Lesson 3 - Introduction to Sets

Learning Outcomes

By the end of this lesson, students will be able to;

- define a set for a given collection of objects.
- representation of sets using builder form and tabular form.
- define finite sets, infinite sets, null sets, subsets, proper set, equality of sets and disjoint sets.
- compute the cardinality of a finite set.

3.1 Introduction

Set is defined as a *collection of well-defined objects*. The objects in a set are called the **elements** or **members**, of the set. The elements in a set are of the same kind and they are distinct with no repetition of the same element in the set. Usually capital letters, A, B, X, Y, ..., are used to denote sets and lowercase letters a, b, x, y, ..., are used to denote elements of sets. The symbol $\in$ is used to express that an element is belongs to a set and its negation is represented by $\notin$. For instance, $10 \in \{12, 24, 10, 53\}$ and $5 \notin \{12, 24, 10, 53\}$.

There are two main ways of specifying a set:

1. Tabular form
   We can specify a set by explicitly listing all its elements.
   Eg: $A = \{\text{red}, \text{yellow}, \text{green}, \text{blue}\}$
2. Set-builder notation
   We can specify a set by giving a property that all elements must satisfy.
   Eg: $B = \{x : x \text{ is a letter in the English alphabet, } x \text{ is a vowel}\}$
   This would be read "B is the set of x such that x is a letter in the English alphabet and x is a vowel". Note that a letter, usually x, is used to denote a typical member of the set; and the colon is read as "such that" and the comma as "and".

3.2 Universal Set and Empty Set

A *universal set* is all the elements, or members, of any group under consideration. For example, in astronomy all the stars of the Milky Way galaxy and in human population studies the universal set consists of all the people in the world. The symbol U is used to denote the universal set. The set with no element is called the *empty set or null set*. It is denoted by $\emptyset$ or $\{\}$. For example,

$$S = \left\{x : x \text{ is a positive integer}, x^2 = 3\right\}.$$

3.3 Some Important Sets

Some sets of numbers that occur frequently in mathematics. These sets and their notations are as follows:

- $\mathbb{N} = \{1, 2, 3, \dots\}$ = the set of natural numbers
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ = the set of natural numbers with zero
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ = the set of integers, both positive and negative, with zero
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ = the set of positive integers
- $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$ = the set of negative integers
- $\mathbb{Q} = \left\{x : x = \frac{n}{m},\ \ n, m \in \mathbb{Z}\right\}$ = the set of rational numbers

  The numbers contained in $\mathbb{Q}$ are exactly those of the form $\frac{n}{m}$, where n and m are integers and $m \neq 0$.

  Eg: $\dfrac{3}{2}(= 1.5) \in \mathbb{Q}$, $\quad \dfrac{8}{4}(= 2) \in \mathbb{Q}$, $\quad \dfrac{136}{100}(= 1.36) \in \mathbb{Q}$, $\quad \dfrac{-1}{1000}(= -0.001) \in \mathbb{Q}$
- $\mathbb{R} = \{x : -\infty < x < \infty\}$ = the set of real numbers

  Eg: $1.5,\ \ -12.3,\ \ 99,\ \ \sqrt{2},\ \ \pi$
- $\mathbb{I}$ = the set of imaginary numbers
- $\mathbb{C} = \{z : z = a + bi,\ \ -\infty < a < \infty,\ \ -\infty < b < \infty\}$ = the set of complex numbers

3.4 Subsets

Let A be a set. If B is a set such that each element of B is also an element of the set A, then B is said to be a subset of the set A, denoted $B \subseteq A$ or $A \supseteq B$.

Two sets are equal if they both have the same elements or, equivalently, if each is contained in the other. That is:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A.$$

Example 3.1

Consider the sets:

$$A = \{1, 3, 4, 7, 8, 9\}, \qquad B = \{1, 2, 3, 4, 5\}, \qquad C = \{1, 3\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 3, the elements of C, are also members of A and B. But $B \nsubseteq A$ since some of the elements of B, e.g., 2 and 5, do not belong to A. Similarly, $A \nsubseteq B$.

Remark 3.1.

1. *For any set A, we have $\emptyset \subseteq A \subseteq U$.*
2. *For any set A, we have $A \subseteq A$.*
3. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Remark 3.2. *Observe that*

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

This is shown in Figure 3.1.

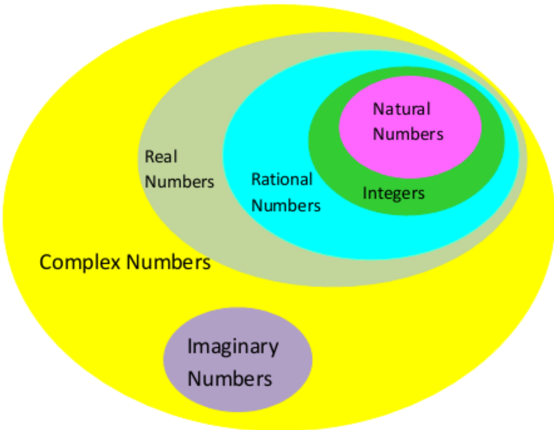

Figure 3.1: Number sets

3.5 Proper Subset

If $A \subseteq B$, but $A \neq B$, we say A is a *proper subset* of B, denoted $A \subset B$.

Example 3.2

Let $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$. Then C is a proper subset of A since C is a subset of A but C does not equal A. We write $C \subset A$.

3.6 Disjoint Sets

Two sets A and B are said to be disjoint if they have *no elements in common.*

Example 3.3

Suppose

$$A = \{1, 2\}, \qquad B = \{4, 5, 6\} \qquad \text{and} \qquad C = \{5, 6, 7, 8\}.$$

Then A and B are disjoint and A and C are disjoint. But B and C are not disjoint since B and C have elements in common, e.g., 5 and 6. We note that if A and B are disjoint, then neither is a subset of the other

3.7 Finite and Infinite Sets

Sets can be finite or infinite. A set S is said to be *finite* if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is *infinite*.

If the set S is finite, its number of elements is called as "cardinality" and it is represented as $|S|$ or $n(S)$.

Example 3.4

(i) If $A = \{12, 24, 10, 53\}$ then $n(A) = 4$. Therefore A is a finite set.

(ii) Let E be the set of even positive integers, that is $E = \{2, 4, 6, \dots\}$. Then E is infinite.

3.8 Venn Diagram

A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane. The universal set U is represented by the interior of a rectangle and the other sets are represented by disks lying within the rectangle. Figure 3.2 shows the Venn diagrams for disjoint sets and proper subsets.

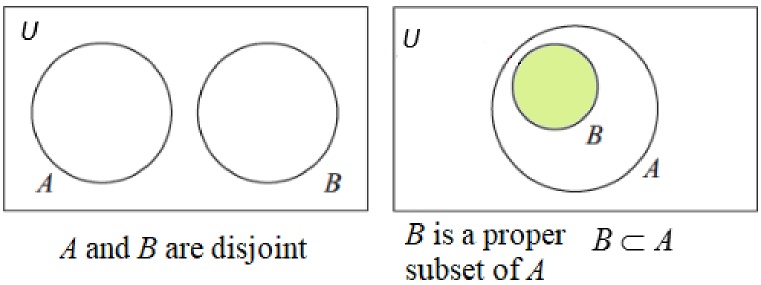

Figure 3.2: Venn diagrams

Self-Assessment Exercises

1. List the elements of each set where $\mathbb{N} = \{1, 2, 3, \dots\}$.

   (a) $A = \{x \in \mathbb{N} : 3 < x < 9\}$

   (b) $B = \{x \in \mathbb{N} : x \text{ is even}, x < 11\}$

   (c) $C = \{x \in \mathbb{N} : 4 + x = 3\}$

2. Decide whether the following statements are true or false.

   (a) $\{2, 4, 7\} \subset \{3, 2, 5, 4, 7\}$

   (b) $\{2, 3, 0\} \subseteq \mathbb{N}$

   (c) $-5 \in \mathbb{N}$

3. Let $A = \{2, 3, 4, 5\}$.
   (a) Show that A is not a subset of $B = \{x \in \mathbb{N} : x \text{ is even}\}$.
   (b) Show that A is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

Suggested Reading

Chapter 2: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7^{th} Edition, McGraw-Hill Education.