Discrete Mathematicsy

Section 9: Combinatorics

Questions

1. Simplify the expression $\frac{(n+2)!}{n!}$ and evaluate it for $n = 5$.
2. If $5! + 6! - \frac{7!}{8!}$ is simplified, what is the result?
3. Using the Fundamental Counting Principle, determine how many ways you can arrange 5 books on a shelf if you have 3 different titles and each book can be repeated.
4. How many different 4-digit numbers can be formed using the digits 1, 2, 3, 4, 5 if repetition of digits is not allowed?
5. In how many ways can a committee of 4 be chosen from a group of 10 people if one specific person must be included?
6. Calculate the number of permutations of the word "MATHEMATICS" when all letters are used.
7. Show that in any group of 13 people, at least two will have been born in the same month.
8. How many ways can 4 different math books and 3 different science books be arranged on a shelf if all math books must be together and all science books must be together?
9. A password consists of 3 English letters followed by 2 digits. How many possible passwords can be created if repetition is allowed for both letters and digits?
10. How many ways can a class of 10 students be divided into 2 groups of 5 students each?
11. Calculate the number of ways to arrange 8 people in a row if two specific people must not sit next to each other.
12. Find the number of ways to choose 5 cards from a standard deck of 52 cards such that there are exactly 3 aces.
13. Calculate the number of distinct permutations of the letters in the word "FACTORIAL".
14. Determine the number of ways to select 4 objects from 7 distinct objects if one particular object must always be included.
15. Using the Fundamental Counting Principle, determine how many ways you can choose a 4-course meal from a menu of 6 appetizers, 8 main courses, 5 desserts, and 3 drinks.
16. How many ways can the digits 1, 2, 3, 4, 5, 6, 7 be arranged so that the digits 1 and 2 are not next to each other?
17. Using the Pigeonhole Principle, show that in any set of 6 numbers chosen from $\{1, 2, 3, ..., 10\}$, there are at least two numbers whose sum is 11.

18. Calculate the number of ways to arrange 5 boys and 5 girls in a line such that no two boys are next to each other.
19. Determine the number of ways to distribute 8 different books among 4 students if each student must receive exactly 2 books.
20. How many ways can you arrange 9 people in a circle such that two specific people are always next to each other?
21. Calculate the number of ways to choose 5 cards from a standard deck of 52 cards such that there are exactly 2 hearts and 3 diamonds.
22. Evaluate the expression $\frac{(2n)!}{(n!)^2}$ for $n = 4$.
23. Determine the number of ways to choose 6 objects from 10 distinct objects if two particular objects must always be included.
24. How many different ways can you assign 4 tasks to 5 people if each task must be assigned to exactly one person?
25. A company needs to distribute 3 promotional red pens and 2 promotional blue pens for an event. If the company has 8 red pens and 6 blue pens available, in how many different ways can they select the 3 red pens and 2 blue pens for distribution? Assume that each pen is uniquely branded, and the selection depends only on which pens are chosen, not the order.

Questions and Answers

1. Simplify the expression $\frac{(n+2)!}{n!}$ and evaluate it for $n = 5$.
   First, simplify the expression:

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$$

   Now, substitute $n = 5$:

$$(5+2)(5+1) = 7 \times 6 = 42$$

2. If $5! + 6! - \frac{7!}{8!}$ is simplified, what is the result?
   First, calculate each factorial:

$$5! = 120, \quad 6! = 720, \quad 7! = 5,040, \quad 8! = 40,320$$

   Now, simplify $\frac{7!}{8!} = \frac{5,040}{40,320} = 0.125$, then calculate:

$$5! + 6! - 0.125 = 120 + 720 - 0.125 = 839.875$$

   Therefore, the final result is approximately 839.88.

3. Using the Fundamental Counting Principle, determine how many ways you can arrange 5 books on a shelf if you have 3 different titles and each book can be repeated.
   First, understand that you have 3 different titles for the books, and each of the 5 positions on the shelf can hold any of these 3 titles. Using the Fundamental Counting Principle, we calculate the number of ways to arrange the books by considering the number of choices for each position independently. For each of the 5 positions on the shelf, you have 3 choices (one for each title). This means for the first position, you have 3 choices; for the second position, you again have 3 choices, and so on. Since the choices are independent, the total number of ways to arrange the books is the product of the number of choices for each position:

$$3 \times 3 \times 3 \times 3 \times 3 = 3^5$$

   Calculate the final result:

$$3^5 = 243$$

   Therefore, there are 243 different ways to arrange 5 books on a shelf if you have 3 different titles and each book can be repeated.

4. How many different 4-digit numbers can be formed using the digits 1, 2, 3, 4, 5 if repetition of digits is not allowed?
   The number of ways to form a 4-digit number without repetition is:

$$5 \times 4 \times 3 \times 2 = 120$$

   Choose the first digit in 5 ways, the second in 4 ways, the third in 3 ways, and the fourth in 2 ways.

5. In how many ways can a committee of 4 be chosen from a group of 10 people if one specific person must be included?
   We use the combination formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to find the number of ways to choose 3 people from 9. If one specific person is included, we need to choose 3 more from the remaining 9 people:

$$\frac{9!}{3!(9-3)!} = \frac{9!}{3!6!} = 84$$

   Therefore the number of ways a committee of 4 can be chosen from a group of 10 people if one specific person is included is 84 ways.

6. Calculate the number of permutations of the word "MATHEMATICS" when all letters are used.
The word "MATHEMATICS" has 11 letters, with some letters repeating. Specifically, the letters M, A, and T each appear twice. Use the formula for permutations of items with repetitions:

$$P = \frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$$

Substituting the values, we get:

$$P = \frac{11!}{2! \times 2! \times 2!}$$

Calculate the factorials:

$$11! = 39,916,800 \quad \text{and} \quad 2! = 2$$

$$P = \frac{39,916,800}{2 \times 2 \times 2} = \frac{39,916,800}{8} = 4,989,600$$

Therefore, the number of permutations of the word "MATHEMATICS" when all letters are used is 4,989,600.

7. Show that in any group of 13 people, at least two will have been born in the same month.
There are 12 months and 13 people. By the Pigeonhole Principle, if more items (in this case, people) are placed into fewer containers (in this case, months) than there are items, at least one container must contain more than one item. Since there are 13 people and only 12 months, at least two people must have been born in the same month. Therefore, in any group of 13 people, at least two will share a birth month.

8. How many ways can 4 different math books and 3 different science books be arranged on a shelf if all math books must be together and all science books must be together?
Treat each set of books (the math books and the science books) as a single "superbook." This reduces the problem to arranging two items (the "superbooks") on the shelf. There are:

$$2! \text{ways to arrange the two sets of books (superbooks).}$$

Next, arrange the books within each "superbook":

$$4! \text{ways to arrange the 4 math books.}$$

$$3! \text{ways to arrange the 3 science books.}$$

Multiply these together to get the total number of arrangements:

$$2! \times 4! \times 3! = 2 \times 24 \times 6 = 288$$

Therefore, the total number of ways to arrange the books is 288.

9. A password consists of 3 English letters followed by 2 digits. How many possible passwords can be created if repetition is allowed for both letters and digits?
Each of the 3 letter positions can be filled with any of the 26 letters in the English alphabet, and repetition is allowed. Therefore, there are 26^3 possible combinations for the letters.

Similarly, each of the 2 digit positions can be filled with any of the 10 digits (0-9), and repetition is allowed. Thus, there are 10^2 possible combinations for the digits.

The total number of possible passwords is the product of the number of choices for the letters and the number of choices for the digits:

$$26^3 \times 10^2 = 17,576 \times 100 = 1,757,600$$

Therefore, there are 1,757,600 possible passwords.

10. How many ways can a class of 10 students be divided into 2 groups of 5 students each?
First, choose 5 students out of 10 to form the first group. The remaining 5 students will automatically form the second group. Since the two groups are indistinguishable (i.e., swapping the groups does not create a new division), we divide by 2 to avoid counting the same division twice:

$$\frac{\binom{10}{5}}{2} = \frac{\frac{10!}{5!5!}}{2} = \frac{252}{2} = 126$$

Therefore, there are 126 ways to divide the class of 10 students into 2 groups of 5 students each.

11. Calculate the number of ways to arrange 8 people in a row if two specific people must not sit next to each other.
First, calculate the total number of arrangements for 8 people:

$$8! = 40,320$$

Next, calculate the number of arrangements where the two specific people are treated as a single unit (as if they are one entity). This reduces the problem to arranging 7 units (the single unit plus the remaining 6 people):

$$7! \times 2! = 5,040 \times 2 = 10,080$$

The 2! accounts for the internal arrangement of the two specific people within their unit (they can switch places).

Finally, subtract the number of invalid arrangements (where the two specific people are next to each other) from the total arrangements to find the number of valid arrangements:

$$40,320 - 10,080 = 30,240$$

Therefore, there are 30,240 ways to arrange 8 people in a row such that two specific people are not sitting next to each other.

12. Find the number of ways to choose 5 cards from a standard deck of 52 cards such that there are exactly 3 aces.
First, choose 3 aces from the 4 available aces:

$$\binom{4}{3} = 4$$

Next, choose 2 non-aces from the remaining 48 cards:

$$\binom{48}{2} = \frac{48 \times 47}{2 \times 1} = 1,128$$

Multiply the two results to get the total number of ways to choose 5 cards with exactly 3 aces:

$$4 \times 1,128 = 4,512$$

Therefore, there are 4,512 ways to choose 5 cards with exactly 3 aces.

13. Calculate the number of distinct permutations of the letters in the word "FACTORIAL".
The word "FACTORIAL" has 9 letters with the letter "A" repeated twice. To find the number of distinct permutations, use the formula:

$$P = \frac{9!}{2!}$$

Calculate the factorials:

$$9! = 362,880 \quad \text{and} \quad 2! = 2$$

Thus, the number of distinct permutations is:

$$P = \frac{362,880}{2} = 181,440$$

Therefore, the number of distinct permutations of the letters in "FACTORIAL" is 181,440.

14. Determine the number of ways to select 4 objects from 7 distinct objects if one particular object must always be included.
First, include the specific object, reducing the problem to choosing 3 additional objects from the remaining 6. The number of ways to choose 3 objects from 6 is given by the combination formula:

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!}$$

Calculate the factorials:

$$6! = 720, \quad 3! = 6$$

Substitute these values back into the formula:

$$\frac{720}{6 \times 6} = \frac{720}{36} = 20$$

Therefore, there are 20 ways to select 4 objects from 7 distinct objects if one particular object must always be included.

15. Using the Fundamental Counting Principle, determine how many ways you can choose a 4-course meal from a menu of 6 appetizers, 8 main courses, 5 desserts, and 3 drinks.
Multiply the number of choices for each course. The number of ways to choose the meal is:

$$6 \times 8 \times 5 \times 3 = 720$$

16. How many ways can the digits 1, 2, 3, 4, 5, 6, 7 be arranged so that the digits 1 and 2 are not next to each other?
First, calculate the total number of arrangements of the digits 1, 2, 3, 4, 5, 6, 7:

$$7! = 5,040$$

Next, calculate the number of arrangements where the digits 1 and 2 are next to each other. To do this, treat the pair (1, 2) as a single unit. This gives us 6 units to arrange (the (1, 2) pair and the remaining 5 digits):

$$6! = 720$$

Since the pair (1, 2) can be arranged as (1, 2) or (2, 1), multiply by 2:

$$6! \times 2 = 720 \times 2 = 1,440$$

Finally, subtract the invalid arrangements (where 1 and 2 are next to each other) from the total arrangements to find the valid arrangements:

$$5,040 - 1,440 = 3,600$$

Therefore, there are 3,600 ways to arrange the digits 1, 2, 3, 4, 5, 6, 7 so that the digits 1 and 2 are not next to each other.

17. Using the Pigeonhole Principle, show that in any set of 6 numbers chosen from $\{1, 2, 3, ..., 10\}$, there are at least two numbers whose sum is 11.
The pairs of numbers that sum to 11 from the set $\{1, 2, 3, \ldots, 10\}$ are: (1, 10), (2, 9), (3, 8), (4, 7), and (5, 6). These are 5 distinct pairs. If we choose 6 numbers from the set $\{1, 2, 3, \ldots, 10\}$, there are only 5 pairs that can sum to 11. By the Pigeonhole Principle, since we have 6 numbers but only 5 pairs, at least one pair must be chosen, meaning that there will be at least two numbers whose sum is 11. This guarantees that among any 6 numbers selected, at least one of these pairs will be included, ensuring a sum of 11.

18. Calculate the number of ways to arrange 5 boys and 5 girls in a line such that no two boys are next to each other.
First, arrange the 5 girls in a line. The number of ways to arrange the 5 girls is:

$$5! = 120$$

After arranging the 5 girls, there are 6 possible gaps where the boys can be placed (one before each girl, four between the girls, and one after the last girl). The 5 boys must occupy 5 out of these 6 gaps. The number of ways to choose 5 gaps out of 6 is given by the combination formula:

$$\binom{6}{5} = 6$$

Now, arrange the boys in the chosen gaps. The number of ways to arrange the 5 boys in these gaps is:

$$5! = 120$$

Thus, the total number of ways to arrange the students are:

$$120 \times 6 \times 120 = 86,400$$

Therefore, the total number of ways to arrange 5 boys and 5 girls in a line such that no two boys are next to each other is 86,400.

19. Determine the number of ways to distribute 8 different books among 4 students if each student must receive exactly 2 books.
First, arrange all 8 different books in a line. The number of ways to arrange 8 books is:

$$8! = 40,320$$

Next, divide the 8 books into 4 groups of 2 books each. The number of ways to do this, accounting for the fact that the order within each group doesn't matter, is:

$$\frac{8!}{(2!)^4} = \frac{40,320}{16} = 2,520$$

Finally, assign the 4 groups to the 4 students. The number of ways to assign the groups is:

$$4! = 24$$

Thus, the total number of ways to distribute the books is:

$$2,520 \times 24 = 60,480$$

Therefore, the number of ways to distribute 8 different books among 4 students, with each student receiving exactly 2 books, is 60,480.

20. How many ways can you arrange 9 people in a circle such that two specific people are always next to each other?
Treat the two specific people as one unit, then arrange the remaining 8 people in a circle:

$$8! = 40,320$$

Arrange the two specific people within their unit:

$$2! = 2$$

Total arrangements:

$$40,320 \times 2 = 80,640$$

21. Calculate the number of ways to choose 5 cards from a standard deck of 52 cards such that there are exactly 2 hearts and 3 diamonds.
To solve this, we need to choose 2 hearts from the 13 hearts and 3 diamonds from the 13 diamonds. First, calculate the number of ways to choose 2 hearts from 13:
$$\binom{13}{2} = \frac{13 \times 12}{2 \times 1} = 78$$
Next, calculate the number of ways to choose 3 diamonds from 13:
$$\binom{13}{3} = \frac{13 \times 12 \times 11}{3 \times 2 \times 1} = 286$$
Multiply the results together to find the total number of ways to choose 5 cards with 2 hearts and 3 diamonds:
$$78 \times 286 = 22,308$$
Therefore, the number of ways to choose 5 cards with exactly 2 hearts and 3 diamonds is 22,308.

22. Evaluate the expression $\frac{(2n)!}{(n!)^2}$ for $n = 4$.
First, substitute $n = 4$ into the expression:
$$\frac{(2 \times 4)!}{(4!)^2} = \frac{8!}{(4!)^2}$$
Calculate the factorials:
$$8! = 40,320$$
$$4! = 24$$
Substitute these values back into the expression:
$$\frac{40,320}{24^2} = \frac{40,320}{576} = 70$$

23. Determine the number of ways to choose 6 objects from 10 distinct objects if two particular objects must always be included.
Since two particular objects must always be included, we are left with choosing the remaining 4 objects from the 8 objects that are left. Now, calculate the number of ways to choose 4 objects from the remaining 8 using the combination formula:
$$\binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$$
Therefore, the number of ways to choose 6 objects from 10 distinct objects, where 2 specific objects are always included, is 70.

24. How many different ways can you assign 4 tasks to 5 people if each task must be assigned to exactly one person?
Each task can be assigned to any of the 5 people independently. This means that for each of the 4 tasks, you have 5 choices, since no task can be shared. Using the multiplication principle (also known as the fundamental counting principle), you multiply the number of choices for each task. For the first task, you have 5 possible people; for the second task, you also have 5 possible people, and so on for all 4 tasks. Therefore, the total number of ways to assign the tasks is:
$$5 \times 5 \times 5 \times 5 = 5^4 = 625$$
Thus, the total number of ways to assign 4 tasks to 5 people is 625.

25. A company needs to distribute 3 promotional red pens and 2 promotional blue pens for an event. If the company has 8 red pens and 6 blue pens available, in how many different ways can they select the 3 red pens and 2 blue pens for distribution? Assume that each pen is uniquely branded, and the selection depends only on which pens are chosen, not the order.
To solve this, we first determine how to select 3 red pens from 8 available red pens. The number of ways to do this is:
$$\binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$
Next, we calculate how to select 2 blue pens from 6 available blue pens:
$$\binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15$$
The total number of ways to choose 3 red pens and 2 blue pens is:
$$56 \times 15 = 840$$
Thus, there are 840 possible ways to select the pens for distribution.

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