Expression for radiant energy absorption for use in energy equation

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This document develops an expression for radiant energy absorption for a planar, infinite flame akin to a laminar counterflow configuration. The absorption of radiant energy involves a non-local integral. This integral may be used in the energy equation using the leading order temperatures as a regular perturbation.

1 Radiation transport equation

The radiation transport equation in the absence of scattering is

$$\frac{dI}{ds} = \kappa(I_b - I) \tag{1}$$

Since this is an entirely symmetric problem, one may alternatively recast the above in terms of x and the polar angle θ. The azimuthal angle ϕ will not appear in the expression because of symmetry, and will be integrated out.

Considering an angle $0 < \theta < \pi/2$, one may write

$$dx = -\frac{ds}{\cos\theta} \tag{2}$$

and

$$-\cos\theta\frac{dI}{dx} = \kappa(I_b - I) \tag{3}$$

Or

$$\int_{\infty}^{x} \frac{d(Ie^{-\frac{\kappa x'}{\cos\theta}})}{dx'}dx' = -\int_{\infty}^{x} \frac{\kappa I_b}{\cos\theta}dx' \tag{4}$$

giving

$$I(x) = \int_{x}^{\infty} e^{-\frac{\kappa(x'-x)}{\cos\theta}} \frac{\kappa I_b}{\cos\theta}dx' \tag{5}$$

This is only the contribution from the *right* of the point x. One may likewise derive an expression for the contribution from the left of x. The subscript $+$ and $-$ shall henceforth be used to mark this distinction.

2 Absorption term in energy equation

In order to derive the absorption energy term, given by the divergence of the radiant heat flux vector, one takes the dot product of the local radiant heat flux vector with the appropriate outward normal in the x direction (depending on whether the ray moves to the left or right), which is the only contributing direction because of symmetry. An integral over solid angle space is to be carried out, in which the azimuthal dependence can be removed (as 2π) because of symmetry.

$$d\vec{q_R} = I d\Omega \hat{n} \tag{6}$$

$$\bigtriangledown \cdot q_R = \int_\Omega dq_R.\hat{n_x} \tag{7}$$

For a ray arriving from the right, one has

$$\begin{aligned} q_+(x) &= \int_0^{\frac{\pi}{2}} I_+(x) \int_\Omega d\Omega & (8)\\ &= \int_0^{\frac{\pi}{2}} I_+(x) \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi & (9)\\ &= 2\pi \int_0^{\frac{\pi}{2}} I_+(x) \cos\theta \sin\theta d\theta & (10)\\ &= 2\pi \int_x^\infty \int_0^{\frac{\pi}{2}} \kappa I_b e^{-\frac{\kappa(x'-x)}{\cos\theta}} dx' d\theta \sin\theta & (11) \end{aligned}$$

Or

$$q_+(x) = 2\pi \int_x^\infty \kappa I_b dx' \int_{\cos\theta=0}^1 e^{-\frac{\kappa(x'-x)}{\cos\theta}} d(\cos\theta) \tag{12}$$

The aforementioned equation (12) features the exponential integral function and may be rewritten, by using the transformation $\kappa(x-x')/\cos\theta = 1/t$ as

$$\begin{aligned} q_+(x) &= 2\pi \int_x^\infty \kappa I_b \kappa(x'-x) dx' \int_{\kappa(x'-x)}^\infty \frac{e^{-t}}{t^2} dt & (13)\\ &= 2\pi \int_x^\infty \kappa I_b \kappa(x'-x) dx' \mathrm{F}(\kappa(x'-x)) & (14) \end{aligned}$$

The aforementioned equation (14) poses some difficulties because the exponential integral $\mathrm{E}_2(\kappa(x'-x)$ diverges when $x' = x$. However, it is seen that the product $\kappa(x'-x)\mathrm{F}(\kappa(x'-x))$ is convergent.

Furthermore, by integrating by parts, one sees that

$$\begin{aligned}
\mathrm{F}(x) &= \int_x^\infty \frac{e^{-t}}{t^2} dt && (15)\\
&= -\int_x^\infty \frac{d}{dt}\left(\frac{1}{t}\right) e^{-t} && (16)\\
&= \left[\frac{1}{t} e^{-t}\right]_x - \int_x^\infty \frac{e^{-t}}{t} dt && (17)\\
&= \frac{1}{x} e^{-x} - \mathrm{E}_1(\mathrm{x}) && (18)
\end{aligned}$$

Or

$$\mathrm{F}(x) = \frac{e^{-x}}{x} - \mathrm{E}_1(x) \tag{19}$$

It can be seen that $x\mathrm{F}(x)$ converges at $x = 0$ because

$$\lim_{x \to 0} x\mathrm{E}_1(x) \sim x(\gamma + \ln x) = 0 \tag{20}$$

where γ is the Euler-Mascheroni constant.

In fact, one can recognize $xF(x)$ to be the the exponential integral $E_2(x)$, which is always convergent.