Lesson 5 - Propositional Logic

Learning Outcomes

By the end of this lesson, students will be able to;

- identify propositions.
- construct and use compound propositions with logical connectives.
- apply conditional and biconditional propositions.

5.1 Introduction to Logic

Logic is the science of reasoning. It is based on several rules. These rules gives a meaning to mathematical statements and help to check the validity of mathematical arguments. The concepts of logic are important in interpretation and developing mathematical proofs. You will learn basic concepts of mathematical logic in this lesson.

5.2 Propositions

A *proposition* is a declarative statement which is true or false, but not both. The lowercase letters, such as p, q, r,. . . , are used to denote propositions. We use the notation

$$p : 2 + 3 = 5,$$

to define p is a proposition such that $2+3 = 5$. The truth value of this proposition is true and it is denoted by T. If the truth value of a given proposition is false it is denoted by F. Statements that are not propositions include questions, exclamations, wishes, requests and commands.

Example 5.1

Which of the following are propositions? Give the truth value of the propositions.

(a) $2 + 1 = 5$.

(b) $x^2 + 1 = 0$ has a solution.

(c) $x^2 + 1 = 0$ has a real solution.

(d) What time is it?

Solution.

(a) A proposition with truth value F.

(b) A proposition with truth value T.

(c) A proposition with truth value F.

(d) Not a proposition since no truth value can be assigned to this statement.

5.3 Compound Statement

The statements in Example 5.1 are all simple propositions. A combination of two or more simple propositions is a *compound proposition*. We can combine simple propositions using *logical operators* in order to construct compound propositions.

5.4 Logical Operators

This section presents the three basic logical operators. They are conjunction, disjunction and negation which correspond to the English words *and*, *or* and *not* respectively.

5.4.1 Negation

Given any proposition p, another proposition, called the negation of p, can be formed by inserting the word “not” before p or by writing “It is false that...” before p. The negation of p denoted by $\neg p$ and read as “not p”. For example, if

$$p : x \text{ is } 5,$$

the negation of p can be written

$$\neg p : x \text{ is not } 5.$$

If the truth value of p is true, then the truth value of $\neg p$ is false and if the truth value of p is false, then the truth value of $\neg p$ is true. Table 5.1 shows the truth table for the negation of a proposition p.

Table 5.1: Truth table for negation.

p	$\neg p$
T	F
F	T

5.4.2 Conjunction

Two simple propositions can be combined using the English word “and” to form a compound proposition. The resulting compound proposition is called the *conjunction*. It is symbolically denoted as $p \wedge q$, where p and q are simple propositions. For example, consider the following two propositions.

$$p : \text{Today is Friday.}$$
$$q : \text{It is raining today.}$$

Then the conjunction of p and q is

$$p \wedge q : \text{Today is Friday and it is raining today.}$$

The truth value of $p \wedge q$ is true when both p and q are true and is false otherwise. Table 5.2 shows the truth table for conjunction of propositions p and q.

Table 5.2: Truth table for the conjunction of two propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

5.4.3 Disjunction

Two simple propositions can be combined using the English word “or” to form a compound proposition. The resulting compound proposition is called the *disjunction*. It is symbolically denoted as $p \vee q$, where p and q are simple propositions. For example, consider the following two propositions.

$$p : \text{Today is Friday.}$$
$$q : \text{It is raining today.}$$

Then the disjunction of p or q is

$$p \vee q : \text{Today is Friday or it is raining today.}$$

The truth value of $p \vee q$ is false when both p and q are false and is true otherwise. Table 5.3 shows the truth table for disjunction of propositions p or q.

Table 5.3: Truth table for the disjunction of two propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5.2

Construct the truth tables for the following compound propositions.

(a) $\neg(p \wedge q)$

(b) $\neg(\neg p)$

Solution.

(a) The truth table of $\neg(p \wedge q)$.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(b) The truth table of $\neg(\neg p)$.

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

5.5 Conditional Propositions

There are propositions, especially in mathematics, are of the form "*If p then q*". These type of propositions are called *conditional propositions*. The conditional propositions are denoted by

$$p \to q,$$

where the proposition p is called the hypothesis and the proposition q is called the conclusion. The notation $p \to q$ is generally read "p implies q" or "p only if q". For example, consider the following two propositions.

$$p : \text{I eat breakfast.}$$
$$q : \text{I don't eat lunch.}$$

Then the conditional proposition is

$$p \to q : \text{If I eat breakfast then I don't eat lunch.}$$

The truth value of conditional proposition $p \to q$ is false when the truth value of p is true and the truth value of q is false and true otherwise. Truth values for the conditional proposition $p \to q$ is shown in Table 5.4.

Table 5.4: The truth table for the conditional proposition $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

5.6 Biconditional Propositions

Another important compound proposition is of the form "p if and only if q". These propositions are called *biconditional propositions.* They are symbolically denoted by

$$p \leftrightarrow q.$$

For example, consider the following two propositions.

$$\begin{aligned} p &: \text{You can take the flight.} \\ q &: \text{You buy a ticket.} \end{aligned}$$

Then the biconditional proposition is

$$p \leftrightarrow q : \text{You can take the flight if and only if you buy a ticket.}$$

The truth value of biconditional statement $p \leftrightarrow q$ is true when the truth value of the propositions p and q have the same truth values and is false otherwise. Truth values for the biconditional proposition $p \leftrightarrow q$ is shown in Table 5.5.

Table 5.5: The truth table for the biconditional proposition $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 5.3

Consider the following propositions:

$$\begin{array}{lcl} p & : & \text{Mathematicians are generous.} \\ q & : & \text{Spiders hate algebra.} \end{array}$$

Write the compound propositions symbolized by:

(a) $p \vee \neg q$

(b) $\neg(q \wedge p)$

(c) $\neg p \rightarrow q$

(d) $\neg p \leftrightarrow \neg q$

Solution.

(a) Mathematicians are generous or spiders don't hate algebra

(b) It is not the case that spiders hate algebra and mathematicians are generous.

(c) If mathematicians are not generous then spiders hate algebra.

(d) Mathematicians are not generous if and only if spiders don't hate algebra.

5.7 Precedence of Logical Operators

The operational priorities of logical connectives are defined according to the following order:

1. Negation ($\neg$)
2. Conjunction ($\wedge$)
3. Disjunction ($\vee$)
4. Conditional Propositions ($\rightarrow$)
5. Biconditional Propositions ($\leftrightarrow$)

Self-Assessment Exercises

1. Construct a truth table for each of these compound propositions.
   (a) $p \wedge \neg p$
   (b) $\neg p \wedge (p \vee \neg q)$
   (c) $\neg q \vee (\neg p \wedge q)$
2. Let p and q be the propositions.

   $$\begin{array}{ccl} p & : & \text{It is below freezing.} \\ q & : & \text{It is snowing.} \end{array}$$

   Write following propositions using p, q and logical connectives.
   (a) It is below freezing and snowing.
   (b) It is below freezing but not snowing.
   (c) It is not below freezing and it is not snowing.
   (d) It is either snowing or below freezing.

Suggested Reading

Chapter 1: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7th Edition, McGraw-Hill Education.