Project Assignment #2

FEM1230 Estimation Theory

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Outline

- Data Model
- Maximum Likelihood Estimation
- Mean Square Error Bounds With Affine Bias
- Mean Square Error Of SNR Affine Biased Estimator

Data Model

- Assume a data model

$$x[n] = A + w[n] \quad n = 1, 2, \ldots, N-1 \tag{1}$$

where A is the unknown DC level and $w[n] \sim \mathcal{N}(0, \sigma^2)$ is additive noise with unknown variance σ^2,

$$\theta = \begin{bmatrix} A \ \sigma^2 \end{bmatrix} \tag{2}$$

- The SNR $\alpha = \frac{A^2}{\sigma^2}$ is lower bounded for any unbiased estimator $\hat{\alpha}$ by CRLB,

$$var(\alpha) \geq \frac{4\alpha + 2\alpha^2}{N}. \tag{3}$$

Problem: Signal–to–noise ratio (SNR) estimation of the data model with unknown A and σ^2.

Maximum Likelihood Estimation (MLE)

- Using the probability density function of the given data model

$$p(\mathbf{x};\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right], \tag{4}$$

where $\mathbf{x} = [x[0]\dots x[N-1]]^T$ and by using the invariance property of the MLE, the MLE of SNR α is given by

$$\hat{\alpha}_{ML} = \widehat{\left(\frac{A^2}{\sigma^2}\right)}_{ML} = \frac{\hat{A}^2_{ML}}{\hat{\sigma}^2_{ML}} = \frac{\left(\frac{1}{N}\sum_{n=0}^{N-1}x[n]\right)^2}{\frac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-\frac{1}{N}\sum_{n=0}^{N-1}x[n]\right)^2}. \tag{5}$$

Maximum Likelihood Estimation (MLE)

- The PDF of $\hat{A}^2_{ML}$ is non-central chi-squared $\mathcal{X}'^2_{\kappa_1}$ with degree of freedom $\kappa_1 = 1$ and non-centrality parameter $\lambda = N\alpha$ because $\hat{A}_{ML} \sim \mathcal{N}(A, \sigma^2/N)$.
- The PDF of $\hat{\sigma}^2$ is central chi-squared $\mathcal{X}^2_{\kappa_2}$ with degree of freedom $\kappa_2 = (N-1)$.
- Therefore the PDF of $\hat{\alpha}_{ML}$ is a non-central chi-squared,

$$f(\hat{\alpha}_{ML}) = \begin{cases} \dfrac{e^{-\frac{N\alpha}{2}}}{N-1} \displaystyle\sum_{i=0}^{\infty} \dfrac{(N\alpha/2)^i \Gamma\left(i+\frac{N}{2}\right)}{i!\Gamma\left(\frac{1+2i}{2}\right)\Gamma\left(\frac{N-1}{2}\right)} \dfrac{\hat{\alpha}^{(i-\frac{1}{2})}}{[1+\hat{\alpha}]^{(i+\frac{N}{2})}}, & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}$$

with the expected value

$$\mathrm{E}\{\hat{\alpha}_{ML}\} = \frac{N}{N-3}\left(\alpha + \frac{1}{N}\right) \quad \text{N}>3 \tag{6}$$

and variance

$$var\{\hat{\alpha}_{ML}\} = \frac{\left[2\alpha^2 + 4\alpha\left(1-\frac{2}{N}\right) + \frac{2}{N}\left(1-\frac{2}{N}\right)\right]}{N(1-\frac{3}{N})^2(1-\frac{5}{N})} \quad \text{N}>5 \tag{7}$$

Maximum Likelihood Estimation (MLE)

- For $N \to \infty$, The ML estimator $\hat{\alpha}_{ML}$ is asymptotically efficient,

$$var\{\hat{\alpha}_{ML}\} \xrightarrow{N \to \infty} CRLB \tag{8}$$

  and the PDF of $\hat{\alpha}_{ML}$ is obey a Gaussian distribution for $N \to \infty$ from the central limit theorem.

- The following figure compare the ML MSE to the CRLB as a function of α for finite N.

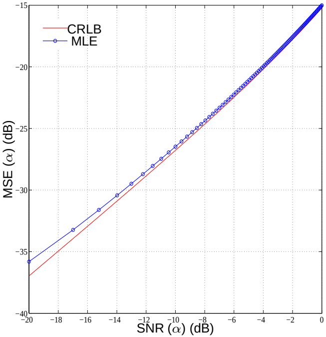


Figure: The CRLB and the ML MSE as the function of α for N $=$ 200.

Mean Square Error Bounds (MSEB) With Affine Bias

- CRLB on the MSE of an unbiased estimator $\hat{\theta}$ of θ is

$$MSEB(0,0,\theta) = E\{\|\hat{\theta}-\theta\|^2\} \geq Tr(J^{-1}(\theta)), \tag{9}$$

  where $J(\theta)$ is the Fisher information matrix.

- Affine MSE bound on the MSE of an estimator with bias $E\{\hat{\theta}\} - \theta = M\theta + u$ is

$$\begin{aligned} MSEB(M,u,\theta) &= \|E\{\hat{\theta}\} - \theta\|^2 + Tr((I + D(\theta))J^{-1}(\theta)(I + D(\theta))^H) \\ &= (M\theta + u)^H(M\theta + u) + Tr(I + M)J^{-1}(\theta)(I + M)^H, \end{aligned} \tag{10}$$

  for some matrix M and $u \in \mathcal{U}$, where $D(\theta) = \frac{\partial(E\{\hat{\theta}\}-\theta)}{\partial\theta}$.

Mean Square Error Bounds (MSEB) With Affine Bias

Theorem 1 [Y. C. Eldar (2008)]

- For a random vector **x** with PDF $p(\mathbf{x}, \theta)$, $\hat{M}$ and $\hat{u}$ are unique and admissible solution of

$$(\hat{M}, \hat{u}) = \arg \min_{M,u} \sup_{\theta \in \mathcal{U}} \{MSEB(M, u, \theta) - MSEB(0, 0, \theta)\}. \tag{11}$$

and for $\hat{M} \neq 0$ or $\hat{u} \neq 0$, we have

$$MSEB(\hat{M}, \hat{u}, \theta) < MSEB(0, 0, \theta), \quad \forall\, \theta \in \mathcal{U}, \tag{12}$$

with affine bias estimator $\hat{\theta}_b$ which achieves the affine MSE bound and have smaller MSE thanc$\hat{\theta}$ $\forall\, \theta \in \mathcal{U}$,

$$\hat{\theta}_b = (1 + M)\hat{\theta} + u. \tag{13}$$

Mean Square Error Bounds (MSEB) With Affine Bias

Proposition 1 [Y. C. Eldar (2008)]

Let a random vector $\mathbf{x}$ with pdf $p(\mathbf{x}, \theta)$ has the Fisher information of form

$$J^{-1}(\theta) = b^2\theta_0^2 + 2c\theta_0 + a. \tag{14}$$

Then the minimax M and u that are the solution to

$$(\hat{M}, \hat{u}) = \arg \min_{M,u} \sup_{\alpha \in \mathcal{U}} \{MSEB(M, u, \theta) - MSEB(0, 0, \theta)\} \tag{15}$$

with $\mathcal{U} = \mathbb{R}$ are given by

$$M = \begin{cases} -\frac{2b^2}{1+b^2}, & |b| < 1 \\ -1, & |b| \geq 1, \end{cases} \quad u = \begin{cases} -\frac{2c}{1+b^2}, & |b| < 1 \\ -\frac{c}{b^2}, & |b| \geq 1. \end{cases} \tag{16}$$

Furthermore, if there exists an efficient estimator $\hat{\theta}$, then

$$\hat{\theta}_b = \begin{cases} \frac{1-b^2}{1+b^2} - \frac{2c}{1+b^2}, & |b| < 1 \\ -\frac{c}{b^2}, & |b| \geq 1, \end{cases} \tag{17}$$

has smaller MSE than $\hat{\theta}\ \forall\ \theta$.

Mean Square Error Of SNR Affine Biased Estimator

- The $\hat{\alpha}_{ML}$ does not achieve the CRLB.
- To achieve lower bound than CRLB, one need to find the scalar $\hat{M}$ and $\hat{u}$ that are the solution to

$$(\hat{M}, \hat{u}) = \arg \min_{M,u} \sup_{\alpha \geq 0} \{(M\alpha + u)^2 + ((1 + M)^2 - 1)J^{-1}(\alpha)\} \tag{18}$$

- Affine biased estimator of the SNR α is

$$\hat{\alpha}_b = (1 + M)\hat{\alpha}_{ML} + u, \tag{19}$$

- The optimal M and u, that lead to lower MSE than CRLB, can be found using the Semi-definite programming formulation.

Mean Square Error Of SNR Affine Biased Estimator

- Since, for finite N, we have

$$var\{\hat{\alpha}_{ML}\} = b^2\alpha^2 + 2c\alpha + a \geq J^{-1}(\alpha) \tag{20}$$

where

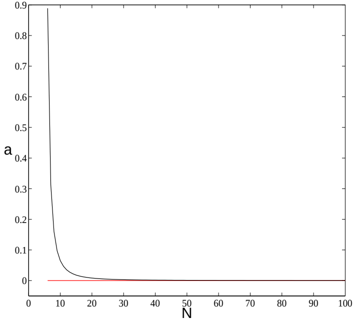
$$a = \frac{\frac{2}{N}(1-\frac{2}{N})}{N(1-\frac{3}{N})^2(1-\frac{5}{N})}$$

$$b = \sqrt{\frac{2}{N(1-\frac{3}{N})^2(1-\frac{5}{N})}} \tag{21}$$

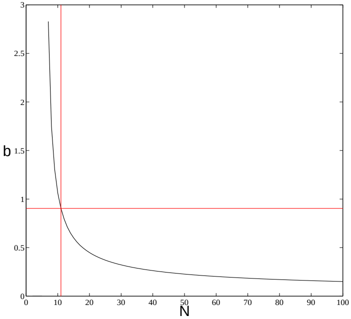
$$c = \frac{2(1-\frac{2}{N})}{N(1-\frac{3}{N})^2(1-\frac{5}{N})} \tag{22}$$

with $a > 0$ for $N > 5$.

Mean Square Error Of SNR Affine Biased Estimator



(a) The parameter a as the function of N.



(b) The parameter b as the function of N.

Mean Square Error Of SNR Affine Biased Estimator

- As $\hat{\alpha}_{ML}$ is an efficient estimator of α, the affine biased estimator of α is, given by Proposition 1,

$$\hat{\alpha}_b = (1+M)\hat{\alpha}_{ML} + u \tag{23}$$
$$= \begin{cases} \frac{(N(1-\frac{3}{N})^2(1-\frac{5}{N})-2)\hat{\alpha}_{ML}-4(1-\frac{2}{N})}{N(1-\frac{3}{N})^2(1-\frac{5}{N})+2}, & |b| < 1 (i.e., N>10) \\ -(1-\frac{2}{N}), & |b| \geq 1, \end{cases}$$

and

$$MSEB(M, u, \hat{\alpha}_b) < MSE(0, 0, \hat{\alpha}_{ML}) \tag{24}$$

for $a > 0$.

♯ Here, readily shown that the affine biased estimator outperforms the ML estimator in terms of MSE for finite N.

Conclusion

- ML estimator is asymptotically efficient for large N.
- Affine biased estimator analytically outperform the ML estimator in terms of MSE for finite N.
- The suggested affine biased estimator is realizable for N > 5.
- One can obtain an MSE that is lower than the (unbiased) CRLB over some range of α and N, as described by **Y. C. Eldar**.

References

S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice Hall, Upper Saddle River, New Jersey 07458, 1993.

Y. C. Eldar, “Uniformly Improving the Cramér-Rao Bound and Maximum-Likelihood Estimation,” *IEEE Trans. Signal Process.*, vol. 54, pp. 2943–2956, Aug. 2006.

Y. C. Eldar, “MSE Bounds With Affine Bias Dominating the Cramér-Rao Bound,” *IEEE Trans. Signal Process.*, Vol. 56, pp. 3824–3836, Aug. 2008.