# Emmons Problem With Radiative Heat Loss

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## 1 Introduction

Solutions to the Emmons boundary layer problem, with radiation heat loss, are presented in Howarth transformed coordinates. The problem of interest is that of burning fuel in a boundary layer, posed by Emmons [3]. By means of the Howarth transformation, the governing equations are rendered in a form that is similar to the incompressible Navier-Stokes equations. The analytical technique makes use of self-similarity, as is customary in flows of the Blasius boundary layer type, and which has been carried out in numerous investigations. Subsequently, an analysis of flame extinction is made, similar to the now classical analysis by Liñan [5] and Fendell [4]. The onus is to identify the effect of radiation mathematically in order to envisage its role in weakening the flame as one goes farther downstream along the boundary layer. One does not necessarily want to isolate extinction conditions, but only extract the relevant quantities that may provide physical insight. Of particular interest are the boundary layer thickness and residence time in the reacting boundary layer. Some comments are also made on self-similarity, or the breaking down of the same, when non-local radiation effects make themselves felt. The motivation to investigate radiation effects in boundary layers comes from previous work of the author's, in turbulent boundary layers [7], where flames were seen to weaken as one moved farther along the boundary layer, owing primarily to radiation heat losses, the extent of which depended - in a manner of speaking - on the level of soot loading in the medium. Nevertheless, it remains to be seen if such phenomena are of practical relevance, in such fire problems as flame spread over solids.

# 2 Blasius Boundary Layer Problem For Compressible Flow

The problem posed is that of a thin strip of burning fuel supplied by a wall, the mixing and burning occurring from interactions with ambient oxidizer (assumed as air). A boundary layer forms over the wall, analogous to the Blasius problem (Emmons [3], Baum [2]). The combustion process

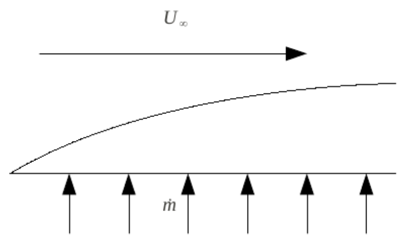


Figure 1: Blasius boundary layer

is described by a one-step reaction in which fuel mixes with oxidizer and burns immediately when stoichiometric conditions occur. In this article, the governing equations are presented in Howarth transformed coordinates. In so doing, the compressible flow equations get transformed to an incompressible form, a procedure that effects considerable simplification and allows further analytical treatment prior to setting up the equations for numerical solution.

# 3 Howarth Transformation and Boundary Layer Equations

The Howarth transformation is carried out by introducing a density weighted coordinate

$$\xi = \int_0^y \frac{\rho}{\rho_\infty} \mathrm{d}y \tag{1}$$

where $\rho = \rho(x, y)$ is the density of the fluid, $\rho_\infty$ is the density at ambient conditions (henceforth the subscript is assumed to denote the same), and the integration being carried out from the wall, located at $y = 0$. Upon introducing the Howarth transformation defined above, and then transforming the coordinates using transformation rules defined as

$$\begin{aligned} t &\to \tau \\ x &\to X \\ y &\to \xi \end{aligned} \tag{2}$$

with $X = x, \tau = t$, replacing $\xi$ with $y$ for notational convenience, and then invoking steady state conditions, one obtains the governing equations for the boundary layer as

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu_\infty \frac{\partial^2 u}{\partial y^2} &= 0 \\ u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} - \mathscr{D}_\infty \frac{\partial^2 F}{\partial y^2} &= \frac{\dot{\omega}}{\rho} \\ u\frac{\partial Y}{\partial x} + v\frac{\partial Y}{\partial y} - \mathscr{D}_\infty \frac{\partial^2 Y}{\partial y^2} &= \frac{\dot{\omega}}{\rho} \\ u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - \mathscr{D}_\infty \frac{\partial^2 h}{\partial y^2} &= -\frac{\dot{\omega}}{\rho} + C_h \frac{\nabla \cdot \vec{q}_R}{\rho} \end{aligned} \tag{3}$$

where $u, v$ are the velocities of the fluid in the $x, y$ directions (in Howarth transformed space); $F, Y, h$ are the stoichiometrically adjusted fuel, oxidizer mass fraction and energy fields respectively; $\nu_\infty$, $\mathscr{D}_\infty$ are the viscosity and diffusivity at ambient conditions (the assumptions $\nu = \mathscr{D}$ and $\rho^2\mathscr{D} = \rho^2\mathscr{D}_\infty$

have been made), with $h = \int_{T_0}^{T} c_p \mathrm{d}T$ ($T$ being the temperature and $c_p$ the constant specific heat); $\dot{\omega}$ is the stoichiometrically adjusted fuel mass burning rate; $\nabla\cdot\vec{q}_R$ is the radiation source term; $C_h$ is a lumping constant that appears after manipulation. It is assumed that the ideal gas law holds, with nearly constant pressure so that

$$\rho T = constant \tag{4}$$

The reaction rate is specified according to Arrhenius chemical kinetics for the one-step reaction

$$\mathrm{F} + \mathrm{rO} \rightarrow \mathrm{P} \tag{5}$$

The form for the reaction term is assumed to be

$$\dot{\omega} = BFY \exp\left(-\frac{\theta}{h}\right) \tag{6}$$

where $\theta$ is the quantity representing the activation energy (assumed to be large compared to $h$), and B is the pre-exponential factor.

# 4 Blasius solution

The Blasius solution is discussed in some detail, having dispensed with the necessity of describing the notation and the Howarth transformation (as it would have been sufficient to use incompressible flow for demonstration purposes). The fluid dynamic portion is first solved, which is a refresher of the Blasius form by invoking similarity so that all quantities are functions of the normalized distance from the wall. Under these assumptions, the two dimensional partial differential equation for momentum is reduced to an ordinary differential equation in the wall normal coordinate. Following this, one examines (again, using the similarity approach) the species and energy equations, to throw pointers on the effect of the radiation source term. For perspective, one is referred to Barenblatt [1].

The momentum equation is given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu_\infty \frac{\partial^2 u}{\partial y^2} = 0 \tag{7}$$

in which $v \ll u$ and $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$. One may therefore neglect neither of the convective terms in the above equation. One posits that both $u$ and $v$ may be expressed in terms of the normalized distance to the wall $\eta$ where

$$\eta = \frac{y}{\delta} \tag{8}$$

where the adjustment parameter $\delta$ is the boundary layer thickness

$$\delta = \sqrt{\frac{\nu_\infty x}{U_\infty}} \tag{9}$$

The system is characterized by a *large* Reynolds number, with which one may define a small parameter $\epsilon$ (it will become apparent that this is the appropriate scaling parameter).

$$\epsilon = \frac{1}{\sqrt{Re}} = \sqrt{\frac{\nu_\infty}{U_\infty x}} \tag{10}$$

One invokes the self-similarity argument that the velocities $u$ and $v$ may be expressed as functions of $\eta$ and $x$. Furthermore, let $v$ be scaled by a small parameter $\varepsilon$, which will be shown to be of the same order of magnitude as $\epsilon$ by dominant balance.

$$\begin{aligned} u &= U_\infty f(\eta) \\ v &= U_\infty \varepsilon g(\eta) \end{aligned} \tag{11}$$

It is apparent from the above that $v \ll u$ owing to the presence of $\varepsilon$ in its expression. Also, $f$ and $g$ are of $O(1)$. Upon applying this to the momentum equation, one gets

$$-\epsilon \frac{\eta}{2} f f' + \varepsilon g f' - \epsilon f'' = 0 \tag{12}$$

Since all the terms should be of the same order in the above equation, one must have $\varepsilon \sim \epsilon$. Set $\varepsilon = \epsilon$. The foregoing equation is solved together with the continuity equation by making use of a stream function $\psi$ defined as follows

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned} \tag{13}$$

so that $f$ and $g$ may be obtained in terms of this stream function. It is seen that choosing the following form of $\psi$ satisfies the equations.

$$\psi = \sqrt{\nu U_\infty x}\,\phi(\eta) = U_\infty \delta(x) \phi(\eta) \tag{14}$$

With this expression, the equations will be cast in terms of the quantity $\phi$, and all pertinent quantities may be derived from it. The momentum equation, upon manipulation, emerges as

$$\frac{1}{2}\phi\phi' + \phi''' = 0 \tag{15}$$

The boundary conditions for this problem are obtained by using the constraints

$$\begin{aligned} u(y=0) &= 0 \\ u(y=\infty) &= U_\infty \\ v(y=0) &= v_0 \end{aligned} \tag{16}$$

The first two conditions in the foregoing give (in the order specified above), together with the definitions in (13)

$$\begin{aligned} \phi'(0) &= 0 \\ \phi'(\infty) &= 1 \end{aligned} \tag{17}$$

The final constraint for $v$ translates to

$$v = \epsilon U_\infty(\eta\phi' - \phi) \tag{18}$$

When applied at $\eta = 0$, where $v = v_0$ is assumed, together with (17) one gets

$$\phi(0) = \frac{2v_0}{U_\infty\epsilon} \equiv \phi_0 \tag{19}$$

It is clear from the foregoing equation that the blowing velocity $v_0$ must be small in comparison with $U_\infty$, i.e. $v_0/U_\infty \sim O(\epsilon)$. The boundary conditions may thus be written as

$$\begin{aligned} \phi(0) &= \phi_0 \\ \phi'(0) &= 0 \\ \phi'(\infty) &= 1 \end{aligned} \tag{20}$$

$f$ and $g$ are cast in terms of $\phi$ as follows

$$\begin{aligned} f &= \phi' \\ g &= \frac{\epsilon}{2}(\phi - \eta\phi') \end{aligned} \tag{21}$$

$f$ and $g$ could be used in subsequent analysis if required.

## 5 The Emmons problem

The foregoing developments are readily extended for species and energy. In the absence of the radiation term $\nabla \cdot \vec{q}_R$, one recovers the classical Emmons problem studied in [3]. One defines a mixture fraction to remove the source term from the species equations as follows

$$Z = F - Y + Y_\infty \tag{22}$$

where $Y_\infty$ is the oxidizer mass fraction at ambient conditions, at $y = \infty$. The mixture fraction evolves as follows

$$\frac{\phi Z'}{2} + Z'' = 0 \tag{23}$$

In this problem, one needs to specify conditions at the wall $y = 0$, while the free stream boundary condition is $Z(\infty) = 0$. Two essentially similar conditions are considered here for the wall boundary condition.

The first case corresponds to that of a boiling liquid or pyrolyzing solid. Here, the wall temperature is set to the boiling or pyrolyzing temperature and the mass fraction of the fuel is set to unity.

$$\begin{aligned} F(0) &= 1 \\ Y(0) &= 0 \\ h(0) &= h_0 \\ Z(0) &= 1 \end{aligned} \tag{24}$$

Here, it is assumed that heat is being supplied to the wall in some way - this could either be a hot ambient (as a result of combustion) or heating from the wall. The velocity at the wall is zero, so that the fluxes are entirely diffusive in nature. A second, fairly similar case may consist of a non-zero blowing velocity $v(0) = v_0$, but with pure fuel and constant temperature. It is moot whether one can set up such a burning configuration, and the author is unsure about the same. However, one proceeds by assuming that the conditions (24) hold at the wall. This may warrant reappraisal in future.

If the coupling function approach is used with the energy equation and one of the species equations, one will get an additional term in the form of the radiation source term, and in this is absorbed a time scale that one may think of as a residence time.

$$\frac{\phi}{2}(h+Y)' + (h+Y)'' = -C_h \frac{x}{U_\infty}\frac{1}{\rho}\nabla \cdot \vec{q}_R \tag{25}$$

The energy equation becomes

$$\frac{\phi}{2}h' + h'' = -\frac{x}{U_\infty}\frac{1}{\rho}\left(-\dot{\omega} + C_h \nabla \cdot \vec{q}_R\right) \tag{26}$$

The equation (26) presents two aspects that bear discussion, on extinction and on self-similarity (or the lack of the same) that will be touched upon here.

# 6 Extinction conditions

The energy equation in Blasius form (26) lends itself to the usual approximation for the flame sheet, using which one may solve on both sides of the flame, together with a stretching transformation in the vicinity of the flame zone for the inner reacting layer. In the outer regions (far from the flame) one has

$$\frac{\phi}{2}h' + h'' = -C_h \frac{x}{U_\infty}\frac{1}{\rho}\nabla \cdot \vec{q}_R \tag{27}$$

If one posits that radiation lowers the enthalpy at any given location by $\Delta h = h_0 - h$ (from it's non-radiating value $h_0$), where $h_0$ satisfies

$$\frac{\phi}{2}(Y+h_0)' + (Y+h_0)'' = 0 \tag{28}$$

one gets for the enthalpy deficit $\Delta h(y)$ far from the flame

$$\frac{\phi}{2}\Delta h' + \Delta h'' = C_h \frac{x}{U_\infty}\frac{1}{\rho}\nabla\cdot\vec{q}_R \tag{29}$$

Using analyses similar to those carried out elsewhere [6], it may be seen that the effect of radiation increases as one moves farther downstream from the leading edge, with increasing $x$. This will manifest as a lower flame temperature $h_*$ (compared to the adiabatic non-radiating case), obtained by using the corresponding Green's function and linearizing the source term $\nabla\cdot\vec{q}_R$. Thus

$$\Delta h_* = \frac{x}{U}\int_0^y C_h \frac{\nabla\cdot\vec{q}_R(y')}{\rho(y')} K(y;y')\mathrm{d}y' \tag{30}$$

where $K(y;y')$ is the Green's function characterizing the energy equation.

When one carries out the extinction analysis for the inner equation, a similar picture as that of the radiative extinction of counterflow flames emerges. One obtains extinction conditions with Liñan's reduced Damköhler number (whose precise functional form is not presented) which scales as

$$Da = Const\frac{x}{U}\exp\left(-\frac{\theta}{h_*}\right) \tag{31}$$

Extinction will occur if $Da < Da_C$, a critical condition that one may derive by rewriting the inner equations. With this picture, one might expect that the flame weakens due to radiative heat losses as one moves farther downstream, owing to the lowering of the temperature $h_*$. Far enough downstream, the flame will lose such a substantial portion of its generated heat to radiation, that it will quench. Alternately, the increased thickness of the boundary layer propitiates conditions where larger amounts of heat is lost to radiation, which is a volumetric quantity.

## 6.1 Comments on self-similarity

One has assumed, in the foregoing, that the governing equations were completely self-similar. While the momentum and species equations present no difficulties in this respect, the energy equation seems to be somewhat problematic - at any rate, it is pertinent to speculate on it, as is done here.

The radiation source term consists of a purely local emissive component, and a non-local absorptive component $G$.

$$\nabla \cdot \vec{q}_R = -\kappa(4\sigma T^4 - G) \tag{32}$$

The enthalpy at any given point - e.g. in Equation (30) - is expressed as a convolution, involving non-local information, collated by means of the propagator $K$. It may be seen that self-similarity is retained if the governing differential equation is still linear, so that $h$ does not differ significantly from its radiation unperturbed value $h_0$. In practice, one will use successive approximations to solve the energy equation, linearizing the source term at each successive iterative step. In this case, it might be reasonable to envisage $h$ to be purely a function of the wall-normal coordinate. However, as the importance of the radiative term increases, and the temperature perturbation from adiabatic conditions becomes more pronounced, it might be conjectured that conditions exist where self-similarity breaks down, if one may not put too fine a point on it. One may similarly observe that non-locality may also be introduced by the presence of the absorption term, under conditions where large amounts of soot far from the flame (or in thick boundary layers) causes significant absorption.

# 7 Conclusions and recommendations

If the - fairly trivial - analysis presented in the foregoing holds water, it might be of interest to investigate the same in more complicated scenarios with numerical simulations. With appropriately fabricated numerical experiments, it might be possible to shed light on the self-similarity aspect, elucidate flame weakening effects from radiation and perhaps differentiate laminar and turbulent scenarios, in addition to validating the findings of this work, which attempts to dilineate radiative flame weakening effects in laminar boundary layers, as a simple extension of Emmons' problem discussed in [3]. The author does not make the case for the pertinence of radiative extinction, but only wishes to touch upon the role of radiative heat losses in such configurations.

## References

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