Lesson 9 - Relations

Learning Outcomes

By the end of this lesson, students will be able to;

- define a relation, inverse relation and composition of relations.
- find the inverse of a relation and composition of relations.
- identify and describe the properties of relations.
- identify equivalence relations.
- compute equivalence classes.
- find the closures of a given relation.

9.1 Introduction

In many naturally occurring phenomena, two variables may be related with some kind of relationship. For instance, a teacher wants to find whether the study time of a student affects the marks of the students at the examination. Table 9.1 shows a correspondence between study time and the marks.

Table 9.1: Relation of study time to marks

Time (Hours) - X	Marks - Y		Ordered Pair
3	60	→	(3,60)
2	80	→	(2,80)
1	55	→	(1,55)
5	100	→	(5,100)
4	90	→	(4,90)

Each data point from the above table may be represented as an **ordered pair**. In this case, the first value represents the study time and the second, the marks. The set of ordered pairs $\{(3,60),(2,80),(1,55),(5,100),(4,90)\}$ defines a relation between study time and marks.

Any set of ordered pairs (x, y) is called a relation in x and y. Furthermore,

- The set of first components in the ordered pairs is called the **domain** of the relation.
- The set of second components in the ordered pairs is called the **range** of the relation.

The symbol R is used to denote the relation and its negation is represented by $\not R$. For instance, $(3,60) \in R$; we then say **"3 is R-related to 60"**, written **3R60**.

Definition 9.1: Relation

A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

Remark 9.1.

- *A relation may consist of a finite number of ordered pairs or an infinite number of ordered pairs.*
- *The x and y components that constitute the ordered pairs in a relation do not need to be numerical.*

Example 9.1

Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by

$$(x, y) \in R \qquad \text{if } x \text{ divides } y,$$

we obtain

$$R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}.$$

Example 9.2

Let R be the relation on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$. Then

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}.$$

Definition 9.2: Inverse

Let R be a relation from X to Y. The inverse of R, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Example 9.3

If we define a relation R from $X = \{2, 3, 4\}$ to $Y = \{3, 4, 5, 6, 7\}$ by

$$(x, y) \in R \qquad \text{if } x \text{ divides } y,$$

we obtain

$$R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}.$$

The inverse of this relation is

$$R^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}.$$

Definition 9.3: Composition of Relations

Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) |\ (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Example 9.4

The composition of the relations

$$R_1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

and

$$R_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$$

is

$$R_2 \circ R_1 = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}.$$

Remark 9.2. Let R be a relation on the set X. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

9.2 Properties of Relations

There are various properties that are used to classify relations on a set.

1. A relation on a set X is reflexive if $(x, x) \in R$ for all $x \in X$.
2. A relation R is symmetric if whenever $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in X$.
3. A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ for all $x, y, z \in X$.
4. A relation R on a set X is antisymmetric if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Example 9.5

Consider the following relations on $\{1, 2, 3, 4\}$.

$$\begin{aligned}
R_1 &= \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}, \\
R_2 &= \{(1, 1), (1, 2), (2, 1)\}, \\
R_3 &= \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}, \\
R_4 &= \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}, \\
R_5 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}, \\
R_6 &= \{(3, 4)\}.
\end{aligned}$$

Discuss reflexive, symmetric, transitive and antisymmetric properties of the above relations.

Solution.
The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (x, x), namely, $(1, 1), (2, 2), (3, 3)$ and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4 and R_6 are not reflexive because $(3, 3)$ is not in any of these relations.

The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2, the only thing to check is that both $(2, 1)$ and $(1, 2)$ are in the relation. For R_3, it is necessary to check that both $(1, 2)$ and $(2, 1)$ belong to the relation and $(1, 4)$ and $(4, 1)$ belong to the relation. The relations R_1, R_5 and R_6 are not symmetric, because $(3, 4)$ is in the rations but $(4, 3)$ is not in any of these relations. However $(4, 3)$ is in the ration R_4 but $(3, 4)$ is not in the relation R_4. Therefore R_4 is not symmetric.

R_4, R_5 and R_6 are all antisymmetric. For each of these relations there is no pair of elements x and y with $x \neq y$ such that both (x, y) and (y, x) belong to the relation. The relations R_1, R_2 and R_3 are not antisymmetric, because $(1, 2)$ in the relation with $1 \neq 2$. such that $(1, 2)$ and $(2, 1)$ are both in the relation.

R_4, R_5 and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (x, y) and (y, z) belong to this relation, then (x, z) also does. For instance, R_4 is transitive, because $(3, 2)$ and $(2, 1)$, $(4, 2)$ and $(2, 1)$, $(4, 3)$ and $(3, 1)$, and $(4, 3)$ and $(3, 2)$ are the only such sets of pairs and $(3, 1)$, $(4, 1)$ and $(4, 2)$ belong to R_4. Similarly, by comparing all the pairs we can verify the transitive property of the relations R_5 and R_6. R_1 is not transitive because $(3, 4)$ and $(4, 1)$ belong to R_1, but $(3, 1)$ does not. R_2 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_2, but $(2, 2)$ does not. R_3 is not transitive because $(4, 1)$ and $(1, 2)$ belong to R_3, but $(4, 2)$ does not.

9.3 Equivalence Relations

9.3.1 Equivalence Relations

Definition 9.4: Equivalence Relation

A relation that is reflexive, symmetric and transitive on a set X is called an equivalence relation on X.

Example 9.6

Consider the relation

$$R = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\},$$

on $\{1, 2, 3, 4, 5\}$. Show that R is an equivalence relation on $\{1, 2, 3, 4, 5\}$.

Solution.
The relation is reflexive because $(1,1),(2,2),(3,3),(4,4),(5,5) \in R$. The relation is symmetric because whenever (x,y) is in R, (y,x) is also in R. Finally, the relation is transitive because whenever (x,y) and (y,z) are in R, (x,z) is also in R. Since R is reflexive, symmetric and transitive, R is an equivalence relation on $\{1,2,3,4,5\}$.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike". The two elements x and y that are related by an equivalence relation are called equivalent. The notation $x \sim y$ is often used to denote that x and y equivalent elements with respect to a particular equivalence relation.

Definition 9.5: Partial Orders

A relation R on a set X is a partial order if R is reflexive, antisymmetric and transitive.

9.3.2 Equivalence Classes

Definition 9.6: Equivalence Classes

Let R be an equivalence relation on a set X. For each $a \in X$, the sets $[a]$ defined as

$$[a] = \{x \in X \mid xRa\},$$

are called the equivalence classes of X given by the relation R.

Example 9.7

Find the equivalence classes for the equivalence relation

$$R = \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\},$$

on $\{1,2,3,4,5\}$.

Solution.
In Example 2.6, we showed that the relation

$$R = \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\},$$

on $\{1,2,3,4,5\}$ is an equivalence relation. The equivalence class $[1]$ containing 1 consists of all x such that $(x,1) \in R$. Therefore, $[1] = \{1,3,5\}$. The remaining equivalence classes are found similarly:

$$[2] = [4] = \{2,4\}, \qquad [3] = [5] = \{1,3,5\}.$$

Therefore, there are two equivalence classes. They are

$$[2] = [4] = \{2,4\} \qquad \text{and} \qquad [1] = [3] = [5] = \{1,3,5\}.$$

9.4 Closure Properties

Consider a given set X and the collection of all relations on X. Let P be a property of such relations, such as being symmetric or being transitive. A relation with property P will be called a P-relation. The P-closure of an arbitrary relation R on X, written $P(R)$, is a P-relation such that

$$R \subseteq P(R) \subseteq S$$

for every P-relation S containing R. We will write

$$reflexive(R), \quad symmetric(R) \quad \text{and} \quad transitive(R)$$

for the reflexive, symmetric and transitive closures of R.

Generally speaking, $P(R)$ need not exist. However, there is a general situation where $P(R)$ will always exist. Suppose P is a property such that there is at least one P-relation containing R and that the intersection of any P-relations is again a P-relation. Then

$$P(R) = \cap \{S \mid S \ is \ a \ P-\text{relation} \ \text{ and } \ R \subseteq S\}.$$

Thus one can obtain $P(R)$ as the intersection of relations. However, one usually wants to find $P(R)$ by adjoining elements to R to obtain $P(R)$.

9.4.1 Reflexive and Symmetric Closures

Theorem 9.1 tells us how to obtain easily the reflexive and symmetric closures of a relation.

9.1 Theorem: *Let R be a relation on a set X. Then:*

(i) $R \cup \Delta_X$ *is the reflexive closure of* R.

(ii) $R \cup R^{-1}$ *is the symmetric closure of* R.

Here $\Delta_X = \{(x,x) \mid x \in X\}$ *is the diagonal or equality relation on* X.

In other words, $reflexive(R)$ is obtained by simply adding to R those elements (x,x) in the diagonal which do not already belong to R and $symmetric(R)$ is obtained by adding to R all pairs (y,x) whenever (x,y) belongs to R.

Example 9.8

Find the reflexive closure and the symmetric closure of the relation $R = \{(1,1),(1,3),(2,4),(3,1),(3,3),(4,3)\}$ on the set $X = \{1,2,3,4\}$.

Solution.
The reflexive closure of R is

$$reflexive(R) = R \cup \{(2,2),(4,4)\} = \{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,3),(4,4)\}.$$

The symmetric closure of R is

$$symmetric(R) = R \cup \{(4,2),(3,4)\} = \{(1,1),(1,3),(2,4),(3,1),(3,3),(3,4),(4,2),(4,3)\}.$$

9.4.2 Transitive Closure

Let R be a relation on a set X. We define

$$R^* = \bigcup_{i=1}^{\infty} R^i.$$

Then the following theorem holds.

9.2 Theorem: *Let R^* is the transitive closure of R. Suppose X is a finite set with n elements. Then*

$$\boxed{R^* = R \cup R^2 \cup \ldots \cdots \cup R^n.}$$

This gives us the following theorem:

9.3 Theorem: *Let R be a relation on a set X with n elements. Then*

$$\boxed{transitive(R) = R \cup R^2 \cup \ldots \cdots \cup R^n.}$$

Example 9.9

Find the transitive closure of the relation $R = \{(1,2),(2,3),(3,3)\}$ on $X = \{1,2,3\}$.

Solution.
The transitive closure of the relation R, since X has three elements, is obtained by taking the union of R with $R^2 = R \circ R$ and $R^3 = R^2 \circ R$. Note that

$$R^2 = R \circ R = \{(1,3),(2,3),(3,3)\}$$

and

$$R^3 = R^2 \circ R = \{(1,3),(2,3),(3,3)\}.$$

Hence,

$$transitive(R) = R \cup R^2 \cup R^3 = \{(1,2),(1,3),(2,3),(3,3)\}.$$

9.5 n-Ary Relations

The relationships among elements from more than two sets are defined using n-ary relations. By an n-ary relation, we mean a set of ordered n-tuples. For any set X, a subset of the product set X^n is called an n-ary relation on X. In particular, a subset of X^3 is called a ternary.

Definition 9.7

Let $X_1, X_2, \ldots, X_n$ be sets. An n-ary relation on these sets is a subset of $X_1 \times X_2 \times \cdots \times X_n$. The sets $X_1, X_2, \ldots, X_n$ are called the domains of the relation and n is called its degree.

Example 9.10

Let R be the relation on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ consisting of triples (x, y, z), where x, y and z are integers with $x < y < z$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

Self-Assessment Exercises

1. Given $A = \{1, 2, 3, 4\}$. Consider the following relation in A:

$$R = \{(1,1), (2,2), (2,3), (3,2), (4,2), (4,4)\}.$$

   Discuss reflexive, symmetric and transitive properties of R.

2. Let R and S be the following relations on $A = \{1, 2, 3\}$:

$$R = \{(1,1), (1,2), (2,3), (3,1), (3,3)\}, \quad S = \{(1,2), (1,3), (2,1), (3,3)\}.$$

   Find

   (a) $R \cup S$, $R \cap S$ and R^C.

   (b) $R \circ S$.

   (c) $S^2 = S \circ S$.

3. Consider the relation $R = \{(a, a), (a, b), (b, c), (c, c)\}$ on the set $A = \{a, b, c\}$.
   Find

   (a) $reflexive(R)$.

   (b) $symmetric(R)$.

   (c) $transitive(R)$.

Suggested Reading

Chapter 7: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7th Edition, McGraw-Hill Education.