Almost Optimal Pseudorandom Generators for Spherical Caps

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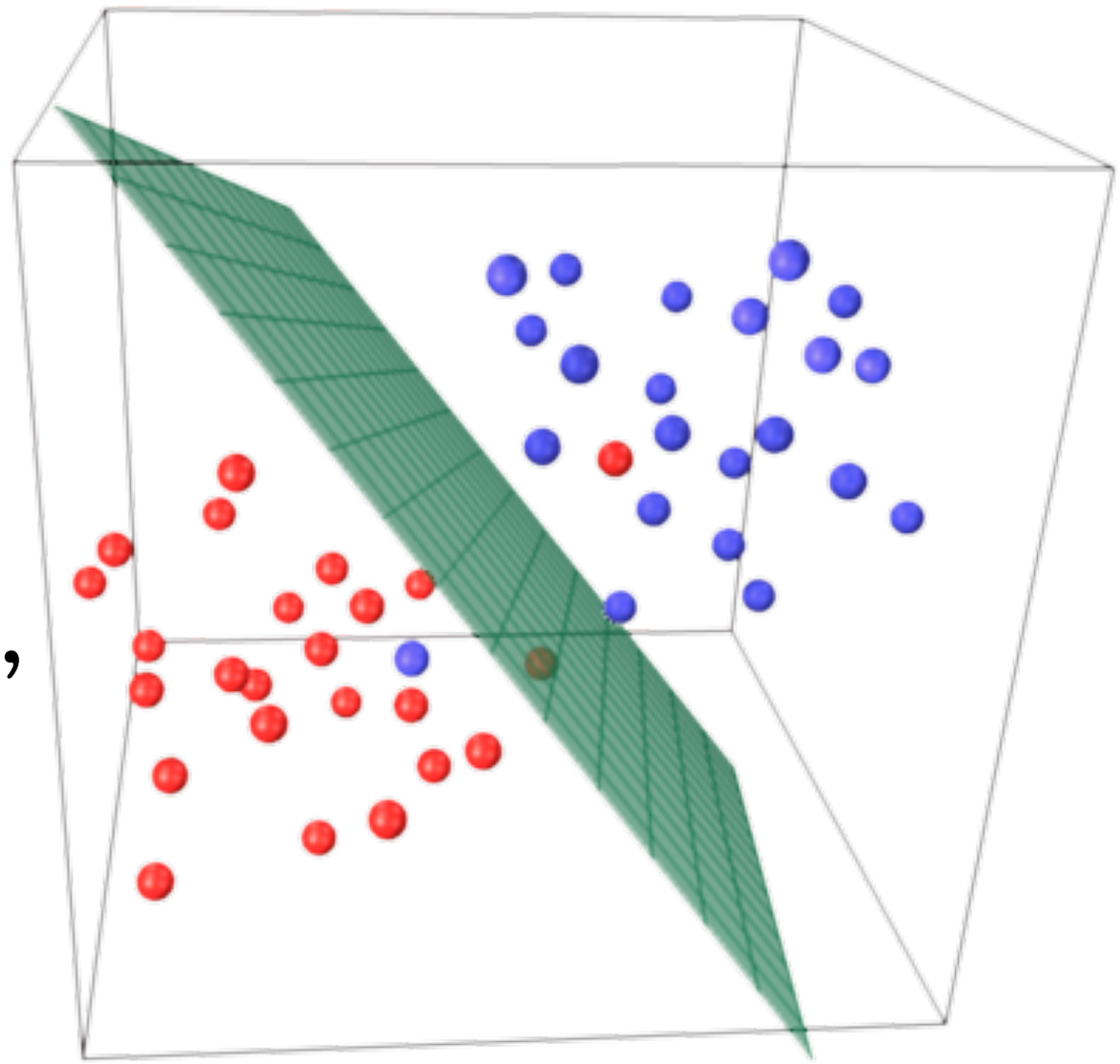
Halfspaces

$$f : \mathbb{R}^n \to \{-1, 1\}$$

$$f(x) = \mathsf{sign}(\langle w, x \rangle - \theta)$$

Applications: Perceptrons, Boosting, Support Vector Machines, Voting, Social Choice



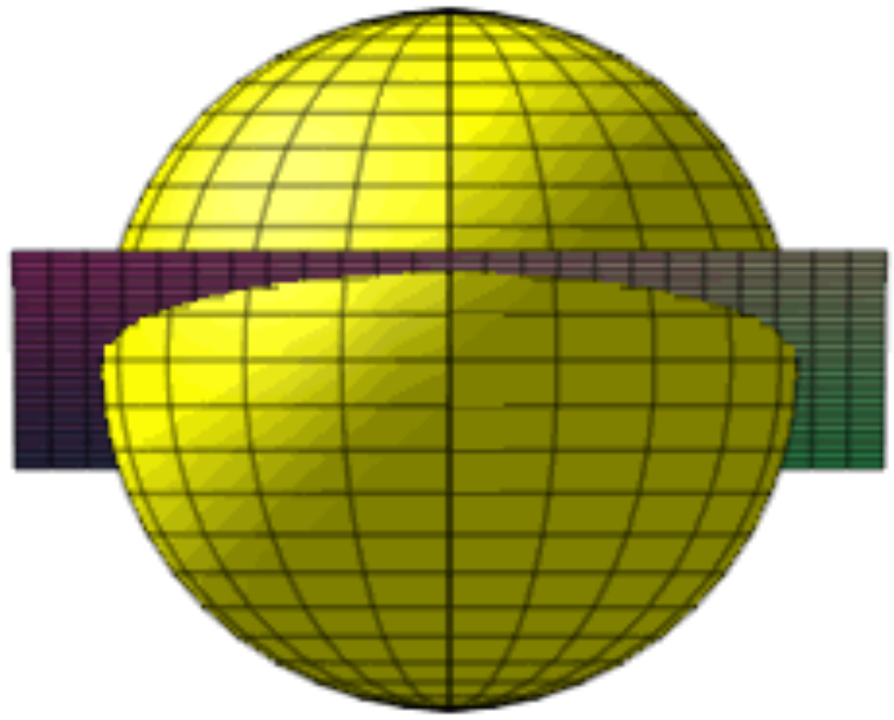
Spherical Caps

halfspaces on the sphere

$\mathbb{S}^{n-1}$:unit sphere $\subseteq \mathbb{R}^n$

$$f : \mathbb{S}^{n-1} \to \{-1, 1\}$$

$$f(x) = \mathsf{sign}(\langle w, x \rangle - \theta)$$



Applications: Perceptrons, Boosting, Support Vector Machines, Voting, Social Choice

```
int getRandomNumber()
{
    return 4;  // chosen by fair dice roll.
               // guaranteed to be random.
}
```

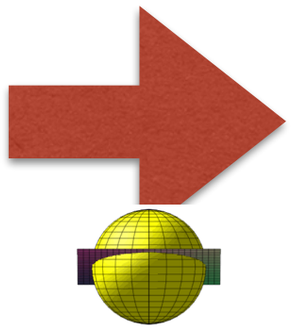
PRGs for Spherical Caps/Halfspaces?

efficiently samplable prob. dist.

1) “looks” uniform on the sphere to **every** halfspace.

2) needs only **a few** purely random bits to sample.

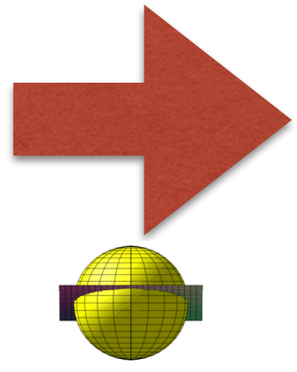
“small set with low spherical cap discrepancy”



PRGs for Spherical Caps/Halfspaces?

1. natural problem in pseudorandomness
2. derandomization of Goemans-Williamson
3. deterministic estimation of accuracy of halfspaces classifiers.

PRGs for Spherical Caps

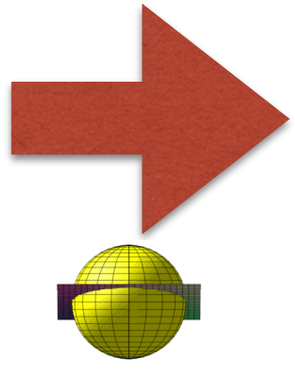


efficiently computable $G : \{0,1\}^r \to \mathbb{S}^{n-1}$

for all spherical caps $f : \mathbb{S}^{n-1} \to \{-1,1\}$

$$\left|\mathbb{E}_{x\sim\mathbb{S}^{n-1}}[f(x)] - \mathbb{E}_{y\sim\{0,1\}^r}[f(G(y))]\right| \leq \varepsilon$$

PRGs for Spherical Caps



efficiently c

for all s

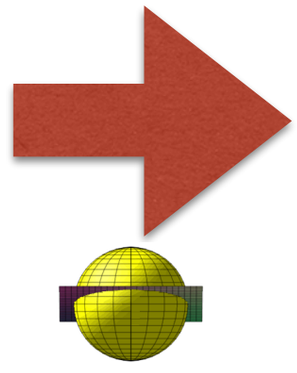
$$G : \{0,1\}^r \to \mathbb{S}^{n-1}$$

$$\mathbb{S}^{n-1} \to \{-1,1\}$$

bias on uniform distribution on the sphere

$$\left|\mathbb{E}_{x \sim \mathbb{S}^{n-1}}[f(x)] - \mathbb{E}_{y \sim \{0,1\}^r}[f(G(y))]\right| \leq \varepsilon$$

PRGs for Spherical Caps



efficiently computable

for all spherical caps

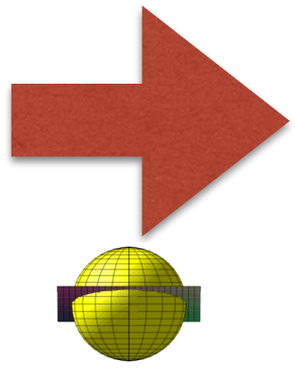
$$G : \{$$

$$f : \mathbb{S}$$

bias on output of G on uniform seed bits

$$\left|\mathbb{E}_{x\sim\mathbb{S}^{n-1}}[f(x)] - \mathbb{E}_{y\sim\{0,1\}^r}[f(G(y))]\right| \leq \varepsilon$$

PRGs for Spherical Caps



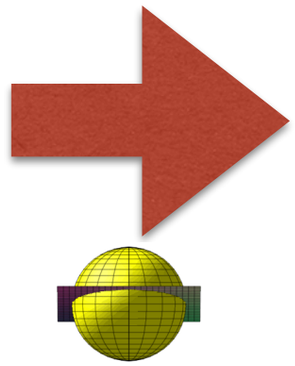
efficiently computable $G : \{0,1\}^r \to \mathbb{S}^{n-1}$

for all spherical caps $f : \mathbb{S}^{n-1} \to \{-1$

error

$$\left|\mathbb{E}_{x\sim\mathbb{S}^{n-1}}[f(x)] - \mathbb{E}_{y\sim\{0,1\}^r}[f(G(y))]\right| \leq \varepsilon$$

PRGs for Spherical Caps



seed length

efficiently computable $G : \{0,1\}^r \to \mathbb{S}^{n-1}$

for all spherical caps $f : \mathbb{S}^{n-1} \to \{-1,1\}$

$$\left|\mathbb{E}_{x\sim\mathbb{S}^{n-1}}[f(x)] - \mathbb{E}_{y\sim\{0,1\}^r}[f(G(y))]\right| \leq \varepsilon$$

PRGs for Spherical Caps

efficiently computable $G : \{0,1\}^r \to \mathbb{S}^{n-1}$

for all spherical caps $f : \mathbb{S}^{n-1} \to \{-1, 1\}$

$$\left|\mathbb{E}_{x \sim \mathbb{S}^{n-1}}[f(x)] - \mathbb{E}_{y \sim \{0,1\}^r}[f(G(y))]\right| \leq \varepsilon$$

Goal: To have a small seed length.

Probabilistic Method: $2\log(n) + 2\log(1/\epsilon) + O(1)$

PRGs for Spherical Caps

Probabilistic Method: $2\log(n) + 2\log(1/\epsilon) + O(1)$

Reference	Class/Distribution	Seed Length
LPS87	3 Dimensional Spherical Caps	$3 \cdot \log(1/\epsilon) + O(1)$
DGJSV10	Halfspaces/Uniform on the hypercube	$O(\log(n)/\epsilon^2)$
KRS12	Spherical Caps	$O(\log(n) + \log^2(1/\epsilon))$
Kan14	Spherical Caps	$O(\log(n) + \log^{3/2}(1/\epsilon))$

PRGs for Spherical Caps

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For $\varepsilon \sim 1/\mathrm{poly}(n)$ best previous result $\Omega(\log^{1.5}(n))$

PRGs for Spherical Caps

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PRGs for Spherical Caps

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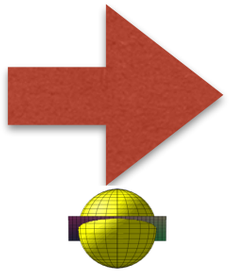
optimal up to a factor of $\log\log(n)$ for inverse poly error!

Main Result



PRG for spherical caps with seed length

$$O(\log(n) + \log(1/\varepsilon) \log\log(1/\varepsilon))$$

optimal up to a factor of $\log\log(n)$ for inverse poly error!

similar result for spherical Gaussian distribution.

General Techniques

Iterated Dimension Reduction

[Kane-Meka-Nelson11, Celis,Reingold,Segev,Wieder11]

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Strong quantitative bounds for the **truncated moment problem** for "mixtures of smooth random variables".

General Techniques

Iterated Dimension Reduction

[Kane-Meka-Nelson11, Celis,Reingold,Segev,Wieder11]

Strong quantitative bounds for the **truncated moment problem** for "mixtures of smooth random variables".

how many moments of a random variable must we match to approximate it?

Order r moment of X: $\mathbb{E}[X^r]$

General Techniques

Iterated Dimension Reduction

[Kane-Meka-Nelson11, Celis,Reingold,Segev,Wieder13]

Strong quantitative bounds for the **truncated moment problem** for “mixtures of smooth random variables”.

Pseudorandom Projection Matrices from **orthogonal designs**.

Proof Sketch

Outline

1. Iterative Dimension Reduction

2. Pseudorandom Projections to Moment Matching

3. Moment Matching

4. Pseudorandom Projections from Orthogonal Designs

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1. Iterative Dimension Reduction

2. Pseudorandom Projections to Moment Matching

3. Moment Matching

4. Pseudorandom Projections from Orthogonal Designs

Goal

A generator G such that:

$w \in \mathbb{R}^n$ unit coeff. vector of a half space

$x \sim \mathbb{S}^{n-1}$ uniform point from the unit sphere

$y \sim \{0,1\}^r$ uniformly random r bits.

$$\langle w, x \rangle \quad \sim_{\varepsilon}^{CDF} \quad \langle w, G(y) \rangle$$

CDF Dist (X,Y) = $\sup_t |\Pr[X \leq t] - \Pr[Y \leq t]|$

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$\varepsilon \sim 1/\mathsf{poly}(n)$

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$w \in \mathbb{R}^n$ unit coeff. vector of a half space

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$y \sim \{0,1\}$ n r bits.

random projection of w on to dim 1

$$\langle w, x \rangle \sim_{\varepsilon}^{CDF} \langle w, G(y) \rangle$$

$\varepsilon \sim 1/\mathsf{poly}(n)$

Two Step Dimension Reduction

$w \in \mathbb{R}^n$

$Q_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

$Q_1 w$

$x_1 \sim \mathbb{S}^{\sqrt{n}-1}$

$\langle Q_1 w, x_1 \rangle$

Q_1:a uniformly random projection matrix.

Two Step Dimension Reduction

$w \in \mathbb{R}^n$

derandomize the first step

$Q_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

niformly random projection matrix.

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$\langle Q_1 w, x_1 \rangle$

$w \in \mathbb{R}^n$

$P_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

$P_1 w$

Two Step Generator

$w \in \mathbb{R}^n$

$Q_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

$w \in \mathbb{R}^n$

$P_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

P_1 :"Pseudorandom Projection"
can be sampled using $\sim$log n random bits.

$x_1 \sim \mathbb{S}^{\sqrt{n}-1}$

$\langle Q_1 w, x_1 \rangle$

Two Step Generator

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$$\langle Q_1 w, x_1 \rangle \quad \sim_{\varepsilon}^{CDF}$$

$$w \in \mathbb{R}^n$$

$$P_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$$

$$P_1 w$$

$$\langle P_1 w, x_1 \rangle$$

Two Step Generator

$$Q_1 w \qquad\qquad P_1 w$$

$$x_1 \sim \mathbb{S}^{\sqrt{n}-1}$$

$$\langle Q_1 w, x_1 \rangle \qquad \sim_{\varepsilon}^{CDF} \qquad \langle P_1 w, x_1 \rangle$$

reduction to a $\sqrt{n}$ dimensional problem!

$w \in \mathbb{R}^n$

$Q_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

$Q_1 w$

$Q_2 : \mathbb{R}^{\sqrt{n}} \to \mathbb{R}^{n^{1/4}}$

$Q_2 Q_1 w$

w_t

$\langle w_t, x \rangle$

recurse!

$t \sim \log\log(n)$ steps

$x_t \sim \mathbb{S}^{\sim \log(n)}$

$w \in \mathbb{R}^n$

$P_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

$P_1 w$

$P_2 : \mathbb{R}^{\sqrt{n}} \to \mathbb{R}^{n^{1/4}}$

$P_2 P_1 w$

v_t

$\langle v_t, x \rangle$

$w \in \mathbb{R}^n$

$Q_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$

$Q_1 w$

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w_t

$\langle w_t, x \rangle$

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P_i :"Pseudorandom Projection"
can be sampled using $\sim$log n random bits.

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$\sim_{\varepsilon}^{CDF}$

Outline

1. Iterative Dimension Reduction

2. Pseudorandom Projections from Moment Matching

3. Moment Matching

4. Pseudorandom Projections from Orthogonal Designs

Two Step Generator

$$w \in \mathbb{R}^n$$

$$Q_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$$

$$Q_1 w$$

$$x_1 \sim \mathbb{S}^{\sqrt{n}-1}$$

$$\langle Q_1 w, x_1 \rangle \quad \sim_{\varepsilon}^{CDF}$$

$$w \in \mathbb{R}^n$$

$$P_1 : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$$

$$P_1 w$$

$$\langle P_1 w, x_1 \rangle$$

Pseudorandom Projection

A distribution on projections

$$P : \mathbb{R}^n \to \mathbb{R}^{\sqrt{n}}$$

such that for

$$x_1 \sim \mathbb{S}^{\sqrt{n}-1}$$

- **P can be sampled with log n random bits.**
- $$\langle Qw, x_1 \rangle \sim_{\varepsilon}^{CDF} \langle Pw, x_1 \rangle$$

Q: uniformly random projection

w: arbitrary unit vector

Pseudorandom Projection

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- **P can be sampled with log n random bits.**
- $\langle Qw, x_1 \rangle$ **and** $\langle Pw, x_1 \rangle$ **have approx. equal low order moments.**

Q: uniformly random projection

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"Moment Matching Projections"

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- $\langle Qw, x_1 \rangle$ **and** $\langle Pw, x_1 \rangle$ **have approx. equal low order moments.**

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such that for

$$x_1 \sim \mathbb{S}^{\sqrt{n}-1}$$

- **P can be sampled with l… its.**
- $\langle Qx$… **… equal lo…**

How many moments to match?

Q: uniformly random projection

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Truncated Moment Problem

Order r moment of X: $\mathbb{E}[X^r]$

Suppose $\forall d \leq r \ , \mathbb{E}[X^r] = \mathbb{E}[Y^r]$

How large is DCDF(X,Y)?

extensively studied in probability

[Akhiezer1965], [Klebanov-Mkrtcjan1980],[Lasserre 2012]

Our Setting

Order r moment of X: $\mathbb{E}[X^r]$

Suppose $\forall d \leq r \ , \mathbb{E}[X^r] = \mathbb{E}[Y^r]$

How large is DCDF(X,Y)?

For us:

$X = \langle Qw, x_1 \rangle, Y = \langle Pw, x_1 \rangle, \ \ x_1 \sim \mathbb{S}^{\sqrt{n}-1}$

Our Setting

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[Klebanov-Mkrtchyan80]

r = $\left(\frac{1}{\varepsilon}\right)^C$ for ε CDF distance bound.

Our Setting

For us:

$$X = \langle Qw, x_1 \rangle, Y = \langle Pw, x_1 \rangle, \quad x_1 \sim \mathbb{S}^{\sqrt{n}-1}$$

Only $||Qw||_2$ and $||Pw||_2$ matter.

Our Setting

For us:

$$X = \langle Qw, x_1 \rangle, Y = \langle Pw, x_1 \rangle, \ x_1 \sim \mathbb{S}^{\sqrt{n}-1}$$

$$M = ||Qw||_2 \qquad N = ||Pw||_2$$

$$(z_1, z_2, \ldots) \sim \mathbb{S}^{\sqrt{n}-1}$$

Our Setting

For us:

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$$M = ||Qw||_2 \qquad N = ||Pw||_2$$

$$(z_1, z_2, \ldots) \sim \mathbb{S}^{\sqrt{n}-1}$$

$$\langle Qw, x_1 \rangle \sim M \cdot z_1 \qquad \langle Pw, x_1 \rangle \sim M \cdot z_1$$

"mixtures of different scalings of z_1"

This Work

Matching $\sim \frac{\log(1/\varepsilon)}{\log(n)}$ **moments enough!**

For random variables of the form

$$M \cdot Z \, , \, N \cdot Z$$

Approximately match moments of M and N!

This Work

Matching $\sim \dfrac{\log(1/\varepsilon)}{\log(n)}$ **moments enough!**

For random variables of the form

$$M \cdot Z \,,\; N \cdot Z$$

1. $\forall d \leq r,\, \mathbb{E}[M^d] \sim \mathbb{E}[N^d]$

This Work

Matching $\sim \frac{\log(1/\varepsilon)}{\log(n)}$ **moments enough!**

For random variables of the form

$$M \cdot Z \,,\, N \cdot Z$$

1. $\forall d \leq r,\, \mathbb{E}[M^d] \sim \mathbb{E}[N^d]$
2. **Moments of M don't grow too fast.**

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For random variables of the form

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1. $\forall d \leq r,\, \mathbb{E}[M^d] \sim \mathbb{E}[N^d]$
2. **Moments of M don't grow too fast.**
3. **Z has symmetric, infinitely differentiable CDF**

This Work

Matching $\sim \frac{\log(1/\varepsilon)}{\log(n)}$ **moments enough!**

For random variables of the form

$$M \cdot Z \,,\, N \cdot Z$$

1. $\forall d \leq r,\ \mathbb{E}[M^d] \sim \mathbb{E}[N^d]$
2. [illegible]
3. [illegible] **CDF**

Crucially uses the smoothening effect of Z.

This Work

Matching $\sim \frac{\log(1/\varepsilon)}{\log(n)}$ **moments enough!**

For random variables of the form

$$M \cdot Z \,,\, N \cdot Z$$

1. $\forall d \leq r,\, \mathbb{E}[M^d] \sim \mathbb{E}[N^d]$
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structure of random variables+plus moment matching:

[Anandkumar,Hsu,Kakade12], [Daskalakis,Papadimitriou13]

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Q: uniformly random projection

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"Moment Matching Projections"

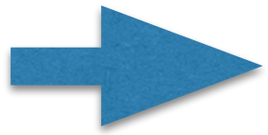
- **P can be sampled with log n random bits.**
- $\langle Qw, x$ **ox. equal low order moments.**

Construction?

Q: uniformly random projection

Approximate Orthogonal Designs

t-wise independence



fools degree t polys on hypercube.

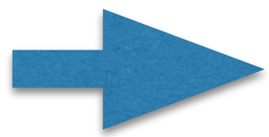
Approximate Orthogonal Designs

t-wise independence → fools degree t polys on hypercube.

orthogonal t-design → fools degree t polys on "**rotation matrices**".

Approximate Orthogonal Designs

orthogonal t-design



fools
degree t polys
on "**rotation matrices**".

A distribution D on a subset of rotation matrices:

For every degree t Polynomial $p : \mathsf{SO}(n) \to \mathbb{R}$

$$|\mathbb{E}_D[p] - \mathbb{E}_{\mathcal{H}}[p]| \leq \epsilon$$

ε **approximate, orthogonal t-design**

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magnitudes of coeffs sum to 1.

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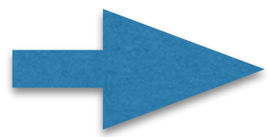

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"fool" low degree polynomials in the entries of the matrices.

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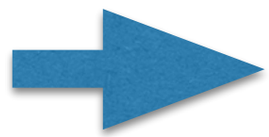

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real analogs of unitary designs studied in quantum computing

[Seymour-Zaslavsky84, Harrow-Low09,Ambainis-Bouda-Winter09, Brandao-Harrow-Horodecki12]

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need explicit construction!

[Brandao-Harrow-Horodecki12]

based on **[Bourgain-Gamburd12]**
"expander walks on Lie Groups"

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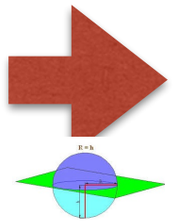
Pseudorandom Projection

1. Draw R according to D.

2. P = first $\sqrt{n}$ rows of R.

Summary



PRG for spherical caps with seed length

$$O(\log(n) + \log(1/\varepsilon) \log\log(1/\varepsilon))$$

- stronger bounds for truncated moment problem.

- pseudorandom projections via the moment method using orthogonal designs.