Asymptotic analysis of radiative extinction in a one-dimensional planar diffusion flame

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1 Introduction

In this study, the role of radiation is analysed to understand its effect on flame structure and extinction. A simplified, one dimensional configuration is adopted, in which pertinent quantities are all one-dimensional, with the exception of the velocity field. The two dimensional Howarth transformation is used to transform the governing equations into mass weighted coordinates in order to simplify the treatment of density. The chemical reaction source term is handled in conventional fashion as a singularity, resolved using matched asymptotic expansions. The manifestation of radiation is in lowering the flame temperature which then affects the reaction rate significantly. The form of the inner expansions is not affected by radiation, and is described by a purely reactive-diffusive structure. However, the strength of the reaction term is dictated by reduction in temperature brought about by radiation. The treatment of the radiatively active outer layer is achieved by observing that radiative length scale, as dictated by the inverse of the absorption coefficient is large compared to the diffusion length scales. Thus radiation can be incorporated as a small perturbation, the functional form of which may be sufficiently described by the leading order (or unperturbed) temperatures in the respective emission and absorption terms. The outer region can then be solved using a Green function as a convolution. The radiation terms may also be treated equivalently using a regular perturbation expansion using the small parameter identified above.

The radiation absorption term in the energy equation is obtained from the radiation transport equation, with use being made of there being azimuthal symmetry about the polar axis formed by the flame normal in spherical coordinates. As mentioned above, absorption is accommodated in the current

framework from the leading order temperatures, and requires no special treatment. However, being a non-local quantity, it appears as a non-local integral over space. It is mentioned in passing (perhaps naively) that the utility of Green's functions is especially apparent in this case with non-singular (and non-local) radiation terms. From the author's cursory perusal of Mathematical Physics literature, the use of Green's functions is a common procedure (notable examples are in electromagnetism, quantum mechanics, and astrophysics) to solve differential equations with source terms.

2 Governing equations

2.1 Howarth transformation

Density weighted coordinates are used, in conformity with Carrier, Fendell and Marble [1] that greatly simplifies the analysis. The transformation is described here in brief. For more extensive notes, Dr. Howard Baum's [3] document may be referred to.

Consider a two dimensional planar flame configuration in which the flame normal is along the x direction; u and v being the velocities in the x and y directions and ρ being the density. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{1}$$

A mass weighted coordinate is defined as

$$\xi = \int_{-\infty}^{x} \frac{\rho}{\rho_{\infty}} \mathrm{d}x \tag{2}$$

where $\rho_{\infty} = \rho(x = -\infty, y) = \rho(x = -\infty)$ because of one-dimensionality in this planar configuration. Transform the equations into this coordinate system comprised of $X = \xi$, $Y = \eta$ and $T = t$ with the following transformation rules

$$x = x(\xi, Y, T) \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial Y} \frac{\partial Y}{\partial x} + \frac{\partial}{\partial T} \frac{\partial T}{\partial x}$$
$$= \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} \tag{3}$$
$$= \frac{\rho}{\rho_{\infty}} \frac{\partial}{\partial \xi}$$

$$y = y(\xi, Y, t) \Rightarrow \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial Y} \frac{\partial Y}{\partial y} + \frac{\partial}{\partial t} \frac{\partial y}{\partial t}$$
$$= \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial Y} \tag{4}$$

$$t = t(\xi, Y, T) \Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial Y}\frac{\partial Y}{\partial t} + \frac{\partial}{\partial T}\frac{\partial T}{\partial t}$$
$$= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial T} \tag{5}$$

The continuity equation becomes, together with equation (2)

$$\frac{\partial}{\partial x}(\rho u + \rho_\infty \frac{\partial \xi}{\partial t}) + \frac{\partial(\rho v)}{\partial y} = 0 \tag{6}$$

Now one may define a stream function $\psi(x, y, t)$ as follows

$$\rho_\infty \frac{\partial \psi}{\partial x} = \rho v; \qquad \rho_\infty \frac{\partial \psi}{\partial x} = -(\rho u + \rho_\infty \frac{\partial \xi}{\partial t}) \tag{7}$$

Together with equation (2) the this gives

$$\frac{\partial \psi}{\partial \xi} = v \tag{8}$$

One can now define a normal transformed velocity $U(\xi, Y, t)$ as

$$\frac{\partial \psi}{\partial Y} = -U \tag{9}$$

A constant density version of the continuity equation may now be written as

$$\frac{\partial v}{\partial Y} + \frac{\partial U}{\partial \eta} = 0 \tag{10}$$

The velocity U is obtained from the equation (7) as

$$U = \frac{1}{\rho_\infty}\frac{\partial \xi}{\partial t} + \frac{\rho}{\rho_\infty}u + v\frac{\partial \xi}{\partial y} \tag{11}$$

The continuity equation may now into a form, after suitable manipulation, to preseve the convective derivative so that one obtains

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial T} + U\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial Y} \equiv \frac{\tilde{\mathrm{D}}}{\tilde{\mathrm{D}}T} \tag{12}$$

2.2 Applying Howarth transform to governing equations

In order to obtain the governing equations, one-dimensionality is assumed, so that all quantities (except the velocities) vary only in the x direction, which

is normal to the flame. The velocity in the direction normal to the flame in this new coordinate system is obtained from the equation (10) as

$$U = -\int_0^\xi \frac{\partial v}{\partial Y} d\xi \tag{13}$$

If a constant strain $\partial v/\partial Y = \varepsilon$, together with a reference velocity of zero at $\xi = 0$ is assumed (which are perfectly reasonable assumptions), one gets

$$U = \int_0^\xi \frac{\partial v}{\partial Y} d\xi = -\varepsilon\xi \tag{14}$$

In the undermentioned, T and Y are replaced *later* by t and y to retain the usual notation. The transport operator is now transformed into mass weighted coordinates. As mentioned in the foregoing, quantities of interest are only assumed to depend on x and t. Furthermore, unity Lewis numbers are assumed so that the diffusion coefficient may be denoted by a quantity D. The untransformed transport operator is

$$\rho L \equiv \rho\frac{\partial}{\partial t} + \rho u\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left(D\rho\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(D\rho\frac{\partial}{\partial y}\right) \tag{15}$$

Upon introducing the transformation, and ignoring Y dependence, one gets

$$\rho\frac{\partial}{\partial t} + \rho u\frac{\partial}{\partial x} = \rho\left(\frac{\partial}{\partial T} - \varepsilon\xi\frac{\partial}{\partial \xi}\right) \tag{16}$$

The diffusion terms are manipulated, using the transformation (2) as follows

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\rho D\frac{\partial}{\partial x}\right) &= \frac{\rho}{\rho_\infty}\frac{\partial}{\partial \xi}\left(D\rho\frac{\rho}{\rho_\infty}\frac{\partial}{\partial \xi}\right) \\
&= \frac{\rho}{\rho_\infty^2}\frac{\partial}{\partial \xi}\left(D\rho^2\frac{\partial}{\partial \xi}\right)
\end{aligned} \tag{17}$$

Now, if one makes the simplification that $D\rho^2$ is nearly constant in flows of interest, one may pull out this quantity as $D\rho^2 = D_\infty\rho_\infty^2$. One may replace T with t to obtain

$$L \equiv \frac{\partial}{\partial t} - \varepsilon\xi\frac{\partial}{\partial \xi} - D_\infty\frac{\partial^2}{\partial \xi^2} \tag{18}$$

2.3   Non-dimensionalization

The governing equations for species mass and energy are

$$\begin{aligned}
L(F) &= -\rho^{-1}\dot{\omega}_F \\
\rho L(Y) &= -\rho^{-1}r_s\dot{\omega}_F \\
\rho L(h) &= \rho^{-1}\Delta h_c\dot{\omega}_F + \rho^{-1}\nabla\cdot\vec{q}_R
\end{aligned} \tag{19}$$

where F, Y are the fuel and oxidizer mass fractions respectively; $h = \int_{T_0}^{T} c_p \mathrm{d}T$ is the enthalpy for material at temperature T, based on some reference enthalpy at temperature T_0; Δh_c is the heat of combustion per unit mass of fuel (a positive quantity); $\dot{\omega}_F$ is the fuel reaction rate per unit volume; r_s is the stoichiometric fuel-air coefficient in the chemical reaction. The non-dimensionalized quantities are superscripted by primes

$$\rho' = \frac{\rho}{\rho_\infty}; \qquad \xi' = \frac{\xi}{l}; \qquad t' = \varepsilon t$$

$$F' = \frac{F}{F_\infty}; \qquad Y' = \frac{Y}{r_s F_\infty}; \qquad h' = \frac{h}{\Delta h_c} \tag{20}$$

In the foregoing equations, l is a reference length scale, and quantities subscripted with ∞ are reference quanties at the boundaries: $F_\infty = F(\xi = -\infty)$, $Y_\infty = Y(\xi = \infty)$, $\rho_\infty = \rho(\xi = -\infty)$. The diffusion coefficient is defined based on the aforementioned length and time-scales as

$$D_\infty = \varepsilon l^2 \tag{21}$$

The reaction term is taken to be of the following form, with an activation energy θ defined *suitably*.

$$\dot{\omega} = B(\rho F)^{\nu_F}(\rho Y)^{\nu_O} \exp\left(-\frac{\theta}{h}\right) \tag{22}$$

In conjunction with the ideal gas law, with nearly constant pressures

$$P = \rho R T = P_\infty \tag{23}$$

one gets for the reaction term as

$$\dot{\omega} = B\left(\frac{P_\infty c_p}{R \Delta h_c}\right)^{\nu_F + \nu_O} h^{-(\nu_F + \nu_O)} \exp\left(-\frac{\theta}{h}\right) \tag{24}$$

On non-dimensionalizing, one gets

$$\begin{aligned}
\frac{\dot{\omega}}{\rho \varepsilon} &= \frac{B}{\rho \varepsilon}\left(\frac{P_\infty c_p}{R \Delta h_c}\right)^{\nu_F + \nu_O} h^{-(\nu_F + \nu_O)} \exp\left(-\frac{\theta}{h}\right) \\
&= B\left(\frac{P_\infty}{R}\frac{c_p}{\Delta h_c}h^{-1}\right)^{-1}\left(\frac{P_\infty c_p}{R \Delta h_c}\right)^{\nu_F + \nu_O} h^{-(\nu_F + \nu_O)} \exp\left(-\frac{\theta}{h}\right) \\
&= \frac{B}{\varepsilon}\left(\frac{P_\infty c_p}{R \Delta h_c}\right)^{\nu_F + \nu_O - 1} h^{-(\nu_F + \nu_O - 1)} \exp\left(-\frac{\theta}{h}\right)
\end{aligned} \tag{25}$$

From the above, one may identify the multiplying quantities on the right hand side as the first Damköhler number D_1 (notation consistent with Carrier, Fendell and Marble).

$$D_1 = \frac{B}{\varepsilon} \left(\frac{P_\infty c_p}{R \Delta h_c} \right)^{\nu_F + \nu_O - 1} \tag{26}$$

The small parameters for the reaction zone may either be taken as the inverse of the first Damköhler number D_1 for large Damköhler number expansions or as h_*^2/θ where h_* is the flame temperature in a large activation energy expansion. The treatment is postponed until later. In either case, it will be seen that the form of the inner expansion is not affected by the presence of radiation. The radiation terms are non-dimensionalized with a reference absorption coefficient $\bar{\kappa}$ and a reference radiation intensity $\bar{I}$. This, in fact, furnishes a small parameter for the radiation source term.

$$\nabla \cdot \vec{q}_R \sim \bar{\kappa} \bar{I_b}^* \sim \bar{\kappa} \sigma T_*^4 \tag{27}$$

where the subscript $*$ denotes quantities at the flame, located at ξ_*; $I_b(\xi) = \pi \sigma T(\xi)^4$ is the black body radiation intensity. The emission term is inspected to provide a suitable small quantity for radiation.

$$\nabla \cdot q_R \sim \bar{\kappa} \sigma T_*^4$$
$$\text{Or} \quad \frac{\nabla \cdot q_R}{\rho \Delta h_c \varepsilon} \sim \frac{\bar{\kappa} \sigma T_*^4}{\rho_* \Delta h_c \varepsilon} = \bar{\kappa} l \frac{\sigma T_*^4}{\rho_* \Delta h_c l \varepsilon} \tag{28}$$

Plugging in typical values taken as $T_* = 2000$ K, $\rho_* = 0.2$ kg/m^3, $\Delta h_c = 32$ MJ/kg, one gets

$$\frac{\nabla \cdot q_R}{\rho \Delta h_c \varepsilon} \sim \frac{\bar{\kappa} l}{l \varepsilon} \frac{(5.67 \times 10^- 8)(2000)^4}{0.2 \times (32 \times 10^6)}$$
$$= \left(\frac{0.14}{\varepsilon l} \right) \kappa l \tag{29}$$

In the current problem, κl is a small number because the radiation length scale ($\sim 1/\kappa$) is expected to be much larger than the diffusion length scale ($\sim l$). Thus one can now define a small parameter ϵ such that

$$\epsilon = \kappa l \ll 1 \tag{30}$$

Furthermore, the quantity multiplying κl in the equation (29) is assumed to be $O(1)$. This is a reasonable assumption for small flames, and large strain rates. The intent is to demonstrate that the product

$$\left(\frac{0.14}{\varepsilon l} \right) \bar{\kappa} l \ll 1 \tag{31}$$

6

and not per se, that $0.14/(\varepsilon l) \sim O(1)$ in that one wants to be able to show that the non-dimensionalized radiation source term is much smaller than unity so that the unperturbed (or radiation free) temperatures may be used as a first approximation for these terms. Radiation is then a known field and may be used in calculating the flame temperature.

To get back to the main discussion it therefore transpires that in the non-dimensional equations the radiation term is multiplied by a quantity of $O(\epsilon)$. This will be used in the analysis later. In order to make the notation consistent, the radiation term is now scaled in terms of enthalpy instead of temperature. If one were to non-dimensionalize κ

$$\kappa' = \frac{\kappa}{\bar{\kappa}} \tag{32}$$

and drop the prime symbol in the equations, one gets for the emission term $q_{R,E}$

$$
\begin{aligned}
-\frac{\nabla \cdot q_{R,E}}{\rho \varepsilon \Delta h_c} &= \bar{\kappa} l \left(\frac{\sigma T^4}{\Delta h_c \varepsilon \rho} \right) = (\bar{\kappa} l) \left(\frac{\sigma}{\Delta h_c \varepsilon l} \frac{\Delta h_c^4}{c_p^4} \frac{\kappa h^4}{\rho} \right) \\
&= (\bar{\kappa} l) \left(\frac{\sigma}{\Delta h_c \varepsilon l} \right) \left(\frac{\Delta h_c^4 h^4}{c_p^4} \right) \left(\frac{RT}{P_\infty} \right) \kappa \\
&= (\bar{\kappa} l) \left(\frac{\sigma}{\Delta h_c \varepsilon l} \right) \left(\frac{\Delta h_c^4 h^4}{c_p^4} \right) \left(\frac{R c_p T}{\Delta h_c} \frac{\Delta h_c}{c_p P_\infty} \right) \kappa \\
&= (\bar{\kappa} l) \left(\frac{\sigma}{\Delta h_c \varepsilon l} \right) \left(\frac{\Delta h_c^4 h^4}{c_p^4} \right) \left(h \frac{\Delta h_c R}{c_p P_\infty} \right) \kappa \\
&= (\bar{\kappa} l) \left[\left(\frac{\sigma}{\Delta h_c \varepsilon l} \right) \left(\frac{\Delta h_c}{c_p} \right)^5 \left(\frac{R}{P_\infty} \right) \kappa h^5 \right] \\
&\sim O(\bar{\kappa} l)
\end{aligned}
\tag{33}
$$

The absorption term is obtained in the document [2].

One may then write the radiation terms (which will now include both emission and absorption) as

$$\frac{\nabla \cdot \vec{q}_R}{\rho \varepsilon \Delta h_c} = (\bar{\kappa} l) f(h(\xi), \kappa(\xi)) = (\bar{\kappa} l) g(\xi) \sim O(\bar{\kappa} l) \tag{34}$$

The energy equation becomes

$$L(h) = D_1 h^{-(\nu_F + \nu_O - 1)} F^{\nu_F} Y^{\nu_O} \exp\left(-\frac{\theta}{h} \right) + (\bar{\kappa} l) g(\xi) \tag{35}$$

The equations for fuel and oxidizer mass fractions are obtained from Scwhab-Zeldovich relationships.

$$L(h + F) = (\bar{\kappa}l)g(\xi) \tag{36}$$

$$L(h + Y) = (\bar{\kappa}l)g(\xi) \tag{37}$$

3 Inner expansions

The inner expansions may be carried out either with a large Activation Energy expansion, or a large Damköhler number expansion. For purposes of illustration, a large Damköhler number expansion is chosen. The large activation energy expansion is explored later, and is handled analogously to that of numerous previous investigations. Consider the *steady* energy equation

$$-\xi\frac{\mathrm{d}h}{\mathrm{d}\xi} - \frac{\mathrm{d}^2h}{\mathrm{d}\xi^2} = D_1 h^{-(\nu_F + \nu_O - 1)} F^{\nu_F} Y_{\nu_O} \exp\left(-\frac{\theta}{h}\right) + (\bar{\kappa}l)g(\xi) \tag{38}$$

As in Carrier, Fendell and Marble, one takes the the inverse of the first Damköhler number $\delta = D_1^{-1}$ as the small parameter. Propose a stretched inner coordinate for the boundary layer as

$$\eta = \frac{\xi - \xi_*}{\delta^b} \tag{39}$$

where ξ_* is the flame location and b is a number to be determined shortly. One writes

$$
\begin{aligned}
F(\xi) &= \mathscr{F}(\eta) \\
Y(\xi) &= \mathscr{Y}(\eta) \\
h(\xi) &= \mathscr{H}(\eta)
\end{aligned}
\tag{40}
$$

All pertinent quantities are now expanded as a power series of δ^b to first order. Note that fuel and oxidizer are vanishing in the inner layer to leading order.

$$
\begin{aligned}
\mathscr{F} &= 0 + \delta^b \mathscr{F}_1 \\
\mathscr{Y} &= 0 + \delta^b \mathscr{Y}_1 \\
\mathscr{H} &= H_* - \delta^b \mathscr{H}_1
\end{aligned}
\tag{41}
$$

Upon inserting the expansion (41) into equation (38) and rearranging (with higher order terms included only where deemed useful for illustration), one

8

gets

$$-\frac{\xi_*}{\delta^b}\frac{\mathrm{d}}{\mathrm{d}\eta}(\mathscr{H}_* - \delta^b\mathscr{H}_1) - \frac{1}{\delta^{2b}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}(\mathscr{H}_* - \delta^b\mathscr{H}_1)$$
$$= \delta^{-1}\delta^{b(\nu_F+\nu_O)}h_*^{-(\nu_F+\nu_O-1)}\mathscr{F}_1^{\nu_F}\mathscr{Y}_1^{\nu_O}\exp\left(-\frac{\theta}{h_*}\right) + (\bar{\kappa}l)g(\xi_*) \tag{42}$$

It emerges from the above on multiplying throughout by δ^b and rearranging, that the radiation term will be $O(\delta^b(\bar{\kappa}l))$ and is negligible compared to the reaction term, which is $O(1)$. Also, the convective term may be neglected compared to the diffusion term. It transpires that the inner equation does not feature the radiation term to leading order. Matching $O(1)$ terms gives

$$b = \frac{1}{\nu_F + \nu_O + 1} \tag{43}$$

and the inner equation becomes

$$\frac{\mathrm{d}^2\mathscr{H}_1}{\mathrm{d}\eta^2} = h_*^{-(\nu_F+\nu_O-1)}\exp\left(-\frac{\theta}{h_*}\right)\mathscr{F}_1^{\nu_F}\mathscr{Y}_1^{\nu_O} \tag{44}$$

One may notice that although the form of the inner equation is unaltered, compared to the radiation free case, the strength of the reaction is reduced because of a drop in temperature in the term

$$h_*^{-(\nu_F+\nu_O-1)}\exp\left(-\frac{\theta}{h_*}\right) \tag{45}$$

Notably, a small decrease in the temperature will lead to a large drop in the chemical reaction rate because of exponential behavior of the function. The exercise therefore boils down to determining the flame temperature given by h_* with radiation. This is obtained from the outer solutions. The flame location $*$ is also obtained from the outer solutions. To leading order, this is unaltered from the case with no radiation.

4   Outer solutions

The outer solutions may be postulated as a perturbation from the solution obtained without radiation. To that end, it the species mass concentrations are unaffected, but the presence of the radiation source term will see an alteration in the flame temperature. One may propose the following expansion for the enthalpy

$$h(\xi) = h^0(\xi;\delta^a) + (\bar{\kappa}l)h^1 \tag{46}$$

9

where $h^0(\xi; \delta^a)$ is the solution obtained without radiation (and is a known function), and a is a quantity that is to be determined accordingly for the fuel and oxidizer sides of the flame (also known). It can be seen that insofar as radiation is concerned, one only needs the leading order behavior of h in the expansion proposed above. This may be inserted in the outer equations to obtain the flame temperature in the outer region. The equations may then be solved using a regular perturbation expansion with $\bar{\kappa}l$ as the expansion parameter, or, more elegantly, by the use of Green's functions.

4.1 Flame location

The flame location is the same as without radiation, because one can match leading order solutions from the local Schwab-Zeldovich expansions. The procedure is identical to that described in Carrier, Fendell and Marble. For the local Schwab-Zeldovich relations one uses

$$
\begin{aligned}
L(F^0 + h^0) &= 0 \\
L(Y^0 + h^0) &= 0
\end{aligned}
\tag{47}
$$

This gives an expression for the flame location as

$$
\operatorname{erf}\left(\frac{\xi_*}{\sqrt{2}}\right) = \frac{F_\infty - Y_\infty}{F_\infty + Y_\infty}
\tag{48}
$$

4.2 Obtaining the flame temperature

4.2.1 Regular Perturbation

The flame temperature, as mentioned in the foregoing, may be determined from a regular perturbation expansion for the temperature as follows

$$
h(\xi) = h^0(\xi; \delta^a) + (\bar{\kappa}l)h^1(\xi)
\tag{49}
$$

In the foregoing, h^1 is a function that is smooth upto its first derivative so that

$$
\begin{aligned}
[h^1]_{\xi_{*-}} &= [h^1]_{\xi_{*+}} \\
\left(\frac{\mathrm{d}h}{\mathrm{d}\xi}\right)_{\xi_{*-}} &= \left(\frac{\mathrm{d}h}{\mathrm{d}\xi}\right)_{\xi_{*+}}
\end{aligned}
\tag{50}
$$

The function $g(\xi)$ is entirely known because it only contains leading order terms. One can then obtain for the $O(\bar{k}l)$ terms

$$
L(h^1) = g_0
\tag{51}
$$

with boundary conditions

$$
\begin{aligned}
h^1_{\xi=-\infty} &= 0 \\
\left(\frac{\mathrm{d}h^1}{\mathrm{d}\xi}\right)_{\xi=-\infty} &= 0 \\
h^1_{\xi=\infty} &= 0 \\
\left(\frac{\mathrm{d}h^1}{\mathrm{d}\xi}\right)_{\xi=\infty} &= 0
\end{aligned}
\tag{52}
$$

In the aforementioned equation, it is expected that the conditions on the derivatives should hold trivially.

4.2.2 Green's function

As an alternative to regular perturbation, one may also use a Green's function approach to solve for the relevant quantities in the outer layers. One of the advantages with this formalism is that it is easy to set up, and involves much less labor than a regular perturbation. It is noted that the approach works because the governing equations are linear. The procedure is briefly illustrated below. For a more elaborate discussion Carrier and Pearson [4] or any other text on ordinary differential equations may be referred to. Consider a *linear* differential equation with the operator $\mathscr{L}$

$$
\mathscr{L} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(p(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\right) + q(\xi)
\tag{53}
$$

acting on the variable y to give the inhomogeneous problem

$$
\mathscr{L}y = f(\xi)
\tag{54}
$$

If one were to find a Green's function $G(\xi, s)$, which is continuous in ξ and differentiable at all points except $\xi = s$, satisfying the boundary conditions of the inhomogeneous problem (54) so that

$$
\mathscr{L}G = \delta(\xi - s)
\tag{55}
$$

where $\delta(\xi - s)$ is the Dirac Delta function, then upon multiplying both sides by $f(\xi)$ (a known function) and integrating with respect to s one gets

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathscr{L}Gf(\xi)\mathrm{d}s &= \int_{-\infty}^{\infty} f(\xi)\delta(s - \xi)\mathrm{d}s \\
&= f(\xi)
\end{aligned}
\tag{56}
$$

11

Since the operator $\mathscr{L}$ is linear one can take it out of the integral to obtain

$$\mathscr{L} \int_{-\infty}^{\infty} G(\xi, s) f(\xi) \mathrm{d}s = f(\xi) \tag{57}$$

Inspection of equation (57) reveals that the integral in the LHS is the sought solution.

$$y = \int_{-\infty}^{\infty} G(\xi, s) f(\xi) \mathrm{d}s \tag{58}$$

One only needs to find a Green's function, which can be done without much difficulty, as is shown in the forthcoming.

4.2.3 Application of Green's function to find outer solutions

In this problem, the function f in the foregoing is replaced by the known function $g(\xi)$. Thus, one can determine the outer solution from the convolution

$$h = \int_{-\infty}^{\infty} G(\xi, s)(\bar{\kappa}l) g(\xi) \mathrm{d}s \tag{59}$$

Since this has known boundary conditions, the problem can be solved. One may therefore obtain h_* for use in the inner equations (which has the same boundary conditions as the case without radiation) and solve for the inner temperature $\mathscr{H}$. Extinction analysis can then be carried out by increasing the radiation losses, which translates to a parametric variation of the coefficient κl.

4.2.4 Obtaining Green's function for given problem

The Green's function for this problem is obtained by solving the equivalent homogeneous problem, defined as

$$L(G(\xi, s)) = \delta(\xi - s) \tag{60}$$

satisfying the boundary conditions

$$\begin{aligned} G(-\infty, s) &= h_{-\infty} \\ G(\infty, s) &= h_{\infty} \end{aligned} \tag{61}$$

It may be added, superfluously, that the Green's function is constructed so as to satisfy the boundary conditions of the original inhomogeneous problem. One is referred to the text by Carrier and Pearson [4] for details. The

Green's function constructed above is continuous over the domain, but has a discontinuous derivative at $\xi = s$. Rewrite the differential equation as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(p(\xi)\frac{\mathrm{d}G}{\mathrm{d}\xi}\right) = 0 \tag{62}$$

where

$$p(\xi) = \exp\left(\frac{\xi^2}{2}\right) \tag{63}$$

Integrate (62) to get

$$\begin{aligned}
G(\xi, s) &= h_{-\infty} + A(s)\int_{-\infty}^{\xi} \exp\left(-\frac{\mu^2}{2}\right)\mathrm{d}\mu & -\infty < \xi < s \\
&= h_{\infty} + B(s)\int_{\xi}^{\infty} \exp\left(-\frac{\mu^2}{2}\right)\mathrm{d}\mu & s < \xi < \infty
\end{aligned} \tag{64}$$

Call the linearly independent solutions obtained above, on either side of $\xi = s$ as ψ_1 and ψ_2.

$$\begin{aligned}
A(s)\psi_1(\xi) &= G(\xi, s) - h_{-\infty}; & -\infty < \xi < s \\
B(s)\psi_2(\xi) &= G(\xi, s) - h_{\infty}; & s < \xi < \infty
\end{aligned} \tag{65}$$

The constants $A(s)$ and $B(s)$ (invariant with respect to ξ) may be determined by imposing continuity of the Green's function, and discontinuity of its derivative at $\xi = s$. Imposing continuity of G at $\xi = s$ one gets

$$h_{-\infty} + A(s)\psi_1(s) = h_{\infty} + B(s)\psi_2(s) \tag{66}$$

The jump condition for the derivative at $\xi = s$ is given by

$$G_\xi(s_+, s) - G_\xi(s_-, s) = \frac{1}{p(s)} = \exp\left(-\frac{s^2}{2}\right) \tag{67}$$

where the subscript ξ refers to differentiation with respect to ξ. Equations (66) and (67) may be solved to give $A(s)$ and $B(s)$ as follows

$$\begin{aligned}
A(s) &= -\frac{h_{-\infty} - h_{\infty} + \psi_2(s_+)}{\sqrt{2\pi}} \\
B(s) &= \frac{h_{-\infty} - h_{\infty} - \psi_1(s_-)}{\sqrt{2\pi}}
\end{aligned} \tag{68}$$

The outer solution to the energy equation may now be written in terms of the Green's function

$$\begin{aligned}
h(\xi) &= \int_{-\infty}^{\infty} G(\xi, s)f(\xi)\mathrm{d}s \\
&= \int_{-\infty}^{\xi} [h_{\infty} + B(s)\psi_2(\xi)]f(\xi)\mathrm{d}s + \int_{\xi}^{\infty} [h_{-\infty} + A(s)\psi_1(\xi)]f(\xi)\mathrm{d}s
\end{aligned} \tag{69}$$

5   Conclusions

A tentative framework has been laid out for the asymptotic analysis of the radiating flame. The problem extends upon the work by Carrier, Fendell and Marble by adding radiation as a small perturbation. The resulting *small* decrease in flame temperature is expected to lead to a more drastic decrease in reaction rate, and eventually cause flame extinction. The formulation is general insofar as to account for both emission and absorption. As further work, the details will be worked out for the regular perturbations and the Green's functions, so as to subsequently perform an analysis of extinction.

Large activation energy asymptotic expansions will also be explored to obtain an extinction criterion. Since only the inner equations are relevant for the extinction analysis, one may see that it is readily amenable to the form proposed by Liñan from which can be extracted a criterion as reported in other investigations. However, it is critical to obtain the flame temperature from the outer expansions in this case as well.

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