AstroWiz

Problems

July 16, 2022

Instructions:

1. The duration of this contest is 4 hours (9a.m. - 1p.m.).
2. You have been provided additional 30 minutes specifically for scanning and emailing the response sheets (1p.m. - 1:30p.m.).
3. There are, in total, 8 problems in this contest and solution to each must be scanned separately and the file name should be **FirstName_LastName_Sol#.pdf**. For example, a valid solution file name for question number 4 would be Angelina_Jolie_Sol4.pdf.
4. Make sure to send a single mail containing the 8 pdfs with the subject line "AstroWiz Submission".
5. Solutions can be typed or handwritten. If handwritten then please ensure that it's legible.
6. Make sure to highlight/box the final answer of each questions. This, in general, is a good practice and will also ensure that we don't miss out any answers while grading.
7. Use of calculators is permitted (for integration too).
8. Remember that though it's a timed contest we feel that we have given the necessary amount of time and therefore expect all submissions to be made on or before 1:30 p.m. No emails/requests for late submissions will be entertained.
9. In case we are not able to get back to you soon on a question try making the most reasonable assumption and move forward.
10. Lastly, the aim of this contest is not to give you trepidations rather give you an enriching experience from which you can (hopefully) get to know how exciting astrophysics is; therefore, keep a smiling face and start the paper.

Boot your brain, shoot your ideas, and loot the booty!!!

Problem 1 [5pts]

A: Flatten the dough to get a disc!

Consider the following simplified model of disc shaped galaxies.

1. Let the galaxy be approximated as a 2-Dimensional disc of radius R with a constant surface density. Find the velocity curve v(r). [1]
2. Find the epicyclic frequency κ, if it is given by the following formula

$$\kappa^2 = \left(\frac{2v}{r}\right)\left(\frac{v}{r} + \frac{dv}{dr}\right)$$

   [1]

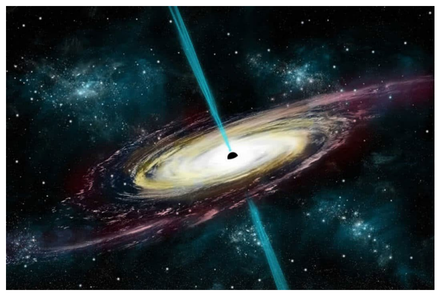
3. A given disc shaped galaxy is said to be Keplerian if it follows the following relation for all orbits $T^2 \propto r^3$. Find the mass surface density distribution $\sigma(r)$ for such a galaxy. [1]

B: Balance in Play

Consider a star made of an ideal gas having molecular mass M_0. Assume that the density of the star is constant throughout. The mass and radius of the star is M and R. Assume $T = k$p where k is a constant and that the radiation pressure is negligible and that hydrostatic pressure (p) is responsible for equilibrium. Find the pressure as a function of distance from centre. [2]

Problem 2: Cosmic Killer Rays [15pts]

Gamma Ray Bursts (GRBs) are the most violent explosions in the Universe, with some releasing more energy in 10 seconds than what the Sun will emit in its entire 10 billion year lifetime! *(treat it as a constant, say $E \approx 10^{45} J$)*
Observations have revealed that GRBs are located in the distant Universe, are accompanied by afterglows at less energetic wavelengths, and that at least some are associated with very energetic supernova explosions called hyper-novae.



It has often been claimed in popular science that one such event with sufficient energy can sterilise life on Earth forever, like a cosmic autoclave!

A. A basic model

A group of astrophysicists wish to determine the chances of us, Earth, to be hit by a killer gamma ray burst. They have made some observations and concluded upon a few facts as follows:

- It is suggested that the energy of the explosion is beamed into a narrow cone of emission along two opposite direction. The average cone angle being Ω steradians.($\approx 10^{-3}$)
- Sources of such Gamma ray bursts (e.g. blackholes, colliding neutron stars, hyper-novae etc.) are scattered uniformly throughout the observed universe. A very rough estimate of this number density (D_n) gives a number in the ball park of 10^6 (mega-parsec)$^{-3}$
- It has been estimated that about N, ($N \approx 500$) GRBs occur within our observed universe *everyday.*
- It can not be predicted at all where a GRB is about to happen, since all observed GRBs were scattered uniformly across our sky.

Assume the radius of observable universe to be $R_u \approx 14$ giga-parsecs. For the sake of simplicity, in this part we don't consider our universe to be expanding. Based on these conjectures, now you have been tasked to evaluate a **Probability Distribution Function** f of us being hit by a GRB of a certain intensity I on a single day

In other words,

$$\int_{I_a}^{I_b} f(I)\, dI$$

should give the probability of us being hit by a GRB within an intensity range of I_a to I_b on a single day.

Note: In case you can't solve any integral leave it in expression form.

A.1 Meet the assassins [3+2+1]

Firstly, let us focus on a single GRB source. Say it is at a distance r mega-parsec (Mpsc) away (1 parsec ≈ 3.2 light-year).

A.1.1 Did he shoot that day? [3]

What is the probability that this potential source went off one fine day, y years ago?
In mathematical terms, find the probability distribution function p_y that this blasted y years ago. $p_y(y_0)dy$ thus gives the probability for year $y = y_0$
The astrophysicist team you are working for has come up with this interesting result that may be useful to you:
Let $p(t)$ be the probability that a GRB source goes off t years since its birth. Then,

$$p(t) = Cte^{-(t-T_{mp})}$$

Here, T_{mp} is the most probable time for the GRB to occur, and C is the normalisation constant (don't worry about it too much).
Assume that a GRB source could have been created at any point of time in the Universe's lifetime before y. ($T_{uni} \approx 14$ billion).

Hint: *Think about the conditional probability of the source being born, say Y years ago,* ***and*** *the burst to happen y years ago. Now consider that Y can be any year$\geq y$. You shall approximate 1 day to be a differential change in a year. You might find this result useful:*

$$\int xe^{(-(x-t))}\, dx = -(x+1)e^{t-x} + C$$

A.1.2 Did he shoot at us? [2]

Now, in the case that this source goes off today, what are the chances that we will be in its path (p_{us})? Assume the axis of the cones of the GRB can be oriented along any direction with uniform probability.
The value $p_{r,y} = p_{us} \cdot p_y$ gives the probability distribution of a source at a distance r to ***Fire!*** at us within a particular day, y years ago.

A.1.3 How hard did he hit? [1]

Estimate the intensity of the death ray he shot at us today, by the time it reaches us.(I)
Also express $p_{r,y}$ as a function of I and y. Call it $p_{I,y}$
Hint: *Assume that the entire energy is released in τ seconds during a gamma ray burst event.*

A.2 Here comes the gang! [1+1+2]

Now we will work out the probability of us getting hit by anyone among this entire clan of assassins.

A.2.1 The ones at the same distance [1]

What is the probability of *any of the sources at distance r* to Fire at us within a particular day y years ago? ($P_{r,y}$)
Also express $P_{r,y}$ as $P_{I,y}$ by using your result from part 1.1.3.

A.2.2 Hidden across the cosmos [1]

Now we are finally ready to measure our threat from the entire lot of cosmic assassins at no matter what distance they stand.
Calculate the probability that Earth will get hit by a GRB from anywhere on the cosmos, ***today!***

Hint: *Find a relation between r and y that must hold for us to get hit today. All possible valid (r, y) combination will contribute to our overall risk. You should get an integral of a function of r by eliminating y.*

A.2.3 Assessing the risk [2]

You are very close to the PDF $f(I)$ that we have been searching for. Try to parameterise r, y with I. Express the final integral in the answer of previous part in terms of I. This, integrand is precisely the $f(I)$ we have been looking for!(briefly justify how)
Of-course all GRB hits aren't lethal (we harmlessly detect about one or two almost everyday). We wish to have a formula that gives how many years must go by from now on wards, for chances of us getting *flat-lined* by the cosmos to exceed a given probability (p_{kill}).

A strike of intensity I_{kill} and above is generally considered lethal. Based on what you have understood while solving this problem, can you derive such a formula?

Of-course, you are free to make reasonable assumptions, but make sure to justify all of them.

B. Addendum: A cute Doppler correction [5]

It is well known that the universe is expanding currently. This causes an average red-shift in the electromagnetic waves that approach us from heavenly bodies across our cosmos. This is called the *Doppler effect*. We are all well familiar with the Doppler effect in our daily life with sound waves, e.g. a passing ambulance.
For the Doppler effect of electromagnetic waves, we will use a more refined version incorporating special relativity. (Again don't worry too much about it!) For your convenience we present you with a result of relativistic Doppler effect:

$$f_r = \frac{f_s}{\gamma\,(1 + \beta \cos\theta_r)}$$
$$\beta = v/c > 0$$

γ being the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$
f_r is the frequency received, f_s is source frequency, v is the velocity of source in receiver's frame and θ_r is the angle the source velocity makes with the radial direction from the receiver.
Theoretical calculation shows that, $\frac{I(f)}{f^3}$ is a *Lorentz invariant*. I.e. it is a constant for both the source and a receiver. Here $I(f)$ is the intensity of radiation at frequency f. In our case this is the intensity of the GRB as it is purely monochromatic.

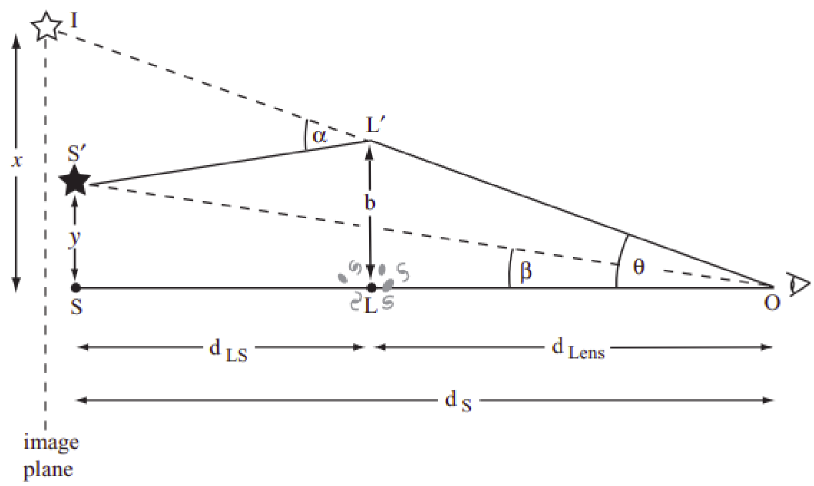
It has been determined that the frequency of an average GRB is ν_0 at the source. Assume that the velocity of all GRB source are radially outwards and this velocity is proportional to the radial distance of the source from us ($v = kR$, k being a constant). Repeat what you did for part 1.1.3, but this time with this correction in place.

Problem 3: Gravitational Lensing [8pts]

Einstein's General Relativity predicts bending of light around massive bodies. This phenomenon is called 'gravitational lensing'. For simplicity, we assume that the bending of light happens at a single point for each light ray. The angle of deflection of the light ray is given by:

$$\alpha \approx \frac{4GM}{bc^2} = \frac{2R_s}{b}$$

where R_s is the Schwarzschild radius associated with the mass M acting as the gravitational lens and b is the distance of the incoming light ray from the parallel x-axis passing through the centre of the body also called the "impact parameter".
Use the image below for reference to answer the following questions:



i) For the above situation, show that for $d_s >> d_{lens}$, $\theta - \beta \approx \alpha$. Moreover, if $d_{lens} = 1AU$, calculate the Einstein radius, θ_E . Einstein radius is given by:

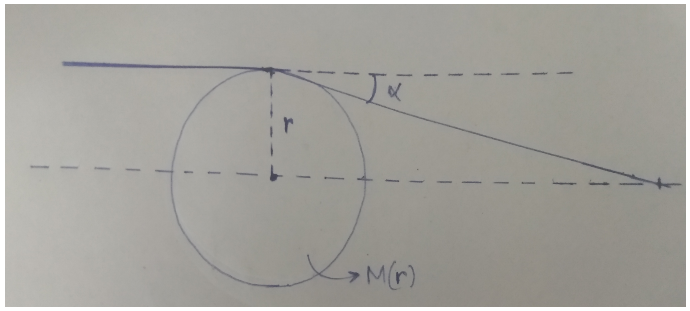
$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{d_{LS}}{d_{Lens} d_S}}$$

[1+1]

ii) If $\beta = 0$ in the original situation, calculate θ in terms of the Einstein radius, θ_E ? Also, how will the image of S' look like from O? Explain briefly. [1+2]

iii) Gravitational lensing can be used to detect dark matter. Even though dark matter doesn't interact with light, it shows gravitational lensing due to its mass. Suppose a spherical region of dark matter, while allowing light to pass through it, is found to concentrate light from very distant sources at a single point in space. Assuming that for the gravitational bending with impact parameter, r, only the mass $M(r)$ enclosed inside the radius r is relevant; $M(r)$ was found to be proportional to r^α. Find α. [3]

The middle line in the figure below represents the x-axis with the centre of the spherical mass distribution as origin.



Problem 4: O' Gravity, thou vile beast [10pts]

Elves are magical creatures who can modify the shape and location of planets but cannot change the density of the material they are made of. Dobby, an elf, wishes to generate maximum gravitational field at a given point using a planet in isolation made of malleable material of uniform density. He poses this conundrum to two of his best friends, Alice and Bob.

Alice and Bob start discussing what should the shape of the planet be. Bob argues that owing to symmetry, the shape should be spherical with the considered point at the surface of sphere. However, Alice is apprehensive of Bob's claim.

i. Give suitable argument to refute Bob's claim . [2]

ii. Since Bob's claim is wrong how should Dobby mould the planet so that he obtains the maximum possible gravitational field. [3]

iii. Compare the dimensions of the shape you obtained in (b) with that of the sphere which Bob proposed. [2]

iv. Given that mass of Earth is 5.97×10^{24} kg and the radius is 6370 km find the maximum acceleration due to gravity felt by a test mass by redistributing the mass of Earth. Compare this value with standard value acceleration due to gravity at the equator i.e. $9.81 m/s^2$. [2+1]

Assume that the mass of Earth is uniformly distributed (i.e. constant density) and that the Earth is isolated. Neglect any rotational effects of Earth if required.

Problem 5: 3...2...1... Launch! [10pts]

Scientists of Pravega Space Agency are discussing the possibility of launching a spaceship from their newly discovered planet
'RUSTUM' which revolves around a star 'ERIDANI' in a planetary system. They want to send their spaceship to another galaxy which is very far away (far enough that you can consider that satellite needs to escape the planetary system). They wish to calculate the minimum velocity with respect to 'RUSTUM' with which spaceship must be launched so that it can reach the other galaxy. (Assume that our planetary system has just one star and one planet). One of the scientists suggests that the requisite velocity should be:

$$\sqrt{2\left(G\frac{M_s}{R} + G\frac{M_p}{r}\right)}$$

where G is the Gravitational Constant; M_s and M_p are the Masses of star and planet respectively; R is the planet-star distance and r is the radius of planet. Neglect rotational effects of planet.

$G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$
$M_s = 2.382 \times 10^{30} kg$
$M_p = 5.57 \times 10^{24} kg$
$R = 2.02 \times 10^{8} km$
$r = 5423 km$

(a) Interpret the expression suggested by the scientist. Reason why her claim is wrong. [2]

(b) After a long debate with her colleagues, the scientist accepts that her claim was erroneous. She then suggests that since we are launching from the planet which revolves around the star with some orbital velocity, the planet's orbital velocity can 'assist' us in launching the spaceship and, thus, the expression for minimum velocity must be equal to

$$\sqrt{2\left(G\frac{M_s}{R} + G\frac{M_p}{r}\right)} - v_o$$

where v_o is the orbital velocity of the planet. Reason why her claim is still wrong. [2]

(c) **Now after two flawed suggestions, we finally intend to reach a conclusion that is tangible ☺**

In which direction must the spaceship be launched so that our purpose is met? Explain in no more than 2 sentences why you chose this direction. [1]

(d) Find the expression and also the minimum velocity correct upto two decimal places required to launch the satellite from
'RUSTUM' planet in the planetary system. [4+1]

OR

We have different orbital transfer mechanisms which are essentially orbital maneuvers, used to transfer space craft between two different orbits. As depicted in Figure 1, one can visualise a satellite orbit transfer. The satellite is being transferred from one circular orbit to another using an arbitrary path. Hohmann transfer is another orbital manoeuvre For the final figure, Figure 2, you are given a satellite orbiting the moon of a planet in a highly elliptical orbit (Use H as the height of atmosphere and radius R' as the the radius of the moon).

a) Refer to Figure 1

Assuming that the arbitrary path is elliptical answer the following questions. Changes in velocities (provided by the boosters) at points 1 and 2 are the required velocities for transfer of orbits; shown by Δv_1 and Δv_2 respectively.

i. Find Δv_1 and Δv_2. **(Compute it under the assumption of instantaneous impulse)** [2]

ii. What is the eccentricity of the ellipse? [1]

iii. What is the time taken in the transfer? [2]

iv. Find an general equation for velocity in terms of r (radius of circle it's orbiting) and a (length of the semi major axis). [1]

*Assume the mass of planet to be M and mass of satellite be m

b) Refer to Figure 2

i. Find the maximum possible value of eccentricity [3]

ii. Find the corresponding apogee distance. [1]

Assume that all the orbits are in same plane

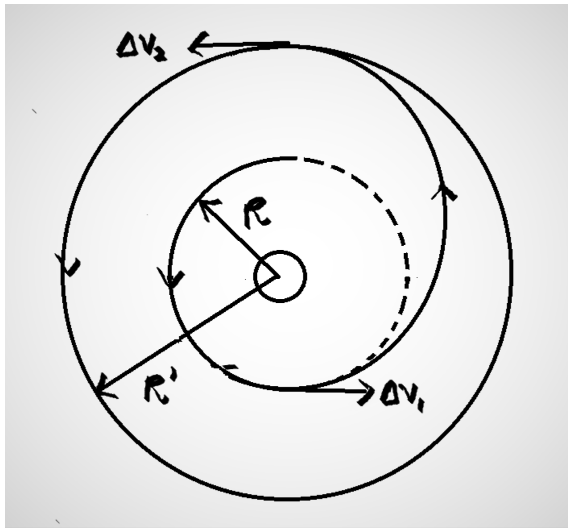


Figure 1: Image for the orbit transfer

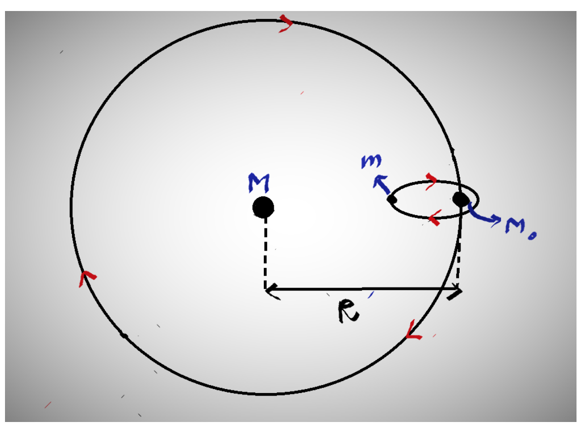


Figure 2: Figure 2

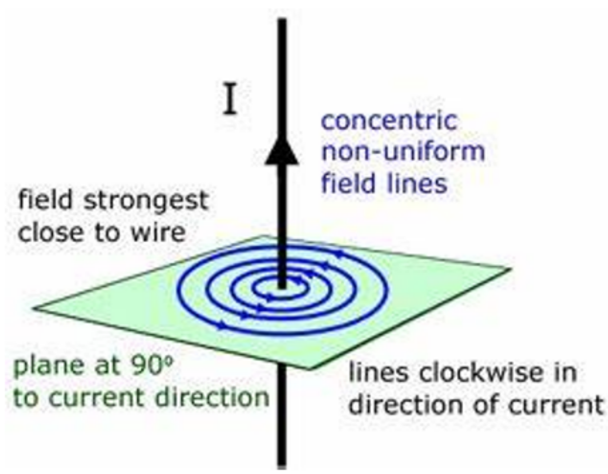
Problem 6: Trouble in Magnetosphere! [9pts]

Magnetic field lines were introduced by Michael Faraday who called them lines of force. They are elegant graphic depictions of magnetic field due to a source. Tracing out magnetic field lines is an interesting task. The basic principle is that the magnetic field vectors are always tangent to the magnetic field line curve. Mathematically, it can be expressed as

$$\mathbf{F}\ (\mathbf{r}(t)) \propto \frac{d\mathbf{r}(t)}{dt}$$

where $\mathbf{F}$ is the field vector and $\mathbf{r}$(t) is the position vector with t being just a parameter (not time!).

i. The given image indicates field lines due to infinitely long wire. They are well known to be concentric circles. Prove that magnetic field lines due to infinitely long current carrying wire are concentric circles. You might wish to exploit the z -axis invariant property of curves in this case. [1]



ii. The given image indicates magnetic field lines due to a bar magnetic or magnetic dipole. Note that field lines are perpendicular to the equatorial plane. Prove that the equation for field lines is given by $r = r_e sin^2(\theta)$, where r is the distance from centre; θ is polar angle assuming dipole along z axis and r_e is perpendicular distance of field line from dipole at equator(indicated in diagram). One must remember that sometimes a geometric argument using a

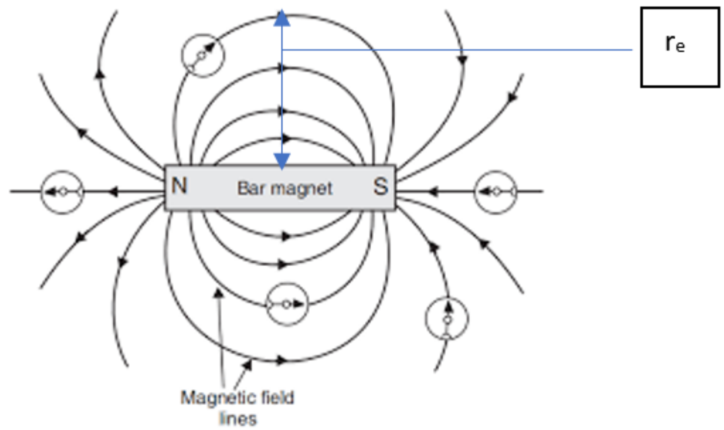


diagram can simplify our task. [3]

iii. Van Allen radiation belt is a zone of energetic charged particles around Earth, named after its discoverer James Van Allen. Using the above question now we wish to investigate the movement of charged particles in Van Allen radiation belt of a newly discovered planet. We study gyro-motion in which a particle gyrates around a magnetic field line i.e. moves in an almost helical pattern such that the axis of curve is along magnetic field line (The axis will be curved and at any instant motion can be considered helical with axis along the tangent). Model the planet as an ideal dipole with dipole moment due to this planet as $4.39 \times 10^{22} Am^2$. The radius of planet is 4983 km. Consider a general magnetic field line and find the frequency and radius of helical motion as the particle gyrates along the magnetic field line.
Assume the mass and charge of the particle as m and q. Assume that speed along the axial direction is v_1 and speed perpendicular to axis is v_2. Define a variable 'pitch angle' denoted by p_θ which is the angle made by net velocity vector with with plane perpendicular to axis . Note that $p_\theta = tan^{-1}(\frac{v_2}{v_1})$ Consider the perpendicular distance of the field line at the equator to be r_e. Neglect the rotational effects of Earth. [1+1]

iv. Now consider the particle to be a proton. It has been observed that the magnetic moment that the proton generates due to its motion perpendicular to axis remains constant. Consider v_1 and v_2 at equator to be v_{1eq} and v_{2eq}. After reaching a specific latitude the proton's axial speed (velocity parallel to earth) becomes zero. Qualitatively explain what might be the reason for this and then determine this latitude. Neglect gravitational interactions of proton with planet. [2]

v. Determine the area of the planet from which the particle would be 'visible' at the point it turns back ($v_1 = 0$) ?(You can neglect effect of Earth's atmosphere on light rays and assume that particle is visible at those locations where rays after getting reflected from particle reach. Assume dimensions of particle are sufficiently larger than wavelength of light) [1]

Problem 7: Can Earth be detected? [10pts]

Transit method of exoplanet detection is a really successful method whereby the periodic dimming of the star is used to detect the presence of an exoplanet transiting the star, even though we can resolve neither the exoplanet or the star.

Stars appear bright, as they emit photons. For nearby stars, the number of individual photons we detect in any attempt to image is generally pretty high. However, for faraway stars, the fact that the photons reach us randomly becomes an issue.

Light from stars and exoplanet detection.

Consider for now that a star of brightness B means that, in an infinitesimal time dt, the probability of our apparatus detecting a photon is Bdt.

i. Consider a finite time interval t. What is the probability of detecting n photons in this interval?

Consider this time interval as t/dt time intervals of dt length, and take appropriate limits in the end [2]

ii. Find the mean number of photons detected in time t and the standard deviation in it. [1+1]

Transit of the Earth

iii. For a faraway alien civilization in the ecliptic plane, how long will the transit of the Earth last? [1]

iv. The radius of the Sun is 109 times that of the Earth. What fraction of the Sun's light will be blocked by the Earth? [1]

We shall call this fraction f_t.

v. Consider a civilization at distance d from the Earth-Sun system. Assume the Sun to emit all photons in wavelength $500nm$ only. The luminosity of the Sun is $3.8 \times 10^{26}W$. Find B if the civilization is using a 100% efficient detector with a circular aperture of $1m$ diameter. [1]

vi. Find the fractional uncertainty in the detection of photons upon observing for the entire length of the transit.

We shall call this f_u. [1]

vii. We can detect dimming only if $f_t > f_u$. What are the constraints on d for the Earth to be detected in a single transit? [2]

Problem 8: Cosmology [11pts]

The scale factor

The scale factor $a(t)$ is a factor representing the scale of the universe under homogeneous expansion. Through this problem, we define the scale factor as the factor by which the universe has expanded compared to our current time. Thus, $a(t_0) = 1$, where t_0 is the current time, $a(0) = 0$ where time 0 corresponds to the big bang. $a(t_1) = 0.5$ means that, at time t_1, the distances between distant objects were half of what they are now. Note that we have used the word 'distant', because, with nearby objects, random motions and gravitational and other forces far dominate the expansion of the universe.

i. Find the Hubble parameter $H(t)$, defined by

$$H(t) = v/d$$

where v is the velocity of recession of a galaxy a distance d away, at time t, in terms of $a(t)$ and its derivatives. [1]

ii. Consider two bodies of mass m. In terms of this mass and the other constants, assuming the Hubble parameter to be unchanging with time, obtain the distance at which the expansion of the universe dominates the gravitational acceleration (Assume Newtonian gravitation). [1]

iii. Find this distance (in kpc) for $m = 10^{42} kg$, the typical mass of a galaxy. Compare this distance to the size of the local group. [1+1]

Note: Nowhere further shall we assume H(t) as a constant. The reason we did it here was because:

$$\frac{\dot{H}}{H} \ll \frac{\dot{a}}{a}$$

$$i.e.\ \dot{H} \ll H^2$$

Redshift

Cosmological redshift is the redshift caused by the expansion of the universe. The relation is simple: the expansion stretches light just as it stretches all of space. Thus, when travelling through vacuum, the wavelength of light goes by:

$$\lambda(t) \propto a(t)$$

Redshift is defined as:

$$z = \frac{f_e - f_o}{f_o}$$

where f_e is the frequency emitted and f_o is the frequency observed.
iv. Find $a(t_e)$, the scale factor at time of emission in terms of z (Observation time is the current time, t_0). [1]
v. How will we see the spectrum of a blackbody at temperature T, placed far off at a redshift z? (Hint: We shall see a blackbody spectrum, but at a different temperature. Show this and find the new temperature) [1]

Brightness and angular size

In this section, we look at how the expansion of the universe messes with the inverse square law of the brightness of objects. We also see how it messes with the angular sizes, which are supposed to fall inversely with distance. Suppose there exists objects with luminosity L and diameter l scattered in space. Consider one such object with redshift z. Assume that the distance between the Earth and the body today is d.
You may notice that redshift is easily measurable, while d is not.
vi. Over what area has the light emitted by the body spread today, when we observe it? [1]
vii. Noting that the light is also redshifted and stretched, find the flux F (energy passing per unit time per unit area) from the object observed today, given its luminosity L. [1]
viii. Use a Earth centric reference frame, and consider light emitted from two ends of the body at time t_e to find an expression for the observed angular size of the body θ, given its diameter l. [1]
ix. Use these and find a relationship between z, F and θ which must be obeyed. [1]
x. Supposing you know L and l, find d in terms of F and θ. [1]