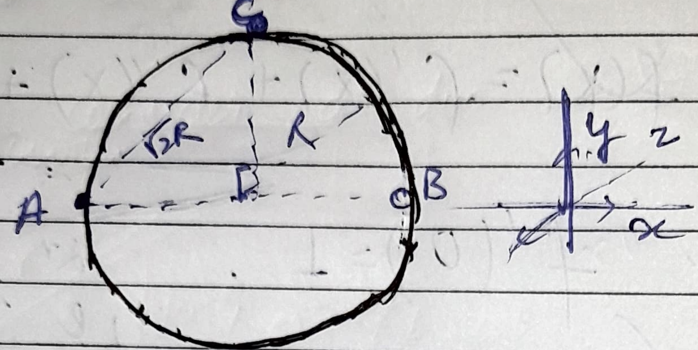
Pravega Astroatz Solutions

Q4

i)



Let the point where field is to be maximised be A

Field at A $= \frac{GM}{R^2} = \frac{4}{3}\pi G\rho R$

Now remove a very tiny sphere of radius r from diametorically opposite point B and place it at C

$$E_x = \frac{4}{3}\pi G\rho R - \frac{4}{3}\frac{G\pi\rho r^3}{(2R)^2} + \frac{4}{3}\frac{G\pi r^3}{(\sqrt{2}R)^2}\cos 45$$

$$= \frac{4}{3}\pi G\rho R - \frac{4}{3}\pi\frac{G\rho r^3}{4R^2} + \frac{4}{3}\pi\frac{G}{2\sqrt{2}}\pi\rho r^3$$

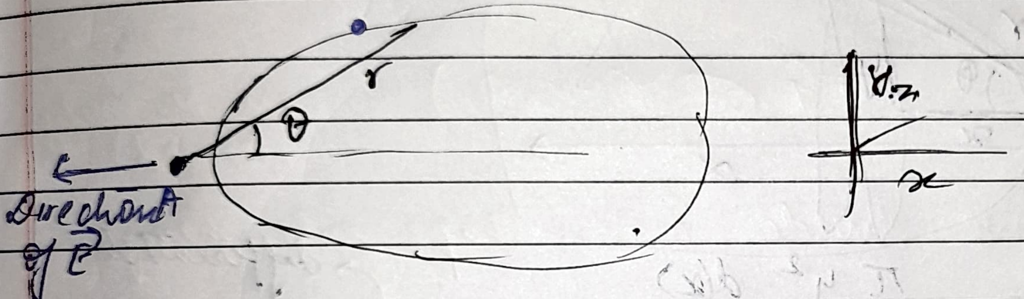
$$> \frac{4}{3}\pi G\rho R$$

$\therefore (E_x)_{final} > E_{initial}$

$\therefore \sqrt{E_x^2 + E_y^2 + E_z^2}_{\,final} > E_{initial}$

$\therefore E_{final} > E_{initial}$.

ii) It can be perceived that body must have an axis of symmetry. } for max gravity field
Now suppose material is designed such that gravitational field is max.



Note that small elements of mass (which have equal ~~weigh~~ mass) must contribute equally to E_x. If not we can slide a very small element of mass from position where mass contributes less to a position where mass contributes more. In that case $E_x \uparrow$

Consider the situation as shown in figure

$$dF_x = \frac{G\,dm\cos\theta}{r^2} = \frac{G\rho\,dV\cos\theta}{r^2}$$

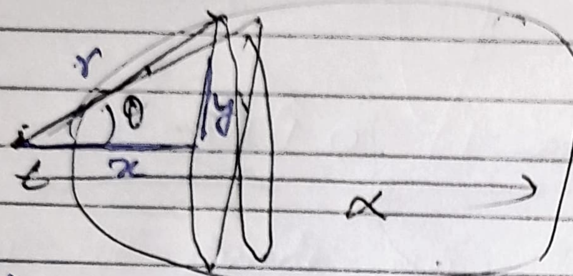
$$= G\rho\,dV\left(\frac{\cos\theta}{r^2}\right)$$

$\therefore \frac{\cos\theta}{r^2} =$ constant

This constant must be positive because it can be seen that it is positive for values (r, θ) shown in figure iii Let $\frac{r^2}{\cos\theta} = \alpha^2$

$r = \alpha\sqrt{\cos\theta}$

Now consider a disc be θ and $\theta + d\theta$ as shown



$$dV = \pi y^2 \, d(x)$$

differential

$$= \pi r^2 \sin^2\theta \, d(r\cos\theta)$$

$$= \pi \alpha^2 \sin^2\theta \cos\theta \, (d(\alpha \cos\theta^{3/2}$$

$$= \pi \alpha^3 \sin^2\theta \cos\theta \, \frac{3}{2} \cos\theta \sin\theta \, d\theta$$

$$= -\pi \alpha^3 \frac{3}{2} \sin^3\theta \cos^2\theta \, d\theta$$

$$\int_0^{\pi/2} |dV| = \frac{4\pi\alpha^3}{15}$$

Now

$$\frac{4}{15}\pi\alpha^3 = V = \frac{M}{\rho}$$

$$\Rightarrow \alpha = \sqrt[3]{\frac{15}{4\pi}\frac{M}{\rho}}$$

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Now Radius of sphere proposed by Bob

$$\frac{4\pi R^3}{3} = \frac{M}{\rho}$$

$$\Rightarrow R = \left(\frac{3}{4\pi}\frac{M}{\rho}\right)^{1/3}$$

The indicated distance $\alpha = \sqrt[3]{\frac{15}{4\pi}\frac{M}{\rho}} = 0.854 \times (2R)$

Thus our shape is compressed by 0.854 compared to sphere along x

Maximum radius of cross section

$\text{Max}\,(r\sin\theta) = \text{Max}\,(\alpha\sqrt{\cos\theta}\sin\theta)$

$\sqrt{\cos\theta}\sin\theta$ max value is $\left(\frac{64}{27}\right)^{1/4}$.

Maximum radius of cross section

$$= \left(\frac{64}{27}\right)^{1/4}\alpha$$

$$= \left(\frac{64}{27}\right)^{1/4} 0.854 \times 2R$$

$$= 1.0608\,(2R)$$

Thus our shape is expanded 1.0608 times the sphere at max cross section

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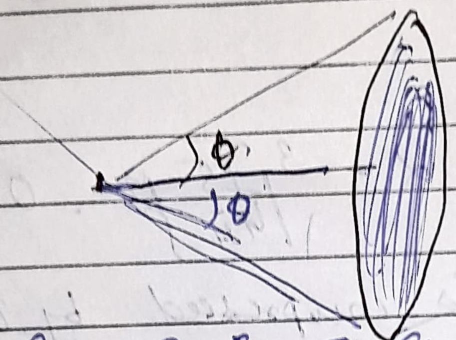
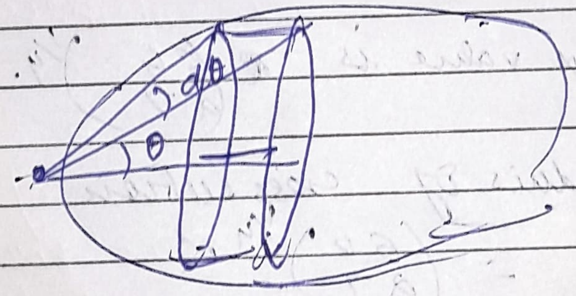
iv) Gravitational field due to disc as shown is given by



$G = 2\pi G \sigma (1-\cos\theta)$ (can be easily derived)

Now we divide our shape into many disc



$dE = 2\pi G \sigma (1-\cos\theta)$

$= 2\pi G \rho\, dx (1-\cos\theta)$

$= 2\pi \rho G\, d(\alpha \sqrt{\cos\theta}\cos\theta) (1-\cos\theta)$

$= 2\pi \rho G \alpha \frac{3}{2} \sqrt{\cos\theta} \sin\theta (1-\cos\theta) d\theta$ assume $\cos\theta = t$

$\int_0^{\pi/2} |dE| = 3\pi \rho G \alpha \int_0^1 \sqrt{t}(1-t) dt$

$3\pi \rho G \alpha \left(\frac{2}{3} - \frac{2}{5}\right)$

$= \frac{9\pi \rho G \alpha\, 12}{15} = \frac{4\pi \rho \alpha\, 4}{5}$

field due to sphere $= \frac{4\pi}{3} G \rho R$

$$\rho = \frac{M}{4/3\,\pi R^3} = 5516.8\ kg/m^3$$

$$\alpha = \sqrt[3]{\frac{15 \times 5.97 \times 10^{24}}{4\pi\,\rho}} = 1.0892 \times 10^7$$

$$\text{Max ficeld} = \frac{6.67 \times 10^{-11} \times 3.14 \times 5516 \times 1.0892 \times 10^7 \times 4}{5}$$

$$= 10.0664\ gf\ m/s^2$$

$$\text{ratio} = \frac{10.0664}{9.81} = 1.0261$$