Image Restoration Subject To A Total Variation Constraint

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Outline

- Image Restoration
- Problem Formulation
- Solution Technique
- Simulation Result

Image Restoration

Degraded Image

Image Restoration

Recovered original Image

- Image restoration attempts to reconstruct or recover an image that has been degraded by using of a priori knowledge of the degradation phenomenon

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Image Restoration

Find the original form of an image x from the observation of a degraded image y,

$$y = Lx + u$$

L : a bounded linear operator modeling the blurring process

u: an additive noise component

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Mathematical Formulation

A general formulation for image restoration problem is given as

$$Find \quad x \in S = \bigcap_{i=1}^{m} S_i \quad such\ that \quad J(x) = \inf J(S)$$

where $J : H \to \left]-\infty, +\infty\right]$ is a ***convex residual energy function*** and $(S_i)_{1 \le i \le m}$ are closed convex subsets of H .

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Limitation

- **Staircase effect:**

  The intensity levels of images produced by **total variation minimization** tend to cluster in patches because total variation of restored image is significantly below that of the original Image.

- **Knowledge of noise environment:**

  The construction of the noise requires specific assumptions and information about the noise. In some problems such conditions may not be met.

Constraints Formulation

From priori knowledge about the restoration problem, the closed convex constraint sets arise,

$$(S_i)_{1\le i\le m} \subset H$$

For $i=1$, $(S_1) = \{x \in H \mid tv(x) \le \tau\}$

Where $tv(x) = \int \left|\nabla x(\omega)\right|_2 d\omega$ is the **total variation** and τ is the upper bound of total variation

For $i = 2,\ldots,m$, $(S_i)_{2\le i\le m}$

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Constraints Formulation

For a given continuous function $f : H \to \Re$ and $\eta \in \Re$, the closed and convex set

$$lev_{\leq\eta} f = \{x \in H \mid f(x) \leq \eta\}$$

is the lower level set of f at height η

> ***Proposition 1:*** Let H be either $H^1(\Omega)$ or $\Re^n$. Then $tv : H \to \Re$ is a continuous convex function.

From **proposition 1**, the restoration constraint set S_1 can be put as,

$$f_1 = tv - \tau$$

and for constraints sets $(s_i)_{2\leq i\leq m}$ as level sets given by as

$$lev_{\leq 0} f_i = S_i (\forall i \in \{2,...,m\})$$

where $(f_i)_{2\leq i\leq m}$ are continuous convex functions from H to $\Re$.

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Problem Formulation

$$\begin{aligned} &\min \ J(x) \\ &\text{subject to } \max_{1\le i\le m} f_i(x) \le 0 \end{aligned}$$

where $J(x)$ is a convex function and $(f_i)_{1\le i\le m}$ are continuous convex function.

Therefore, it is a ***convex optimization problem***.

Solution Technique

Assumptions

- Suppose that $S \neq \phi$ and that $J : \Re^N \to \Re$ is strictly convex , differentiable and coercive.
- There exist $z \in S$ such that $J(z) < +\infty, C = lev_{\leq J(z)} J$ is bounded and J is differentiable and strictly convex on C.
- If $t_{i,n}$ is a subgradient of f_i at $x_n \in \Re^N$,then **subgradient projection** of x_n onto S_i associated with $t_{i,n}$ is $p_{i,n}$. Given $I \subset \{1,\dots,m\}$, conevx weight $(\omega_{i,n})_{i \in I}$ and define λ_n .

Solution Technique

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Techniques: Subgradient projections and Block-iterative method

1. Fix $\varepsilon \in \left]0, 1/m\right[$. Let x_0 be the minimizer of J over $\mathbf{R}^n$ and set $n = 0$.
2. Take a non-empty index set $I_n \subset \{1, \cdots, m\}$.
3. Set $z_n = x_n + \lambda_n \left(\sum_{i \in I_n} \omega_{i,n} p_{i,n} - x_n \right)$, where
   (a) for every $i \in I_n$, $p_{i,n}$ is as in (1);
   (b) $(\omega_{i,n})_{i \in I_n}$ lies in $\left]\varepsilon, 1\right]$ and $\sum_{i \in I_n} \omega_{i,n} = 1$;
   (c) λ_n is as in (2).
4. Set $D_n = \{ x \in \mathbf{R}^N \mid \langle x_n - x \mid \nabla J(x_n) \rangle \le 0 \}$ and $H_n = \{ x \in \mathbf{R}^N \mid \langle x - z_n \mid x_n - z_n \rangle \le 0 \}$.
5. Let $x_{n+1} (= z_n)$ be the minimizer of J over $D_n \cap H_n$.
6. Set $n = n + 1$ and go to step 2.

$$p_{i,n} = \begin{cases} x_n - \dfrac{f_i(x_n) t_{i,n}}{\|t_{i,n}\|^2} & \text{if } f_i(x_n) > 0 \\ x_n & \text{if } f_i(x_n) < 0 \end{cases} \quad \ldots(1)$$

$$\lambda_n = \begin{cases} \dfrac{\sum_{i \in I_n} \omega_{i,n} \|p_{i,n} - x_n\|^2}{\left\| x_n - \sum_{i \in I_n} \omega_{i,n} p_{i,n} \right\|^2}, & \text{if } \max_{i \in I_n} f_i(x_n) > 0 \\ 1, & \text{otherwise} \end{cases} \quad \ldots(2)$$

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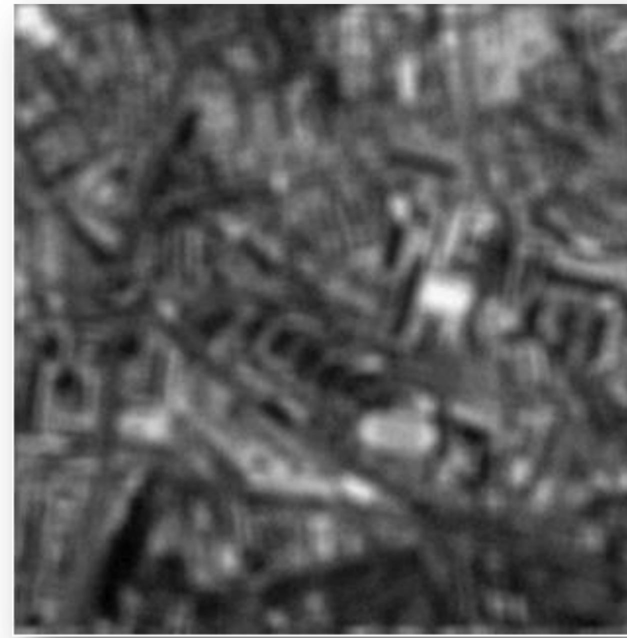
Simulation

Original Image

Convolving 7x7 uniform blur

Adding white Gaussian noise



Degraded Image

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Simulation

$$\min \; J : x \mapsto \|Lx - y\|^2 + \alpha \|x\|^2$$

$$\text{subject to } S_1 = \{x \in H \,|\, tv(x) \leq \tau\}$$

$$S_2 = [0,255]^N$$

$$S_3 = \{x \in \Re^N \,|\, \langle x | \vec{1} \rangle = \mu\}$$

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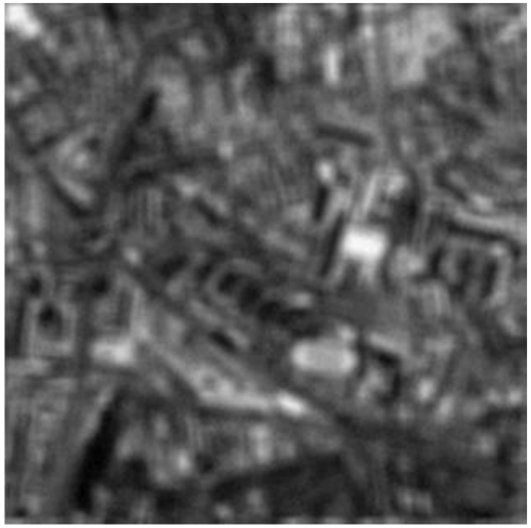
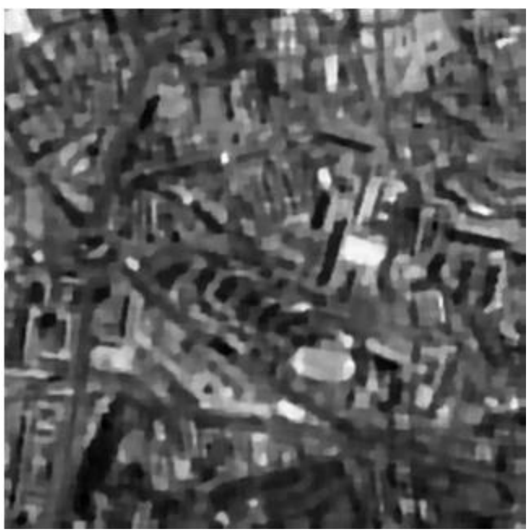
Simulation Results



Degraded Image

With total variation constraint only



Adaptive level set method

With total variation and additional constraints

Thank You