Lesson 8 - Predicates and Quantifiers

Learning Outcomes

By the end of this lesson, students will be able to;

- use algebra of propositions to solve some algebraic propositions.
- describe and use duality principle and replacement rule.
- express the meaning of wide range of statements in mathematics using predicate logic.
- apply universal quantifiers for the proposition.
- apply existential quantifiers for the proposition.

8.1 The Algebra of Propositions

We can use propositional logic to draw logical conclusions for various real world problems. In this sense, we have to convert a given description into propositional logic, and simplify them to obtain a desired conclusion. The simplification process of propositional logical statements needs some rules.

Theorem 8.1 gives a list of some important logical equivalences, which comes as rules to simplify logical expressions.

8.1 Theorem: *Let p and q be any two propositions. Then following laws hold.*

(i) *Idempotent laws*

(a) $p \vee p \equiv p$

(b) $p \wedge p \equiv p$

(ii) *Associative laws*

(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(iii) *Commutative laws*

(a) $(p \vee q) \equiv (q \vee p)$

(b) $(p \wedge q) \equiv (q \wedge p)$

(iv) *Distributive laws*

(a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(v) *Identity laws*

(a) $p \wedge T \equiv p$

(b) $p \vee F \equiv p$

(vi) *Domination laws*

(a) $p \vee T \equiv T$

(b) $p \wedge F \equiv F$

(vii) *Complement laws*

(a) $p \vee \neg p \equiv T$

(b) $p \wedge \neg p \equiv F$

(viii) *Double negation laws*
$\neg(\neg p) \equiv p$

(ix) *Absorption laws*

(a) $p \vee (p \wedge q) \equiv p$

(b) $p \wedge (p \vee q) \equiv p$

(x) *Contrapositive law*
$(p \to q) \equiv (\neg q \to \neg p)$

(xi) *De Morgan's laws*

(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Here T and F are restricted to the truth values 'True' and 'False' respectively.

Example 8.1

Prove the following by developing a series of logical equivalences.

(a) $\neg(p \vee (\neg p \wedge q)) \equiv (\neg p \wedge \neg q)$

(b) $(p \vee q) \equiv \neg(\neg p \wedge \neg q)$

Solution.

(a)
$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{(by the De Morgan's law)}\\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{(by the De Morgan's law)}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{(by the double negation law)}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{(by the distributive law)}\\
&\equiv F \vee (\neg p \wedge \neg q) && \text{(by the complement law)}\\
&\equiv (\neg p \wedge \neg q) \vee F && \text{(by the commutative law)}\\
&\equiv \neg p \wedge \neg q && \text{(by the identity law)}
\end{aligned}$$

$$\begin{array}{llll} \text{(b)} & \neg(\neg p \wedge \neg q) & \equiv & \neg(\neg p) \vee \neg(\neg q) \quad \text{(by the De Morgan's law)} \\ & & \equiv & p \vee q \quad \text{(by the double negation law)} \end{array}$$

8.2 The Dual

For any given compound proposition P involving only the logical connectives denoted by $\wedge$ and $\vee$, the dual of that proposition is obtained by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$, $\boldsymbol{T}$ by $\boldsymbol{F}$ and $\boldsymbol{F}$ by $\boldsymbol{T}$.
For instance the dual of $[(p \wedge q) \vee \neg p]$ is $[(p \vee q) \wedge \neg p]$.

8.2.1 Duality Principle

If two propositions are logically equivalent, then their duals are also logically equivalent.

8.2.2 Replacement Rule

Suppose that we have two logically equivalent propositions P_1 and P_2, so that $P_1 \equiv P_2$. Also assume that we have a compound proposition Q in which P_1 appears. The replacement rule says that, we may replace P_1 by P_2 and the resulting proposition is logically equivalent to Q. This says that substituting a logically equivalent proposition for another in a compound proposition does not change the truth values of that proposition.

Example 8.2

Given the conditional proposition $p \to q$, we define the converse of $p \to q$ as $q \to p$, the inverse of $p \to q$ as $\neg p \to \neg q$, the contrapositive of $p \to q$ as $\neg q \to \neg p$.

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

A conditional proposition $p \to q$ and its contrapositive $\neg q \to \neg p$ are logically equivalent.

8.3 Predicate Logic

Statement involving variables, such as

$$p: \quad x > 3,$$

are often found in mathematical assertions. Such sentences are called *predicates*. They are not propositions because they do not have a definite truth value. The truth value depends on the unknown variable x. For example, p is true if $x = 10$ and false if $x = 1$. Since most of the statements in mathematics and computer science use variables, we must extend the system of logic to include such statements.

8.3.1 Propositional Function

We can denote the statement "is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the propositional function P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

We can write $P(x), Q(x), \ldots$ for predicates involving an unknown x, $P(x,y), Q(x,y), \ldots$ when there are unknowns x and y, $P(A,B), Q(A,B), \ldots$, when there are unknowns A and B.

Definition 8.1: Predicate

Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a propositional function or predicate (with respect to D) if for each $x \in D$, $P(x)$ is a proposition. We call D the domain of discourse of P.

Example 8.3

(a) Let $P(x)$ denote the predicate $x > 5$. Then $P(1)$ denotes the proposition $1 > 5$ with truth value "F", while $P(10)$ denotes the proposition $10 > 5$ with truth value "T".

(b) Let $Q(x,y)$ denote $x^2 + y^2 = 1$. Then $Q(0,1)$ denotes the proposition $0^2 + 1^2 = 1$ with truth value "T", while $Q(1,1)$ denotes the proposition $1^2 + 1^2 = 1$ with truth value "F".

8.3.2 Compound Predicates

The logical connectives $\wedge, \vee, \neg, \rightarrow$ and $\leftrightarrow$ can be used to combine predicates to form compound predicates.

Example 8.4

(a) Let $P(x)$ denote $x^2 > 10$ and let $Q(x)$ denote 'x is positive'. Then $P(x) \wedge Q(x)$ denotes the predicate '$x^2 > 10$ and x is positive'.

(b) Let $Q(x,y)$ denote $x = 2y^2$. Then $\neg Q(x,y)$ denotes $x \neq 2y^2$.

Example 8.5

Let $P(x,y)$ denote $x > y$ and let $Q(x)$ denote $x < 2$. Find the truth value of the predicate $\neg(P(x,y) \wedge Q(x))$ when $x = 3$ and $y = 1$.

Solution.

We need to find the truth value of the statement $\neg(P(3,1) \wedge Q(3))$. Truth value of $P(3,1)$ is "T" and $Q(3)$ is "F". Therefore truth value of $(P(3,1) \wedge Q(3))$ is "F". Hence truth value of $\neg(P(3,1) \wedge Q(3))$ is "T".

8.4 Quantifiers

Many statements in mathematics involve the phrase "for all" and "for some". For example, in mathematics we have the following theorem:

For every triangle T, the sum of the angles of T is equal to $180°$.

We now extend the concepts of logic, so that we can handle statements that include "for all" and "for some".

8.4.1 Universal Quantifier

Let P be a propositional function with domain of discourse D. The statement

$$\text{for all } x,\ P(x)$$

is said to be a universally quantified statement. The symbol $\forall$ means "for all". Therefore the statement

$$\text{for all } x,\ P(x)$$

may be written

$$\forall x P(x).$$

The symbol $\forall$ is called a *universal quantifier*.

The statement $\forall x P(x)$ is true if $P(x)$ is true for every x in D. The statement $\forall x P(x)$ is false if $P(x)$ is false for at least one x in D.

Example 8.6

Consider the universally quantified statement

$$\forall x(x^2 = 0).$$

The domain of discourse is $\mathbb{R}$. The statement is true because, for every real number x, it is true that the square of x is positive or zero.

Example 8.7

Determine whether the universally quantified statement $\forall x(x^2 - 1 > 0)$ is true or false. The domain of discourse is $\mathbb{R}$.

Solution.

The statement is false since, if $x = 1$, the proposition $1^2 - 1 > 0$ is false. The value 1 is a counterexample to the statement $\forall x(x^2 - 1 > 0)$ is false. Although there are values of x that make the propositional function true, the counter example provided shows that the universally quantified statement is false.

8.4.2 Existential Quantifier

Let P be a propositional function with domain of discourse D. The statement

$$\text{there exists } x,\ P(x)$$

is said to be an *existentially quantified* statement. The symbol $\exists$ means "there exists". Thus the statement

$$\text{there exists } x,\ P(x)$$

may be written

$$\exists x P(x).$$

The symbol $\exists$ is called an *existential quantifier*.

The statement $\exists x P(x)$ is true if $P(x)$ is true for at least one x in D. The statement $\exists x P(x)$ is false if $P(x)$ is false for every x in D.

Example 8.8

Consider the existentially quantified statement

$$\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right).$$

The domain of discourse is $\mathbb{R}$. The statement is true because it is possible to find at least one real number x for which the proposition

$$\left(\frac{x}{x^2+1} = \frac{2}{5} \right)$$

is true. For example, if $x = 2$, we obtain the true proposition

$$\left(\frac{2}{2^2+1} = \frac{2}{5} \right).$$

It is not the case that every value of x results in a true proposition. For example, if $x = 1$, the proposition

$$\left(\frac{1}{1^2+1} = \frac{2}{5} \right)$$

is false.

Example 8.9

Verify that the existentially quantified statement

$$\exists x \in \mathbb{R} \left(\frac{1}{x^2+1} > 1 \right)$$

is false.

Solution.
We must show that

$$\frac{1}{x^2+1} > 1$$

is false for every real number x. Now

$$\frac{1}{x^2+1} > 1$$

is false precisely when

$$\frac{1}{x^2+1} \leq 1$$

is true. Thus, we must show that $\frac{1}{x^2+1} \leq 1$ is true for every real number x. To this end, let x be any real number whatsoever. Since $0 \leq x^2$, we may add 1 to both sides of this inequality to obtain $1 \leq x^2+1$. If we divide both sides of this last inequality by x^2+1, we obtain $\frac{1}{x^2+1} \leq 1$. Therefore, the statement $\frac{1}{x^2+1} \leq 1$ is true for every real number x. Thus the statement

$$\frac{1}{x^2+1} > 1$$

is false for every real number x. We have shown that the existentially quantified statement

$$\exists x \left(\frac{1}{x^2+1} > 1 \right)$$

is false.

Remark 8.1. *The statement* $(\forall x)(\exists y)P(x)$ *is not the same as* $(\exists y)(\forall x)P(x)$.

Example 8.10

Consider the statements

(a) $(\forall x)(\exists y)(x < y)$

(b) $(\exists y)(\forall x)(x < y)$

The domain of discourse is $\mathbb{R}$.
Statement (a) is true but statement (b) is false. Note that (a) states that whatever number x we choose we can find a number y which is greater than x (e.g. $y = x+1$). But (b) states that there is a number y which is simultaneously greater than every number x: this is impossible because, with $x = y$, $x < y$ does not hold.

Self-Assessment Exercises

1. Show that each of the following implication is a tautology without using truth tables.
   (a) $[\neg p \wedge (p \vee q)] \rightarrow q$
   (b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
   (c) $[p \wedge (p \rightarrow q)] \rightarrow q$
2. (a) Let $A(x)$ denote the statement "$x < 5$". What are the truth values of $A(7)$ and $A(3)$?
   (b) Let $P(x, y)$ denote the statement "$x = 5 - y$". What are the truth values of the propositions $P(4, 1)$ and $P(6, 2)$?

Suggested Reading

Chapter 1: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7^{th} Edition, McGraw-Hill Education.