# APEC 8601: Problem set 1

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## Question 1: 2.10.2 Conard and Clarke

Compute the MSY and corresponding effort levels for the following two examples, using the Schafer model. Also, find the level of effort just sufficient to drive the populations to extinction.
**Solution:**
For Schaefer model,

$$Y = qEX = qKE(1 - \frac{qE}{r})$$

Maximum sustainable yield (MSY) is obtained by maximizing this yield function w.r.t to E:

$$\begin{aligned}\frac{\delta Y}{\delta E} &= \frac{\delta(qKE - fracq^2KE^2r}{\delta E} \\ &= qK - \frac{q^2K2E}{r} = 0 \\ qK &= \frac{q^2K2E}{r} \\ E_{MSY} &= \frac{r}{2q}\end{aligned}$$

Substituting $E_{MSY}$ to get $Y_{MSY}$:

$$Y_{MSY} = qk\frac{r}{2q}\left(1 - \frac{qr}{2qr}\right)$$

$$Y_{MSY} = \frac{rK}{4}$$

Likewise, level of effort to drive the population to extinction:

$$\begin{aligned}Y &= 0 \\ E_{ext} &= \frac{r}{q}\end{aligned}$$

## 1(a)

**Antarctic baleen whales: r= 0.05 per year, K = 400,000 BWU (Blue whale units),** $q = 1.6 \times 10^{-3}$ **per whale-catcher year.**

**Solution:** Now, we just use the relations we derived above:

$$\begin{aligned} Y_{MSY} =& \frac{rK}{4} \\ \text{Solving, } Y_{MSY} =& \frac{0.05 \times 400,000}{4} \\ =& \frac{20,000}{4} \\ \therefore Y_{MSY} =& 5,000 \end{aligned}$$

Now, we compute the effort level given the stock levels and yield,

$$\begin{aligned} E_{MSY} =& \frac{r}{2q} \\ =& \frac{0.05}{2 \times 1.6 * 10^{-3}} \\ \therefore E_{MSY} =& 15.63 \text{catchers} \end{aligned}$$

Likewise, the effort level just sufficient to drive the population to extinction:

$$\begin{aligned} E_{ext} &= \frac{r}{q} \\ E_{ext} &= 2E_{MSY} = 31.25 \text{catchers} \end{aligned}$$

## 1(b)

**Pacific yellowfin tuna: r= 2.6 per year, K = 250,000 metric tons,** $q = 3.8 \times 10^{-5}$ **per standard fishing day (SFD).**

**Solution:**

$$\begin{aligned} Y_{MSY} =& \frac{rK}{4} \\ \text{Solving, } Y_{MSY} =& \frac{2.6 \times 250,000}{4} \\ \therefore Y_{MSY} =& 162,500 \quad \text{metric tons/yr} \end{aligned}$$

$$\begin{aligned} E_{MSY} =& \frac{r}{2q} \\ =& \frac{2.6}{2 \times 3.8 * 10^{-5}} \\ \therefore E_{MSY} =& 17100 \quad \text{SFD/ yr} \end{aligned}$$

$$E_{ext} = \frac{r}{q}$$
$$E_{ext} = 2E_{MSY} = 34200 SFD/year$$

## Question 2: 2.10.3 Conard and Clarke

Continuing with Problem 2.10.2, compute the optimal stock levels of $X^*$, optimal sustained biological yields $Y^*$, optimal effort levels $E^*$, optimal sustained economic rents $R^* = pY^* - cE^*$, and optimal initial economic yields $Y_0$ assuming that $X(0) = K$ (and that maximum effort is unconstrained), given:
**Solution:**
Computing optimal harvest level for Scahefer model with logistic growth function:

$$\int_0^\infty \left(pY_t - \frac{cY_t}{qY_t}\right) e^{-\delta t} \qquad s.t. \quad X_t = aX_t\left(1 - \frac{X_t}{k}\right)$$

where, $X_0 =$ given, $0 \leq Y_t \leq Y_{max} \quad \forall t$ and $X_t \geq 0 \forall t$.

Current value Hamiltonian,

$$\tilde{Y} = pY_t - \frac{cY_t}{qY_t} + \mu_t\left(aX_t\left(1 - \frac{X_t}{K}\right) - Y_t\right)$$

Necessary conditions:
1)

$$\frac{\delta\tilde{Y}}{\delta Y} = p - \frac{c}{qX_t} - \mu_t$$

2)

$$\frac{\delta\tilde{Y}}{\delta X} = \frac{cY_t}{q{X_t}^2} + \mu_t a\left(1 - \frac{2X_t}{k}\right) = -\mu_t + \delta\mu_t$$

3)

$$\frac{\delta\tilde{Y}}{\delta\mu} = X_t = aX_t\left(1 - \frac{X_t}{K}\right) - Y_t$$

Solving for optimal state, we get:

$$Y^* = aX^*\left(1 - \frac{X^*}{K}\right)$$

Using quadratic formula:

$$X^* = \frac{1}{4}\left[\frac{c}{pq} + K\left(1 - \frac{\delta}{a}\right) + \sqrt{\left(\frac{c}{pq} + K\left(1 - \frac{\delta}{a}\right)\right)^2 + 8K\frac{c\delta}{pqa}}\right]$$

For ease:
define $X_\infty = \frac{c}{pq}$

$$X^* = \frac{1}{4}\left[X_\infty + K\left(1-\frac{\delta}{a}\right) + \sqrt{\left(X_\infty + K\left(1-\frac{\delta}{a}\right)\right)^2 + 8KX_\infty\frac{\delta}{a}}\right]$$

## 2(a)

Whales: p$= 7,000$ per BWU, c$=600,000$ per whale-catcher year. Use $\delta = 0$ and $\delta = 10\%$ per year. Explain your results.
**Solution:**

$$X_\infty = \frac{c}{pq} = \frac{600000}{7000 * 1.6 * 10^{-3}} = \frac{375000}{7}$$

**Case I: When** $\delta = 0$,

$$\begin{aligned} X^* &= \frac{1}{4}\left[\frac{375000}{7} + 400,000 + \sqrt{\left(\frac{375000}{7} + 400,000\right)^2}\right] \\ &= 226,785.714 BWU \end{aligned}$$

Now, solving for the optimal sustained yields,

$$\begin{aligned} Y^* =& aX^*\left(1-\frac{X^*}{k}\right) \\ =& 0.05 \times 226899\left(1-\frac{226,899}{400,000}\right) \\ =& 4909.55 = 4910 \quad BWU/yr \end{aligned}$$

Optimal effort levels:

$$\begin{aligned} E^* =& \frac{Y^*}{q\cdot X^*} \\ =& \frac{4910}{1.6\times 10^{-3}\cdot 226899} \\ =& 13.52 \quad catchers \end{aligned}$$

Optimal sustained economic rents,

$$\begin{aligned} R^* =& pY^* - cE^* \\ =& 7000\times 4910 - 600,000\times 13.52 \\ =& 26258000 = 26.25 \quad million/yr \end{aligned}$$

Optimal initial economic yields ($Y_0$) assuming $X(0) = K$.

$$\begin{aligned} Y_0 = & \int_{X^*}^{X(0)} \left(p - \frac{c}{qX} dx\right) = p\left(X(0) - X^*\right) - \frac{c}{q} log \frac{X(0)}{X^*} \\ = & 7,000(400,000 - 226,899) - \frac{600,000}{1.6 \times 10^{-3}} log \frac{400,000}{226,899} \\ = & 99909716.4 \\ = & 100 \quad million \end{aligned}$$

**Case II: When** $\delta = 10\% = 0.1$,

$$\begin{aligned} X^* = & \frac{1}{4}\left[\frac{375000}{7} + 400,000\left(1 - \frac{0.1}{0.05}\right) + \sqrt{\left(\frac{375000}{7} + 400,000\left(1 - \frac{0.1}{0.05}\right)\right)^2 + 8 \times 400,000 \frac{375000}{7} \frac{0.10}{0.05}}\right] \\ = & 83479.22 BWU \end{aligned}$$

Now, solving for the optimal sustained yields,

$$\begin{aligned} Y^* = & aX^*\left(1 - \frac{X^*}{k}\right) \\ = & 0.05 \times 83500\left(1 - \frac{83500}{400,000}\right) \\ = & 3303 BWU/yr \end{aligned}$$

Optimal effort levels:

$$\begin{aligned} E^* = & \frac{Y^*}{q \cdot X^*} \\ = & \frac{3303}{1.6 \times 10^{-3} \cdot 83500} \\ = & 24.73 \quad catchers \end{aligned}$$

Optimal sustained economic rents,

$$\begin{aligned} R^* = & pY^* - cE^* \\ = & 7000 \times 3304 - 600,000 \times 24.73 \\ = & 8286500 = 8.28 million/yr \end{aligned}$$

Optimal initial economic yields ($Y_0$) assuming $X(0) = K$.

$$\begin{aligned} Y_0 = & \int_{X^*}^{X(0)} \left(p - \frac{c}{qX} dx\right) = p\left(X(0) - X^*\right) - \frac{c}{q} log \frac{X(0)}{X^*} \\ = & 7,000(400,000 - 83500) - \frac{600,000}{1.6 \times 10^{-3}} log(\frac{400,000}{83500}) \\ = & 162801828 \\ = & 162.8 \quad million \end{aligned}$$

## 2(b)

Tuna: p= 600 per matric ton, c=$2,500$ per SFD. Use $\delta = 0$ and $\delta = 10\%$ per year. Explain your results.

$$X_\infty = \frac{c}{pq} = \frac{2500}{600 * 1.6 * 10^{-3}}$$

**Case I: When** $\delta = 0$,

$$X^* = \frac{1}{4}\left[\frac{2500}{600 \times 3.8 \times 10^{-5}} + 250,000 + \sqrt{\left(\frac{2500}{600 \times 3.8 \times 10^{-5}} + 250,000\right)^2}\right]$$
$$= 179,880 \quad MT$$

Now, solving for the optimal sustained yields,

$$\begin{aligned}Y^* =& aX^*\left(1 - \frac{X^*}{k}\right)\\ =& 2.6 \times 179,880\left(1 - \frac{179,880}{250,000}\right)\\ =& 131177 MT/yr\end{aligned}$$

Optimal effort levels:

$$\begin{aligned}E^* =& \frac{Y^*}{q \cdot X^*}\\ =& \frac{131177}{3.8 \times 10^{-5} \cdot 179,880}\\ =& 19190.72 \quad SFD/yr\end{aligned}$$

Optimal sustained economic rents,

$$\begin{aligned}R^* =& pY^* - cE^*\\ =& 600 \times 131177 - 2500 \times 19191\\ =& 30728700 = 30.73 \quad million/yr\end{aligned}$$

Optimal initial economic yields ($Y_0$) assuming $X(0) = K$.

$$\begin{aligned}Y_0 =& \int_{X^*}^{X(0)}\left(p - \frac{c}{qX} dx\right) = p\left(X(0) - X^*\right) - \frac{c}{q} log\frac{X(0)}{X^*}\\ =& 600(250,000 - 179,880) - \frac{2,500}{3.8 \times 10^{-5}} log(\frac{250,000}{179,880})\\ =& 20416016\\ =& 20.4 \quad million\end{aligned}$$

**Case II: When** $\delta = 10\% = 0.1$,

$$X^* = \frac{1}{4}\left[\frac{2500}{600 \times 3.8 \times 10^{-5}} + 250,000 \times \left(1 - \frac{0.1}{2.6}\right)\right] +$$
$$\frac{1}{4}\left[\sqrt{\left(\frac{2500}{600 \times 3.8 \times 10^{-5}} + 250,000 \times \left(1 - \frac{0.1}{2.6}\right)\right)^2 + 8 \times 250,000 \frac{2500 \times 0.1}{600 \times 3.8 \times 10^{-5} \times 2.6}}\right]$$
$$= 178,000 \quad MT$$

Now, solving for the optimal sustained yields,

$$\begin{aligned} Y^* =& aX^*\left(1 - \frac{X^*}{k}\right) \\ =& 2.6 \times 178,000\left(1 - \frac{178,000}{250,000}\right) \\ =& 133286.4 = 133286 \quad MT/yr \end{aligned}$$

Optimal effort levels:

$$\begin{aligned} E^* =& \frac{Y^*}{q \cdot X^*} \\ =& \frac{133286}{3.8 \times 10^{-5} \cdot 178,000} \\ =& 19700 \quad SFD/yr \end{aligned}$$

Optimal sustained economic rents,

$$\begin{aligned} R^* =& pY^* - cE^* \\ =& 600 \times 133286 - 2500 \times 19700 \\ =& 30728700 = 30.73 \quad million/yr \end{aligned}$$

Optimal initial economic yields ($Y_0$) assuming $X(0) = K$.

$$\begin{aligned} Y_0 =& \int_{X^*}^{X(0)} \left(p - \frac{c}{qX} dx\right) = p\left(X(0) - X^*\right) - \frac{c}{q} log \frac{X(0)}{X^*} \\ =& 600(250,000 - 178,000) - \frac{2,500}{3.8 \times 10^{-5}} log(\frac{250,000}{178,000}) \\ =& 20416016 \\ =& 20.9 \quad million \end{aligned}$$

Discounting at 10% per annum has a strong effect on the whale results because $r = 5\%$ per annum for whales, but there is a little effect on tuna, for which $r = 260\%$ per annum.

# Question 3: 2.10.4 Conard and Clarke

Find the stock level $X_\infty$, effort level $E_\infty$, and sustained yield $Y_\infty$ corresponding to the bionomic equilibrium for the whale and tuna examples above. Explain why $Y_\infty > Y^*$, for the case of tuna.

## 3(a)

Whales : Stock level($X_\infty$):

$$\begin{aligned} X_\infty &= \frac{c}{pq} \\ &= \frac{600,000}{7,000 \times 1.6 \times 10^{-3}} \\ &= 53571.4 \quad BHU \end{aligned}$$

Sustained Yield($Y_\infty$):

$$\begin{aligned} Y_\infty =& F(X_\infty) \\ =& rX_\infty \left(1 - \frac{X_\infty}{K}\right) \\ =& 0.05 \times 53571 \left(1 - \frac{53571}{400,000}\right) \\ =& 2319.82 \\ =& 2320 \quad BHU/yr \end{aligned}$$

Effort level,

$$\begin{aligned} E_\infty &= \frac{Y_\infty}{qX_\infty} \\ &= \frac{230}{1.6 \times 10^{-3} \times 53571} \end{aligned}$$

$$\therefore E_\infty = 27.06 \quad catchers$$

## 3(b)

Tuna :

$$\begin{aligned} X_\infty &= \frac{c}{pq} \\ &= \frac{2,500}{600 \times 3.8 \times 10^{-5}} \\ &= 109649 \quad MT \end{aligned}$$

$$\begin{aligned} Y_\infty =& F(X_\infty) \\ =& rX_\infty \left(1 - \frac{X_\infty}{K}\right) \\ =& 2.6 \times 109649 \left(1 - \frac{109649}{250,000}\right) \\ =& 160049.2 \\ =& 160,049 \quad MT/yr \end{aligned}$$

Effort level,

$$
\begin{aligned}
E_\infty &= \frac{Y_\infty}{qX_\infty} \\
&= \frac{160,049}{3.8 \times 10^{-5} \times 109649} \\
\therefore E_\infty &= 38411.8 \quad SFD/yr
\end{aligned}
$$

Tuna fishing is relatively expensive =, with the result that $X_{infty}$ is only slightly less than $X_{MSY}$. Thus this process explains the reason for $Y_\infty > Y^*$, for the case of tuna.

# Question 4:

The people of Suckerville harvest the long-nosed sucker fish from Big Sucker Lake. Suckerville residents get utility both from harvesting fish and from the stock of fish in the lake, represented by a utility function $U(h_t,\ X_t)$, where $h_t$ is fish harvest at time $t$ and $X_t$ is fish stock at time $t$. Assume that $U_h > 0$, $U_{hh} < 0$, $U_X > 0$, $U_{XX} < 0$, $U_{hX} = 0$, where these expressions represent the first and second derivatives of the utility function with respect to $h$ or $X$. The long-nosed sucker fish has a biological growth function given by $G(X_t)$, with $G(0) = G(K) = 0$, where $K$ is natural carrying capacity, $G(X_t) > 0$ and G"(Xt) $<$ 0 for $X_t$ (0, $K$), G'(Xt) $>$ 0, for $X_t$ [0, $X_{msy}$), G'(Xt) $<$ 0 for $X_t$ ($X_{msy}$, $K$), where $X_{msy}$ is the maximum sustainable yield stock level. The change in the fish population is given by Xt=GXt-ht. The initial stock of the fish is $X_0 > 0$. The people of Suckerville have a discount rate applied to utility of $>$ 0.

## 4(a)

Write down the dynamic optimization problem where the goal is to maximize the present value of utility from harvest over an infinite time horizon. Please include relevant constraints in the statement of the problem.

**Solution:** Assuming that the value comes from harvest of the stock, and possibly from the stock itself (existence value, tourism value, provision of ecosystem services, or cost of harvest dependent on stock size), the utility function is jointly concave in $(h_t, s_t)$:

$$U(t) = U(h_t, X_t)$$

For some $\delta > 0$, the maximization problem of the present value of net benefits can now be written as:

$$\max \int_0^T U(h_t, X_t, t)e^{-\delta t} dt$$

Now, given our constraint (state equation),

$$\dot{X}_t = G(X_t) - h_t$$

where $\dot{X}_t$ id the time-dependent or tie derivative of $X_t$. We have the initial stock of the fish as: $X_0 > 0$.
Finally, the statement of the optimal harvesting problem subject to the constraint is:

$$\max \int_0^T U(h_t, X_t) e^{-\delta t} dt \quad \text{subject to } \dot{X}_t = G(X_t) - h_t$$

## 4(b)

Write down the current value Hamiltonian and derive the necessary conditions for an optimal path.
**Solution:** Current value Hamiltonian:

$$\tilde{H} = U(h_t, X_t) + \quad \mu_t(G(X_t) - h_t)$$

Solving for the necessary conditions for the optimal solution:

$$\frac{\partial \tilde{H}}{\partial h} = \frac{\delta U(h_t, s_t)}{dh_t} - \mu_t = 0$$
$$\text{or,} \quad U_h = \mu(t) \cdots\cdots (1)$$

$$\frac{\partial \tilde{H}}{\partial X} = -\dot{\mu}(t) + \delta \mu_t$$
$$\text{or,} \frac{\delta U(h_t, s_t)}{dX_t} + \mu_t G^{'}(X_t) = \dot{\mu}(t) + \delta \mu(t)$$
$$\dot{\mu}(t) = \mu_t(\delta - G^{'}(X_t)) - U_X \cdots (2)$$

$$\frac{\partial \tilde{H}}{\partial \mu} = \dot{X}_t = G(X_t) - h_t \cdots (3)$$

Here, equations (1), (2), and (3) are the necessary conditions for an optimal path.
Note: Here, $U_h = \frac{\delta U(h_t,s_t)}{dh_t}$; $U_X = \frac{\delta U(h_t,s_t)}{dX_t}$; $U_{hh} = \frac{\delta^2 U(h_t,s_t)}{dh_t^2}$

## 4(c)

Solve for conditions that define an optimal steady state.
**Solution:** Now using equations (1),

$$U_h = \mu(t)$$
$$\text{Differentiating w.r.t } t: \quad U^{"}_{hh} \dot{h}_t + U_{hX} \dot{X}_t = \dot{\mu}(t)$$

Now, bringing in the equation(2),

$$
\begin{aligned}
U_X - \mu(t)G^{'}(X_t) &= -\mu(t)e^{-\delta t} + \delta\mu(t)e^{-\delta t} \\
U_X - \mu(t)G^{'}(X_t) &= -U^{"}_{hh}\dot{h}_t - U_{hX}\dot{X}_t \\
U^{"}_{hh}\dot{h}_t &= \mu(t)G^{'}(X_t) - U_X - U_{hX}(G(X_t) - h_t)) \\
\dot{h}_t &= \frac{U_h(G^{'}(X_t) - \delta) + U_X + U_{hX}(G(X_t) - h_t))}{-U^{"}_{ht}}
\end{aligned}
$$

So, for optimal steady state, assuming $U_X > 0$, we get, at steady state,

$$G(X_t) - h_t = \dot{X}_t = 0$$

So, the numerator is also zero. With $U_x > 0$, we must have $G^{'}(X_t) - \delta < 0 \rightarrow G^{'}(X_t) < \delta$ to make $\dot{h}_t = 0$ since utility increases with the harvest. i.e. $U_h > 0$.

## 4(d)

How does the stock level in the optimal steady state compare to stock level that satisfies the golden rule of growth? Is it greater than, equal, less than, or ambiguous? **Answer:**
The stock level that satisfies the golden rule of growth is defined as:

$$\delta = G^{'}(X_t) + \frac{cG(X_t)}{X_t(pqX_t - c)}$$

Stock level in optimal steady state:

$$\delta > G^{'}(X_t)$$

If we have a singular solution for the stock level at optimal steady state then these two will be equal. Without further information, the solution at this stage is ambiguous.

## 4(e)

How does the stock level in the optimal steady state compare to the maximum sustained yield stock level? Is it greater than, equal, less than, or ambiguous?
**Answer:** We do not have a definite conclusion to this. The comparison between optimal steady state and $S_{MSY}$ can go either way depending on the strength of various effects. Hence, it is ambiguous.

## 4(f)

Clark (1973) showed that it may be optimal to harvest stock to extinction. Write down conditions that would give rise to optimal extinction and explain the economic logic for your result.

**Answer:** Condition for extinction::
If the discount rate is sufficiently high, and the growth rate is sufficiently low
**Economic logic**: As a harvester looking to maximize his profit, it makes sense to harvest in this fashion - since waiting for future harvest does not guarantee more returns i.e. the rate of return on saving the resource in place is lower than rate of return on harvesting the resource and investing in an alternative asset
However, as we discussed in the class this will not be socially optimal. The stock may have public value (existence value) which we did not consider in our optimization problem, so even though the harvester wants to harvest to extinction social optimum may be to save some stock. Likewise, the whole notion of cost and benefit analysis has no value if there is some kind of ethics and species right to exist (like the ESA) which will prohibit harvesting to extinction.