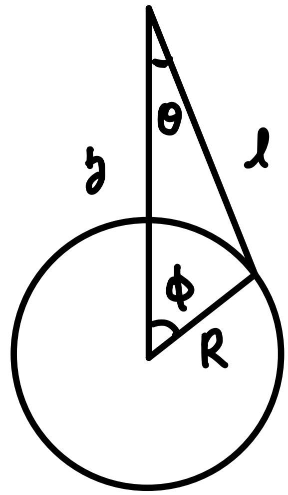
1a) We first consider the field due to a circular ring mass.
For a point outside the ring,

$$\cos\phi = \frac{z^2+R^2-\ell^2}{2zR}$$

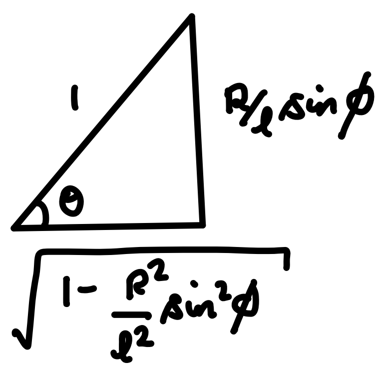
$$\ell^2 = 2zR\cos\phi - z^2 - R^2$$



$$\frac{\sin\theta}{R} = \frac{\sin\phi}{\ell} \quad \left(\text{Sine Rule}\right)$$

$$\theta = \sin^{-1}\left(\frac{R}{\ell}\sin\phi\right)$$

$$\cos\theta = \sqrt{1-\frac{R^2}{\ell^2}\sin^2\phi}$$



$$dE = 2\,\frac{G\,dm}{\ell^2}\cos\theta$$

$$= 2\left(\frac{G\lambda R\,d\phi}{2zR\cos\phi - z^2 - R^2}\right)\sqrt{1-\frac{R^2}{\ell^2}\sin^2\phi}$$

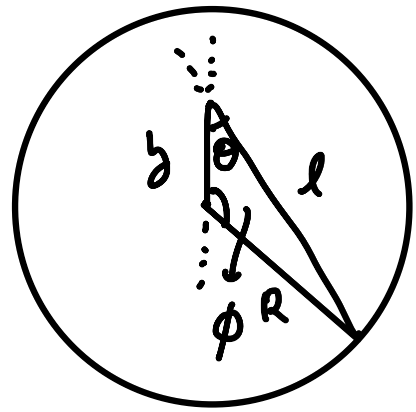
$$E_1 = 2\int_0^{\pi}\left(\frac{G\lambda R\,d\phi}{2zR\cos\phi - z^2 - R^2}\right)\sqrt{1-\frac{R^2}{\ell^2}\sin^2\phi}$$

For a point inside the ring, (Here, λ is linear mass density)

$$\cos\phi = \frac{z^2+R^2-\ell^2}{2zR}$$

$$\ell^2 = 2zR\cos\phi - z^2 - R^2$$

$$\theta = \sin^{-1}\left(\frac{R}{l}\sin\phi\right)$$



$$dE = 2\,\frac{G\,dm}{l^2}\cos\theta$$

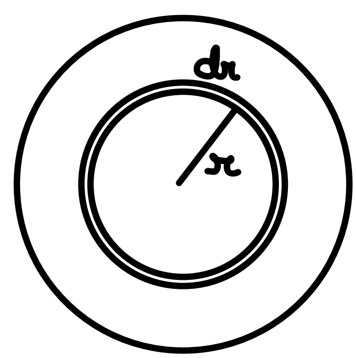
$$= 2\left(\frac{G\,\lambda R\,d\phi}{2yR\cos\phi - y^2 - R^2}\right)\sqrt{1-\frac{R^2}{l^2}\sin^2\phi}$$

$$\frac{\sin\theta}{R} = \frac{\sin\phi}{l}$$

$$E_2 = 2\int_0^{\pi}\left(\frac{G\,\lambda R\,d\phi}{2yR\cos\phi - y^2 - R^2}\right)\sqrt{1-\frac{R^2}{l^2}\sin^2\phi}$$

∴ E_1 and E_2 are dependent on R.
Let $E_1 = f_1(R)$ and $E_2 = f_2(R)$

∴ For the disc,
we have the following integral



$$\lambda = \frac{\sigma\,\cancel{2\pi r}\,dr}{\cancel{2\pi r}}$$
$$= \sigma\,dr$$

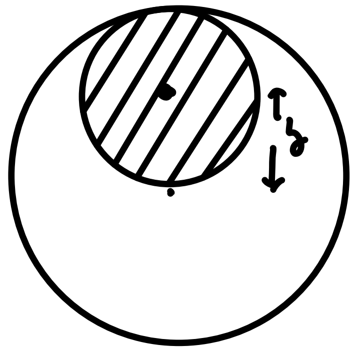
Substituting σ for λ, and integrating
we get field as follows at distance 'd' as follows

$$E(d) = \int_0^d f_1(r)\,dr + \int_d^R f_2(r)\,dr$$

The above integral is an example of 'Elliptic integral' and analytic solutions exist only in special cases. Thus, grace points will be awarded to all contestants for Part A (a, b, c) of the 1st Question

A Qualitative note -

For a point y distance away, we can neglect the contribution of the circle centred at that point of radius $(R-y)$. This can be shown by symmetry.



The mass that can be neglected is larger as we move towards the centre.

$\therefore$ The gravitational field is stronger as we move away from the centre.

From field E, we get $E = v^2/r$

$$\therefore \quad v(r) = \sqrt{Er}$$

$\because$ E and r both increase with r

$\therefore$ $v(r)$ also increases.

1b)
$$K^2 = \left(\frac{2v}{r}\right)\left(\frac{v}{r} + \frac{dv}{dr}\right)$$

$$\frac{dv}{dr} = \left(\frac{dE}{dr} + E\right)$$

This can be solved by substituting for $E(r)$ and $v(r)$

1c) We know that Kepler's law holds for a field obeying inverse square law.

$\because$ in a Keplerian disc, the relation $r^3 \propto T^2$ is true for <u>ALL</u> orbits the only mass distribution that satisfying this condition is a mass concentrated at the centre of the galaxy.

This can be properly expressed by the delta function

The original intent of the problem was to provide a situation with sufficient symmetries to calculate the gravitational field. Many dark model halo models indeed have the necessary symmetries that exploit Gauss' Law in order to calculate the field strength