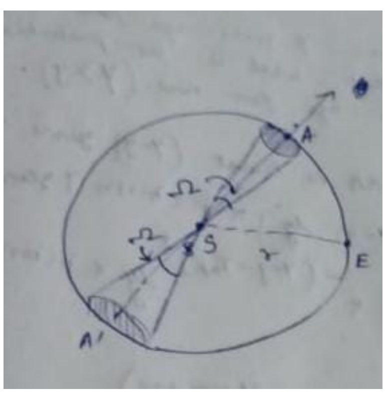
P3

1 A basic model (10 pt)

A group of astrophysicists wish to determine the chances of us, Earth, to be hit by a killer gamma ray burst. They have made some observations and concluded upon a few facts as follows:

- It is suggested that the energy of the explosion is beamed into a narrow cone of emission along two opposite direction. The average cone angle being Ω steradians.($\approx 10^{-3}$)
- Sources of such Gamma ray bursts (e.g. blackholes, colliding neutron stars, hyper-novae etc.) are scattered uniformly throughout the observed universe. A very rough estimate of this number density (D_n) gives a number in the ball park of 10^6(mega-parsec)$^{-3}$
- It has been estimated that about N, ($N \approx 500$) GRBs occur within our observed universe *everyday*.
- It can not be predicted at all where a GRB is about to happen, since all observed GRBs were scattered uniformly across our sky.



Assume the radius of observable universe to be $R_u \approx 14$ giga-parsecs. For the sake of simplicity, in this part we don't consider our universe to be expanding. Based on these conjectures, now you have been tasked to evaluate a **Probability Distribution Function** f of us being hit by a GRB of a certain intensity I on a single day
In other words,

$$\int_{I_a}^{I_b} f(I)\, dI$$

should give the probability of us being hit by a GRB within an intensity range of I_a to I_b on a single day.

Note: In case you can't solve any integral leave it in expression form.

1.1 Meet the assassins (6 pt)

Firstly, let us focus on a single GRB source. Say it is at a distance r mega-parsec(Mpsc) away.(1 parsec $\approx$ 3.2 light-year)

1.1.1 Did he shoot that day? (3 pt)

What is the probability that this potential source went off one fine day, y years ago?
In mathematical terms, find the probability distribution function p_y that this blasted y years ago. $p_y(y_0)dy$ thus gives the probability for year $y = y_0$
The astrophysicist team you are working for has come up with this interesting result that may be useful to you:
Let $p(t)$ be the probability that a GRB source goes off t years since its birth. Then,

$$p(t) = Cte^{-(t-T_{mp})}$$

Here, T_{mp} is the most probable time for the GRB to occur, and C is the normalisation constant (don't worry about it too much).
Assume that a GRB source could have been created at any point of time in the Universe's lifetime before y. ($T_{uni} \approx 14$ billion).

Hint: *Think about the conditional probability of the source being born, say Y years ago,* ***and*** *the burst to happen y years ago. Now consider that Y can be any year$\geq y$. You shall approximate 1 day to be a differential change in a year. You might find this result useful:*

$$\int xe^{(-(x-t))}\,dx = -(x+1)e^{t-x} + C$$

Solution:
Let this source be born say Y years ago from now. We wish to first know the probability of it blowing y years ago from present. Or, $Y - y$ years after its birth.
This probability is

$$\delta P(\text{GRB y years ago}\| \text{ source born Y years ago}) = p(Y-y)dy \times P(\text{birth Y years ago})$$

Now, since we can assume that the source can be born anytime before y years with equal probability,

$$P(\text{birth Y years ago}) = \frac{dY}{T_{uni} - y}$$

Therefore,

$$\Rightarrow \delta P = C(Y-y)e^{-(Y-y-T_{mp})}dy \times \frac{dY}{T_{uni}-y}$$

Now, to get the probability distribution over y, we add up the probability for all possible values of Y.

$$p_y dy = \int_y^{T_{uni}} \delta P \, dY$$

$$\Rightarrow p_y dy = \int_y^{T_{uni}} C(Y-y)e^{-(Y-y-T_{mp})}dy \times \frac{1}{T_{uni}-y}\, dY$$

$$= \frac{Ce^{T_{mp}}dy}{T_{uni}-y} \times \int_y^{T_{uni}} (Y-y)e^{-(Y-y)}\, d(Y-y)$$

$$= \frac{Ce^{T_{mp}}dy}{T_{uni}-y} \times [-(T_{uni}+1)e^{-T_{uni}} + (y+1)e^{-y}]$$

Therefore,

$$p_y = \frac{Ce^{T_{mp}}[(y+1)e^{-y} - (T_{uni}+1)e^{-T_{uni}}]}{T_{uni}-y}$$

1.1.2 Did he shoot at us? (2 pt)

Now, in the case that this source goes off today, what are the chances that we will be in its path (p_{us})? Assume the axis of the cones of the GRB can be oriented along any direction with uniform probability.
The value $p_{r,y} = p_{us} \cdot p_y$ gives the probability distribution of a source at a distance r to ***Fire!*** at us within a particular day, y years ago.

Solution:
Let E denote the position of Earth and S be the position of the source. Length $SE = r$. $A'SA$ is the axis of the cone of emission. A' and A are points on the sphere centred at S with radius $SE = r$.
For a fixed S and r there are only a few limited position of A and A' w.r.t the SE axis so that we will come within the death ray cone. By symmetry A or A' points must lie within a spherical cap at E with Ω solid angle.

$$p_{us} = 2 \times \frac{\text{Area of this region}}{\text{Area of entire sphere}}$$

$$\Rightarrow p_{us} = \frac{2\Omega r^2}{4\pi r^2}$$

Therefore,

$$p_{us} = \frac{\Omega}{2\pi}$$

Also,

$$p_{r,y} = \frac{\Omega}{2\pi} \times \frac{Ce^{T_{mp}}[(y+1)e^{-y} - (T_{uni}+1)e^{-T_{uni}}]}{T_{uni}-y}$$

Observe, that based on our model so far, the probability of a GRB firing at us y years ago is independent of how far is it from us.

1.1.3 How hard did he hit? (1 pt)

Estimate the intensity of the death ray he shot at us today, by the time it reaches us.(I)
Also express $p_{r,y}$ as a function of I and y. Call it $p_{I,y}$

Hint: *Assume that the entire energy is released in τ seconds during a gamma ray burst event.*

Solution:
Total energy released $= E$
Power of beam $= P = \frac{E}{\tau}$
Power in any of the two opposite beams $= \frac{E}{2\tau}$
Intensity at Earth $= \frac{\text{Power}}{\text{Surface area of spherical cap}}$
So, intensity $I = \frac{E}{2\Omega\tau r^2}$
Since, $p_{r,y}$ is independent of r, it is independent of I as well.

1.2 Here comes the gang! (4 pt)

Now we will work out the probability of us getting hit by anyone among this entire clan of assassins.

1.2.1 The ones at the same distance (1 pt)

What is the probability of *any of the sources at distance r* to Fire at us within a particular day y years ago? ($P_{r,y}$)
Also express $P_{r,y}$ as $P_{I,y}$ by using your result from part 1.1.3.

Solution:
Number density of potential GRB sources across the universe is D_n (given).
So, in as spherical shell at distance r with thickness dr, number of sources are

$$D_n \times 4\pi r^2 \times dr$$

Therefore,

$$P_{r,y}dr = \sum_{\textit{for all sources at } r} p_{r,y}$$

$$\Rightarrow P_{r,y}dr = (D_n \times 4\pi r^2 \times dr) \times p_{r,y}$$

$$\Rightarrow P_{r,y} = 4\pi D_n r^2 p_{r,y}$$

Now using the relation between I and r in 1.1.3 we get $P_{I,y}$

$$P_{I,y} = \frac{2\pi D_n E p_{r,y}}{\Omega\tau I}$$

1.2.2 Hidden across the cosmos (1 pt)

Now we are finally ready to measure our threat from the entire lot of cosmic assassins at no matter what distance they stand.
Calculate the probability that Earth will get hit by a GRB from anywhere on the cosmos, ***today!***

Hint: *Find a relation between r and y that must hold for us to get hit today. All possible valid (r, y) combination will contribute to our overall risk. You should get an integral of a function of r by eliminating y.*

Solution:
Since the Gamma rays will reach us at the speed of light, for us to be hit today, a source r distance away should blow off y years ago such that:

$$y = \frac{r}{c}$$

Making this substitution in $P_{r,y}$ we obtain P'_r. We can approximate one day to be a differential change in y. So we write $p_y dy \longrightarrow p_y \Delta y$
Therefore,

$$P'_r = 4\pi D_n r^2 \frac{\Omega}{2\pi} \times \frac{Ce^{T_{mp}}[(\frac{r}{c}+1)e^{-\frac{r}{c}} - (T_{uni}+1)e^{-T_{uni}}]}{T_{uni} - \frac{r}{c}} \Delta y$$

$$\Rightarrow P'_r = 2D_n r^2 \Omega \times \frac{Ce^{T_{mp}}[(r+c)e^{-\frac{r}{c}} - c(T_{uni}+1)e^{-T_{uni}}]}{cT_{uni} - r} \Delta y$$

So, probability that Earth is hit today is,

$$\int_0^\infty P'_r \, dr$$

We did not expect you to evaluate this integral. Coming to this step would have fetch you full marks for this section.

1.2.3 Assessing the risk (2 pt)

You are very close to the PDF $f(I)$ that we have been searching for. Try to parameterise r, y with I. Express the final integral in the answer of previous part in terms of I. This, integrand is precisely the $f(I)$ we have been looking for!(briefly justify how)
Of-course all GRB hits aren't lethal (we harmlessly detect about one or two almost everyday). We wish to have a formula that gives how many years must go by from now on wards, for chances of us getting *flat-lined* by the cosmos to exceed a given probability (p_{kill}).

A strike of intensity I_{kill} and above is generally considered lethal. Based on what you have understood while solving this problem, can you derive such a formula?

Of-course, you are free to make reasonable assumptions, but make sure to justify all of them.

Solution:
First we rewrite our formula for P_r' in terms of I to get P_I'.

$$P_I' = \frac{ED_n}{\tau I} \times \frac{Ce^{T_{mp}}[(\sqrt{\frac{E}{2\Omega\tau I}} + c)e^{-\frac{\sqrt{\frac{E}{2\Omega\tau I}}}{c}} - c(T_{uni}+1)e^{-T_{uni}}]}{cT_{uni} - \sqrt{\frac{E}{2\Omega\tau I}}}\Delta y$$

$$\text{Probability of a lethal hit } today(p_k) = \int_{I_{kill}}^{\infty} P_I'\, dI$$

$$P(\text{survive 1 day}) = 1 - p_k$$
$$P(\text{survive n days}) = (1 - p_k)^n$$
$$P((survivenyrs)) = (1 - p_k)^{365n}$$

Therefore, $P(\text{killed within n yrs}) = 1 - (1 - p_k)^{365n}$
For $P(\text{killed within n yrs}) \geq p_{kill}$

$$365n \geq \log_{1-p_k}(1 - p_{kill})$$

Where, n is the number of years to go by.

2 Addendum: a cute Doppler correction (5 pt)

It is well known that the universe is expanding currently. This causes an average red-shift in the electromagnetic waves that approach us from heavenly bodies across our cosmos. This is called the *Doppler effect.* We are all well familiar with the Doppler effect in our daily life with sound waves, e.g. a passing ambulance.
For the Doppler effect of electromagnetic waves, we will use a more refined version incorporating special relativity. (Again don't worry too much about it!) For your convenience we present you with a result of relativistic Doppler effect:

$$f_r = \frac{f_s}{\gamma\,(1 + \beta\cos\theta_r)}$$
$$\beta = v/c > 0$$

γ being the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$
f_r is the frequency received, f_s is source frequency, v is the velocity of source in receiver's frame and θ_r is the angle the source velocity makes with the radial direction from the receiver.

Theoretical calculation shows that, $\frac{I(f)}{f^3}$ is a *Lorentz invariant*. I.e. it is a constant for both the source and a receiver. Here $I(f)$ is the intensity of radiation at frequency f. In our case this is the intensity of the GRB as it is purely monochromatic.

It has been determined that the frequency of an average GRB is ν_0 at the source. Assume that the velocity of all GRB source are radially outwards and this velocity is proportional to the radial distance of the source from us ($v = kR$, k being a constant). Repeat what you did for part 1.1.3, but this time with this correction in place.

Solution:
The intensity of the beam in the frame of the source is same as what we derived in 1.1.3.

$$I_{source} = \frac{E}{2\Omega\tau r^2}$$

Now,

$$\frac{I_{source}}{f^3_{source}} = \frac{I_{Earth}}{f^3_{Earth}}$$

$$I_{Earth} = I_{source}\left(\frac{f_{source}}{f_{earth}}\right)^3$$

From the given equation of relativistic Doppler effect,

$$\frac{f_{source}}{f_{earth}} = \gamma\left(1 + \beta\cos\theta_r\right)$$

Since, velocity of all source is along radial direction $\theta_r = 0$.

$$\beta = v/c = \frac{kr}{c}$$

Hence,

$$\gamma = 1/\sqrt{1-\beta^2} = \frac{c}{\sqrt{c^2 - k^2r^2}}$$

Therefore,

$$I_{Earth} = \frac{E}{2\Omega\tau r^2}\frac{c+kr}{\sqrt{c^2-k^2r^2}}$$

Or,

$$I_{Earth} = \frac{E}{2\Omega\tau r^2}\frac{c+kr}{\sqrt{c^2-k^2r^2}}$$