Lesson 6 - Tautologies and Contradictions

Learning Outcomes

By the end of this lesson, students will be able to;

- identify tautologies and contradictions.
- distinguish tautologies and contradictions.

6.1 Tautologies

The compound propositions

$$p \vee \neg p \qquad \text{and} \qquad p \vee \neg(p \wedge q)$$

have the truth tables shown in Table 6.1 and Table 6.2.

Table 6.1: Truth table for $p \vee \neg p$.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Table 6.2: Truth table for $p \vee \neg(p \wedge q)$.

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

In Table 6.1 and Table 6.2 only truth value 'T' occurs in the last column. In other words, the truth value of the compound proposition is always 'T', regardless of the truth values of its simple propositions $p, q, r, \ldots$. A statement with this property is called a *tautology*.

Example 6.1

Let p: It is raining. Then $p \vee \neg p$ is "Either it is raining or it is not raining". This is true regardless of whether it is raining or not.

6.2 Contradictions

The truth value of a compound proposition which is always 'F', regardless of the truth values of its components $p, q, r \ldots$ is called a *contradiction*. If a compound proposition is a contradition only 'F' occurs in the last column of the truth table.

Example 6.2

Show that the proposition $p \wedge \neg p$ is a contradiction.

Solution. Consider the below truth table.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

In the truth table only truth value 'F' occurs in the last column. Therefore $p \wedge \neg p$ is a contradiction.

Remark 6.1. *The negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.*

Remark 6.2. *A proposition that is neither a tautology nor a contradiction is called a contingency.*

Self-Assessment Exercises

Determine whether each of the following is a tautology, a contradiction or a contingency.

(a) $p \rightarrow (q \wedge p)$

(b) $(p \vee q) \rightarrow \neg p$

(c) $(p \wedge q) \rightarrow \neg p$

(d) $(p \rightarrow \neg p) \rightarrow \neg p$

(e) $(p \vee q) \leftrightarrow (q \vee p)$

Suggested Reading

Chapter 1: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7^{th} Edition, McGraw-Hill Education.