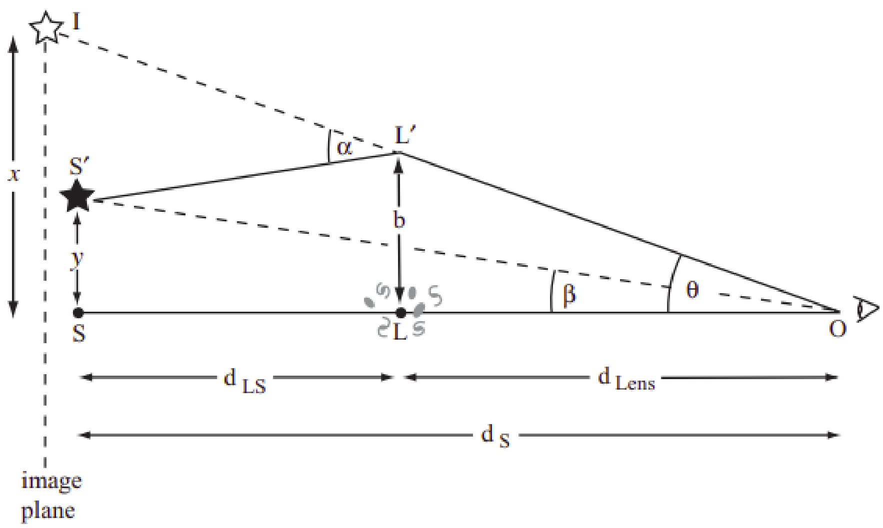
Problem 3: Gravitational Lensing



i) The formula for α is valid as long as the bending is small, with $\alpha \ll 1$. Moreover, we can assume the angles to be very small (often they are of the order of seconds which is just $\frac{1}{3600}$ of a degree).
From the figure, we can write:

$$x - y \approx \alpha d_{LS} \tag{1}$$

Now, we can write:

$$
\begin{aligned}
\theta - \beta &\approx \frac{x-y}{d_S} \\
&\approx \frac{\alpha d_{LS}}{d_S} && \text{[From 1]} \\
&\approx \frac{\alpha d_S}{d_S} && [\because d_{lens} \ll d_S \therefore d_{LS} \approx d_S] \\
&= \alpha
\end{aligned}
$$

Now,

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{d_{LS}}{d_{lens}d_S}} \approx \sqrt{\frac{4GM}{c^2}\frac{1}{d_{lens}}}$$

Putting $d_{lens} = 1AU$ and the value of other constants, we get,

$$\theta_E \approx 1.406 \times 10^{-19}\sqrt{M}$$

in SI units or

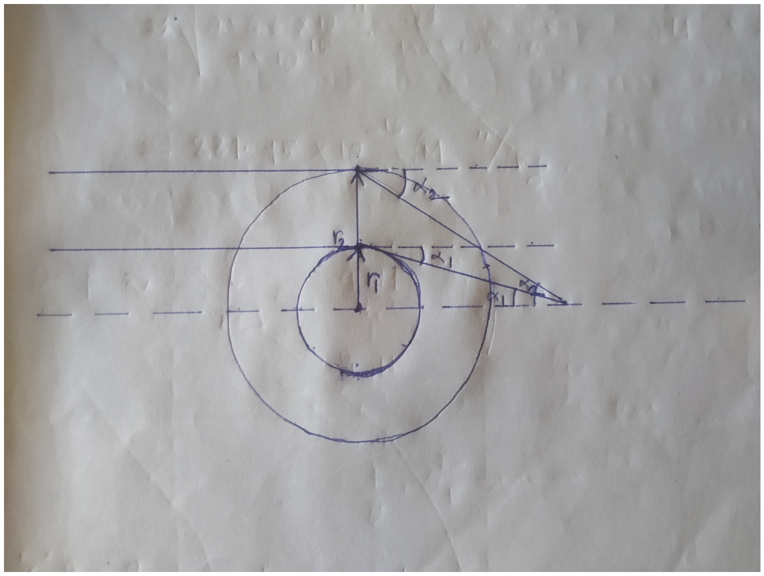
$$\theta_E \approx 2.899 \times 10^{-14}\sqrt{M}$$

in "$/\sqrt{kg}$. Both answers are accepted.

ii) We have,

$$
\begin{aligned}
\theta - \beta &\approx \frac{\alpha d_{LS}}{d_S} \\
&= \frac{4GM}{bc^2}\frac{d_{LS}}{d_S} && [\because \alpha = \frac{4GM}{bc^2}] \\
&= \frac{1}{\theta}\frac{4GM}{c^2}\frac{d_{LS}}{d_S d_{lens}} && [\because b \approx \theta d_{lens}] \\
&= \frac{1}{\theta}\theta_E^2 \\
\therefore \theta^2 &= \theta_E^2 && [\beta = 0] \\
\therefore \theta &= \pm\theta_E
\end{aligned}
$$

For $\beta = 0$, the star S' will be seen as a circle of light on the sky, with radius θ_E. For $\beta > 0$, multiple images are formed, one image being brighter and further away from the lens than the other.

iii) 

We can write:

$$\alpha_1 = \frac{4GM(r_1)}{r_1 c^2}$$

$$\alpha_2 = \frac{4GM(r_2)}{r_2 c^2}$$

Now, $r_1 \approx \alpha_1 d$ and $r_2 \approx \alpha_2 d$ (Here, d is the distance of the point of focus from the centre of the spherical dark matter distribution).

$$\therefore \frac{r_1}{\alpha_1} = \frac{r_2}{\alpha_2}$$

$$\Rightarrow \frac{r_1}{\frac{4GM(r_1)}{r_1 c^2}} = \frac{r_2}{\frac{4GM(r_2)}{r_2 c^2}}$$

$$\Rightarrow \frac{M(r_1)}{M(r_2)} = \frac{r_1^2}{r_2^2}$$

$$\therefore M(r) \propto r^2$$

$$\therefore \boxed{\alpha = 2}$$