Lesson 4 - Set Operations and Identities

Learning Outcomes

By the end of this lesson, students will be able to;

- apply set operations over given sets.
- apply basic theorems in set theory for problem solving.
- apply duality principle for the sets.
- use Venn diagrammatic representation of sets for real world applications.

4.1 Set Operations

Given two sets A and B, there are various set operations involving A and B that can produce a new set. There are several set operations. Some of them are union, intersection, difference, complement and symmetric difference.

4.1.1 Union

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Figure 4.1 is a Venn diagram in which $A \cup B$ is shaded.

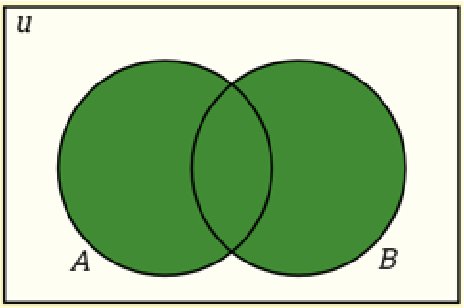

Figure 4.1: Venn diagram for $A \cup B$

4.1.2 Intersection

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Figure 4.2 is a Venn diagram in which $A \cap B$ is shaded.

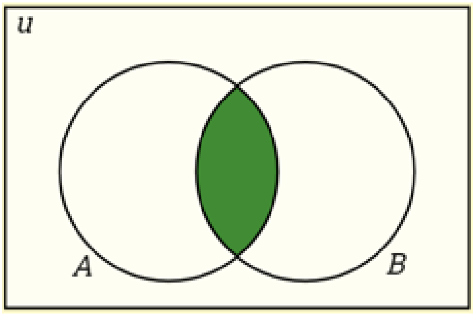

Figure 4.2: Venn diagram for $A \cap B$

Example 4.1

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{2, 3, 8, 9\}$. Then

$$\begin{array}{lll} A \cup B = \{1, 2, 3, 4, 5, 6, 7\}, & A \cup C = \{1, 2, 3, 4, 8, 9\}, & B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}, \\ A \cap B = \{3, 4\}, & A \cap C = \{2, 3\} \quad \text{and} & B \cap C = \{3\}. \end{array}$$

4.1.3 Complement

The *complement* of a set A, denoted by A^C, is the set of elements which belong to U but which do not belong to A.

$$A^C = \{x : x \in U, \ \ x \notin A\} .$$

Figure 4.3 is a Venn diagram in which A^C is shaded.

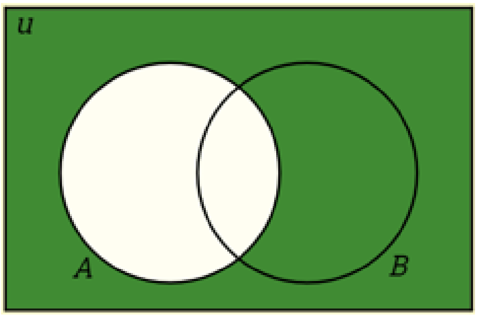

Figure 4.3: Venn diagram for A^C

4.1.4 Difference

The *difference* of A and B, denoted by $A \setminus B$, is the set of elements which belong to A but which do not belong to B.

$$A \setminus B = \{x : x \in A, \; x \notin B\}\,.$$

Figure 4.4 is a Venn diagram in which $A \setminus B$ is shaded.

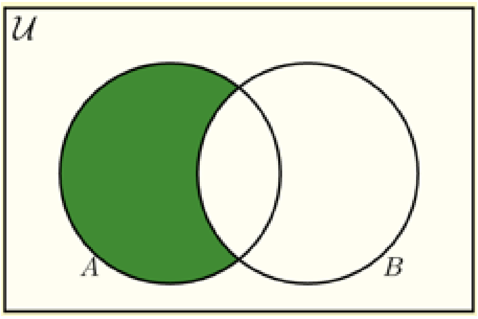

Figure 4.4: Venn diagram for $A \setminus B$

4.1.5 Symmetric Difference

The *symmetric difference* of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both.

$$A \oplus B = (A \cup B) \setminus (A \cap B) \qquad \text{or} \qquad A \oplus B = (A \setminus B) \cup (B \setminus A)\,.$$

Figure 4.5 is a Venn diagram in which $A \oplus B$ is shaded.

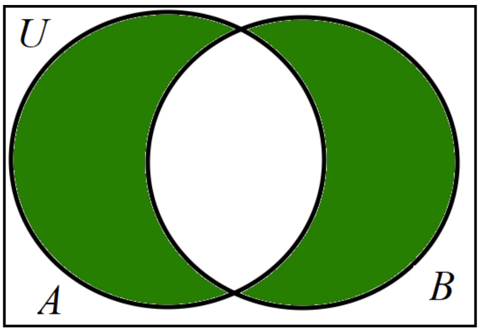

Figure 4.5: Venn diagram for $A \oplus B$

Example 4.2

Suppose $U = \mathbb{N} = \{1, 2, 3, \dots\}$ is the universal set. Let

$$A = \{1, 2, 3, 4\}\,, \quad B = \{3, 4, 5, 6, 7\}\,, \quad C = \{2, 3, 8, 9\} \text{ and } E = \{2, 4, 6, \dots\}\,.$$

Here E denote the set of even integers. Then

$$A^C = \{5, 6, 7, \dots\}\,, \quad B^C = \{1, 2, 8, 9, 10, \dots\} \text{ and } E^C = \{1, 3, 5, 7, \dots\}\,.$$

That is, E^C is the set of odd positive integers. Also:

$$\begin{array}{llll} A \setminus B = \{1, 2\}, & A \setminus C = \{1, 4\}, & B \setminus C = \{4, 5, 6, 7\}, & A \setminus E = \{1, 3\}, \\ B \setminus A = \{5, 6, 7\}, & C \setminus A = \{8, 9\}, & C \setminus B = \{2, 8, 9\} \text{ and} & E \setminus A = \{6, 8, 10, 12, ...\}\,. \end{array}$$

Furthermore:

$$\begin{aligned} A \oplus B &= (A \setminus B) \cup (B \setminus A) = \{1, 2, 5, 6, 7\}\,, \\ B \oplus C &= (B \setminus C) \cup (C \setminus B) = \{2, 4, 5, 6, 7, 8, 9\}\,, \\ A \oplus C &= (A \setminus C) \cup (B \setminus C) = \{1, 4, 8, 9\} \\ \text{and } A \oplus E &= (A \setminus E) \cup (E \setminus A) = \{1, 3, 6, 8, 10, \dots\}\,. \end{aligned}$$

4.2 Algebras of Sets and Duality

4.2.1 Laws of the Algebras

Sets under the operations of union, intersection and complement satisfy various laws which are listed in Table 4.1.

Table 4.1: Laws of the algebra of sets

Idempotent law	$A \cup A = A$	$A \cap A = A$
Associative law	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative law	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive law	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity law	$A \cup \emptyset = A$	$A \cap U = A$
	$A \cup U = U$	$A \cap \emptyset = \emptyset$
Involution law	$\left(A^C\right)^C = A$	
Complement law	$A \cup A^C = U$	$A \cap A^C = \emptyset$
	$U^C = \emptyset$	$\emptyset^C = U$
DeMorgan's law	$(A \cup B)^C = A^C \cap B^C$	$(A \cap B)^C = A^C \cup B^C$

4.2.2 Duality

Suppose E is an equation of set algebra. The dual E^* of E is the equation obtained by replacing each occurrence of $\cup, \cap, U$ and $\emptyset$ in E by $\cap, \cup, \emptyset$, and U respectively.

For example, the dual of

$$(U \cap A) \cup (B \cap A) = A \qquad \text{is} \qquad (\emptyset \cup A) \cap (B \cup A) = A.$$

Note that the pairs of laws in Table 4.1 are duals of each other.

4.3 Counting Elements in Finite Sets

Lemma 4.1

Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and

$$n\,(A \cup B) = n(A) + n(B).$$

Proof
In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

For any sets A and B, the set A is the disjoint union of $A \setminus B$ and $A \cap B$. This leads to the Corollary 4.1.

Corollary 4.1

Let A and B be finite sets. Then

$$n(A \setminus B) = n(A) - n(A \cap B).$$

Given any set A, recall that the universal set U is the disjoint union of A and A^C. Accordingly, Lemma 4.1 we can prove the Corollary 4.2.

Corollary 4.2

Let A be a subset of a finite universal set U. Then

$$n(A^C) = n(U) - n(A).$$

There is a expression for $n(A \cup B)$ even when they are not disjoint, called the inclusion-exclusion principle.

4.1 Theorem (Inclusion-Exclusion Principle): *Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and*

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B).}$$

That is, we find the number of elements in A or B by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to obtain a similar expression for three sets.

Corollary 4.3

Suppose A, B and C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

Example 4.3

Suppose a list A contains the 30 students in a mathematics class and a list B contains the 35 students in an English class and suppose there are 20 names on both lists. Find the number of students:

(a) only on list A.

(b) only on list B.

(c) on list A or B.

(d) on exactly one list.

Solution.
We can draw a Venn diagram for the problem as in Figure 4.6.

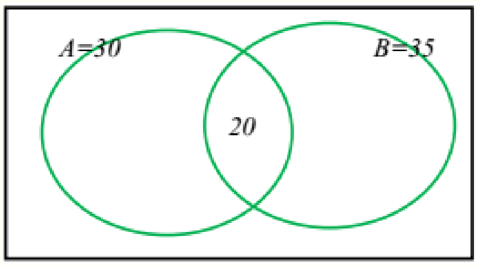

Figure 4.6: Venn diagram for the Example 9.3

Given that $n(A) = 30$, $n(B) = 35$ and $n(A \cap B) = 20$.

(a) The number of students only on list $A = n(A) - n(A \cap B) = 30 - 20 = 10$.

(b) The number of students only on list $B = n(B) - n(A \cap B) = 35 - 20 = 15$.

(c) By inclusion-exclusion principle

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 30 + 35 - 20 \\ &= 45 \end{aligned}$$

Therefore, the number of students on list A or B is 45.

(d) By using (a) and (b), $10 + 15 = 25$. Therefore number of students on exactly one list is 25.

Self-Assessment Exercises

1. Draw the Venn diagrams and shade the regions representing each of the following sets:
   (a) $(A \cup B) \setminus C$
   (b) $A^C \cap (B^C \cup C)$
   (c) $(B^C \cap C^C) \cup (A^C \cap B^C) \cup (C^C \cap A^C)$
2. In a class of 80 students, 50 students know English, 55 know French and 46 know German language. 37 students know English and French, 28 students know French and German, 25 students know English and German and 7 students know none of the languages. Find out
   (a) How many students know all the 3 languages?
   (b) How many students know exactly 2 languages?
   (c) How many know only one language?
3. It is known that in university 60% of the students play tennis, 50% of them play netball, 70% badminton, 20% play tennis and netball, 40% play netball and badminton and 30% play tennis and badminton. If someone claimed that 20% students play badminton and tennis and netball, would you believe his claim? Why?

Suggested Reading

Chapter 2: Kenneth Rosen, (2011) Discrete Mathematics and Its Applications, 7$^{\text{th}}$ Edition, McGraw-Hill Education.