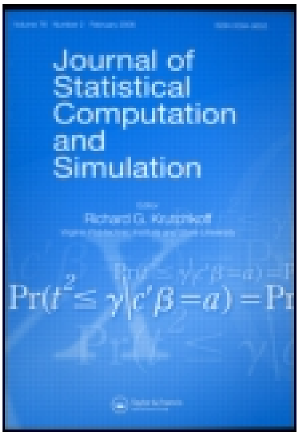


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# The gamma-exponentiated exponential distribution

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# The gamma-exponentiated exponential distribution

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In this paper, we introduce a new distribution generated by gamma random variables. We show that this distribution includes as a special case the distribution of the lower record value from a sequence of i.i.d. random variables from a population with the exponentiated (generalized) exponential distribution. The properties of this distribution are derived and the estimation of the model parameters is discussed. Some applications to real data sets are finally presented for illustration.

**Keywords:** generalized exponential distribution; exponentiated exponential distribution; lower record values; gamma-generated distribution

*AMS Subject Classification*: 62E99



## 1. Introduction

Recently, Zografos and Balakrishnan [1] have introduced a new family of distributions generated by gamma random variables. This family of distributions has its cumulative distribution function (cdf) as

$$G(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log \bar{F}(x)} t^{\delta-1} \mathrm{e}^{-t} \, \mathrm{d}t, \quad x \in \mathbb{R}, \ \delta > 0,$$

where $\bar{F}(x)$ is a survival function which is used to generate a new distribution. The cdf $F(x)$ is referred to as the parent distribution. The corresponding probability density function (pdf) is given by

$$g(x) = \frac{1}{\Gamma(\delta)} (-\log \bar{F}(x))^{\delta-1} f(x), \quad x \in \mathbb{R}, \ \delta > 0, \tag{1}$$

where $f(x)$ is the pdf of the parent distribution $F(x)$. Zografos and Balakrishnan [1] then showed that for $\delta = n$, a positive integer, the pdf given by Equation (1) represents the pdf of the upper record value from a sequence of i.i.d. random variables from a population with pdf $f(x)$. They also showed that if a random variable $X$ has the pdf in Equation (1), then a random variable

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$Z = -\log \bar{F}(X)$ has the gamma distribution $G(\delta, 1)$ with pdf $f(z) = (1/\Gamma(\delta))z^{\delta-1}\,\mathrm{e}^{-z}$, $z > 0$. On the other hand, if a random variable $Z$ has the gamma distribution $G(\delta, 1)$, then a random variable $X = F^{-1}(1 - \mathrm{e}^{-Z})$ has the pdf in Equation (1).

Using a similar approach, we can introduce a new family of distributions with survival function given by

$$\bar{G}(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log F(x)} t^{\delta-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \quad x \in \mathbb{R},\ \delta > 0,$$

where $F(x)$ is the parent distribution. The corresponding pdf is given by

$$g(x) = \frac{1}{\Gamma(\delta)}(-\log F(x))^{\delta-1} f(x), \quad x \in \mathbb{R},\ \delta > 0. \tag{2}$$

Now we give some motivations for this new family of distributions:

*Motivation 1.* Let $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ be lower record values from a sequence of i.i.d. random variables from a population with pdf $f(x)$. Then, the pdf of the $n$th lower record value is

$$g_{X_{L(n)}}(x) = \frac{(-\log F(x))^{n-1}}{(n-1)!} f(x), \quad x \in \mathbb{R}.$$

Thus, if $\delta$ in Equation (2) is a positive integer, $n$, then Equation (2) is the distribution of the $n$th lower record value from a sequence of i.i.d. random variables from a population with pdf $f(x)$.

*Motivation 2.* Let $Z$ be a random variable with gamma distribution $G(\delta, 1)$. Then, the random variable $X = F^{-1}(\mathrm{e}^{-Z})$ has the pdf in Equation (2). Thus, the proposed family of distributions may be regarded as a dual family of the Zografos–Balakrishnan family of distributions.

*Motivation 3.* Let $Z$ be a random variable with the log-gamma distribution LG($\delta$). Then, the random variable $X = F^{-1}(\mathrm{e}^{-\exp(Z)})$ has the pdf in Equation (2).

Here, we introduce a distribution with survival function as

$$\begin{aligned}\bar{G}(x) &= \frac{1}{\Gamma(\delta)} \int_0^{-\alpha\log(1-\mathrm{e}^{-\lambda x})} t^{\delta-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \quad x, \lambda, \alpha, \delta > 0\\ &= \frac{\gamma(\delta, -\alpha\log(1-\mathrm{e}^{-\lambda x}))}{\Gamma(\delta)}, \quad x, \lambda, \alpha, \delta > 0,\end{aligned} \tag{3}$$

where $\gamma(\cdot,\cdot)$ is the incomplete gamma function and $F(x) = (1 - \mathrm{e}^{-\lambda x})^{\alpha}$ is the cdf of a random variable with the exponentiated exponential distribution introduced by Ahuja and Nash [2]. Gupta and Kundu [3] studied properties of this distribution and discussed the estimation of the unknown parameters. We will say that a random variable with survival function in Equation (3) has the gamma-exponentiated exponential (GEE) distribution with parameters $\lambda$, $\alpha$ and $\delta$, which we will denote by GEE($\lambda, \alpha, \delta$). This distribution includes as a special case, the distribution of the lower record value from a sequence of i.i.d. random variables from a population with the exponentiated exponential distribution. Properties of the distribution of the lower record values have been studied by Raqab [4]. In Section 2, we discuss the shape characteristics of the pdf and hazard rate function (hrf) of the GEE distribution. Next, the moments, the Rényi and Shannon entropies are derived in Section 3. In Section 4, we discuss the order statistics. The maximum-likelihood estimates (MLEs) of the unknown parameters are derived in Section 5. Finally, some applications to real data sets are presented in Section 6 for illustrative purposes.

## 2. Shape characteristics of the pdf and hrf

In this section, we consider the distribution with cdf as in Equation (3). Its pdf is

$$g(x) = \frac{\lambda\alpha^{\delta}}{\Gamma(\delta)}\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}(-\log(1-\mathrm{e}^{-\lambda x}))^{\delta-1}, \quad x, \lambda, \alpha, \delta > 0. \tag{4}$$

Several distributions can be obtained as special cases of the GEE distribution in Equation (4). For $\delta = 1$, we obtain the exponentiated exponential distribution with parameters $\lambda$ and $\alpha$. For $\alpha = 1$ and $\delta = 1$, we obtain the exponential distribution with parameter $\lambda$. For $\delta \in \mathbb{N}$, we obtain the distribution of the lower record value from a sequence of i.i.d. random variables from a population with the exponentiated exponential distribution.

*Remark 2.1* It can be easily shown that the pdf of GEE distribution is an infinite sum of pdfs of beta exponential distributed random variables introduced by Nadarajah and Kotz [5].

*Remark 2.2* It is easy to show that if a random variable $X$ has GEE$(\lambda, \alpha, \delta)$ distribution, then the random variables $-\log(1-\mathrm{e}^{-\lambda X})$ and $\log\alpha + \log(-\log(1-\mathrm{e}^{-\lambda X}))$ have gamma $G(\delta, \alpha)$ and log-gamma LG$(\delta)$ distributions, respectively.

Let us now examine the shape characteristics of the pdf. The first derivative of the pdf $g(x)$ is

$$g'(x) = \frac{\lambda^2\alpha^{\delta}}{\Gamma(\delta)}\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-2}(-\log(1-\mathrm{e}^{-\lambda x}))^{\delta-2}u(x),$$

where $u(x) = (1-\alpha\,\mathrm{e}^{-\lambda x})\log(1-\mathrm{e}^{-\lambda x}) - (\delta-1)\,\mathrm{e}^{-\lambda x}$. The first derivative of the function $u(x)$ can be represented as $u'(x) = \lambda\,\mathrm{e}^{-\lambda x}v(x)$, where the function $v(x)$ is of the form

$$v(x) = \alpha\log(1-\mathrm{e}^{-\lambda x}) + (1-\alpha\,\mathrm{e}^{-\lambda x})(1-\mathrm{e}^{-\lambda x})^{-1} + \delta - 1.$$

For $0 < \alpha < \frac{1}{2}$, the function $v(x)$ is positive, which implies that the function $u(x)$ is an increasing function. Since $u(\infty) = 0$, it follows that the function $u(x)$ is negative. So, the function $g'(x)$ is negative and thus we have in this case the pdf to be a decreasing function. Furthermore, $g(0) = \infty$ and $g(\infty) = 0$.

Consider now the case $\frac{1}{2} < \alpha < 1$. If $0 < \delta < 1 - 2\alpha - \alpha\log((1-\alpha)/\alpha)$, then the function $v(x)$ has the minimum at $x_0 = -\lambda^{-1}\log(2-\alpha^{-1})$. Also, we have $v(0) = \infty$, $v(x_0) < 0$ and $v(\infty) = \delta$. So, the function $v(x)$ has two real roots, say $x_1 < x_2$. This implies that the function $u(x)$ increases on $(0, x_1] \cup (x_2, \infty)$ and decreases on $(x_1, x_2]$. We have $u(0) = -\infty$, $u(\infty) = 0$ and $u(x_2) < 0$. If $u(x_1) > 0$, then the function $u(x)$ has two real roots, say $a < b$, which implies that the pdf $g(x)$ decreases on $(0, a] \cup (b, \infty)$ and increases on $(a, b]$. Moreover, $g(0) = \infty$ and $g(\infty) = 0$. If at least one of the above conditions is not satisfied, then we have the pdf to be a decreasing function as in the case $0 < \alpha < \frac{1}{2}$.

Finally, let us consider the case $\alpha > 1$. In this case, we have the function $v(x)$ to be an increasing function with $v(0) = -\infty$ and $v(\infty) = \delta$, which implies that the function $v(x)$ has only one root, say $x_0$. This implies that the function $u(x)$ decreases for $x < x_0$ and increases for $x > x_0$. Since $u(0) = \infty$ and $u(\infty) = 0$, it follows that the function $u(x)$ has only one root, say $x_1$, and that it is positive for $x < x_1$ and negative for $x > x_1$. Thus, it follows that the pdf $g(x)$ is unimodal with mode at $x = x_0$. Also, $g(0) = g(\infty) = 0$.

Now, we consider the hrf of the GEE distribution, which is given by

$$h(x) = \frac{\lambda\alpha^{\delta}\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}(-\log(1-\mathrm{e}^{-\lambda x}))^{\delta-1}}{\gamma(\delta, -\alpha\log(1-\mathrm{e}^{-\lambda x}))}, \quad x, \lambda, \alpha, \delta > 0.$$

The shape characteristics of the hrf of the GEE distribution are as given in the following theorem.

THEOREM 2.1

(a) *Let* $0 < \delta < 1$. *Then, the hrf of the GEE distribution is a decreasing function for* $0 < \alpha < (1+\delta)/2$, *bathtub-shaped function for* $(1+\delta)/2 < \alpha < 1$, *and an increasing function for* $\alpha > 1$.
(b) *Let* $\delta > 1$. *Then, the hrf of the GEE distribution is a decreasing function for* $0 < \alpha < 1$, *upside-down bathtub-shaped function for* $1 < \alpha < (1+\delta)/2$, *and an increasing function for* $\alpha > (1+\delta)/2$.

*Proof* We will discuss the shape characteristics of the hrf by using a theorem of Glaser [6]. We consider the function $\eta(x) = -g'(x)/g(x)$. In the case of the GEE distribution, the function $\eta(x)$ is given by

$$\eta(x) = \frac{\lambda(1-\alpha\,\mathrm{e}^{-\lambda x})}{1-\mathrm{e}^{-\lambda x}} + \frac{\lambda(1-\delta)\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})\log(1-\mathrm{e}^{-\lambda x})}.$$

The first derivative of the function $\eta(x)$ can be expressed as

$$\eta'(x) = \frac{\lambda^2\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})^2\log^2(1-\mathrm{e}^{-\lambda x})} w(x),$$

where

$$w(x) = (\alpha-1)\log^2(1-\mathrm{e}^{-\lambda x}) + (\delta-1)[\log(1-\mathrm{e}^{-\lambda x}) + \mathrm{e}^{-\lambda x}].$$

We can see that for $\delta > 1$ and $0 < \alpha < 1$, the functions $w(x)$ and $\eta'(x)$ are negative. Then, from Part (b) of Glaser's theorem, it follows that the hrf is a decreasing function. Also, for $0 < \delta < 1$ and $\alpha > 1$, we have the functions $w(x)$ and $\eta'(x)$ to be positive. Then, from Part (a) of Glaser's theorem, it follows that the hrf is an increasing function.

Now, let us derive the other shapes of the hrf. The first derivative of the function $w(x)$ is $w'(x) = \lambda\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{-1}z(x)$, where

$$z(x) = 2(\alpha-1)\log(1-\mathrm{e}^{-\lambda x}) + (\delta-1)\,\mathrm{e}^{-\lambda x}.$$

For $0 < \delta < 1$ and $0 < \alpha < (1+\delta)/2$, the function $z(x)$ is positive and this implies that the function $w(x)$ is increasing. Since $w(\infty) = 0$, the function $w(x)$ is negative and then from Part (b) of Glaser's theorem, it follows that the hrf is decreasing.

For $0 < \delta < 1$ and $(1+\delta)/2 < \alpha < 1$, the function $z(x)$ has exactly one root, say $x_0$. Also, the function $z(x)$ is positive for $x < x_0$ and negative for $x > x_0$. This implies that the function $w(x)$ increases on $(0, x_0]$ and decreases on $(x_0, \infty)$. From this and the fact that $w(0) = -\infty$ and $w(\infty) = 0$, we note that the function $w(x)$ has exactly one root, say $x_1$, with $w(x) < 0$ for $x < x_1$ and $w(x) > 0$ for $x > x_1$. The same conclusion holds for the function $\eta'(x)$. Since $g(0) = \infty$, it follows from Glaser's theorem and lemma that the hrf is bathtub-shaped.

Let $\delta > 1$ and $1 < \alpha < (1+\delta)/2$. Then, the function $z(x)$ has exactly one root, say $x_0$. Also, the function $z(x)$ is negative for $x < x_0$ and positive for $x > x_0$. This implies that the function $w(x)$ decreases on $(0, x_0)$ and increases on $(x_0, \infty)$. From this and the fact that $w(0) = \infty$ and $w(\infty) = 0$, we note that the function $w(x)$ has exactly one root, say $x_1$, with $w(x) > 0$ for $x < x_1$ and $w(x) < 0$ for $x > x_1$. The same conclusion holds for the function $\eta'(x)$ and since $g(0) = 0$, it follows from Glaser's theorem and lemma that the hrf is upside-down bathtub-shaped.

Finally, let us consider the case $\delta > 1$ and $\alpha > (1+\delta)/2$. Then, the function $z(x)$ is negative and this implies that the function $w(x)$ is decreasing. Since $w(\infty) = 0$, the function $w(x)$ is positive and then from Part (a) of Glaser's theorem, it follows that the hrf is increasing. ■

In Figure 1, we have plotted the pdf and hrf for $\lambda = 1$ and different values of the parameters $\alpha$ and $\delta$, which supports the properties established above.

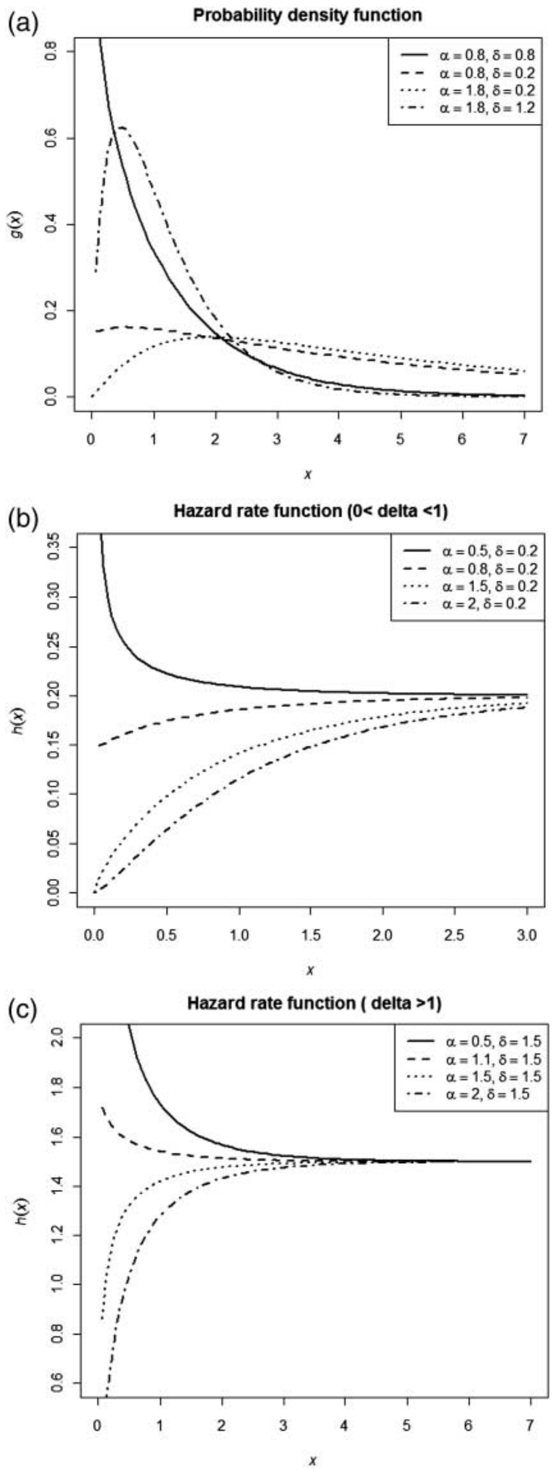


Figure 1. (a) The pdf for $\lambda = 1$ and different values of the parameters $\alpha$ and $\delta$. (b) The hrf for $\lambda = 1$ and different values of the parameter $\alpha$ when $\delta \in (0, 1)$. (c) The hrf for $\lambda = 1$ and different values of the parameter $\alpha$ when $\delta > 1$.

## 3. Moments and some entropies

In this section, we derive the moments of the GEE($1, \alpha, \delta$) distribution. Since $\lambda$ is a scale parameter, we immediately have that if a random variable $X$ has the GEE($\lambda, \alpha, \delta$) distribution, then the random variable $\lambda X$ has the GEE($1, \alpha, \delta$) distribution. Now, let us derive the moment generating function $M(t) = E(\mathrm{e}^{tX})$ for the GEE($1, \alpha, \delta$) distribution, given by

$$M(t) = \frac{\alpha^\delta}{\Gamma(\delta)} \int_0^\infty \mathrm{e}^{-(1-t)x} (1-\mathrm{e}^{-x})^{\alpha-1} (-\log(1-\mathrm{e}^{-x}))^{\delta-1}\, \mathrm{d}x.$$

Making the substitution $u = -\log(1-\mathrm{e}^{-x})$, we obtain the moment generating function as

$$M(t) = \frac{\alpha^\delta}{\Gamma(\delta)} \int_0^\infty u^{\delta-1}\, \mathrm{e}^{-\alpha u} (1-\mathrm{e}^{-u})^{-t}\, \mathrm{d}u. \tag{5}$$

By using the series expansion $(1-\mathrm{e}^{-u})^{-t} = \sum_{k=0}^{\infty} \binom{t+k-1}{k} \mathrm{e}^{-ku}$, Equation (5) reduces to

$$\begin{aligned} M(t) &= \frac{\alpha^\delta}{\Gamma(\delta)} \sum_{k=0}^{\infty} \binom{t+k-1}{k} \int_0^\infty u^{\delta-1}\, \mathrm{e}^{-(\alpha+k)u}\, \mathrm{d}u \\ &= \alpha^\delta \sum_{k=0}^{\infty} \binom{t+k-1}{k} (\alpha+k)^{-\delta} \\ &= 1 + \alpha^\delta \sum_{k=1}^{\infty} \frac{1}{k!} (t)_k (\alpha+k)^{-\delta}, \end{aligned}$$

where $(t)_k = t(t+1)\cdots(t+k-1)$ is the ascending factorial. Now, taking the $n$th derivative of the moment generating function and then evaluating it at $t = 0$, we obtain the $n$th moment of the GEE($1, \alpha, \delta$) distribution as

$$E(X^n) = \alpha^\delta \sum_{k=1}^{\infty} \frac{1}{k!} (\alpha+k)^{-\delta} \frac{\mathrm{d}^n (t)_k}{\mathrm{d}t^n},$$

where the $n$th derivative of the ascending factorial is defined recursively as

$$\frac{\mathrm{d}^n (t)_k}{\mathrm{d}t^n} = \sum_{j=0}^{n-1} \binom{n-1}{j} [\Psi^{(n-1-j)}(t+k) - \Psi^{(n-1-j)}(t)] \frac{\mathrm{d}^j (t)_k}{\mathrm{d}t^j}$$

with $\Psi(t)$ being the digamma function. In particular, the first two moments can be derived as

$$E(X) = \alpha^\delta \sum_{k=1}^{\infty} \frac{1}{k} (\alpha+k)^{-\delta} \tag{6}$$

and

$$E(X^2) = 2\alpha^\delta \sum_{k=1}^{\infty} \frac{1}{k} (\alpha+k)^{-\delta} (-\Psi(1) + \Psi(k)). \tag{7}$$

If we set $\delta = n$, then we obtain the expression for the first moment as in Raqab [4].

Let us now consider the entropy. The entropy represents a measure of uncertainty of a random variable. First, we derive the Rényi entropy defined by $I_{\mathrm{R}}(\gamma) = (1/(1-\gamma)) \log \int_{\mathrm{R}} g^{\gamma}(x)\,\mathrm{d}x$, $\gamma > 0$ and $\gamma \neq 1$. Let us consider the integral $\int_0^{\infty} g^{\gamma}(x)\,\mathrm{d}x$ given by

$$\int_0^{\infty} g^{\gamma}(x)\,\mathrm{d}x = \frac{\lambda^{\gamma}\alpha^{\delta\gamma}}{(\Gamma(\delta))^{\gamma}} \int_0^{\infty} \mathrm{e}^{-\lambda\gamma x}(1-\mathrm{e}^{-\lambda x})^{(\alpha-1)\gamma}(-\log(1-\mathrm{e}^{-\lambda x}))^{(\delta-1)\gamma}\,\mathrm{d}x. \tag{8}$$

Making the substitution $u = -\log(1-\mathrm{e}^{-\lambda x})$, Equation (8) reduces to

$$\int_0^{\infty} g^{\gamma}(x)\,\mathrm{d}x = \frac{\lambda^{\gamma-1}\alpha^{\delta\gamma}}{(\Gamma(\delta))^{\gamma}} \int_0^{\infty} u^{(\delta-1)\gamma}\,\mathrm{e}^{-[(\alpha-1)\gamma+1]u}(1-\mathrm{e}^{-u})^{\gamma-1}\,\mathrm{d}u. \tag{9}$$

Upon using the series expansion $(1-\mathrm{e}^{-u})^{\gamma-1} = \sum_{j=0}^{\infty}((-1)^j\Gamma(\gamma)/(j!\Gamma(\gamma-j)))\,\mathrm{e}^{-ju}$ for $\gamma > 0$, we obtain for $(\delta-1)\gamma+1>0$ that

$$\int_0^{\infty} g^{\gamma}(x)\,\mathrm{d}x = \frac{\lambda^{\gamma-1}\alpha^{\delta\gamma}}{(\Gamma(\delta))^{\gamma}} \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(\gamma)}{j!\Gamma(\gamma-j)} \cdot \frac{\Gamma(1+(\delta-1)\gamma)}{((\alpha-1)\gamma+j+1)^{(\delta-1)\gamma+1}}.$$

Using this expression, the Rényi entropy can be expressed as

$$I_{\mathrm{R}}(\gamma) = \frac{1}{1-\gamma}\log\left(\frac{\lambda^{\gamma-1}\alpha^{\delta\gamma}}{(\Gamma(\delta))^{\gamma}} \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(\gamma)}{j!\Gamma(\gamma-j)} \cdot \frac{\Gamma(1+(\delta-1)\gamma)}{((\alpha-1)\gamma+j+1)^{(\delta-1)\gamma+1}}\right).$$

The particular case of the Rényi entropy for $\gamma \uparrow 1$ is the Shannon entropy. It is defined as $E(-\log g(X))$. For the GEE$(\lambda, \alpha, \delta)$ distribution, the Shannon entropy is given by

$$\begin{aligned} E(-\log g(X)) &= -\log\frac{\lambda\alpha^{\delta}}{\Gamma(\delta)} + \lambda E(X) + (1-\alpha)E(\log(1-\mathrm{e}^{-\lambda X})) \\ &\quad + (1-\delta)E(\log(-\log(1-\mathrm{e}^{-\lambda X}))). \end{aligned}$$

Since, as mentioned earlier, the random variables $-\log(1-\mathrm{e}^{-\lambda X})$ and $\log\alpha + \log(-\log(1-\mathrm{e}^{-\lambda X}))$ have gamma $G(\delta, \alpha)$ and log-gamma LG$(\delta)$ distributions, respectively, it follows that $E(-\log(1-\mathrm{e}^{-\lambda X})) = \delta/\alpha$ and $E(\log(-\log(1-\mathrm{e}^{-\lambda X}))) = \Psi(\delta) - \log\alpha$. Using these results and Equation (6), we obtain the Shannon entropy as

$$E(-\log g(X)) = \log\frac{\Gamma(\delta)}{\lambda\alpha} + \frac{(\alpha-1)\delta}{\alpha} + (1-\delta)\Psi(\delta) + \alpha^{\delta}\sum_{k=1}^{\infty}\frac{1}{k}(\alpha+k)^{-\delta}.$$

## 4. Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the GEE$(\lambda, \alpha, \delta)$ distribution, and let $X_{i:n}$ denote the $i$th order statistic. The pdf of the $i$th order statistic $X_{i:n}$ is given by (see [7])

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} g(x)(1-\bar{G}(x))^{i-1}(\bar{G}(x))^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} g(x)\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\left(\frac{\gamma(\delta, -\alpha\log(1-\mathrm{e}^{-\lambda x}))}{\Gamma(\delta)}\right)^{n+j-i}. \end{aligned}$$

Using the series expansion from Nadarajah and Pal [8]

$$\gamma(\delta, x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m+\delta}}{m!(m+\delta)},$$

we can express the pdf $g_{i:n}(x)$ as

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} g(x) \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \times \left( \frac{1}{\Gamma(\delta)} \sum_{m=0}^{\infty} \frac{(-1)^m \alpha^{m+\delta}}{m!(m+\delta)} (-\log(1-\mathrm{e}^{-\lambda x}))^{m+\delta} \right)^{n+j-i}. \tag{10}$$

Now, upon using a result of Nadarajah and Pal [8], it can be shown that the pdf of the $i$th order statistic $X_{i:n}$ can be expressed as an infinite sum of GEE pdfs.

Let us now consider the asymptotic distributions of the sample maxima $X_{n:n}$ and the sample minima $X_{1:n}$. We begin with the asymptotic distribution of the sample maxima $X_{n:n}$. We will use Theorem 8.3.3 of [7]. Using L'Hôpital's rule twice, we have

$$\lim_{x\to\infty} h(x) = \lambda \lim_{x\to\infty} \frac{(1-\alpha\,\mathrm{e}^{-\lambda x})\log(1-\mathrm{e}^{-\lambda x}) - (\delta-1)\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})\log(1-\mathrm{e}^{-\lambda x})} = \lambda\delta,$$

which implies that $xh(x) \to \infty$ when $x \to \infty$. So, the asymptotic distribution of $X_{n:n}$ is not of Fréchet type. Also, since $G^{-1}(1) = \infty$, it follows that the asymptotic distribution of $X_{n:n}$ is not of Weibull type. The hrf $h(x)$ is non-zero and differentiable for large $x$. Also, since $g'(x)/g(x) \to -\lambda\delta$, when $x \to \infty$, we obtain

$$\lim_{x\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{h(x)}\right\} = -1 + \lim_{x\to\infty} \frac{g'(x)}{g(x)} \frac{1}{h(x)} = 0,$$

which implies that the asymptotic distribution of the sample maxima $X_{n:n}$ is indeed of extreme value type.

In order to derive the asymptotic distribution of the sample minima $X_{1:n}$, we use Theorem 8.3.6 of [7]. Since $G^{-1}(0) = 0$, it follows that the asymptotic distribution of $X_{1:n}$ is not of Fréchet type. The asymptotic distribution of $X_{1:n}$ will be of Weibull type with shape parameter $\theta > 0$ if

$$\lim_{\varepsilon\to 0_+} \frac{G(\varepsilon x)}{G(\varepsilon)} = x^{\theta},$$

for all $x > 0$. Then, by using L'Hôpital's rule, it follows that

$$\lim_{\varepsilon\to 0_+} \frac{G(\varepsilon x)}{G(\varepsilon)} = x \lim_{\varepsilon\to 0_+} \frac{g(\varepsilon x)}{g(\varepsilon)} = x \lim_{\varepsilon\to 0_+} \frac{(1-\mathrm{e}^{-\lambda x\varepsilon})^{\alpha-1}(-\log(1-\mathrm{e}^{-\lambda x\varepsilon}))^{\delta-1}}{(1-\mathrm{e}^{-\lambda\varepsilon})^{\alpha-1}(-\log(1-\mathrm{e}^{-\lambda\varepsilon}))^{\delta-1}}.$$

Finally, since

$$\lim_{\varepsilon\to 0_+} \frac{(1-\mathrm{e}^{-\lambda x\varepsilon})^{\alpha-1}}{(1-\mathrm{e}^{-\lambda\varepsilon})^{\alpha-1}} = x^{\alpha-1}$$

and

$$\lim_{\varepsilon\to 0_+} \frac{(-\log(1-\mathrm{e}^{-\lambda x\varepsilon}))^{\delta-1}}{(-\log(1-\mathrm{e}^{-\lambda\varepsilon}))^{\delta-1}} = 1,$$

we obtain that the asymptotic distribution of the sample minima $X_{1:n}$ is of the Weibull type with shape parameter $\alpha$.

## 5. Maximum-likelihood estimation

In this section, we consider the estimation of the unknown parameters by the method of maximum likelihood. Let $x_1, x_2, \dots, x_n$ be a sample from the GEE$(\lambda, \alpha, \delta)$ distribution. Then, the log-likelihood function based on the given random sample is

$$\log L = n\log\lambda + n\delta\log\alpha - n\log\Gamma(\delta) - \lambda\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda x_i}) + (\delta-1)\sum_{i=1}^{n}\log(-\log(1-\mathrm{e}^{-\lambda x_i})).$$

The first partial derivatives of the log-likelihood function with respect to the parameters $\lambda$, $\alpha$ and $\delta$ are, respectively,

$$\frac{\partial\log L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\frac{x_i\,\mathrm{e}^{-\lambda x_i}}{1-\mathrm{e}^{-\lambda x_i}} + (\delta-1)\sum_{i=1}^{n}\frac{x_i\,\mathrm{e}^{-\lambda x_i}}{(1-\mathrm{e}^{-\lambda x_i})\log(1-\mathrm{e}^{-\lambda x_i})}, \tag{11}$$

$$\frac{\partial\log L}{\partial\alpha} = \frac{n\delta}{\alpha} + \sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda x_i}), \tag{12}$$

$$\frac{\partial\log L}{\partial\delta} = n\log\alpha - n\Psi(\delta) + \sum_{i=1}^{n}\log(-\log(1-\mathrm{e}^{-\lambda x_i})). \tag{13}$$

The maximum-likelihood estimates of the parameters $\lambda$, $\alpha$ and $\delta$ can be obtained in the following manner. First, we solve Equation (12) for $\delta$ and obtain

$$\delta = -\frac{\alpha}{n}\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda x_i}) = \alpha A_1(\lambda), \tag{14}$$

where $A_1(\lambda) = -(1/n)\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda x_i})$. By using Equation (14) in Equation (11) and solving for $\alpha$, we obtain

$$\alpha = \frac{A_2(\lambda) - A_3(\lambda) + \bar{x} - \lambda^{-1}}{A_2(\lambda) - A_1(\lambda)A_3(\lambda)}, \tag{15}$$

where $A_2(\lambda) = (1/n)\sum_{i=1}^{n}(x_i\,\mathrm{e}^{-\lambda x_i}/(1-\mathrm{e}^{-\lambda x_i}))$, $A_3(\lambda) = -(1/n)\sum_{i=1}^{n}(x_i\,\mathrm{e}^{-\lambda x_i}/((1-\mathrm{e}^{-\lambda x_i})\log(1-\mathrm{e}^{-\lambda x_i})))$, and $A_2(\lambda) \neq A_1(\lambda)A_3(\lambda)$. Now, by using Equations (14) and (15) in

Equation (13), we obtain the equation

$$0 = \log\left(\frac{A_2(\lambda) - A_3(\lambda) + \bar{x} - \lambda^{-1}}{A_2(\lambda) - A_1(\lambda)A_3(\lambda)}\right) + \frac{1}{n}\sum_{i=1}^{n}\log(-\log(1-\mathrm{e}^{-\lambda x_i}))$$
$$- \Psi\left(\frac{A_2(\lambda) - A_3(\lambda) + \bar{x} - \lambda^{-1}}{A_2(\lambda) - A_1(\lambda)A_3(\lambda)}A_1(\lambda)\right).$$

By solving this non-linear equation for $\lambda$, we obtain the maximum-likelihood estimate $\hat{\lambda}$. Finally, replacing $\hat{\lambda}$ in Equations (15) and (14), we obtain the maximum-likelihood estimates of the parameters $\alpha$ and $\delta$.

Under suitable regularity conditions, the maximum-likelihood estimator $\hat{\theta} = (\hat{\lambda}, \hat{\alpha}, \hat{\delta})$ has the trivariate normal distribution with mean vector $\theta$ and covariance matrix $I^{-1}(\theta)$, where $I(\theta)$ is the Fisher information matrix given by

$$I(\theta) = -\left[E\left(\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\right)\right]_{i,j=1}^{3}.$$

The elements of the Fisher information matrix can be easily obtained by using the following Lemma.

Lemma 5.1 *Let $X$ be a random variable with GEE$(\lambda, \alpha, \delta)$ distribution, and let*

$$\mu_i(j,l) = E\left(\frac{X^i}{(1-\mathrm{e}^{-\lambda X})^j \log^l(1-\mathrm{e}^{-\lambda X})}\right), \quad i \in \{1,2\},\ j,l \geq 0.$$

*Then, for $\alpha > j$ and $\delta > l$,*

$$\mu_1(j,l) = \frac{(-1)^l \alpha^\delta \Gamma(\delta - l)}{\lambda \Gamma(\delta)} \sum_{k=1}^{\infty} k^{-1}(\alpha - j + k)^{-\delta + l},$$

$$\mu_2(j,l) = \frac{2(-1)^l \alpha^\delta \Gamma(\delta - l)}{\lambda^2 \Gamma(\delta)} \sum_{k=1}^{\infty} k^{-1}(\alpha - j + k)^{-\delta + l}(-\Psi(1) + \Psi(k)).$$

*Proof* Let $Y_{\lambda,\alpha,\delta}$ be a random variable with GEE$(\lambda, \alpha, \delta)$ distribution. Then,

$$E\left(\frac{X^i}{(1-\mathrm{e}^{-\lambda X})^j \log^l(1-\mathrm{e}^{-\lambda X})}\right) = \frac{(-1)^l \alpha^\delta \Gamma(\delta - l)}{\Gamma(\delta)(\alpha - j)^{\delta - l}} E(Y^i_{\lambda,\alpha-j,\delta-l})$$
$$= \frac{(-1)^l \alpha^\delta \Gamma(\delta - l)}{\lambda^i \Gamma(\delta)(\alpha - j)^{\delta - l}} E(Y^i_{1,\alpha-j,\delta-l}).$$

Using Equations (6) and (7), we then obtain the expressions for $\mu_1(j,l)$ and $\mu_2(j,l)$. ■

Now, using the above Lemma, we obtain the elements $E(\partial^2 \log L/\partial\theta_i\partial\theta_j)$ as follows:

$$E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = -\frac{n}{\lambda^2} + (1-\alpha)nE\left(\frac{X^2\,\mathrm{e}^{-\lambda X}}{(1-\mathrm{e}^{-\lambda X})^2}\right)$$
$$+ (1-\delta)nE\left(\frac{X^2\,\mathrm{e}^{-\lambda X}(\mathrm{e}^{-\lambda X} + \log(1-\mathrm{e}^{-\lambda X}))}{(1-\mathrm{e}^{-\lambda X})^2 \log^2(1-\mathrm{e}^{-\lambda X})}\right)$$

$$
\begin{aligned}
&= -\frac{n}{\lambda^2} + (1-\alpha)n(-\mu_2(1,0) + \mu_2(2,0)) \\
&\quad + (1-\delta)n(-\mu_2(1,1) - 2\mu_2(1,2) + \mu_2(2,1) + \mu_2(2,2) \\
&\quad + \mu_2(0,2)), \quad \text{for } \alpha > 2, \delta > 2,
\end{aligned}
$$

$$
E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\lambda}\right) = nE\left(\frac{X\,\mathrm{e}^{-\lambda X}}{1-\mathrm{e}^{-\lambda X}}\right) = n(-\mu_1(0,0) + \mu_1(1,0)), \quad \text{for } \alpha > 1,
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 \log L}{\partial\delta\partial\lambda}\right) &= nE\left(\frac{X\,\mathrm{e}^{-\lambda X}}{(1-\mathrm{e}^{-\lambda X})\log(1-\mathrm{e}^{-\lambda X})}\right) \\
&= n(-\mu_1(0,1) + \mu_1(1,1)), \quad \text{for } \alpha > 1, \delta > 1,
\end{aligned}
$$

$$
E\left(\frac{\partial^2 \log L}{\partial\alpha^2}\right) = -\frac{n\delta}{\alpha^2},
$$

$$
E\left(\frac{\partial^2 \log L}{\partial\delta\partial\alpha}\right) = \frac{n}{\alpha},
$$

$$
E\left(\frac{\partial^2 \log L}{\partial\delta^2}\right) = -n\Psi'(\delta).
$$

Now, let us consider the maximum-likelihood estimators of the shape parameter when the other two parameters are known. First, we start with the MLE of the parameter $\alpha$ when the parameters $\lambda$ and $\delta$ are known. The MLE is given by

$$
\hat{\alpha} = \delta\left(-\frac{1}{n}\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda X_i})\right)^{-1}.
$$

Let us derive the distribution of the estimator $\hat{\alpha}$. Since the random variables $-\log(1-\mathrm{e}^{-\lambda X_i})$ have gamma $G(\delta,\alpha)$ distributions, it follows that the random variable $Z = -(1/n)\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda X_i})$ has gamma $G(n\delta, n\alpha)$ distribution. This implies that the random variable $\hat{\alpha} = \delta Z^{-1}$ has the inverse gamma distribution with parameters $n\delta$ and $n\alpha\delta$. Consequently, the mean and the variance of the MLE of the parameter $\alpha$ are, respectively,

$$
\begin{aligned}
E(\hat{\alpha}) &= \frac{n\alpha\delta}{n\delta - 1}, \quad n\delta > 1, \\
\mathrm{Var}(\hat{\alpha}) &= \frac{(n\alpha\delta)^2}{(n\delta-1)^2(n\delta-2)}, \quad n\delta > 2.
\end{aligned}
$$

Hence, we can see that the estimator $\hat{\alpha}$ is not an unbiased estimator of the parameter $\alpha$. Also, we can see that this estimator is asymptotically unbiased. An unbiased estimator of the parameter $\alpha$, say $\hat{\alpha}_U$, is given by

$$
\hat{\alpha}_U = (n\delta - 1)\left(-\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\lambda X_i})\right)^{-1}.
$$

The variance of this estimator is $\mathrm{Var}(\hat{\alpha}_U) = \alpha^2/(n\delta - 2)$.

Now, we derive the MLE of the parameter $\delta$ when the parameters $\lambda$ and $\alpha$ are known. The MLE is given by

$$
\hat{\delta} = \Psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}(\log(-\log(1-\mathrm{e}^{-\lambda X_i})) + \log\alpha)\right),
$$

where $\Psi^{-1}(\cdot)$ is the inverse digamma function. Note that the random variables $\log(-\log(1-\mathrm{e}^{-\lambda X_i}))+\log\alpha$ have the log-gamma LG($\delta$) distributions.

## 6. Illustrations with data

In this section, we compare the GEE distribution with five other distributions:

- Weibull distribution with cdf $F(x)=1-\mathrm{e}^{-(\lambda x)^{\beta}}$, $x>0$;
- Gamma distribution with cdf $F(x)=(\lambda^{\nu-1}/\Gamma(\nu))\int_0^x t^{\nu-1}\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t$, $x>0$;
- Beta exponential distribution [5] with cdf $F(x)=(1/B(a,b))\int_0^{1-\mathrm{e}^{-\lambda x}} w^{a-1}(1-w)^{b-1}\,\mathrm{d}w$, $x>0$;
- Exponentiated Weibull distribution [9] with cdf $F(x)=(1-\mathrm{e}^{-(\lambda x)^{\beta}})^{\alpha}$, $x>0$;
- Marshall–Olkin Weibull distribution [10] with survival function $\bar{F}(x)=((1-\mathrm{e}^{-(\lambda x)^{\beta}})/(1-(1-\alpha)\,\mathrm{e}^{-(\lambda x)^{\beta}}))$ , $x>0$.

For illustrative purposes, we consider three data sets. For each data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method, and with these obtained estimates, we obtain the values of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) as well as Kolmogorov–Smirnov statistic and the corresponding $p$-value. The AIC and BIC values are given by $-2\log\hat{L}+2r$ and $-2\log\hat{L}+r\log n$, respectively, where $\hat{L}$ is the value of the likelihood function for obtained estimates of the unknown parameters, $r$ is the number of the estimated parameters and $n$ is the sample size.

*Data Set 1:* As first example, we consider the real data set from Proschan [11]. These data consist of 213 observations on the number of successive failures of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. The results for these data are presented in Table 1. We observe that the GEE distribution is a competitive distribution compared with other three-parameter distributions. In fact, based on the values of the AIC and BIC criteria as well as the value of the KS-statistic and the corresponding $p$-value, we observe that the GEE distribution provides the best fit for these data among all the models considered. In Figure 2, we have plotted probability plots for all considered distributions for these data.

*Data Set 2:* As second example, we consider the real data set from Doksum [12]. These data consist of 58 survival times (in days) of guinea pigs that were assigned to a treatment group that received a dose of tubercle bacilli. The results for these data are presented in Table 2. From these results we can observe that gamma, exponentiated Weibull, beta exponential and GEE distributions provide smallest AIC and BIC values and hence best fits for these data among all the models considered. In Figure 3, we have plotted probability plots for all considered distributions for these data.

*Data Set 3:* As third example, we consider the real data set from Colorado Climate Center, Colorado State University (http://ulysses.atmos.colostate.edu). These data consist of 100 annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado, from 1900 through 1999: 239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241. The results for these data are presented in Table 3. From these results, we can observe that the GEE distribution provides a very good fit to these data along with exponential Weibull and beta exponential distributions. In Figure 4, we have plotted probability plots for all considered distributions for these data.

Table 1. Maximum-likelihood estimates, AIC and BIC values, Kolmogorov–Smirnov statistics and $p$-values for the data on the number of successive failures of the air conditioning system.

| Distribution | Estimates | | | AIC | BIC | K–S | $p$-value |
|---|---|---|---|---|---|---|---|
| Gamma($\nu$, $\lambda$) | 0.9216 | 0.0099 | | 2360.6 | 2367.3 | 0.0623 | 0.3808 |
| Weibull($\lambda$, $\beta$) | 0.0112 | 0.9245 | | 2359.2 | 2365.9 | 0.0509 | 0.6384 |
| MOW($\lambda$, $\alpha$, $\beta$) | 0.0049 | 0.2215 | 1.2176 | 2354.4 | 2364.4 | 0.0395 | 0.8944 |
| EW($\alpha$, $\lambda$, $\beta$) | 2.5923 | 0.0344 | 0.5778 | 2355.2 | 2365.3 | 0.0389 | 0.9041 |
| BE($a$, $b$, $\lambda$) | 0.9217 | 4.8078 | 0.0021 | 2362.6 | 2372.7 | 0.0583 | 0.4644 |
| GEE($\alpha$, $\lambda$, $\delta$) | 2.7723 | 0.0003 | 11.6263 | 2354.4 | 2364.5 | 0.0357 | 0.9487 |

Note: MOW, Marshall–Olkin Weibull; EW, Exponentiated Weibull and BE, beta-exponential.

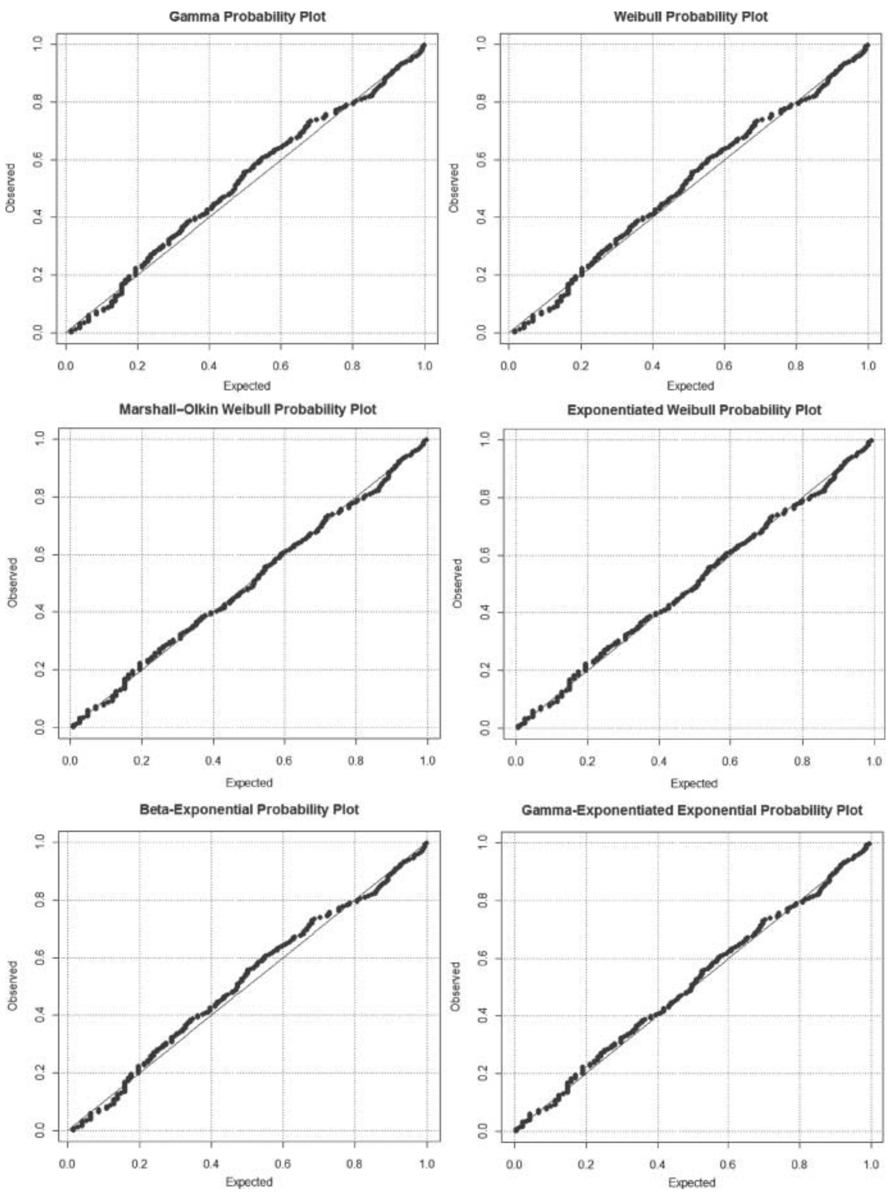


Figure 2. Probability Plot for the data set of Proschan [11].

Table 2. Maximum-likelihood estimates, AIC and BIC values, Kolmogorov–Smirnov statistics and $p$-values for the data on survival times (in days) of guinea pigs that were assigned to a treatment group that received a dose of tubercle bacilli.

| Distribution | Estimates | | | AIC | BIC | K–S | $p$-value |
|---|---|---|---|---|---|---|---|
| Gamma($\nu$, $\lambda$) | 4.7238 | 0.0195 | | 706.9 | 711.0 | 0.0878 | 0.7299 |
| Weibull($\lambda$, $\beta$) | 0.0036 | 2.2130 | | 712.5 | 716.6 | 0.1030 | 0.5357 |
| MOW($\lambda$, $\alpha$, $\beta$) | 0.0020 | 0.0642 | 3.4222 | 708.3 | 714.5 | 0.0858 | 0.7542 |
| EW($\alpha$, $\lambda$, $\beta$) | 17.7170 | 0.0228 | 0.7412 | 707.1 | 713.3 | 0.0618 | 0.9699 |
| BE($a$, $b$, $\lambda$) | 14.1776 | 0.4114 | 0.0211 | 706.8 | 713.0 | 0.0509 | 0.9964 |
| GEE($\alpha$, $\lambda$, $\delta$) | 13.9053 | 0.0212 | 0.4110 | 706.8 | 713.0 | 0.0533 | 0.9936 |

Note: MOW, Marshall–Olkin Weibull; EW, Exponentiated Weibull and BE, beta-exponential.

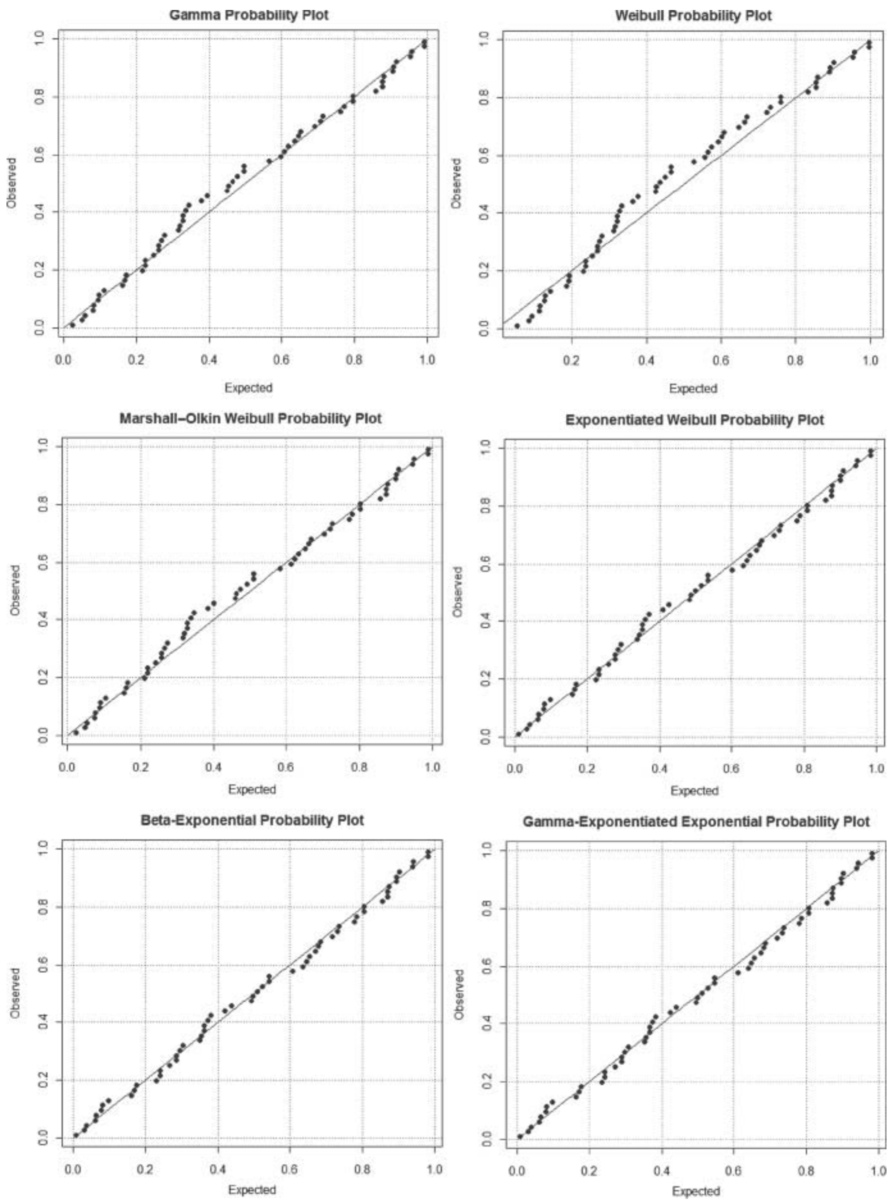


Figure 3. Probability Plot for the data set of Doksum [12].

Table 3. Maximum-likelihood estimates, AIC and BIC values, Kolmogorov–Smirnov statistics and $p$-values for the data on annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado from 1900 through 1999.

| Distribution | Estimates | | | AIC | BIC | K–S | $p$-value |
|---|---|---|---|---|---|---|---|
| Gamma($\nu$, $\lambda$) | 5.2763 | 0.0300 | | 1141.9 | 1147.2 | 0.0619 | 0.8376 |
| Weibull($\lambda$, $\beta$) | 0.0050 | 2.2509 | | 1156.2 | 1161.4 | 0.0973 | 0.2995 |
| MOW($\lambda$, $\alpha$, $\beta$) | 0.0024 | 0.0249 | 3.8444 | 1140.1 | 1148.0 | 0.0561 | 0.9119 |
| EW($\alpha$, $\lambda$, $\beta$) | 174.4058 | 0.2216 | 0.4842 | 1136.3 | 1144.1 | 0.0407 | 0.9964 |
| BE($a$, $b$, $\lambda$) | 34.1184 | 0.3147 | 0.0390 | 1135.5 | 1143.3 | 0.0347 | 0.9997 |
| GEE($\alpha$, $\lambda$, $\delta$) | 33.7857 | 0.0390 | 0.3147 | 1135.5 | 1143.3 | 0.0347 | 0.9997 |

Note: MOW, Marshall–Olkin Weibull; EW, Exponentiated Weibull and BE, beta-exponential.

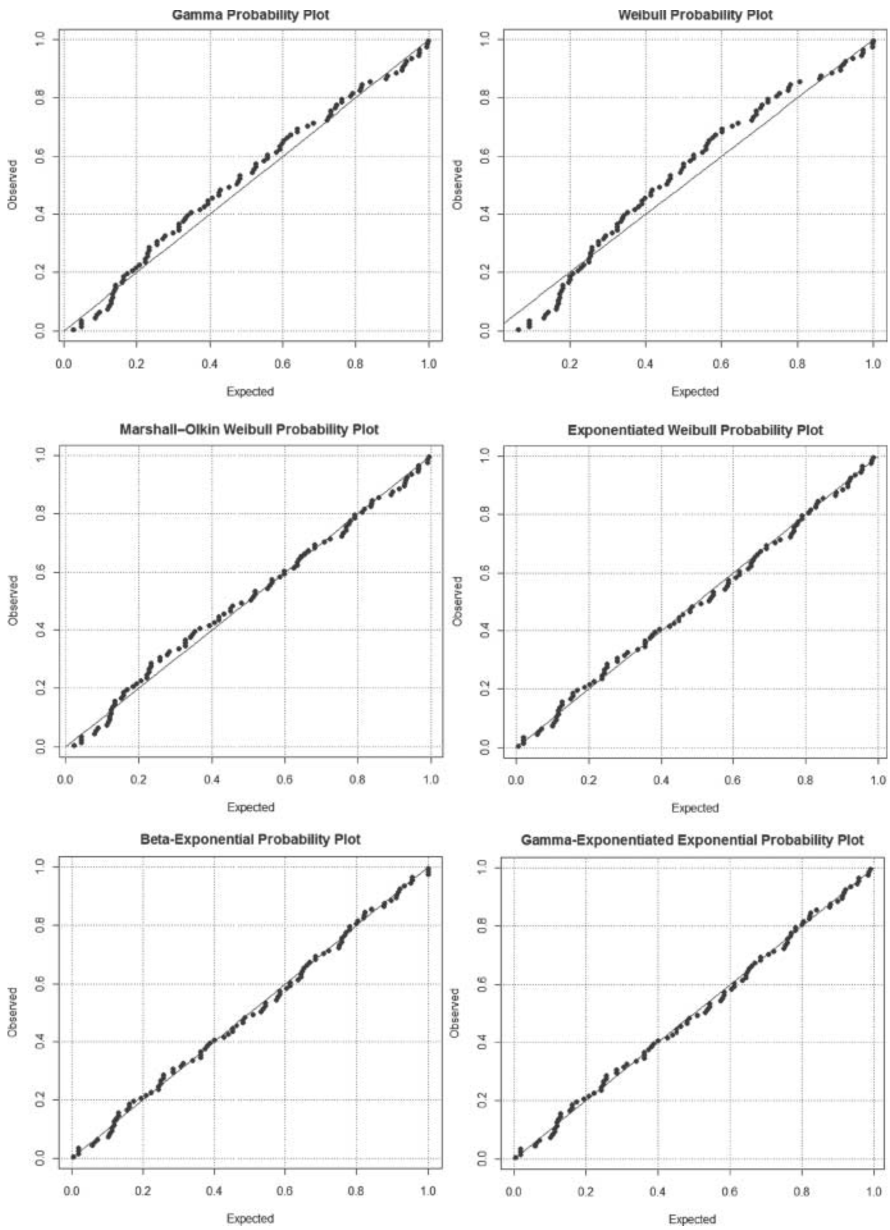


Figure 4. Probability plot for the data set of annual maximum precipitation.

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