Discrete Mathematics

Section 3: Number Theory

Questions

1. Use the definitions of even and odd to justify your answers to the following questions:

   (a) Explain why the product of an even integer and any other integer is even.

   (b) Explain why the sum of an odd integer and an even integer is odd.

   (c) If a and b are integers, explain why $a^2 + b^2$ is even if and only if both a and b are even or both are odd.

   (d) Explain why for any integer n, n^3 has the same parity as n.

   (e) If a and b are integers, explain why $a^2 + 2ab + b^2$ is even if and only if a and b have the same parity.

2. Explain why the square of any odd integer is odd.

3. Explain why the product of any two odd integers is odd.

4. Explain why the sum of any three consecutive integers is divisible by 3.

5. Explain why if n is an even integer, then $n^2 - 1$ is odd.

6. Let n be an integer. Determine if $n^3 + n$ is even or odd for any integer n.

7. Let a, b, and c be integers. Show that if a and b are both even, and c is odd, then $a^2 + b^2 - c^2$ is odd.

8. Let m and n be integers. Given that $m|n$ and $m \neq 0$, determine whether $m^2|n^2$ and justify your answer.

9. Let a and b be integers. Given that $4|(a + b)$ and $4|(a - b)$, determine what can be said about the parity of a and b.

10. Given two integers a and b, show that if their greatest common divisor (GCD) is 1, then the product of a and b is equal to their least common multiple (LCM).

11. Determine if there exists a pair of distinct prime numbers whose sum is also a prime number. Provide examples to justify your answer.

12. Let p be a prime number and a, b, and c be integers. Given that $p|(a^2 + b^2)$ and $p|(a^2 + c^2)$, determine what can be said about $p|(b^2 - c^2)$.

13. Given that n is a product of the first k prime numbers, analyze whether $n + 1$ can be a prime number, providing examples.

14. Determine if the product of any two prime numbers can ever be a prime number. Provide reasoning for your answer.

15. Show that if a and b are two integers such that $\gcd(a, b) = 1$, then $\gcd(a^2, b^2) = 1$. Provide reasoning and an example.

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16. Determine the prime factorization of 540.

17. Explain why the number 1 is not considered a prime number.

18. Find all prime factors of 2310.

19. Identify the largest prime factor of 529.

20. Given the prime factorizations of two numbers, $n = 2^4 \times 3 \times 5$ and $m = 2^3 \times 3^2 \times 7$, determine the highest power of 2 in the factorization of $n \times m$.

21. Find the greatest common divisor (GCD) of 48, 180, and 252 by finding their prime factorizations.

22. Find the least common multiple (LCM) of 12, 15, and 20 by finding their prime factorizations.

23. If the GCD of two numbers a and b is 12 and their LCM is 360, find the product $a \times b$.

24. Explain why the GCD of two consecutive integers is always 1.

25. Given that the GCD of a and b is 5 and their LCM is 100, find a and b given that $a \leq b$.

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Questions and Answers

1. Use the definitions of even and odd to justify your answers to the following questions:

    (a) Explain why the product of an even integer and any other integer is even.

    An even integer can be written as $2k$ for some integer k. If we multiply it by any integer n, we get $2k \cdot n = 2(kn)$. Since kn is an integer, $2(kn)$ is even.

    (b) Explain why the sum of an odd integer and an even integer is odd.

    An odd integer can be written as $2m + 1$ and an even integer as $2n$ for some integers m and n. Their sum is $2m + 1 + 2n = 2(m + n) + 1$. Since $m + n$ is an integer, $2(m + n) + 1$ is odd.

    (c) If a and b are integers, explain why $a^2 + b^2$ is even if and only if both a and b are even or both are odd.

    To determine whether $a^2 + b^2$ is even or odd, we will consider the cases where a and b are either both even, both odd, or one is even and the other is odd.

    Case 1: If a and b are both even, then $a = 2m$ and $b = 2n$ for some integers m and n. Then:

    $$a^2 + b^2 = (2m)^2 + (2n)^2 = 4m^2 + 4n^2 = 4(m^2 + n^2),$$

    which is even.

    Case 2: If a and b are both odd, then $a = 2m + 1$ and $b = 2n + 1$. Then:

    $$a^2 + b^2 = (2m + 1)^2 + (2n + 1)^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 2(2m^2 + 2m + 2n^2 + 2n + 1),$$

    which is also even.

    Case 3: If one is even and the other is odd, say $a = 2m$ and $b = 2n + 1$, then:

    $$a^2 + b^2 = (2m)^2 + (2n + 1)^2 = 4m^2 + 4n^2 + 4n + 1,$$

    which is odd.

    (d) Explain why for any integer n, n^3 has the same parity as n.

    If n is even, then $n = 2k$ for some integer k, so $n^3 = (2k)^3 = 8k^3$, which is even. If n is odd, then $n = 2k + 1$ for some integer k, so $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1$, which is odd.

    (e) If a and b are integers, explain why $a^2 + 2ab + b^2$ is even if and only if a and b have the same parity.

    To determine whether $a^2 + 2ab + b^2$ is even or odd, we will consider the cases where a and b have the same parity (both even or both odd), or different parity (one even, one odd).

    Case 1: If a and b are both even, then $a = 2m$ and $b = 2n$ for some integers m and n. Then:

    $$a^2 + 2ab + b^2 = 4m^2 + 8mn + 4n^2 = 4(m^2 + 2mn + n^2),$$

    which is even.

    Case 2: If a and b are both odd, then $a = 2m + 1$ and $b = 2n + 1$. Then:

    $$\begin{aligned} a^2 + 2ab + b^2 &= (2m + 1)^2 + 2(2m + 1)(2n + 1) + (2n + 1)^2 \\ &= 4m^2 + 4m + 1 + 8mn + 4n^2 + 4n + 1 \\ &= 2(2m^2 + 2m + 4mn + 2n^2 + 2n + 1) \end{aligned}$$

    which is even.

    Case 3: If one is even and the other is odd, say $a = 2m$ and $b = 2n + 1$, then:

    $$a^2 + 2ab + b^2 = (2m)^2 + 2(2m)(2n + 1) + (2n + 1)^2 = 4m^2 + 8mn + 4m + 4n^2 + 4n + 1,$$

    which is odd.

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2. Explain why the square of any odd integer is odd.

An odd integer can be written as $2k + 1$ for some integer k. Squaring it gives:

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

This expression is of the form $2m + 1$ where m is an integer, hence it is odd.

3. Explain why the product of any two odd integers is odd.

Let a and b be odd integers. They can be written as $2m + 1$ and $2n + 1$ respectively for some integers m and n. The product is:

$$(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

This expression is of the form $2k + 1$ where k is an integer, hence it is odd.

4. Explain why the sum of any three consecutive integers is divisible by 3.

Let the three consecutive integers be $n - 1$, n, and $n + 1$. Their sum is $(n - 1) + n + (n + 1) = 3n$. Since $3n$ is divisible by 3, the sum of any three consecutive integers is divisible by 3.

5. Explain why if n is an even integer, then $n^2 - 1$ is odd.

Let n be an even integer, which can be written as $2k$ for some integer k. Then $n^2 - 1 = (2k)^2 - 1 = 4k^2 - 1 = 2(2k^2) - 1$. This expression is of the form $2m - 1$ where m is an integer, hence it is odd.

6. Let n be an integer. Determine if $n^3 + n$ is even or odd for any integer n.

We will check if $n^3 + n$ is even or odd for both cases when n is even and when n is odd.

Case 1: Let n be even. Then, $n = 2k$ for some integer k. We compute $n^3 + n$:

$$n^3 + n = (2k)^3 + 2k = 8k^3 + 2k = 2(4k^3 + k),$$

which is clearly even.

Case 2: Let n be odd. Then, $n = 2k + 1$ for some integer k. We compute $n^3 + n$:

$$n^3 + n = (2k + 1)^3 + (2k + 1) = 8k^3 + 12k^2 + 8k + 2 = 2(4k^3 + 6k^2 + 4k + 1),$$

which is also even. Therefore, $n^3 + n$ is even for any integer n.

7. Let a, b, and c be integers. Show that if a and b are both even, and c is odd, then $a^2 + b^2 - c^2$ is odd.

We will analyze the expression $a^2 + b^2 - c^2$ in two cases:

Case 1: Both a and b are even.

If a and b are both even, then $a = 2m$ and $b = 2n$ for some integers m and n. Their squares are:

$$a^2 = 4m^2 \quad \text{and} \quad b^2 = 4n^2$$

Thus, the sum is:

$$a^2 + b^2 = 4(m^2 + n^2),$$

which is even.

Case 2: c is odd.

If c is odd, then $c = 2p + 1$ for some integer p. Its square is:

$$c^2 = (2p + 1)^2 = 4p^2 + 4p + 1,$$

which is odd.

Finally, subtracting the odd c^2 from the even sum $a^2 + b^2$:

$$a^2 + b^2 - c^2 = (\text{even}) - (\text{odd}) = \text{odd}.$$

Thus, $a^2 + b^2 - c^2$ is always odd.

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8. Let m and n be integers. Given that $m|n$ and $m \neq 0$, determine whether $m^2|n^2$ and justify your answer.

If $m|n$, then there exists an integer k such that $n = mk$. Therefore, $n^2 = (mk)^2 = m^2k^2$. Hence, $m^2|n^2$.

9. Let a and b be integers. Given that $4|(a + b)$ and $4|(a - b)$, determine what can be said about the parity of a and b.

If $4|(a+b)$, then there exists an integer k such that $a+b = 4k$. If $4|(a-b)$, then there exists an integer l such that $a - b = 4l$.

Adding these two equations, we get $2a = 4(k + l)$, so $a = 2(k + l)$, which means a is even.

Subtracting the second equation from the first, we get $2b = 4(k - l)$, so $b = 2(k - l)$, which means b is even.

10. Given two integers a and b, show that if their greatest common divisor (GCD) is 1, then the product of a and b is equal to their least common multiple (LCM).

If $\text{GCD}(a, b) = 1$, then a and b are relatively prime, meaning they have no common factors other than 1. From the known relationship:

$$\text{GCD}(a, b) \times \text{LCM}(a, b) = a \times b$$

we substitute $\text{GCD}(a, b) = 1$, giving:

$$1 \times \text{LCM}(a, b) = a \times b$$

Thus, $\text{LCM}(a, b) = a \times b$. For example, if $a = 5$ and $b = 7$, since $\text{GCD}(5, 7) = 1$, the $\text{LCM}(5, 7) = 35$, which is simply the product $5 \times 7 = 35$.

11. Determine if there exists a pair of distinct prime numbers whose sum is also a prime number. Provide examples to justify your answer.

The sum of two distinct prime numbers is generally not prime, except in cases where one of the primes is 2. For example, $2 + 3 = 5$ (prime), but $3 + 5 = 8$ (not prime).

12. Let p be a prime number and a, b, and c be integers. Given that $p|(a^2 + b^2)$ and $p|(a^2 + c^2)$, determine what can be said about $p|(b^2 - c^2)$.

We are given that $p|(a^2 + b^2)$ and $p|(a^2 + c^2)$. We want to show that $p|(b^2 - c^2)$.

If $p|(a^2 + b^2)$, then $a^2 + b^2 = kp$ for some integer k. If $p|(a^2 + c^2)$, then $a^2 + c^2 = lp$ for some integer l. Subtracting these two equations, we get:

$$(a^2 + b^2) - (a^2 + c^2) = b^2 - c^2 = (k - l)p.$$

Therefore, $p|(b^2 - c^2)$.

For examples, let $p = 5$, $a = 1$, $b = 2$, and $c = 3$. We will verify that $p|(a^2 + b^2)$, $p|(a^2 + c^2)$, and that $p|(b^2 - c^2)$.

$$a^2 + b^2 = 1^2 + 2^2 = 1 + 4 = 5.$$

Since $5|5$, it is true that $p|(a^2 + b^2)$.

$$a^2 + c^2 = 1^2 + 3^2 = 1 + 9 = 10.$$

Since $5|10$, it is true that $p|(a^2 + c^2)$.

$$b^2 - c^2 = 2^2 - 3^2 = 4 - 9 = -5.$$

Since $5| - 5$, this confirms what the theorem predicts: $p|(b^2 - c^2)$.

Thus, as the theorem asserts, if p divides both $a^2 + b^2$ and $a^2 + c^2$, then p also divides $b^2 - c^2$.

13. Given that n is a product of the first k prime numbers, analyze whether $n+1$ can be a prime number, providing examples.

If n is a product of the first k prime numbers, $n+1$ can sometimes be prime. For example, if $n = 2 \times 3 = 6$, then $n+1 = 7$, which is prime. Similarly, if $n = 2 \times 3 \times 5 = 30$, then $n+1 = 31$, which is also prime. However, $n+1$ is not always prime. For instance, $n = 2 \times 3 \times 5 \times 7 \times 11 \times 13 = 30,030$, and $n+1 = 30,031$, is not prime.

14. Determine if the product of any two prime numbers can ever be a prime number. Provide reasoning for your answer.

The product of any two prime numbers cannot be a prime number. By definition, a prime number has exactly two distinct positive divisors: 1 and itself. If two prime numbers p and q are multiplied, their product pq has four distinct positive divisors: 1, p, q, and pq. Thus, pq cannot be a prime number.

15. Show that if a and b are two integers such that $\gcd(a,b) = 1$, then $\gcd(a^2, b^2) = 1$. Provide reasoning and an example.

If $\gcd(a,b) = 1$, this means that a and b share no common prime factors. For any prime p, if p divides both a^2 and b^2, then p must divide both a and b (since if $p|a^2$, then $p|a$, and similarly for b). This would contradict the assumption that $\gcd(a,b) = 1$, as a and b share no common divisors greater than 1. Therefore, $\gcd(a^2, b^2) = 1$.

For example, let $a = 8$ and $b = 15$. We know $\gcd(8, 15) = 1$. Now, calculate $a^2 = 64$ and $b^2 = 225$. The prime factorizations are:
$$64 = 2^6 \quad \text{and} \quad 225 = 3^2 \times 5^2.$$

Since 64 and 225 have no common prime factors, $\gcd(64, 225) = 1$. Hence, $\gcd(a^2, b^2) = 1$ as expected.

16. Determine the prime factorization of 540.

The prime factorization of 540 is $540 = 2^2 \times 3^3 \times 5$.

17. Explain why the number 1 is not considered a prime number.

1 is not considered a prime number because it has only one positive divisor, itself. A prime number is defined as having exactly two distinct positive divisors: 1 and itself.

18. Find all prime factors of 2310.

The prime factors of 2310 are $2, 3, 5, 7,$ and 11. The prime factorization is $2310 = 2 \times 3 \times 5 \times 7 \times 11$.

19. Identify the largest prime factor of 529.

We begin by testing smaller prime numbers and find that $529 \div 23 = 23$, which shows that $529 = 23^2$. Hence, the largest prime factor of 529 is 23.

20. Given the prime factorizations of two numbers, $n = 2^4 \times 3 \times 5$ and $m = 2^3 \times 3^2 \times 7$, determine the highest power of 2 in the factorization of $n \times m$.

The highest power of 2 in the factorization of $n \times m$ is 2^7, since $n \times m = 2^4 \times 3 \times 5 \times 2^3 \times 3^2 \times 7 = 2^7 \times 3^3 \times 5 \times 7$.

21. Find the greatest common divisor (GCD) of 48, 180, and 252 by finding their prime factorizations.

To find the GCD of 48, 180, and 252, we first find their prime factorizations:

$48 = 2^4 \times 3$
$180 = 2^2 \times 3^2 \times 5$
$252 = 2^2 \times 3^2 \times 7$

The GCD is found by taking the lowest powers of the common prime factors. The common prime factors of 48, 180, and 252 are 2 and 3. The lowest power of 2 is 2^2. The lowest power of 3 is 3^1. Therefore, the GCD of 48, 180, and 252 is:

$$2^2 \times 3 = 4 \times 3 = 12.$$

So, the GCD of 48, 180, and 252 is 12.

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22. Find the least common multiple (LCM) of 12, 15, and 20 by finding their prime factorizations.
To find the LCM of 12, 15, and 20, we first find their prime factorizations:

$12 = 2^2 \times 3$
$15 = 3 \times 5$
$20 = 2^2 \times 5$

The LCM is found by taking the highest powers of all prime factors. The prime factors of 12, 15, and 20 are 2, 3, and 5. The highest power of 2 is 2^2. The highest power of 3 is 3^1. The highest power of 5 is 5^1. Therefore, the LCM of 12, 15, and 20 is:

$$2^2 \times 3^1 \times 5^1 = 4 \times 3 \times 5 = 60.$$

So, the LCM of 12, 15, and 20 is 60.

23. If the GCD of two numbers a and b is 12 and their LCM is 360, find the product $a \times b$.
The product $a \times b$ is 4320. This is because the product of the GCD and LCM of two numbers is equal to the product of the numbers:

$$\text{GCD}(a, b) \times \text{LCM}(a, b) = a \times b.$$

Therefore,
$$12 \times 360 = 4320.$$

Thus, the product $a \times b = 4320$.

24. Explain why the GCD of two consecutive integers is always 1.
The GCD of two consecutive integers is always 1 because consecutive integers are coprime, meaning they share no common divisors other than 1. For example, the integers 14 and 15 have no common divisors other than 1.

25. Given that the GCD of a and b is 5 and their LCM is 100, find a and b given that $a \leq b$.
Given that GCD(a, b) = 5 and LCM(a, b) = 100, we have $a \times b = 5 \times 100 = 500$. Therefore, a and b must be 20 and 25 respectively since $20 \times 25 = 500$ and 20 is less than 25.

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