Project Assignment #1

FEM1230 Estimation Theory

January 31, 2012

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Outline

- Signal Model
- Cramer-Rao Lower Bound
- Maximum Likelihood Estimation
- Simulation Results

Signal Model

- Assume a signal model

$$y[n] = s[n] + w[n] \quad n = 1, 2, \ldots, N-1 \tag{1}$$

where the signal of interest $s[n]$ modeled by

$$s[n; A, B, C, \omega] = A\cos(\omega n) + B\sin(\omega n) + C \tag{2}$$

with A, B, C as unknown constants and $w[n]$ the additive zero mean noise.

- Angular frequency ω may be known, or not, leading to models with three or four unknown parameters, respectively.

- Consider the leakage parameter L, given as

$$L = \frac{2C^2}{A^2 + B^2}. \tag{3}$$

Cramer-Rao Lower Bound

- Regularity conditions hold
  - The Gaussian probability density function for the signal model

$$p(\mathbf{y}, \theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}(\mathbf{y} - \mathbf{s}(\theta))^T(\mathbf{y} - \mathbf{s}(\theta))\right], \tag{4}$$

satisfies the *regularity* conditions for all θ, i.e.,

$$\mathrm{E}\left[\frac{\partial \ln p(\mathbf{y}; \theta)}{\partial \theta}\right] = 0 \quad \forall \theta$$

where $\mathbf{y} = [y[0] \dots y[N-1]]^T$ and $\mathbf{s}(\theta) = [s[0; \theta] \dots s[N-1; \theta]]^T$.

- Calculate the Fisher information matrix

$$\mathbf{I} = \begin{bmatrix} \frac{N}{2\sigma^2} & 0 & 0 & \frac{BN(N-1)}{4\sigma^2} \\ 0 & \frac{N}{2\sigma^2} & 0 & -\frac{AN(N-1)}{4\sigma^2} \\ 0 & 0 & \frac{N}{\sigma^2} & 0 \\ \frac{BN(N-1)}{4\sigma^2} & -\frac{AN(N-1)}{4\sigma^2} & 0 & \frac{(A^2+B^2)(2N^3-3N^2+N)}{12\sigma^2} \end{bmatrix} \tag{5}$$

where

$$[\mathbf{I}(\theta)]_{i,j} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1} \frac{\partial s[n;\theta]}{\partial \theta_i}\frac{\partial s[n;\theta]}{\partial \theta_j}.$$

Cramer-Rao Lower Bound

- The asymptotic expression CRLB is given by

$$CRLB(\theta) = \mathbf{I}(\theta)^{-1} \approx 2\sigma^2 \mathbf{I}_1^{-1}, \tag{6}$$

where

$$\mathbf{I}_1 = \begin{bmatrix} N & 0 & 0 & \frac{BN^2}{2} \\ 0 & N & 0 & -\frac{AN^2}{2} \\ 0 & 0 & 2N & 0 \\ \frac{BN^2}{2} & -\frac{AN^2}{2} & 0 & \frac{(A^2+B^2)N^3}{3} \end{bmatrix}. \tag{7}$$

Cramer-Rao Lower Bound

- For the three parameter model

$$var(\hat{A}) \geq CRB(A) \approx \frac{2\sigma^2}{N} \tag{8}$$

$$var(\hat{B}) \geq CRB(B) \approx \frac{2\sigma^2}{N} \tag{9}$$

$$var(\hat{C}) \geq CRB(C) \approx \frac{\sigma^2}{N} \tag{10}$$

- For the four parameter model

$$var(\hat{A}) \geq CRB(A) \approx \frac{2\sigma^2}{N}(1 + \frac{3B^2}{\alpha^2}) \tag{11}$$

$$var(\hat{B}) \geq CRB(B) \approx \frac{2\sigma^2}{N}(1 + \frac{3A^2}{\alpha^2}) \tag{12}$$

$$var(\hat{C}) \geq CRB(C) \approx \frac{\sigma^2}{N} \tag{13}$$

$$var(\hat{\omega}) \geq CRB(\omega) \approx \frac{24\sigma^2}{\alpha^2 N^3} \tag{14}$$

- For the Leakage parameter

$$var(\hat{L}) \geq CRB(L) = 2\sigma^2 \left(\frac{16C^4}{\alpha^6 N} + \frac{8C^2}{\alpha^4 N} \right) \tag{15}$$

Maximum Likelihood Estimation(MLE)

Motivation

- MLE is *asymptotically optimal* because of its asymptotic properties of being unbiased, achieving the CRLB and having a Gaussian PDF.

- MLE is efficient for the estimation of L and the MLE of the unknown vector parameter θ is asymptotically distributed according to

$$\hat{\theta} \overset{a}{\sim} \mathcal{N}(\theta, \mathbf{I}^{-1}(\theta)), \tag{16}$$

because PDF $p(y;\theta)$ hold *regularity conditions* (**Theorem 7.3 [2]**).

Three Parameter Model

- $s[n, \theta]$ is linear in parameters θ, i.e.,

$$\mathbf{s} = \mathbf{H}\theta \tag{17}$$

where $\theta = [A\ B\ C]^T$ and

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ \cos(\omega) & \sin(\omega) & 1 \\ \vdots & \vdots & \vdots \\ \cos(\omega(N-1)) & \sin(\omega(N-1)) & 1 \end{bmatrix}. \tag{18}$$

- Under Gaussian assumption, the MLE of θ is found by minimizing

$$J(\theta) = (\mathbf{y} - \mathbf{H}\theta)^T(\mathbf{y} - \mathbf{H}\theta). \tag{19}$$

- The minimizing solution $\hat{\theta} = [\hat{A}\ \hat{B}\ \hat{C}]^T$ is

$$\hat{\theta} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}. \tag{20}$$

Four Parameter Model

- $s[n,\theta]$ is nonlinear in parameters θ.
- Assuming $\hat{\omega}_i$ is an estimate in iteration i of ω, a Taylor approx. around $\hat{\omega}_i$ gives

$$\cos(\omega n) \approx \cos(\hat{\omega}_i n) - n\cos(\hat{\omega}_i n)\cdot\Delta\omega_i \tag{21}$$
$$\sin(\omega n) \approx \sin(\hat{\omega}_i n) + n\cos(\hat{\omega}_i n)\cdot\Delta\omega_i. \tag{22}$$

This gives

$$s[n,\theta] = A\cos(\hat{\omega}_i n) + B\sin(\hat{\omega}_i n) + C - An\Delta\hat{\omega}_i\sin(\hat{\omega}_i n) + Bn\Delta\hat{\omega}_i\cos(\hat{\omega}_i n) \tag{23}$$

where $\Delta\hat{\omega}_i = \omega - \hat{\omega}_i$.

- Using above approximations, we can write

$$\mathbf{s} = \mathbf{H}_i\theta_i \tag{24}$$
$$J(\theta) = (\mathbf{y} - \mathbf{H_i}\theta_\mathbf{i})^T(\mathbf{y} - \mathbf{H_i}\theta_\mathbf{i}). \tag{25}$$

where $\theta_i = [A \;\; B \;\; C \;\; \Delta\hat{\omega}_i]^T$ and

$$\mathbf{H}_i = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos(\omega_i) & \sin(\omega_i) & 1 & -A_{i-1}\sin(\omega_i) + B_{i-1}\cos(\omega_i) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(\omega_i(N-1)) & \sin(\omega_i(N-1)) & 1 & -A_{i-1}(N-1)\sin(\omega_i(N-1)) + B_{i-1}(N-1)\cos(\omega_i(N-1)) \end{bmatrix}.$$

Four Parameter Model

- **Algorithm I** [IEEE Std 1057-1994 (R2001)]

  Set index $i = 0$
  Initialize: $\hat{\omega}_0, \hat{\theta}_0 = [A_0 \; B_0, C_0, \; 0]^T$
  repeat
  $\quad i \leftarrow i + 1$
  $\quad \hat{\omega}_i \leftarrow \hat{\omega}_{i-1} + \Delta\hat{\omega}_{i-1}$
  $\quad$ Create $\mathbf{H}_i$
  $\quad \hat{\theta}_i \leftarrow (\mathbf{H}_i^T \mathbf{H}_i)^{-1} \mathbf{H}_i^T \mathbf{y}$
  until convergence

- The algorithm iteratively minimize J in order to obtain a new set of estimates of unknown parameters.

- In each iteration, an updated frequency estimate $\hat{\omega}_i$ is obtained based on $\hat{\omega}_{i-1}$ and $\Delta\hat{\omega}_{i-1}$, andestimates that also depend on estimated values of A, B, and C.

Simulation Results

- The performance of parameter estimations are evaluated by the Monte-Carlo simulations using the empirical mean-squared error criteria,

$$EMSE = \frac{1}{M}\sum_{m=0}^{M-1}(\hat{L}_m - \mathrm{E}(\hat{L}_m)) \tag{26}$$

  where M is number of independent realization.

- In experiment,

$$M = 2000 \tag{27}$$

$$L \ll (A^2 + B^2) \tag{28}$$

$$SNR \gg 20\ dB \tag{29}$$

  .

- An initial estimate of the angular frequency is obtained by using number of sample (N) and an initial guess of sampling frequency (kπ); in experiment k=1000.

Signal-to-noise ratio

- By increasing SNR, difference between the EMSE and the CRLB decreases.
- By increasing the number of samples, difference between the EMSE and the CRLB further decreased.

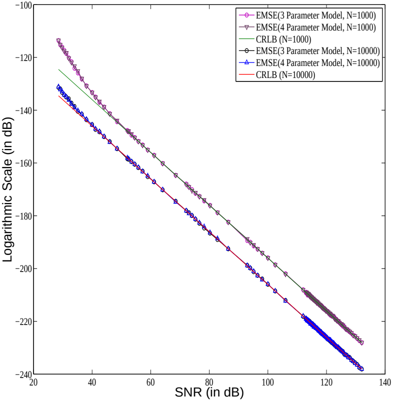

Figure: The CRLB and the EMSE as the function of SNR.

Leakage parameter

- Reducing noise level by decreasing L, increases the difference between the EMSE and the CRLB.
- By increasing the number of samples, difference between the EMSE and the CRLB decreases.

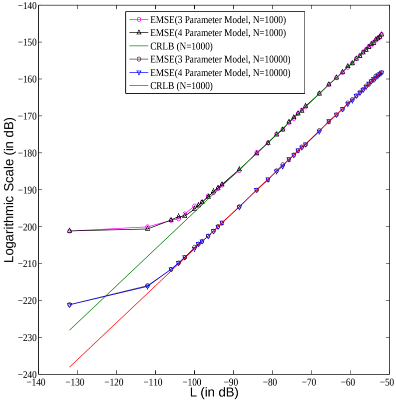

Figure: The CRLB and the EMSE as the function of L.

Number of samples

- Results indicates that the estimator is asymptotically efficient after N > 3000.

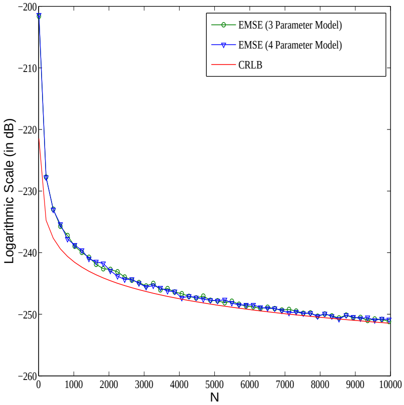

Figure: The CRLB and the EMSE as the function of N.

Conclusion

- The experimental studies revealed that MLE is an asymptotically optimal estimator for the considered signal model.
- The difference between estimated values of parameters from both the three and four parameter model are negligible as suggested by the theory.
- The convergence properties of estimation depend on the initial estimates.