Discrete Mathematics

Section 1: Sets

Questions

1. List the elements in the following sets:
   (a) $\{x \in \mathbb{R} \mid x^2 = 4\}$
   (b) $\{x \in \mathbb{Z}^+ \mid x < 8\}$
   (c) $\{x \in \mathbb{Z}^+ \mid x \text{ is the largest positive integer such that } x^2 < 100\}$

2. Let $A = \{2, 4, 6, 8\}$, $B = \{4, 6\}$, and $C = \{4, 6, 8\}$. Determine which of the sets are subsets of the others.

3. For each of the following sets, determine if the integer 1 is an element of that set:
   (a) $A = \{x \in \mathbb{R} \mid x \text{ is an integer greater than or equal to } 1\}$
   (b) $B = \{1, 1\}$
   (c) $C = \{1, \{1\}\}$
   (d) $D = \{\{1\}, \{1, \{1\}\}\}$

4. Determine whether each of the following statements is true or false:
   (a) $0 \in \varnothing$
   (b) $0 \in \{\varnothing\}$
   (c) $\{\varnothing\} \subseteq \{0\}$
   (d) $\{\varnothing\} \subseteq \{\varnothing\}$
   (e) $\varnothing \in \{0, \varnothing\}$

5. Using set-builder notation, describe the following sets:
   (a) $\{0, 2, 4, 6, 8\}$
   (b) $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\}$

6. Find the power set of each of these sets:
   (a) $\{1\}$
   (b) $\{1, 2\}$
   (c) $\{1, \{1\}\}$

7. Determine whether each of these sets is the power set of another set.
   (a) $\varnothing$
   (b) $\{\varnothing, \{1\}, \{1, \varnothing\}\}$
   (c) $\{\varnothing, \{1\}\}$

8. Let $A = \{1, 2, 3\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find:
   (a) $A \times B$
   (b) $B \times C$
   (c) $A \times B \times C$

9. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.
   (a) $A \cup B$
   (b) $A \cap B$
   (c) $A - B$
   (d) $B - A$

10. Find the sets A and B if $A - B = \{1, 7, 8\}$, $B - A = \{2, 4\}$, and $A \cap B = \{3, 6, 9\}$.

11. Are $\mathbb{N}$ and $\mathbb{Z}$ disjoint sets?

12. Draw the Venn diagrams for the following set combinations:
   (a) $A \cap (B \cup C)$
   (b) $(A - B) \cup (A - C) \cup (B - C)$

13. Let the universal set be the real numbers. Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 0\}$, $B = \{x \in \mathbb{R} : -3 < x < 2\}$, and $C = \{x \in \mathbb{R} : 0 < x \leq 4\}$. Find:
   (a) $A \cup B$
   (b) $A \cap B$
   (c) $(A \cup C)^c$
   (d) $A^c \cap B$

14. Let $A = \{0, 2, 4, 6, 8, 19\}$, $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and $C = \{4, 5, 6, 7, 8, 9, 10, 11\}$. Find:
   (a) $(A \cup B) \cap C$
   (b) $(A \cap B) \cup C$
   (c) $A \cap B \cap C$

15. Draw the Venn diagram to represent the set combination $(A \cap B) \cup (C - (A \cup B))$. Describe the region that this set combination represents.

16. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. What is $P(A \cup B)$, where P denotes the power set?

17. Which of the following is an example of an empty set?
   (a) The set of odd natural numbers divisible by 2.
   (b) $\{x \mid x \in \mathbb{N}, 9 < x < 10\}$
   (c) $A = \{\varnothing\}$

18. For all sets A, B, and C, determine if $(A - B) \cap (C - B) = (A \cap C) - B$.

19. Using Venn Diagrams, show if the following statement is equal or not:

$$(A \cap B') \cup (A \cap B) = A.$$

20. State which of the following is true or false:
    (a) $7,747 \in \{x \mid x \text{ is a multiple of } 37\}$.
    (b) $A \cup \varnothing = \{A, \varnothing\}$.
    (c) $\varnothing \cup (\{A\} \cap B) = \varnothing \cup \{A\}$, where $A \in B$.

21. Let $A = \{a \mid a \text{ is divisible by } 2\}$ and $B = \{b \mid b \text{ is divisible by } 3\}$. Determine if the following is true or false: $A \cap B = \varnothing$.

22. Is $\{\{1,5\},\{4,7\},\{2,3,6,8\}\}$ a partition of $\{1,2,3,4,5,6,7,8\}$?

23. Given that $A = \{1,2,3,4,5\}$, if x represents any member of A, then find the following sets containing all the numbers represented by:
    (a) $x+1$.
    (b) x^2.

24. Let E be the set of even integers and O be the set of odd integers. Is $\{E, O\}$ a partition of $\mathbb{Z}$ (the set of all integers)?

25. Determine the set defined by $((A \cup B) \cap (C \cup D)) - (E \cup (A \cap B \cap C))$ if:
    $A = \{0,2,4,6,8,19\}$
    $B = \{0,1,2,3,4,5,6,7\}$
    $C = \{4,5,6,7,8,9,10,11\}$
    $D = \{x \in \mathbb{N} \mid x \text{ is a prime number less than } 20\}$
    $E = \{x \in \mathbb{N} \mid x \text{ is an even number less than or equal to } 20\}$

Questions and Answers

1. List the elements in the following sets:

   (a) $\{x \in \mathbb{R} \mid x^2 = 4\}$
   The elements of the set $\{x \in \mathbb{R} \mid x^2 = 4\}$ are $\{2, -2\}$, since $2^2 = 4$ and $(-2)^2 = 4$.

   (b) $\{x \in \mathbb{Z}^+ \mid x < 8\}$
   The elements of the set $\{x \in \mathbb{Z}^+ \mid x < 8\}$ are $\{1, 2, 3, 4, 5, 6, 7\}$, since these are the positive integers less than 8.

   (c) $\{x \in \mathbb{Z}^+ \mid x \text{ is the largest positive integer such that } x^2 < 100\}$
   The element of the set $\{x \in \mathbb{Z}^+ \mid x \text{ is the largest positive integer such that } x^2 < 100\}$ is $\{9\}$, since $9^2 = 81 < 100$ and $10^2 = 100$.

2. Let $A = \{2, 4, 6, 8\}$, $B = \{4, 6\}$, and $C = \{4, 6, 8\}$. Determine which of the sets are subsets of the others.
   By definition, the set Q is a subset of set P if all the elements of set Q are in set P, and every set is a subset of itself. Hence, $B \subseteq A$, $B \subseteq C$, and $C \subseteq A$.

3. For each of the following sets, determine if the integer 1 is an element of that set:

   (a) $A = \{x \in \mathbb{R} \mid x \text{ is an integer greater than or equal to } 1\}$
   A is a set that contains all integers greater than or equal to 1, represented as $\{1, 2, 3, \ldots\}$. Thus, 1 is an element of A.

   (b) $B = \{1, 1\}$
   1 is an element of the set B.

   (c) $C = \{1, \{1\}\}$
   The set C has two elements, 1 and $\{1\}$, hence 1 is an element of C.

   (d) $D = \{\{1\}, \{1, \{1\}\}\}$
   The elements of D are $\{1\}$ and $\{1, \{1\}\}$, and neither of these elements is equal to 1. Hence, 1 is not an element of D.

4. Determine whether each of the following statements is true or false:

   (a) $0 \in \varnothing$
   False. The empty set has no elements, hence 0 can't be an element of the empty set.

   (b) $0 \in \{\varnothing\}$
   False. The set $\{\varnothing\}$ contains the empty set as an element, but not the number 0.

   (c) $\{\varnothing\} \subseteq \{0\}$
   False. $\{\varnothing\}$ is a set containing the empty set, whereas $\{0\}$ is a set containing the number 0. Therefore, $\{\varnothing\}$ cannot be a subset of $\{0\}$.

   (d) $\{\varnothing\} \subseteq \{\varnothing\}$
   True. Every set is a subset of itself, hence the set containing the empty set is a subset of itself.

   (e) $\varnothing \in \{0, \varnothing\}$
   True. The set $\{0, \varnothing\}$ contains the elements 0 and $\varnothing$, so the empty set is an element of this set.

5. Using set-builder notation, describe the following sets:

   (a) $\{0, 2, 4, 6, 8\}$
   All the elements in this set are of the form $2n$, where n is an integer between 0 and 4 inclusive. Hence, the set-builder notation is $\{2n \mid n \in \mathbb{Z}, 0 \leq n \leq 4\}$.

   (b) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right\}$
   All the elements in this set are of the form $\frac{1}{2n}$, where n is a positive integer. Hence, the set-builder notation is $\left\{\frac{1}{2n} \mid n \in \mathbb{Z}^+, 1 \leq n\right\}$.

6. Find the power set of each of these sets:

   (a) $\{1\}$
   The power set of $\{1\}$ is $\{\varnothing, \{1\}\}$.

   (b) $\{1, 2\}$
   The power set of $\{1, 2\}$ is $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

   (c) $\{1, \{1\}\}$
   The power set of $\{1, \{1\}\}$ is $\{\varnothing, \{1\}, \{\{1\}\}, \{1, \{1\}\}\}$.

7. Determine whether each of these sets is the power set of another set.

   (a) $\varnothing$
   The power set of a set contains subsets; however, $\varnothing$ is not a set containing subsets. Hence, $\varnothing$ is not the power set of any set.

   (b) $\{\varnothing, \{1\}, \{1, \varnothing\}\}$
   The number of subsets in a power set is 2^n, where n is the number of distinct elements in the original set. Based on this, a power set will have 1, 2, 4, 8, 16, ... elements. Therefore, the mentioned set can't be the power set of any set since it has 3 elements.

   (c) $\{\varnothing, \{1\}\}$
   The subsets of the set $\{1\}$ are $\{1\}$ and $\varnothing$. Hence, $\{\varnothing, \{1\}\}$ is the power set of the set $\{1\}$.

8. Let $A = \{1, 2, 3\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find:

   (a) $A \times B$
   $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

   (b) $B \times C$
   $B \times C = \{(x, 0), (x, 1), (y, 0), (y, 1)\}$

   (c) $A \times B \times C$
   $$A \times B \times C = \left\{ \begin{array}{llll} (1, x, 0), & (1, x, 1), & (1, y, 0), & (1, y, 1), \\ (2, x, 0), & (2, x, 1), & (2, y, 0), & (2, y, 1), \\ (3, x, 0), & (3, x, 1), & (3, y, 0), & (3, y, 1) \end{array} \right\}$$

9. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.

   (a) $A \cup B$
   $A \cup B = \{1, 2, 3, 4, 6\}$

   (b) $A \cap B$
   $A \cap B = \{2, 4\}$

   (c) $A - B$
   $A - B = \{1, 3\}$

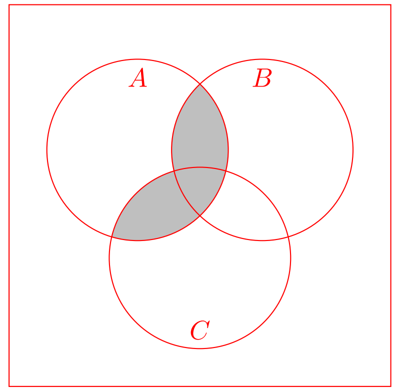
   (d) $B - A$
   $B - A = \{6\}$

10. Find the sets A and B if $A - B = \{1, 7, 8\}$, $B - A = \{2, 4\}$, and $A \cap B = \{3, 6, 9\}$.
    If $A - B = \{1, 7, 8\}$, then we know that A at least has $1, 7, 8$. We also know that $A \cap B = \{3, 6, 9\}$, which means $3, 6, 9$ are in both A and B. Hence, $A = \{1, 3, 6, 7, 8, 9\}$. $B - A = \{2, 4\}$. Hence, $B = \{2, 3, 4, 6, 9\}$.

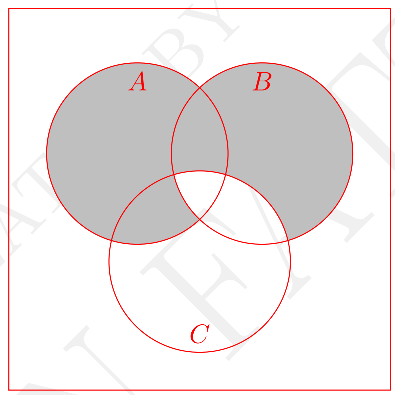
11. Are $\mathbb{N}$ and $\mathbb{Z}$ disjoint sets?
    No. By definition, two sets are disjoint if their intersection is the empty set. The intersection of the set of natural numbers and the set of integers is not the empty set. For example, 1 is in both $\mathbb{N}$ and $\mathbb{Z}$.

12. Draw the Venn diagrams for the following set combinations:

(a) $A \cap (B \cup C)$



(b) $(A - B) \cup (A - C) \cup (B - C)$



13. Let the universal set be the real numbers. Let $A = \{x \in \mathbb{R} : -1 \le x \le 0\}$, $B = \{x \in \mathbb{R} : -3 < x < 2\}$, and $C = \{x \in \mathbb{R} : 0 < x \le 4\}$. Find:

(a) $A \cup B$
The universal set is the set of all real numbers. A is the set of all real numbers x such that $-1 \le x \le 0$, and B is the set of all real numbers x such that $-3 < x < 2$. Therefore,
$A \cup B = \{x \in \mathbb{R} : -3 < x < 2\}$.

(b) $A \cap B$
The intersection of sets A and B is the set of all real numbers x such that $-1 \le x \le 0$. Hence,
$A \cap B = \{x \in \mathbb{R} : -1 \le x \le 0\}$.

(c) $(A \cup C)^c$
The union of sets A and C is the set of all real numbers x such that $-1 \le x \le 4$. Hence, the complement of $A \cup C$, $(A \cup C)^c$, is the set of all real numbers x such that $x < -1$ or $4 < x$. Therefore, $(A \cup C)^c = \{x \in \mathbb{R} : x < -1 \text{ or } 4 < x\}$.

(d) $A^c \cap B$
The complement of set A, is the set of all real numbers x such that $x < -1$ or $0 < x$. Set B is the set of all real numbers x such that $-3 < x < 2$. Hence, $A^c \cap B$ is the set of all real numbers x such that $-3 < x < -1$ or $0 < x < 2$.

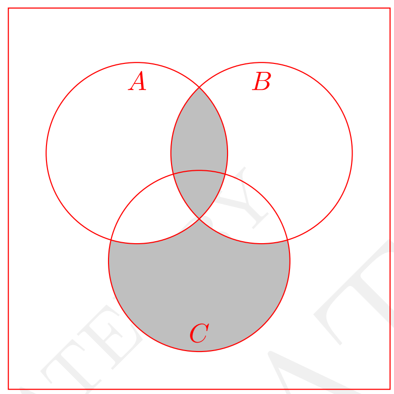
14. Let $A = \{0, 2, 4, 6, 8, 19\}$, $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and $C = \{4, 5, 6, 7, 8, 9, 10, 11\}$. Find:

   (a) $(A \cup B) \cap C$
   $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 19\}$. Hence, $(A \cup B) \cap C = \{4, 5, 6, 7, 8\}$.

   (b) $(A \cap B) \cup C$
   $A \cap B = \{0, 2, 4, 6\}$. Hence, $(A \cap B) \cup C = \{0, 2, 4, 5, 6, 7, 8, 9, 10, 11\}$.

   (c) $A \cap B \cap C$
   $A \cap B \cap C = \{4, 6\}$.

15. Draw the Venn diagram to represent the set combination $(A \cap B) \cup (C - (A \cup B))$. Describe the region that this set combination represents.
The Venn diagram for $(A \cap B) \cup (C - (A \cup B))$ is as follows:



The region $(A \cap B) \cup (C - (A \cup B))$ represents the union of the intersection of sets A and B, and the elements in C that are not in the union of sets A and B.

16. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. What is $P(A \cup B)$, where P denotes the power set?
$A \cup B = \{1, 2, 3\}$. $P(A \cup B)$ is the set of all the subsets of $\{1, 2, 3\}$.
Hence, $P(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

17. Which of the following is an example of an empty set?

   (a) The set of odd natural numbers divisible by 2.
   The above statement exemplifies the empty set since no odd natural number is divisible by 2.

   (b) $\{x \mid x \in \mathbb{N}, 9 < x < 10\}$
   The set $\{x \mid x \in \mathbb{N}, 9 < x < 10\}$ exemplifies the empty set, as there are no natural numbers between 9 and 10.

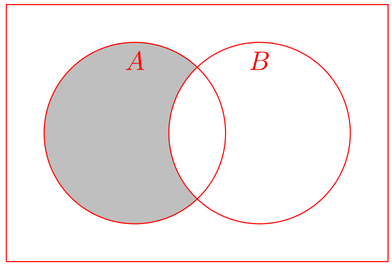
   (c) $A = \{\varnothing\}$
   The set $A = \{\varnothing\}$ is not an example of the empty set because $A = \{\varnothing\}$ is a set that contains one element, which is $\varnothing$.

18. For all sets A, B, and C, determine if $(A - B) \cap (C - B) = (A \cap C) - B$.
Yes. Using Venn diagrams, you can illustrate that the left-hand side equals the right-hand side. However, suppose: $x \in (A - B) \cap (C - B)$, then $x \in (A - B)$ and $x \in (C - B)$. This implies $x \in A$ and $x \in C$, but $x \notin B$. If x is in both A and C but not in B, then $x \in (A \cap C) - B$.

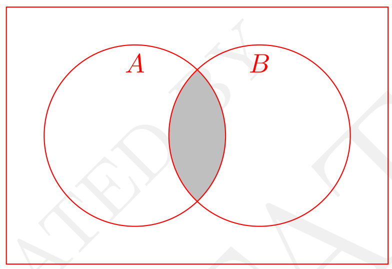
19. Using Venn Diagrams, show if the following statement is equal or not:

$$(A \cap B') \cup (A \cap B) = A.$$

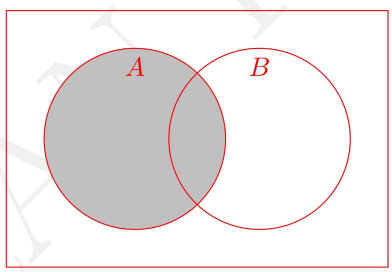
For the set A and B, B' has all those elements that are in A but not in B, as shown below:



The Venn diagram for $A \cap B$ is as below:



The Venn diagram for $(A \cap B') \cup (A \cap B)$ is as below and it equals A:



Hence, the left-hand side equals the right-hand side.

20. State which of the following is true or false:

    (a) $7,747 \in \{x \mid x \text{ is a multiple of } 37\}$.
    False. $7,747$ is not a multiple of 37. The correct multiple of 37 nearest to $7,747$ is $7,755$ (37×209).

    (b) $A \cup \varnothing = \{A, \varnothing\}$.
    False. The union of any set with the empty set is the set itself, so $A \cup \varnothing = A$. The expression $\{A, \varnothing\}$ represents a set containing two elements: A and the empty set.

    (c) $\varnothing \cup (\{A\} \cap B) = \varnothing \cup \{A\}$, where $A \in B$.
    True. The intersection $\{A\} \cap B = \{A\}$, since $A \in B$. The union of the empty set with any set S is simply S, so $\varnothing \cup \{A\} = \{A\}$.

21. Let $A = \{a \mid a \text{ is divisible by } 2\}$ and $B = \{b \mid b \text{ is divisible by } 3\}$. Determine if the following is true or false: $A \cap B = \varnothing$.
False. A is the set of all positive integers that can be divided by 2 with no remainder, thus $A = \{2, 4, 6, 8, \ldots\}$. B is the set of all positive integers that can be divided by 3 with no remainder, thus $B = \{3, 6, 9, 12, \ldots\}$. The intersection of A and B is not the empty set, as it contains at least one element, which is 6.

22. Is $\{\{1, 5\}, \{4, 7\}, \{2, 3, 6, 8\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?
Yes. By definition, the set $\{\{1, 5\}, \{4, 7\}, \{2, 3, 6, 8\}\}$ is a partition of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

23. Given that $A = \{1, 2, 3, 4, 5\}$, if x represents any member of A, then find the following sets containing all the numbers represented by:

    (a) $x + 1$.
    If $A = \{1, 2, 3, 4, 5\}$, then $B = \{b \mid b = x + 1, x \in A\}$. For $x = 1$, $b = 2$, and for $x = 5$, $b = 6$. Hence, $B = \{2, 3, 4, 5, 6\}$.

    (b) x^2.
    If $A = \{1, 2, 3, 4, 5\}$, then $B = \{b \mid b = x^2, x \in A\}$. For $x = 1$, $b = 1$, and for $x = 5$, $b = 25$. Hence, $B = \{1, 4, 9, 16, 25\}$.

24. Let E be the set of even integers and O be the set of odd integers. Is $\{E, O\}$ a partition of $\mathbb{Z}$ (the set of all integers)?
To determine if $\{E, O\}$ is a partition of $\mathbb{Z}$ (the set of all integers), we consider the following: No even integer is odd. Therefore, the intersection of E and O is $\varnothing$. Additionally, the union of E and O is $\mathbb{Z}$. Hence, by definition, $\{E, O\}$ is a partition of $\mathbb{Z}$.

25. Determine the set defined by $((A \cup B) \cap (C \cup D)) - (E \cup (A \cap B \cap C))$ if:
$A = \{0, 2, 4, 6, 8, 19\}$
$B = \{0, 1, 2, 3, 4, 5, 6, 7\}$
$C = \{4, 5, 6, 7, 8, 9, 10, 11\}$
$D = \{x \in \mathbb{N} \mid x \text{ is a prime number less than } 20\}$
$E = \{x \in \mathbb{N} \mid x \text{ is an even number less than or equal to } 20\}$
To solve this problem, we evaluate each component step by step:

    - First, find the union of sets A and B:
    $$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 19\}$$
    - Next, find the union of sets C and D:
    $$C \cup D = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 19\}$$
    - Find the intersection of the above two unions:
    $$(A \cup B) \cap (C \cup D) = \{2, 3, 4, 5, 6, 7, 8, 19\}$$
    - Find the intersection of sets A, B, and C:
    $$A \cap B \cap C = \{4, 6\}$$
    - Given set E:
    $$E = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$
    - Find the union of set E and the intersection found earlier:
    $$E \cup (A \cap B \cap C) = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$
    - Finally, subtract the union from the intersection:
    $$((A \cup B) \cap (C \cup D)) - (E \cup (A \cap B \cap C)) = \{3, 5, 7, 19\}$$

List of primes less than 20: $\{2, 3, 5, 7, 11, 13, 17, 19\}$

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