^2 + 1 = 5.$
   $(q \circ p \circ q)(5) = q(p(q(5))) = q(p(\sqrt{5-1})) = q(p(2)) = q(2^2 + 1) = q(5) = \sqrt{5-1} = 2.$

9. Given the functions $r(x) = \frac{1}{x}$ and $s(x) = x - 3$:

   (a) Find the expression for $(r \circ s)(x)$.
   $(r \circ s)(x) = r(s(x)) = \frac{1}{x-3}.$

   (b) Find the expression for $(s \circ r)(x)$.
   $(s \circ r)(x) = s(r(x)) = \frac{1}{x} - 3.$

   (c) Evaluate $(r \circ s)(4)$.
   $(r \circ s)(4) = r(s(4)) = r(1) = 1.$

   (d) Evaluate $(s \circ r)(1)$.
   $(s \circ r)(1) = s(r(1)) = s(1) = -2.$

   (e) Evaluate $(r \circ s \circ r)(2)$ and $(s \circ r \circ s)(4)$.
   $(r \circ s \circ r)(2) = r(s(r(2))) = r(s(\frac{1}{2})) = r(\frac{1}{2} - 3) = r(-\frac{5}{2}) = -\frac{2}{5},$
   $(s \circ r \circ s)(4) = s(r(s(4))) = s(r(1)) = s(1) = -2.$

10. Given the functions $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$:

   (a) Find the expression for $(f \circ g)(x)$.
   $(f \circ g)(x) = f(g(x)) = x + 2.$

   (b) Find the expression for $(g \circ f)(x)$.
   $(g \circ f)(x) = g(f(x)) = \sqrt{x^2 + 2}.$

   (c) Evaluate $(f \circ g)(4)$.
   $(f \circ g)(4) = f(g(4)) = f(\sqrt{4}) = 6.$

   (d) Evaluate $(g \circ f)(3)$.
   $(g \circ f)(3) = g(f(3)) = \sqrt{11}.$

   (e) Evaluate $(f \circ g \circ f)(2)$ and $(g \circ f \circ g)(9)$.
   $(f \circ g \circ f)(2) = f(g(f(2))) = f(g(6)) = f(\sqrt{6}) = (\sqrt{6})^2 + 2 = 8,$
   $(g \circ f \circ g)(9) = g(f(g(9))) = g(f(3)) = g(11) = \sqrt{11}.$

11. Given the functions $f(x) = 2x + 3$ and $g(x) = x^2 - 1$:

(a) Find the expression for $(f + g)(x)$.
$(f + g)(x) = x^2 + 2x + 2.$

(b) Find the expression for $(f - g)(x)$.
$(f - g)(x) = -x^2 + 2x + 4.$

(c) Find the expression for $(f \cdot g)(x)$.
$(f \cdot g)(x) = 2x^3 + 3x^2 - 2x - 3.$

(d) Determine the domain of $\left(\frac{f}{g}\right)(x)$.

The domain of $\left(\frac{f}{g}\right)(x)$ is $\mathbb{R} \setminus \{-1, 1\}$ because the denominator $g(x) = x^2 - 1$ cannot be zero. Solving $g(x) = 0$ gives $x^2 - 1 = 0$, which results in $x = \pm 1$. These values must be excluded from the domain. Since $f(x) = 2x + 3$ is defined for all real numbers, the only restrictions come from $g(x)$.

12. Given the functions $h(x) = x^3$ and $k(x) = 4x - 5$:

(a) Find the expression for $(h + k)(x)$.
$(h + k)(x) = x^3 + 4x - 5.$

(b) Find the expression for $(h - k)(x)$.
$(h - k)(x) = x^3 - 4x + 5.$

(c) Find the expression for $(h \cdot k)(x)$.
$(h \cdot k)(x) = 4x^4 - 5x^3.$

(d) Find the expression for $\left(\frac{h}{k}\right)(x)$.
$\left(\frac{h}{k}\right)(x) = \frac{x^3}{4x-5}.$

(e) Determine the domain of $\left(\frac{h}{k}\right)(x)$.
The domain of $\left(\frac{h}{k}\right)(x)$ is $\mathbb{R} \setminus \left\{\frac{5}{4}\right\}$ because the function $k(x) = 4x - 5$ must not be zero in the denominator. Setting $k(x) = 0$ gives $4x - 5 = 0$, solving for x gives $x = \frac{5}{4}$. Thus, $x = \frac{5}{4}$ is excluded from the domain. Since $h(x) = x^3$ is defined for all real numbers, the only restriction is from $k(x)$.

13. Given the functions $p(x) = \sqrt{x}$ and $q(x) = x^2 + 1$:

(a) Find the expression for $(p + q)(x)$.
$(p + q)(x) = \sqrt{x} + x^2 + 1.$

(b) Find the expression for $(p - q)(x)$.
$(p - q)(x) = \sqrt{x} - (x^2 + 1) = \sqrt{x} - x^2 - 1.$

(c) Find the expression for $(p \cdot q)(x)$.
$(p \cdot q)(x) = \sqrt{x} \cdot (x^2 + 1) = x^{5/2} + \sqrt{x}.$

(d) Find the expression for $\left(\frac{p}{q}\right)(x)$.

$\left(\frac{p}{q}\right)(x) = \frac{\sqrt{x}}{x^2+1}.$

(e) Determine the domain of $\left(\frac{p}{q}\right)(x)$.

The domain of $\left(\frac{p}{q}\right)(x)$ is $[0, \infty)$ because the function $p(x) = \sqrt{x}$ requires $x \geq 0$ (since square roots are defined only for non-negative numbers), and the denominator $q(x) = x^2 + 1$ is always positive for all real x, meaning there are no additional restrictions.

14. Given the functions $r(x) = \frac{1}{x}$ and $s(x) = x + 2$:

   (a) Find the expression for $(r + s)(x)$.
   $(r + s)(x) = \frac{1}{x} + x + 2.$

   (b) Find the expression for $(r - s)(x)$.
   $(r - s)(x) = \frac{1}{x} - (x + 2) = \frac{1}{x} - x - 2.$

   (c) Find the expression for $(r \cdot s)(x)$.
   $(r \cdot s)(x) = \frac{1}{x} \cdot (x + 2) = 1 + \frac{2}{x}.$

   (d) Find the expression for $\left(\frac{r}{s}\right)(x)$.
   $\left(\frac{r}{s}\right)(x) = \frac{\frac{1}{x}}{x+2} = \frac{1}{x(x+2)}.$

   (e) Determine the domain of $\left(\frac{r}{s}\right)(x)$.
   The domain of $\left(\frac{r}{s}\right)(x)$ is $\mathbb{R} \setminus \{0, -2\}$ because the denominator $x(x + 2)$ cannot be zero. Solving $x(x + 2) = 0$ gives $x = 0$ or $x = -2$. These values must be excluded from the domain.

15. Given the functions $f(x) = x^2 + 2$ and $g(x) = x - 3$:

   (a) Find the expression for $(f + g)(x)$.
   $(f + g)(x) = x^2 + x - 1.$

   (b) Find the expression for $(f - g)(x)$.
   $(f - g)(x) = x^2 - x + 5.$

   (c) Find the expression for $(f \cdot g)(x)$.
   $(f \cdot g)(x) = x^3 - 3x^2 + 2x - 6.$

   (d) Find the expression for $\left(\frac{f}{g}\right)(x)$.
   $\left(\frac{f}{g}\right)(x) = \frac{x^2+2}{x-3}.$

   (e) Determine the domain of $\left(\frac{f}{g}\right)(x)$.
   The domain of $\left(\frac{f}{g}\right)(x)$ is $\mathbb{R} \setminus \{3\}$ because the denominator $g(x) = x - 3$ must not be zero. Solving $x - 3 = 0$ gives $x = 3$, which must be excluded from the domain. Since $f(x) = x^2 + 2$ is defined for all real numbers, the only restriction comes from $g(x)$.

16. Determine whether the function $f(x) = x^3 - x$ is even, odd, or neither.
   A function $f(x)$ is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. For $f(x) = x^3 - x$:
   $$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$
   Since $f(-x) = -f(x)$, the function $f(x) = x^3 - x$ is odd.

17. Determine whether the function $g(x) = x^4 - x^2 + 1$ is even, odd, or neither.
   For $g(x) = x^4 - x^2 + 1$:
   $$g(-x) = (-x)^4 - (-x)^2 + 1 = x^4 - x^2 + 1 = g(x).$$
   Since $g(-x) = g(x)$, the function $g(x) = x^4 - x^2 + 1$ is even.

18. Determine whether the function $h(x) = x^3 + x^2$ is even, odd, or neither.
   For $h(x) = x^3 + x^2$:
   $$h(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2.$$
   Now, calculate $-h(x)$:
   $$-h(x) = -(x^3 + x^2) = -x^3 - x^2.$$
   Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, the function $h(x) = x^3 + x^2$ is neither even nor odd.

19. Given the functions $f(x) = \frac{2x-1}{x+2}$ and $g(x) = \frac{1}{x^2-4}$, find the expression for $(f \circ g)(x)$ and determine its domain.
First, calculate $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2-4}\right)$. This gives:

$$f\left(\frac{1}{x^2-4}\right) = \frac{2\left(\frac{1}{x^2-4}\right) - 1}{\frac{1}{x^2-4} + 2} = \frac{\frac{2}{x^2-4} - 1}{\frac{1}{x^2-4} + 2}.$$

Now simplify the numerator and denominator:

$$\text{Numerator: } \frac{2}{x^2-4} - 1 = \frac{2 - (x^2-4)}{x^2-4} = \frac{-x^2+6}{x^2-4},$$

$$\text{Denominator: } \frac{1}{x^2-4} + 2 = \frac{1 + 2(x^2-4)}{x^2-4} = \frac{2x^2-7}{x^2-4}.$$

Thus,

$$(f \circ g)(x) = \frac{-x^2+6}{2x^2-7}.$$

For the domain, we exclude values that make the denominator zero in both $g(x)$ and $f(g(x))$.

Note: When finding the domain of the composition, it is essential to exclude the values that make the inner function $g(x)$ undefined, as these inputs don't yield valid outputs for $g(x)$ and thus cannot be passed to $f(x)$. Therefore, we must exclude $x = \pm 2$, as these values make the denominator of $g(x)$ zero. Additionally, we exclude the values that make the denominator of $f(g(x))$ zero (i.e., $2x^2 - 7 = 0$, or $x^2 = \frac{7}{2}$), which gives $x \neq \pm\sqrt{\frac{7}{2}}$. Thus, the domain is:

$$x \in \mathbb{R} \setminus \left\{\pm 2, \pm\sqrt{\frac{7}{2}}\right\}.$$

20. Determine whether the function $h(x) = \frac{x^4-4x^2+1}{x^2+1}$ is even, odd, or neither.
First, find $h(-x)$:

$$h(-x) = \frac{(-x)^4 - 4(-x)^2 + 1}{(-x)^2+1} = \frac{x^4 - 4x^2 + 1}{x^2+1} = h(x).$$

Since $h(-x) = h(x)$, the function is even.

21. Given the functions $f(x) = 3x^2 - 1$ and $g(x) = \sqrt{x-1}$ (for $x \geq 1$), find the inverse of the composite function $(g \circ f)(x)$.
First, find $(g \circ f)(x) = g(f(x)) = g(3x^2-1) = \sqrt{3x^2-2}$. Now solve for x in terms of y:

$$y = \sqrt{3x^2-2} \implies y^2 = 3x^2 - 2 \implies 3x^2 = y^2 + 2 \implies x^2 = \frac{y^2+2}{3} \implies x = \sqrt{\frac{y^2+2}{3}}.$$

Thus, the inverse function is:

$$(g \circ f)^{-1}(y) = \sqrt{\frac{y^2+2}{3}}.$$

22. Find the domain of the function $f(x) = \frac{\sqrt{3x-2}}{x^2-9}$.
First, the square root in the numerator requires $3x - 2 \geq 0$, which gives $x \geq \frac{2}{3}$. The denominator must not be zero, so $x \neq \pm 3$. Thus, the domain is:

$$x \in \left[\frac{2}{3}, \infty\right) \setminus \{3, -3\}.$$

Therefore, the domain is all real numbers greater than or equal to $\frac{2}{3}$, except $x = 3$ and $x = -3$.

23. Perform the long division of the functions $f(x) = x^3 - 6x^2 + 11x - 6$ by $g(x) = x - 2$.

	x^2	$-4x$	$+3$	
$x-2$	x^3	$-6x^2$	$+11x$	-6
	x^3	$-2x^2$		
		$-4x^2$	$+11x$	-6
		$-4x^2$	$+8x$	
			$+3x$	-6
			$+3x$	-6
				0

Quotient: $x^2 - 4x + 3$, Remainder: 0.
Therefore:

$$\frac{x^3 - 6x^2 + 11x - 6}{x - 2} = x^2 - 4x + 3$$

24. Divide the function $f(x) = 3x^5 - 2x^4 + x^3 - 4x^2 + x - 6$ by $g(x) = x^2 - 2x + 1$ using long division.

	$3x^3$	$+4x^2$	$+6x$	$+4$		
x^2-2x+1	$3x^5$	$-2x^4$	x^3	$-4x^2$	x	-6
	$3x^5$	$-6x^4$	$+3x^3$			
		$4x^4$	$-2x^3$	$-4x^2$	x	-6
		$4x^4$	$-8x^3$	$+4x^2$		
			$6x^3$	$-8x^2$	x	-6
			$6x^3$	$-12x^2$	$+6x$	
				$4x^2$	$-5x$	-6
				$4x^2$	$-8x$	$+4$
					$3x$	-10

Quotient: $3x^3 + 4x^2 + 6x + 4$, Remainder: $3x - 10$.
Therefore:

$$\frac{3x^5 - 2x^4 + x^3 - 4x^2 + x - 6}{x^2 - 2x + 1} = 3x^3 + 4x^2 + 6x + 4 + \frac{3x - 10}{x^2 - 2x + 1}$$

25. Use long division to divide $f(x) = 4x^3 - 5x^2 + 2x - 3$ by $g(x) = x - 1$.

	$4x^2$	$-x$	$+1$	
$x-1$	$4x^3$	$-5x^2$	$+2x$	-3
	$4x^3$	$-4x^2$		
		$-x^2$	$+2x$	-3
		$-x^2$	$+x$	
			$+x$	-3
			$+x$	-1
				-2

Quotient: $4x^2 - x + 1$, Remainder: -2.
Therefore:

$$\frac{4x^3 - 5x^2 + 2x - 3}{x - 1} = 4x^2 - x + 1 + \frac{-2}{x - 1}$$

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