# Pravega Astrowiz Solutions

Q6

(i) $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Let field be $\frac{\mu_0 i}{2\pi(x^2+y^2)}(y\hat{i} - x\hat{j})$ $\left(= \frac{\mu_0 i}{2\pi r}\hat{n}\right.$ where $\hat{n}$ is Tangential $)$

$$\frac{\mu_0 i}{2\pi(x^2+y^2)}(y\hat{i} - x\hat{j}) = \underset{\text{Constant}}{k}\left(\frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j}\right)$$

$$\Rightarrow \frac{\mu_0 i\, y\hat{i}}{2\pi(x^2+y^2)} = \frac{k\,dx}{dt} \quad \text{— i)}$$

$$\& \; -\frac{\mu_0 i\, x}{2\pi(x^2+y^2)} = \frac{k\,dy}{dt} \quad \text{— ii)}$$

i) / ii)

$$-\frac{y}{x} = \frac{dx}{dy}$$

$$\Rightarrow -y\,dy = dx \cdot x$$

$$\Rightarrow -\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$x^2 + y^2 = -2c \quad (c \text{ is } -ve)$$

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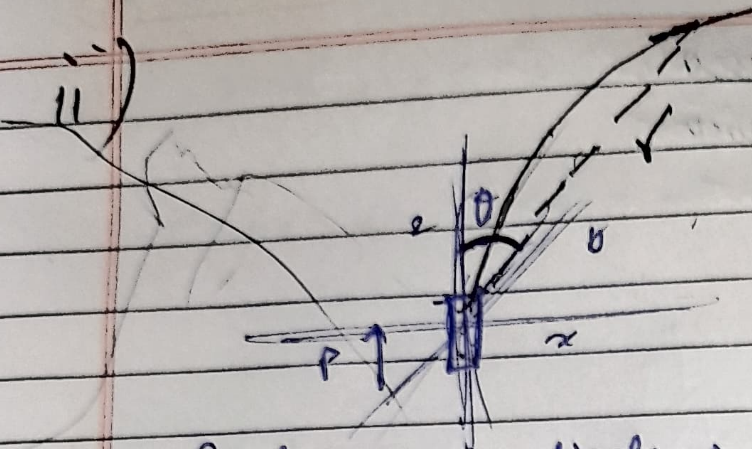
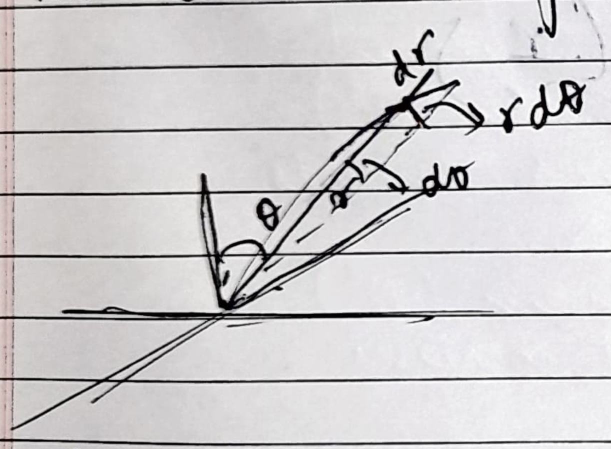
ii)



Field due to dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{p}{r^3} \left( 2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)$$

Now consider the following figure



The field line is tilted with $\vec{r}$ by angle $\alpha$

Then clearly $\tan\alpha = \dfrac{r\,d\theta}{dr}$.

Also this must be $\parallel$ to field

Angle made by field with with $\vec{r}$

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$$\tan\alpha = \frac{B_\theta}{B_r} = \frac{\sin\theta}{2\cos\theta}$$

$$\therefore \frac{r\,d\theta}{dr} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{2\cos\theta\,d\theta}{\sin\theta} = \frac{dr}{r}$$

$$\Rightarrow \int_{\frac{\pi}{2}}^{\theta} \frac{2\cos\theta\,d\theta}{\sin\theta} = \int_{r_e}^{r} \frac{dr}{r}$$

$$\Rightarrow 2\ln(\sin\theta) = \ln\left(\frac{r}{r_e}\right)$$

$$\Rightarrow r = \sin^2\theta\, r_e$$

iii) Assume a field line $r = r_e \sin^2\theta$

Magnitude magnetic field

$$= \frac{\mu_0 P}{4\pi r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

$$= \frac{\mu_0 P}{4\pi r^3}\sqrt{1 + 3\cos^2\theta}$$

$$= \frac{\mu_0 P \sqrt{1 + 3\cos^2\theta}}{4\pi (r_e)^3 \sin^6\theta} = B(\theta) \longrightarrow \text{(iii)}$$

Now if speed of particle ⊥ to field line is $v_2$, we have

$$\frac{m v_2^2}{r} = q v_2 B$$

$$\Rightarrow \boxed{r = \frac{m v_2}{q B}} = \frac{m v_2 \, 4\pi (r_e)^3 \sin^6\theta}{q \mu_0 P \sqrt{1 + 3\cos^2\theta}}$$

Pitch = $v_1 \times$ Time period

$$= v_1 \times \frac{2\pi r}{v_2}$$

$$\boxed{= v_1 \frac{2\pi m}{q B}}$$

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iv Magnetic moment generated due to motion ⊥ to field line

$= \text{area} \times \text{effective current}$

$= \pi r^2 \times \frac{q}{T}$

$= \frac{\pi m^2 v_2^2}{q^2 B^2} \frac{q(q)B}{2\pi m} = \frac{m v_2^2}{2B}$

Now $\frac{m v_2^2}{2B}$ remains constant — iv)

But note that magnetic field does no work $\therefore v_1^2 + v_2^2 = \text{constant}$

Also note as proton moves towards poles $\theta \downarrow$, $Q$ iii)

$\therefore$ from (iii) $B \uparrow$

$\therefore$ from iv $v_2^2$ must increase

But $v_1^2 + v_2^2 =$ constant,

$\therefore v_1$ should decrease and possible could become zero some where

Let latitude be $\alpha$

$\therefore \theta = \frac{\pi}{2} - \alpha$.

When proton stops $v_1 = 0$

$v_2^2 = (v_{1eq})^2 + (v_{2eq})^2$ — V)

$$\frac{m v_2^2}{2B(\theta)} = \frac{m (v_{2eq})^2}{2 B(\frac{\pi}{2})}$$

$$\Rightarrow (v_{1eq})^2 + (v_{2eq})^2 = (v_{2eq})^2 \left(\frac{B(\theta)}{B(\frac{\pi}{2})}\right)$$

$$\Rightarrow 1 + \tan^2(p_\theta) = \tan^2(p_\theta) \frac{\frac{\mu_0 p \sqrt{1+3\cos^2\theta}}{4\pi (r_e)^3 \sin^6\theta}}{\frac{\mu_0 p}{4\pi (r_e)^3}}$$

$$\Rightarrow 1 + \tan^2(p_\theta) = \tan^2(p_\theta) \frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta} \longrightarrow \text{vi)}$$

$$\theta = \frac{\pi}{2} - \alpha$$

$$\Rightarrow \cancel{1+} \; 1 + \cot^2(p_\theta) = \frac{\sqrt{1+3\sin^2\alpha}}{\cos^6(\alpha)} \longrightarrow \text{vii)}$$

The RHS is clearly ↑ in $(0, \frac{\pi}{2})$

∴ We get only one solution for $\alpha$ in $(0, \frac{\pi}{2})$

This can be solved numerically for any given value of $p_\theta$

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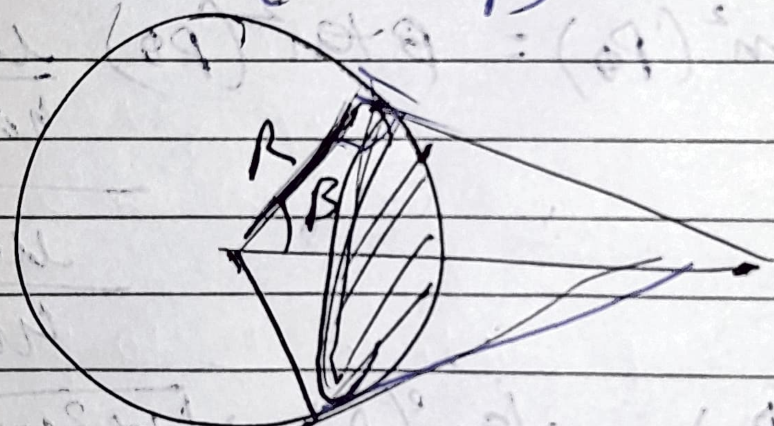
v) Let us say value of $\theta$ obtained from previous part is $\theta'$
we have $r = r_e \sin^2(\theta')$

Now area of spherical cap is given by $2\pi(1-\cos\beta)$




But $\cos\beta = \dfrac{R}{r} = \dfrac{R}{r_e \sin^2(\theta')}$

$\therefore$ Area $= 2\pi\left(1 - \dfrac{R}{r_e \sin^2(\theta')}\right)$