Indian Institute of Science

Pravega 2022

Solutions to Enumeration Prelims

July 16, 2022

Objectives

1. 1012
2. 22
3. 2020
4. 23
5. 23
6. 4042
7. 2173
8. 489
9. 448
10. 17

Subjectives

1. *(by Sudharshan K V)* Bhavya cannot restore the triangle ABC. There may be more than one triangle ABC having the same A, I_B, I_C. See Figure 1.

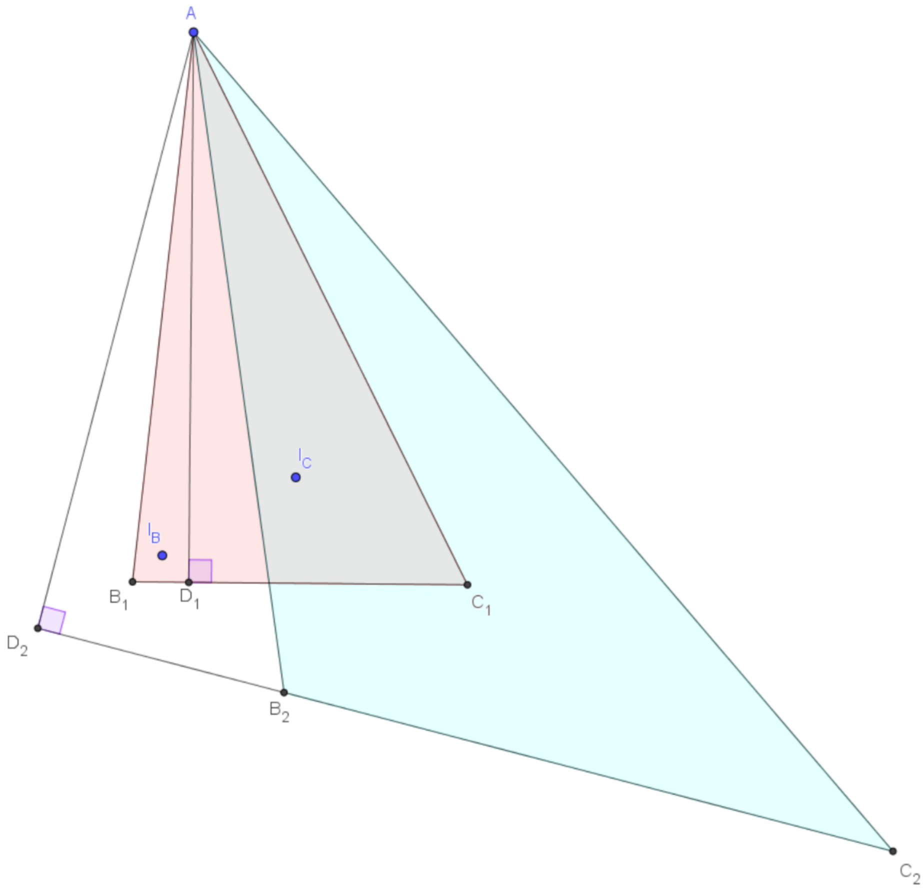

Figure 1: Two triangles with same A, I_B, I_C

2. *(by Sakjit Das)*
For, $\omega = e^{\frac{2\pi i}{3}}$
$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (x+y)(x+y\omega)(x+y\omega^2)$
$(x+y\omega) = (a+b\omega)^2 = (a^2-b^2) + (2ab-b^2)\omega$
$(x+y\omega^2) = (a+b\omega^2)^2$
Thus, setting $x = (a^2-b^2)$ and $y = (2ab-b^2)$, we see that,
$(x^2 - xy + y^2) = (a^2 - ab + b^2)^2$

Now, we want $(x+y)$ to be a perfect square.
$(x+y) = a^2 + 2ab - 2b^2 = (a+b)^2 - 3b^2$
Substitute, $a = n + 1 - b$ and $n = \frac{3b^2-1}{2}$
Then a, n are integers for odd b and $(x+y) = n^2$
Note that for $b \geq 3$, x and y both are positive. Also,
$gcd(x,y) = gcd((a-b)(a+b), b(2a-b))$
Now, $gcd(a,b) = gcd(n+1-b, b) = gcd(n+1, b) = \frac{1}{2}gcd(3b^2+1, 2b)$
$= \frac{1}{2}gcd(3b^2+1, 2) = 1$ since, b is odd.
Observe, $1 = gcd(a,b) = gcd(a-b,b) = gcd(a+b,b) = gcd(a-b, 2a-b)$
Also, $gcd(a+b, 2a-b) = gcd(a+b, 3a) = gcd(a+b, 3) = gcd(\frac{3b^2-2b+1}{2}, 3)$
$= gcd(3b^2 - 2b + 1, 3) = gcd(2b-1, 3)$
$= 1$ for $b \equiv 1, 3 \pmod 6$.
Hence, $gcd(x,y) = 1$ and
$gcd(x, z^2) = gcd(x, x^3 + y^3) = gcd(x, y^3) = 1$
$\implies gcd(x,z) = 1$
Similarly, $gcd(y,z) = 1$ and thus, $gcd(x,y,z) = 1$.
For, $b = 3, 7, 9, 13, 15, 19, ...$ we get an infinite sequence of solutions with the required properties. To list a few we have,
$(x,y,z) \equiv (112, 57, 1261), (4440, 889, 297037), (12688, 1953, 1431793), ...$ **(10)**

3. *(by Bhavya Agrawalla)*
We are given that,

$$u(a,b,c,d,e) - u(b,a,c,d,e) + u(b,c,a,d,e) - u(b,c,d,a,e) + u(b,c,d,e,a) = 0 \quad (1)$$

and

$$u(a,b,c,d,e) - u(a,c,b,d,e) + u(a,c,d,b,e) - u(a,c,d,e,b) + u(c,a,b,d,e) - u(c,a,d,b,e) + u(c,a,d,e,b) + u(c,d,a,b,e) - u(c,d,a,e,b) + u(c,d,e,a,b) = 0 \quad (2)$$

Interchange b and a in 2 and add it to 1 to get

$$u(a,b,c,d,e) + u(c,b,a,d,e) - u(c,b,d,a,e) + u(c,b,d,e,a) + u(c,d,b,a,e) - u(c,d,b,e,a) + u(c,d,e,b,a) = 0 \quad (3)$$

Interchange b and c in 3 and subtract it from 1 to get

$$u(a,b,c,d,e) - u(b,a,c,d,e) - u(a,c,b,d,e) - u(b,d,c,a,e) + u(b,d,c,e,a) - u(b,d,e,c,a) = 0 \quad (4)$$

Interchange c and d, a and e in 4 and add it to 1 to get

$$u(a,b,c,d,e) - u(b,a,c,d,e) + u(e,b,d,c,a) - u(b,e,d,c,a) - u(e,d,b,c,a) = 0 \quad (5)$$

Change a to e, e to b, b to a, and interchange c and d in 5, and add it to 1 to get

$$u(e,a,d,c,b) - u(a,e,d,c,b) + u(b,c,d,e,a) - u(b,c,d,a,e) = 0 \quad (6)$$

Change a to b, b to e, e to a, and interchange c and d in 6 to get

$$u(a,b,c,d,e) - u(b,a,c,d,e) = u(e,d,c,b,a) - u(e,d,c,a,b) \quad (7)$$

Interchange b and e, c and d in 1 to get

$$u(a,e,d,c,b) - u(e,a,d,c,b) + u(e,d,a,c,b) - u(e,d,c,a,b) + u(e,d,c,b,a) = 0 \quad (8)$$

6 + 7 - 1 gives

$$u(a,e,d,c,b) - u(e,a,d,c,b) + u(b,c,a,d,e) - u(e,d,c,a,b) + \quad (9)$$
$$u(e,d,c,b,a) = 0 \quad (10)$$

8 and 9 imply

$$u(e,d,a,c,b) = u(b,c,a,d,e) \quad (11)$$

Finally replacing e with a, d with b, a with c, c with d, b with e in 11

$$u(a,b,c,d,e) = u(e,d,c,b,a) \quad (12)$$

as required. **(10)**

4. *(by Bhavya Agrawalla)* Let $\binom{n}{m}$ denote the binomial coefficient. We will show the following:

   - $d(a,b) = 0$ if a and b are both odd.
   - $d(a,b) = \binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor}$ otherwise

   An (a,b) shuffle is completely determined by $\sigma(1), \sigma(2), \ldots, \sigma(a)$. In fact, it's parity is the same as that of

$$\sigma(1) + \sigma(2) + \cdots + \sigma(a) - \frac{a(a+1)}{2} \quad (13)$$

Thus the number of even shuffle permutations is exactly the number of a element subsets of $\{1, 2, \ldots, a+b\}$ with sum congruent to $\frac{a(a+1)}{2}$ modulo 2, and the number of odd shuffle permutations are equal to the number of a element subsets with sum congruent to $\frac{a(a+1)}{2}+1$ modulo 2.
The difference between these is equal to the coefficient of x^a in the expansion of

$$(-1)^{\frac{a(a+1)}{2}} \prod_{i=1}^{a+b} (1+(-1)^i x) \tag{14}$$

- If $a+b$ is even, the above product is equal to

$$(-1)^{\frac{a(a+1)}{2}} (1-x^2)^{\frac{a+b}{2}} \tag{15}$$

- If $a+b$ is odd, the above product is equal to

$$(-1)^{\frac{a(a+1)}{2}} (1-x^2)^{\frac{a+b-1}{2}} (1-x) \tag{16}$$

In either case, the claim made at the start can be easily checked. **(10)**

5. *(by Sudharshan K V)* No! Suppose such P, Q, R, S exist. Then, Q, R, S are the reflections of P about the sides of $\triangle BCD$. The circumcenter A of $\triangle QRS$ is thus the isogonal conjugate of P WRT $\triangle BCD$. If $ABCD$ is cyclic, then the isogonal conjugate of A WRT BCD is at infinity - so such P, Q, R, S won't exist. However, if $ABCD$ is not cyclic, then we can show that taking as P, Q, R, S the isogonal conjugates of A, B, C, D WRT the triangle formed by the other three satisfies the condition.

6. *(by Kazi Aryan Amin)* Let k be a positive integer, and let ϵ be a positive real number. Prove that there are infinitely many positive integers n, such that the largest prime factor of n^k+1 is less than n^ϵ. **(10)**

   Let $k = \prod p_i^{a_i}$. And let $r \gg \max a_i + 1$. For a large positive integer N. define y to be the product of all the primes less than N. Let $n = x^{\frac{y}{k}}$ for some positive integer x. Then we have :

$$n^k + 1 = x^y + 1 = \frac{x^{2y}-1}{x^y-1} = \frac{\prod_{d|2y} \Phi_d(x)}{\prod_{d|y} \Phi_d(x)} = \prod_{d|y} \Phi_{2d}(x)$$

   , where $\Phi_d(x)$ denotes the d-th cyclotomic polynomial.

Therefore the largest prime factor of n^k+1 must be atmost $\max_{d|y}|\Phi_{2d}(x)|$. By the triangle inequality $\Phi_{2d}(x) \leq (x+1)^{\varphi(2d)} \leq x^{2\varphi(2d)}$.So the largest prime factor of n^k+1 is atmost $\max_{d|y} x^{2\varphi(2d)} = x^{2\varphi(2y)}$. Now we may bound:

$$x^{2\varphi(2y)} = n^{2k\cdot\frac{\varphi(2y)}{y}}$$

Its well known that for large enough N, $\frac{\varphi(2y)}{y}$ goes to zero, so we can a N'. such that for all $N > N'$, the exponent in the previous expression is less than ϵ. This completes the problem.

Best wishes