Lesson 7 - Logical Equivalence and Logical Implication

Learning Outcomes

By the end of this lesson, students will be able to;

- identify logically equivalent propositions.
- describe logical implications.
- use truth tables to determine logically equivalence propositions.

7.1 Logical Equivalences

Let P and Q be two statements made up from the propositions $p, q, r, \ldots$. If the truth value of P is the same as the truth value of Q for every combination of truth values of $p, q, r, \ldots$ then P and Q are said to be *logically equivalent*, and write

$$P \equiv Q.$$

In other words P and Q are logically equivalent if the final columns of their truth tables are the same.

Example 7.1

Show that the negation of $p \to q$ is logically equivalent to $p \wedge \neg q$.

Solution.
Consider the below truth table for $P \equiv \neg(p \to q)$ and $Q \equiv p \wedge \neg q$.

p	q	$\neg(p\to q)$	$p\wedge\neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

Comparing the columns of truth table for $\neg(p \to q)$ and for $p \wedge \neg q$ we note that the truth values are same for each row of the table. Thus $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Remark 7.1. *If the biconditional statement $P \leftrightarrow Q$ is a tautology, then it is called as an equivalence and it is denoted by $P \Leftrightarrow Q$. It is read as "P is equivalent to Q".*

Example 7.2

Verify that the following two propositions, (a) and (b), are logically equivalent.

(a) If I go to the university then, if I study a lot, I'll be a graduate.

(b) If I go to the university and I study a lot then I'll be a graduate.

Solution.
We can define the following three propositions.

$$\begin{aligned} p &: \text{ I go to the university.} \\ q &: \text{ I study a lot.} \\ r &: \text{ I'll be a graduate.} \end{aligned}$$

We want to show that $p \to (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent.

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Comparing the columns for $p \to (q \to r)$ and for $(p \wedge q) \to r$ we note that the truth values are same for each raw. Therefore, $p \to (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent.

7.2 Logical Implication

Let P and Q be two statements, compound or simple. If the conditional statement $P \to Q$ is a tautology, it is called an *implication* and is denoted by $P \Rightarrow Q$.

Example 7.3

Show that $(p \wedge q) \Rightarrow (q \wedge p)$.

Solution.
To prove $(p \wedge q) \Rightarrow (q \wedge p)$ we have to show that $(p \wedge q) \rightarrow (q \wedge p)$ is a tautology.

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \rightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

According to the above truth table only truth value 'T' occurs in the last column. Therefore, $(p \wedge q) \rightarrow (q \wedge p)$ is a tautology.

Self-Assessment Exercises

1. Show that each of the following compound propositions are logically equivalent.
   (a) $p \rightarrow q \equiv \neg q \rightarrow \neg p$
   (b) $\neg(p \rightarrow q) \equiv p \wedge (\neg q)$
   (c) $(p \leftrightarrow \neg q) \equiv (q \leftrightarrow \neg p)$
   (d) $p \rightarrow q \equiv \neg p \vee q$
2. Prove the following.
   (a) $p \Rightarrow (q \rightarrow p)$
   (b) $[(p \vee q) \wedge \neg q] \Rightarrow p$
   (c) $(p \wedge q) \Rightarrow p$
   (d) $p \wedge (p \rightarrow q) \Rightarrow q$

Suggested Reading

Chapter 1: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7^{th} Edition, McGraw-Hill Education.