The Zografos–Balakrishnan–G Family of Distributions: Mathematical Properties and Applications

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Abstract

For any continuous baseline G distribution, Zografos and Balakrishnan (2009) proposed a generalized gamma-generated distribution (denoted here with the prefix "gamma-G") with an extra positive parameter. They studied some of its mathematical properties and presented some special cases. Here, we provide a comprehensive treatment of general mathematical properties of gamma-G distributions. We discuss estimation of the model parameters by maximum likelihood and give an application to a real data set. We also introduce bivariate generalizations.
Keywords: Estimation; Gamma distribution; Generating function; Gamma-G distribution; Mean Deviation; Moment.

1 Introduction

Zografos and Balakrishnan (2009) and Ristic and Balakrishnan (2011) proposed a family of univariate distributions generated by gamma random variables. For any baseline cumulative distribution function (cdf) $G(x)$, and $x \in \mathbb{R}$, they defined the gamma-G distribution with probability density function (pdf) $f(x)$ and cdf $F(x)$ given by

$$f(x) = \frac{1}{\Gamma(a)} \left\{ -\log[1 - G(x)] \right\}^{a-1} g(x) \tag{1}$$

and

$$F(x) = \frac{\gamma\left(a, -\log\left[1 - G(x)\right]\right)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log[1-G(x)]} t^{a-1} \exp(-t) dt, \tag{2}$$

respectively, for $a > 0$, where $g(x) = dG(x)/dx$, $\Gamma(a) = \int_0^\infty t^{a-1}\, \mathrm{e}^{-\mathrm{t}} \mathrm{dt}$ denotes the gamma function, and $\gamma(a, z) = \int_0^z t^{a-1}\, \mathrm{e}^{-\mathrm{t}} \mathrm{dt}$ denotes the incomplete gamma function. The corresponding hazard rate function (hrf) is

$$h(x)=\frac{\{-\log[1-G(x)]\}^{a-1}\,g(x)}{\Gamma\left(a,-\log\left[1-G(x)\right]\right)},\tag{3}$$

where $\Gamma(a,z)=\int_z^\infty t^{a-1}\,\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ denotes the complementary incomplete gamma function. The gamma-G distribution has the same parameters of the G distribution plus an additional shape parameter $a>0$. If X is a random variable with pdf, (1), we write $X\sim$ gamma-$G(a)$. Each new gamma-G distribution can be obtained from a specified G distribution. For $a=1$, the G distribution is a basic exemplar of the gamma-G distribution with a continuous crossover towards cases with different shapes (for example, a particular combination of skewness and kurtosis).

Zografos and Balakrishnan (2009) and Ristic and Balakrishnan (2011) presented several motivations for gamma-G family of distributions: if $X_{L(1)},X_{L(2)},\ldots,X_{L(n)}$ are lower record values from a sequence of independent random variables with common pdf $g(\cdot)$ then the pdf of the nth lower record value takes the form (1); if Z is a gamma random variable with unit scale parameter and shape parameter $a>0$ then $X=F^{-1}(\exp(Z))$ has the pdf (1); and, if Z is a log-gamma random variable then $X=F^{-1}(\exp\{-\exp(Z)\})$ has the pdf (1).

However, very little is known in terms of general mathematical properties of (1) and (2). Zografos and Balakrishnan (2009) derived specific formulae for moments of special gamma-G models which hold only for natural a, a general expression for Shannon entropy and also provided a maximum entropy characterization. But the expressions given involve unevaluated integrals and complicated terms of the form $E[\log\{f(F^{-1}(1-\exp(-Z)))\}]$. No general properties are derived in Ristic and Balakrishnan (2011).

The aim of this paper is to derive all possible mathematical properties of (1) and (2) in the most simple, explicit and general forms. We obtain general expressions for: shape and asymptotic properties of (1), (2) and (3), quantile function, ordinary and incomplete moments, moment generating function (mgf), characteristic function, mean deviations, Bonferroni and Lorenz curves, asymptotic distribution of the extreme values, Shannon entropy, Rényi entropy, reliability and some properties of order statistics.

The rest of the paper is organized as follows. In Section 2, we present some new distributions. A range of mathematical properties of (1) is derived in Sections 3 to 12. Estimation by the method of maximum likelihood – including the case of censoring and the Fisher information matrix – is presented in Section 13. An application to a real data set is illustrated in Section 14. Some bivariate generalizations of (1) and (2) and their properties are explored in Section 15. Finally, some conclusions and future work are noted in Section 16.

2 Special gamma-G distributions

The gamma-G family of density functions (1) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. Here, we present and study some special cases of this family because it extends several widely-known distributions in the literature. The density (1) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

2.1 Gamma-normal distribution

The gamma-normal (GN) distribution is defined from (1) by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the normal $N(\mu,\sigma^2)$ distribution. Its density function is given by

$$f_{\mathcal{GN}}(x)=\frac{1}{\Gamma(a)}\Big\{-\log\Big[1-\Phi\Big(\frac{x-\mu}{\sigma}\Big)\Big]\Big\}^{a-1}\phi\Big(\frac{x-\mu}{\sigma}\Big),\tag{4}$$

where $x\in\mathbb{R}$, $\mu\in\mathbb{R}$ is a location parameter, $\sigma>0$ is a scale parameter, $a>0$ is a shape parameter, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively.

A random variable with density (4) is denoted by $X \sim \mathrm{GN}(a, \mu, \sigma^2)$. For $\mu = 0$ and $\sigma = 1$, we obtain the standard GN distribution. Further, the GN distribution with $a = 1$ coincides with the normal distribution. Plots of the GN density function for selected parameter values are given in Figure 1.

(a) (b) (c)

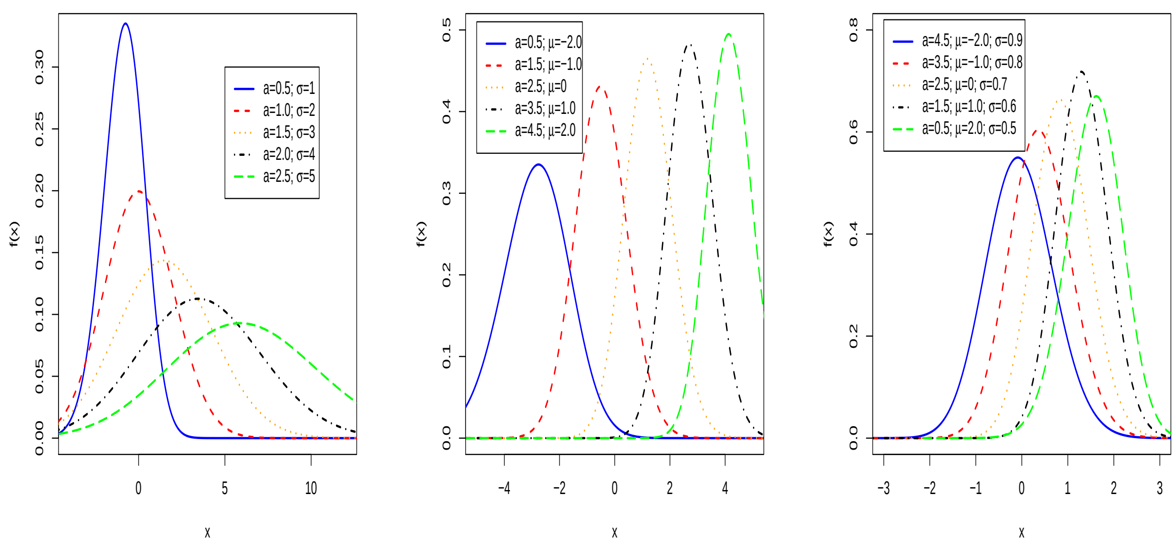

Figure 1: The GN density function for some parameter values. (a) For $\mu = 0$; (b) For $\sigma = 1$.

2.2 Gamma-Weibull distribution

Taking $G(x)$ to be a Weibull cumulative distribution with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$, say $G(x) = 1 - \exp\{-(\beta x)^\alpha\}$, it follows the gamma-Weibull (GW) density function (for $x > 0$)

$$f_{\mathcal{GW}}(x) = \frac{\alpha\,\beta^{\alpha a}}{\Gamma(a)}\,x^{a\alpha-1}\,\exp\{-(\beta x)^\alpha\}. \tag{5}$$

Equation (5) is important because it extends many distributions previously considered in the literature. In fact, it is identical to the generalized gamma distribution (Stacy, 1962). The Weibull distribution (with parameters β and α) is a basic exemplar for $a = 1$, whereas the gamma distribution follows as a special case when $\alpha = 1$. The half-normal distribution is obtained for $a = 3$ and $\alpha = 2$. In addition, the log-normal distribution is a limiting special case when a tends to infinity. Figure 2 illustrates some of the possible shapes of the density function (5).

The cdf and hrf corresponding to (5) are

$$F_{\mathcal{GW}}(x) = \frac{\gamma[a, (\beta\,x)^\alpha]}{\Gamma(a)}$$

and

$$h_{\mathcal{GW}}(x) = \frac{\alpha\,\beta^{a\,\alpha}\,x^{a\alpha-1}\exp\{-(\beta x)^\alpha\}}{\Gamma(a)\left\{1 - \frac{\gamma[a,(\beta\,x)^\alpha]}{\Gamma(a)}\right\}}, \tag{6}$$

respectively.

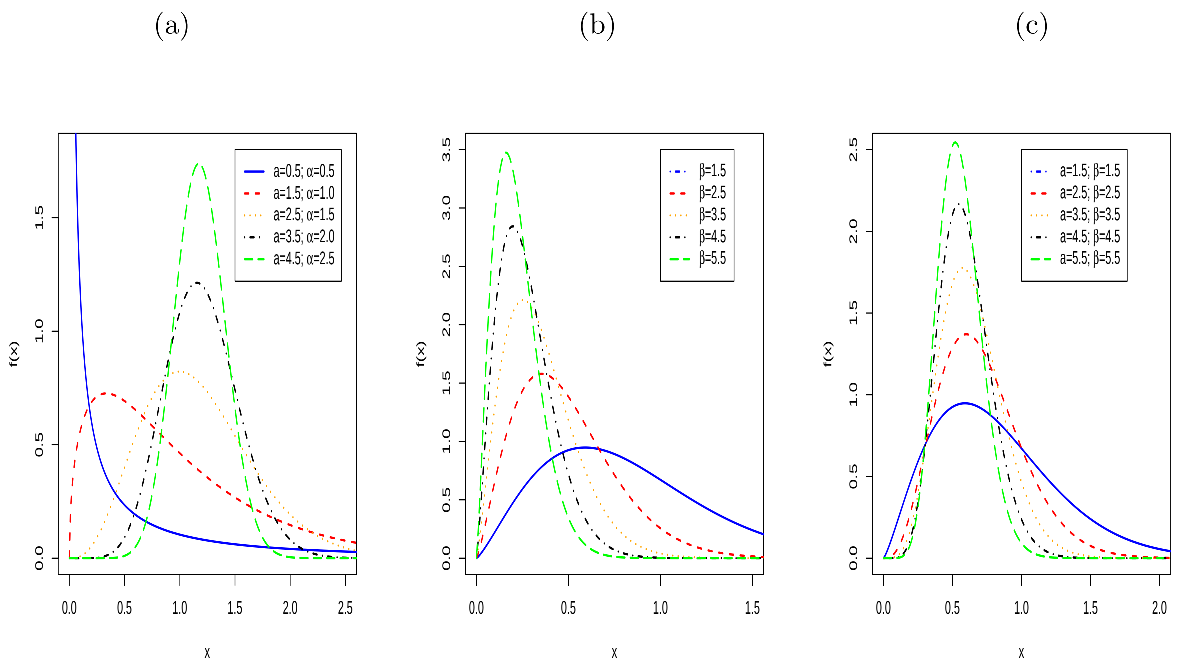

Figure 2: The GW density function: (a) For $\beta = 1.5$; (b) For $a = 1.5$ and $\alpha = 1.5$; (c) For $\alpha = 1.5$.

2.3 Gamma-Gumbel distribution

Consider the Gumbel distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, where the pdf and cdf (for $x \in \mathbb{R}$) are

$$g(x) = \frac{1}{\sigma} \exp\left\{\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right\}$$

and

$$G(x) = 1 - \exp\left\{- \exp\left(\frac{x-\mu}{\sigma}\right)\right\},$$

respectively. The mean and variance are equal to $\mu - \gamma\sigma$ and $\pi^2\sigma^2/6$, respectively, where γ is the Euler's constant ($\gamma \approx 0.57722$). Inserting these expressions into (1) yields the gamma-Gumbel (GGu) density function

$$f_{\mathcal{G}\mathcal{G}u}(x) = \frac{1}{\sigma\Gamma(a)} \exp\left\{(a-1)\left(\frac{x-\mu}{\sigma}\right) + \left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right\}, \tag{7}$$

where $x, \mu \in \mathbb{R}$ and $a, \sigma > 0$. The Gumbel distribution corresponds to $a = 1$. Plots of (7) for selected parameter values are given in Figure 3.

2.4 Gamma-log-normal distribution

Let $G(x)$ be the log-normal distribution with cdf

$$G(x) = 1 - \Phi\left(\frac{-\log(x) + \mu}{\sigma}\right)$$

for $x > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$. The gamma-log-normal (GLN) density function (for $x > 0$) reduces to

$$f_{\mathcal{G}\mathcal{L}\mathcal{N}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma\,\Gamma(a)\,x} \exp\left\{-\frac{1}{2}\left[\frac{\log(x)+\mu}{\sigma}\right]^2\right\}\left\{-\log\left[\Phi\left(\frac{-\log(x)+\mu}{\sigma}\right)\right]\right\}^{a-1}.$$

(a) (b) (c)

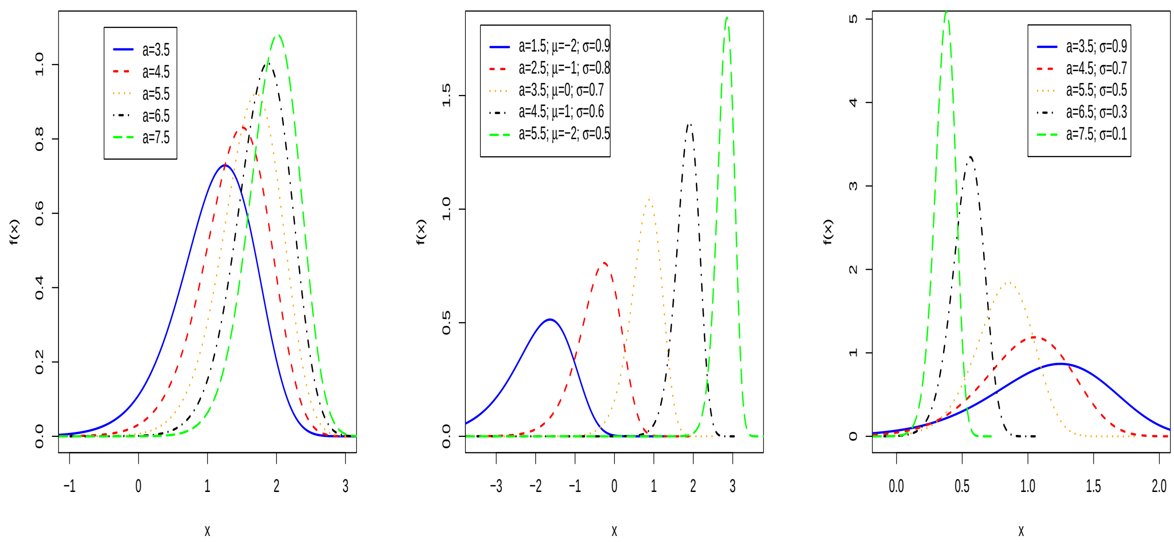

Figure 3: The GGu density function for some parameter values: (a) $\mu = 0$ and $\sigma = 1$; (c) $\mu = 1.0$.

For $a = 1$, we obtain the log-normal distribution. Figure 4 illustrates some possible shapes of the GLN density function for selected parameter values.

(a) (b) (c)

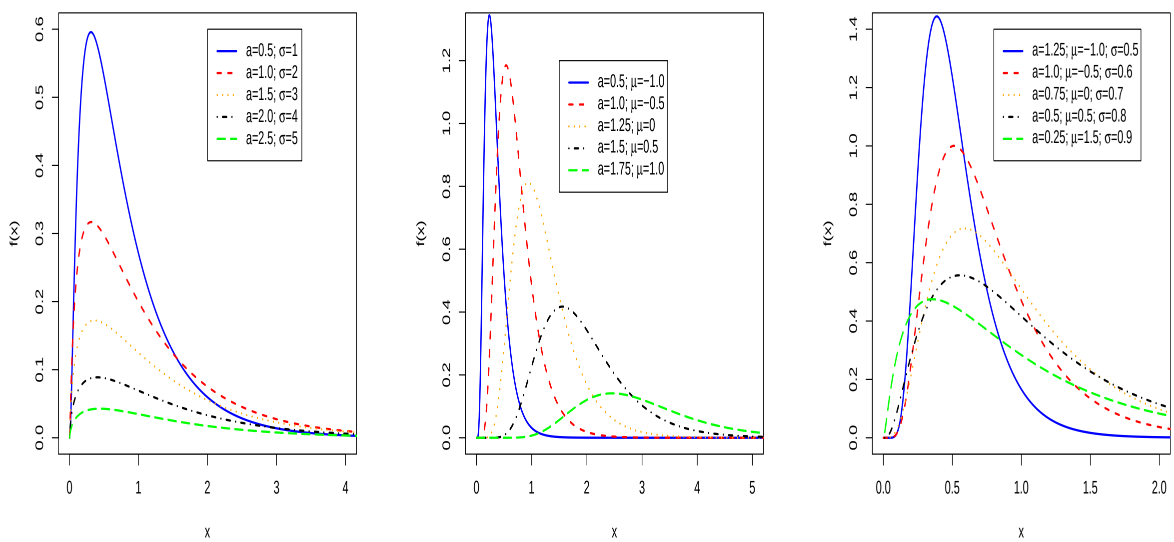

Figure 4: The GLN density function for some parameter values: (a) $\mu = 0$; (b) $\sigma = 1.0$.

2.5 Gamma-log-logistic distribution

The pdf and cdf of the log-logistic (LL) distribution are (for $x, \alpha, \beta > 0$)

$$g(x) = \frac{\beta}{\alpha^\beta}\, x^{\beta-1} \left[1 + \left(\frac{x}{\alpha}\right)\right]^{-2},$$

and

$$G(x) = 1 - \left[1 + \left(\frac{x}{\alpha}\right)^\beta\right]^{-1}.$$

Inserting these expressions into (1) gives the gamma-log-logistic (GLL) density function (for $x > 0$)

$$f_{\mathcal{GLL}}(x) = \frac{\beta}{\alpha^\beta\, \Gamma(a)}\, x^{\beta-1} \left[1 + \left(\frac{x}{\alpha}\right)\right]^{-2} \left\{\log\left[1 + \left(\frac{x}{\alpha}\right)^\beta\right]\right\}^{a-1}.$$

The LL distribution is obtained for $a = 1$. Plots of the GLL density function for selected parameter values are given in Figure 5.

(a) (b) (c)

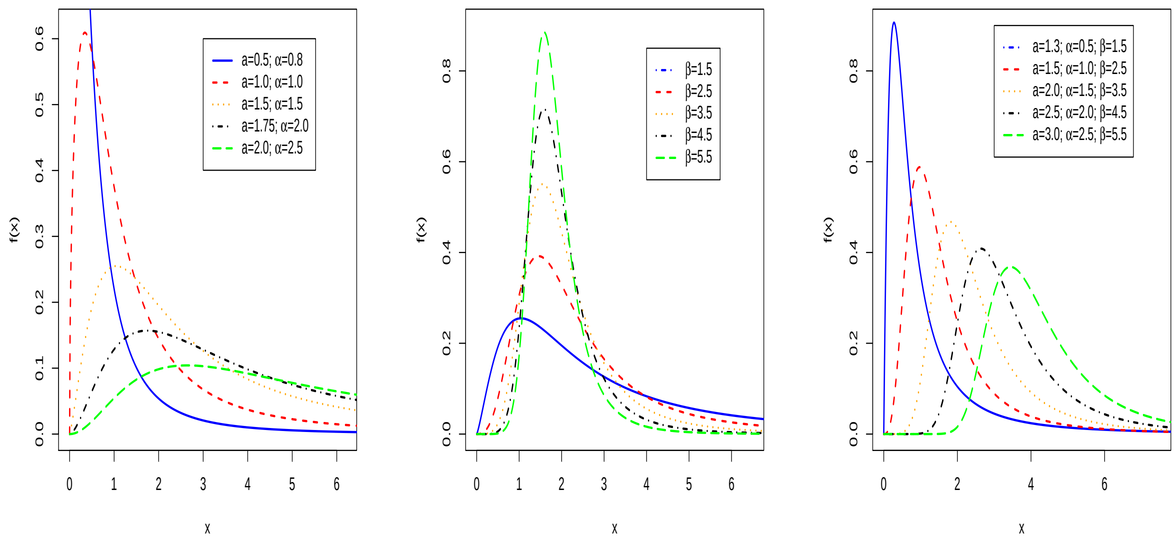

Figure 5: The GLL density function for some parameter values: (a) $\beta = 1.5$; (b) $a = 1.5$ and $\alpha = 1.5$.

3 Expansions for pdf and cdf

Some useful expansions for (1) and (2) can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G distribution with parameter $a > 0$, say $X \sim \text{exp-}G(a)$, if its pdf and cdf are

$$h_a(x) = aG^{a-1}(x)g(x) \tag{8}$$

and

$$H_a(x) = G^a(x), \tag{9}$$

respectively. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) for exponentiated Weibull, Gupta *et al.* (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for exponentiated Gumbel, Kakde and Shirke (2006) for exponentiated lognormal, and Nadarajah and Gupta (2007) for exponentiated gamma distributions. We note that for $a > 1$ and $a < 1$ and for larger values of x, the multiplicative factor $aG(x)^{a-1}$ is greater and smaller than one, respectively. The reverse assertion is also true for smaller values of x. The latter immediately implies that the ordinary moments associated with the density $h_a(x)$ are strictly larger (smaller) than those associated with the density $g(x)$ when $a > 1$ ($a < 1$).

The binomial coefficient generalized to real arguments is given by $\binom{x}{y} = \Gamma(x+1)/[\Gamma(y+1)\Gamma(x-y+1)]$. For any real parameter $a > 0$, the following formula holds (http:// functions.wolfram.com/ ElementaryFunctions/ Log/ 06/ 01/ 04/ 03/)

$$\{-\log\left[1-G(x)\right]\}^{a-1} = (a-1)\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{(a-1-j)}\,G(x)^{a+k-1}, \tag{10}$$

where the constants $p_{j,k}$ can be calculated recursively by

$$p_{j,k} = k^{-1}\sum_{m=1}^{k}[k - m(j+1)]\,c_m\,p_{j,k-m} \tag{11}$$

for $k = 1, 2, \ldots$, $p_{j,0} = 1$ and $c_k = (-1)^{k+1}\,(k+1)^{-1}$.

For a real parameter $a > 0$, we define

$$b_k = \frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{(a-1-j)},$$

and then (1) can be expressed as

$$f(x) = \sum_{k=0}^{\infty} b_k\,h_{a+k}(x), \tag{12}$$

where $h_{a+k}(x)$ denotes the pdf of the exp-$G(a+k)$ distribution. The corresponding (2) can be expressed as

$$F(x) = \sum_{k=0}^{\infty} b_k\,H_{a+k}(x), \tag{13}$$

where $H_{a+k}(x)$ denotes the cdf of the exp-$G(a+k)$ distribution. So, several properties of the gamma-G distribution can be obtained by knowing those of the exp-G distribution, see, for example, Mudholkar *et al.* (1995), Gupta and Kundu (2001) and Nadarajah and Kotz (2006), among others.

4 Asymptotes and shapes

The asymptotes of (1), (2) and (3) as $x \to 0, \infty$ are given by

$$f(x) \sim \frac{1}{\Gamma(a)}G^{a-1}(x)\;g(x)$$

as $x \to 0$,

$$f(x) \sim \frac{1}{\Gamma(a)}\,\{-\log[1-G(x)]\}^{a-1}\,g(x)$$

as $x \to \infty$,

$$F(x) \sim \frac{1}{\Gamma(a+1)} \{-\log[1-G(x)]\}^a \tag{14}$$

as $x \to 0$,

$$1 - F(x) \sim \frac{1}{\Gamma(a)} \{-\log[1-G(x)]\}^{a-1} [1-G(x)] \tag{15}$$

as $x \to \infty$,

$$h(x) \sim \frac{1}{\Gamma(a)} G^{a-1}(x)\ g(x)$$

as $x \to 0$, and

$$f(x) \sim \frac{g(x)}{1-G(x)}$$

as $x \to \infty$. So, the gamma-G density function behaves like the G hrf for very large x, whereas its hrf is proportional to the exp-G density for very small x.

The shapes of (1) and (3) can be described analytically. The critical points of the pdf are the roots of the equation:

$$\frac{g'(x)}{g(x)} = \frac{(1-a)\, g(x)}{\log[1-G(x)]\ [1-G(x)]}. \tag{16}$$

There may be more than one root to (16). If $x = x_0$ is a root of (16) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\lambda(x) = \frac{(a-1)\, g^2(x)\, \{2 - a + \log[1-G(x)]\}}{\log^2[1-G(x)]\, [1-G(x)]^2}.$$

The critical points of the hrf are the roots of the equation:

$$\frac{g'(x)}{g(x)} = (1-a)\, \frac{g(x)}{\log[1-G(x)]\, [1-G(x)]} - \frac{\{-\log[1-G(x)]\}^{a-1}\, g(x)}{\Gamma\left(a, -\log\left[1-G(x)\right]\right)}. \tag{17}$$

There may be more than one root to (17). If $x = x_0$ is a root of (17) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\begin{aligned}
\lambda(x) \quad = \quad & \frac{(a-1) g'(x)}{\log[1-G(x)]\, [1-G(x)]} + \frac{(a-1) g^2(x)\, \{1 + \log[1-G(x)]\}}{\log^2[1-G(x)]\, [1-G(x)]^2} + \frac{g^{''}(x)}{g(x)} - \left[\frac{g^{'}(x)}{g(x)}\right]^2 \\
& + \frac{\{-\log[1-G(x)]\}^{a-2}}{\Gamma\left(a, -\log\left[1-G(x)\right]\right)} \left\{ \frac{(a-1) g^2(x)}{1-G(x)} - g^{'}(x) \log\left[1-G(x)\right] \right\} \\
& - \frac{g^2(x)\, \{-\log\left[1-G(x)\right]\}^{2a-2}}{\Gamma^2\left(a, -\log\left[1-G(x)\right]\right)}.
\end{aligned}$$

5 Quantile function

Here and henceforth, we use an equation by Gradshteyn and Ryzhik (2000, Section 0.314) for a power series raised to a positive integer n

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{18}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily determined from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m\,(n+1) - i]\, a_m\, c_{n,i-m}, \tag{19}$$

where $c_{n,0} = a_0^n$. The coefficient $c_{n,i}$ can be calculated from $c_{n,0}, \ldots, c_{n,i-1}$ and hence from the quantities $a_0, \ldots, a_i$. In fact, $c_{n,i}$ can be given explicitly in terms of the coefficients a_i, although it is not necessary for programming numerically our expansions in any algebraic or numerical software.

Further, from now on, let X denote a gamma-G random variable, say $X \sim$gamma-$G(a)$, given by (1) and (2). The simulation of X is very easy: if V is a gamma random variable with shape parameter a and unit scale parameter then

$$X = G^{-1}\left\{1 - \exp\left(-V\right)\right\}$$

will also be a gamma-G random variable. Further, inverting $F(x) = u$, we obtain

$$F^{-1}(u) = G^{-1}\left\{1 - \exp\left[-Q^{-1}(a, 1-u)\right]\right\} \tag{20}$$

for $0 < u < 1$, where $Q^{-1}(a, u)$ is the inverse function of $Q(a, x) = 1 - \gamma(a, x)/\Gamma(a)$, see http:// functions.wolfram.com/ GammaBetaErf/ InverseGammaRegularized/ for details. The asymptotes of (20) can be determined using known properties of $Q^{-1}(a, u)$. Using http:// functions.wolfram.com/ GammaBetaErf/ InverseGammaRegularized/ 06/ 02/ 01/, one can see that

$$F^{-1}(u) \sim G^{-1}\left\{1 - \exp\left[-(1-a)W_{-1}\left(-\frac{(1-u)^{1/(a-1)}\Gamma 1/(a-1)(a)}{a-1}\right)\right]\right\}$$

as $u \to 0$, where $W_{-1}(\cdot)$ denotes the product log function. Using http:// functions.wolfram.com/ GammaBetaErf/ InverseGammaRegularized/ 06/ 01/ 03/ 0001/, one can see that

$$F^{-1}(u) \sim G^{-1}\left\{1 - \exp\left[-(1-u)^{1/a}\,\Gamma^{1/a}(a+1) + \frac{(1-u)^2\,\Gamma^{2/a}(a+1)}{(a+1)}\right]\right\}$$

as $u \to 1$.

Quantiles of interest can be obtained from (20) by substituting appropriate values for u. In particular, the median of X is

$$\mathrm{Median}(X) = G^{-1}\left\{1 - \exp\left[-Q^{-1}(a, 1/2)\right]\right\}.$$

One can also use (20) for simulating gamma-G variates: if U is a uniform random variable on the unit interval $[0, 1]$ then

$$X = G^{-1}\left\{1 - \exp\left[-Q^{-1}(a, 1-U)\right]\right\}$$

will be a gamma-G random variable.

If V is a gamma random variable with shape parameter a and unit scale parameter, the quantile function of V, say $Q_V(u)$, admits a power series expansion given by

$$Q_V(u) = \sum_{i=0}^{\infty} m_i\, \left[\Gamma(a+1)\, u\right]^{i/a},$$

where $m_0 = 0$, $m_1 = 1$ and any coefficient m_{i+1} (for $i \geq 1$) is determined by the cubic recurrence equation

$$\begin{aligned} m_{i+1} \;\; = \;\; & \frac{1}{i\,(a+i)}\left\{\sum_{r=1}^{i}\sum_{s=1}^{i-s+1} s\,(i-r-s+2)\, m_r\, m_s\, m_{i-r-s+2}\right. \\ & \left. -\Delta(i)\sum_{r=2}^{i} r\,\left[r - a - (1-a)(i+2-r)\right]\, m_r m_{i-r+2}\right\}, \end{aligned}$$

where $\Delta(i)=0$ if $i<2$ and $\Delta(i)=1$ if $i\geq 2$. The first few coefficients are $m_2=1/(a+1)$, $m_3=(3a+5)/[2(a+1)^2(a+2)],\ldots$ We use the fact that $m_0=0$ and define (for $i=0,1,2\ldots$) $m_i^\star=m_{i+1}\,\Gamma(a+1)^{(i+1)/2}$ to express the gamma-G quantile function $F^{-1}(u)=G^{-1}(1-\exp\{-Q_V(u)\})$ as

$$F^{-1}(u)=G^{-1}\left(1-\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[\sum_{i=0}^{\infty}m_i^\star\,u^{(i+1)/a}\right]^k\right).$$

Using equations (18) and (19), we have

$$F^{-1}(u)=G^{-1}\left(1-\sum_{k,i=0}^{\infty}\frac{(-1)^k\,d_{k,i}}{k!}\;u^{(i+k)/a}\right), \tag{21}$$

where $d_{k,0}=m_0^{\star k}$ and, for $i=1,2\ldots,$

$$d_{k,i}=(i\,m_0^\star)^{-1}\sum_{j=1}^{i}[j(k+1)-i]\,m_j^\star\,d_{k,i-k}.$$

Hence, equation (21) reveals that the gamma-G quantile function can be expressed as the G quantile function applied to a double power series. This expansion holds for any gamma-G model.

6 Moments

From now on, let $Y_k\sim$exp-$G(a+k)$. A first formula for the nth moment of X can be obtained from (12) as

$$E(X^n)=\sum_{k=0}^{\infty}b_k\,E(Y_k^n). \tag{22}$$

Expressions for moments of several exponentiated distributions are given by Nadarajah and Kotz (2006), which can be used to produce $E(X^n)$.

A second formula for $E(X^n)$ can be obtained from (22) in terms of the baseline quantile function $Q_G(x)=G^{-1}(x)$. We obtain

$$E(X^n)=\sum_{k=0}^{\infty}(a+k)\,b_k\,\tau(n,a+k-1), \tag{23}$$

where the integral

$$\tau(n,a)=\int_{-\infty}^{\infty}x^n\,G(x)^a\,g(x)dx$$

can be expressed in terms of the G quantile function

$$\tau(n,a)=\int_0^1 Q_G(u)^n\,u^a du. \tag{24}$$

The ordinary moments of several gamma-G distributions can be calculated directly from equations (23) and (24). Here, we give three examples. For the gamma-standard logistic, where $G(x)=\{1+\exp(-x)\}^{-1}$, using a result from Prudnikov *et al.* (1986, Section 2.6.13, equation 4), we have

$$E\left(X^n\right)=\sum_{k=0}^{\infty}(a+k)\,b_k\left(\frac{\partial}{\partial t}\right)^n B(t+a+k,1-t)\Big|_{t=0},$$

where $B(a,b)=\int_0^1 t^{a-1}\,(1-t)^{b-1}dt$ is the beta function. The moments of the gamma-exponential (with parameter $\lambda>0$) are

$$E(X^n)=n!\,\lambda^n\sum_{k,j=0}^{\infty}\frac{(-1)^{n+j}\,(a+k)\binom{a+k-1}{j}\,b_k}{(j+1)^{n+1}}.$$

For the gamma-Pareto distribution, where $G(x)=1-(1+x)^{-\nu}$ and $\nu>0$, we have

$$E(X^n)=\sum_{k,j=0}^{\infty}(-1)^{n+j}\,(a+k)\binom{n}{j}\,B(a+k-1,1-j\nu^{-1})\,b_k.$$

For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X is calculated as

$$m_n(y)=E\left(X^n|X<y\right)=\sum_{k=0}^{\infty}(a+k)b_k\int_0^{G(y)}Q_G(u)^n u^{a+k-1}du.$$

The last integral can be computed for most baseline G distributions.

Let $\mu'_n=E(X^n)$ be the nth ordinary moment of X calculated from (22) or (23). The nth descending factorial moment of X is

$$\mu'_{(n)}=E(X^{(r)})=E\left[X(X-1)\times\cdots\times(X-r+1)\right]=\sum_{k=0}^{r}s(r,k)\,\mu'_k,$$

where

$$s(r,k)=(k!)^{-1}\left[\frac{d^k}{dx^k}x^{(r)}\right]_{x=0}$$

is the Stirling number of the first kind which counts the number of ways to permute a list of r items into k cycles. So, we can obtain the factorial moments from the ordinary moments given before.

Further, the central moments (μ_r) and cumulants (κ_r) of X can be calculated as

$$\mu_r=\sum_{k=0}^{r}(-1)^k\binom{r}{k}\mu_1'^k\,\mu'_{r-k}\qquad\text{and}\qquad\kappa_r=\mu'_r-\sum_{k=1}^{r-1}\binom{r-1}{k-1}\kappa_k\,\mu'_{r-k},$$

respectively, where $\kappa_1=\mu'_1$. Then, $\kappa_2=\mu'_2-\mu_1'^2$, $\kappa_3=\mu'_3-3\mu'_2\mu'_1+2\mu_1'^3$, $\kappa_4=\mu'_4-4\mu'_3\mu'_1-3\mu_2'^2+12\mu'_2\mu_1'^2-6\mu_1'^4$, etc. The skewness $\gamma_1=\kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2=\kappa_4/\kappa_2^2$ follow from the second, third and fourth cumulants. Other kinds of moments such L-moments may also be obtained in closed-form, but we consider only the previous moments for reasons of space.

7 Generating functions

Here, we provide three formulae for the mgf $M(t)=E[\exp(t\,X)]$ of X. Clearly, the first one is simply

$$M(t)=\sum_{k=0}^{\infty}\frac{\mu'_k}{k!}\,t^k, \tag{25}$$

where $\mu'_k = E(X^k)$ is calculated from (22) or (23). A second formula for $M(t)$ comes from (12) as

$$M(t) = \sum_{k=0}^{\infty} b_k \, M_k(t), \tag{26}$$

where $M_k(t)$ is the mgf of Y_k. Hence, $M(t)$ can be immediately determined from the generating function of the exp-G distribution.

A third formula for $M(t)$ can be derived from (12) as

$$M(t) = \sum_{i=0}^{\infty} (a+k) \, b_k \, \rho(t, a+k-1), \tag{27}$$

where

$$\rho(t,a) = \int_{-\infty}^{\infty} \exp(tx) \, G(x)^a \, g(x) dx$$

can be calculated from the baseline quantile function $Q_G(x) = G^{-1}(x)$ by

$$\rho(t,a) = \int_0^1 \exp\left\{t \, Q_G(u)\right\} u^a du. \tag{28}$$

We can obtain the mgf of several gamma-G distributions directly from equations (27) and (28). For example, the mgf's of the gamma-exponential (with parameter λ), gamma-standard logistic and gamma-Pareto (with parameter $\nu > 0$) are

$$M(t) = \sum_{k=0}^{\infty} (a+k) \, B(a+k, 1-\lambda t) \, b_k,$$

$$M(t) = \sum_{k=0}^{\infty} (a+k) \, B(t+a+k, 1-t) \, b_k,$$

and

$$M(t) = \exp(-t) \sum_{k,r=0}^{\infty} \frac{(a+k) \, B(a+k, 1-r\nu^{-1}) \, b_k}{r!} \, t^r,$$

respectively.

Clearly, three representations for the characteristic function (chf) $\phi(t) = E[\exp(\mathrm{i}tX)]$ of the gamma-G distribution are derived from (25)-(27) by $\phi(t) = M(\mathrm{i}\,t)$, where $\mathrm{i} = \sqrt{-1}$.

8 Mean deviations

The mean deviations about the mean ($\delta_1(X) = E(|X - \mu'_1|)$) and about the median ($\delta_2(X) = E(|X - M|)$) of X can be expressed as

$$\delta_1(X) = 2\mu'_1 \, F\left(\mu'_1\right) - 2m_1\left(\mu'_1\right) \qquad \text{and} \qquad \delta_2(X) = \mu'_1 - 2m_1(M), \tag{29}$$

respectively, where $\mu'_1 = E(X)$, $M = Median(X)$ is the median given in Section 5, $F(\mu'_1)$ is easily calculated from the cdf in (2) and $m_1(z) = \int_{-\infty}^{z} x f(x) dx$ is the first incomplete moment.

In this section, we provide two alternative ways to compute $\delta_1(X)$ and $\delta_2(X)$. A general equation for $m_1(z)$ can be derived from (12) as

$$m_1(z) = \sum_{k=0}^{\infty} b_k \, J_k(z), \tag{30}$$

where

$$J_k(z) = \int_{-\infty}^{z} x\, h_{a+k}(x)dx. \tag{31}$$

Equation (31) is the basic quantity to compute the mean deviations of the exp-G distributions. Hence, the mean deviations in (29) depend only on the mean deviations of the exp-G distribution. So, alternative representations for $\delta_1(X)$ and $\delta_2(X)$ are

$$\delta_1(X) = 2\mu_1' F\left(\mu_1'\right) - 2\sum_{k=0}^{\infty} b_k J_k\left(\mu_1'\right) \qquad \text{and} \qquad \delta_2(X) = \mu_1' - 2\sum_{k=0}^{\infty} b_k J_k(M).$$

A simple application of (30) and (31) comes for the GW distribution. The exponentiated Weibull with parameter $a+k$ has pdf (for $x>0$) given by

$$h_{a(k+1)}(x) = c\,(a+k)\,\beta^c\, x^{c-1} \exp\left\{-(\beta x)^c\right\}\, \left[1-\exp\left\{-(\beta x)^c\right\}\right]^{a+k-1}$$

and then

$$\begin{aligned} J_k(z) &= c\,(a+k)\,\beta^c \int_0^z x^c \exp\left\{-(\beta x)^c\right\}\left[1-\exp\left\{-(\beta x)^c\right\}\right]^{a+k-1} dx \\ &= c\,(a+k)\,\beta^c \sum_{r=0}^{\infty}(-1)^r \binom{(a+k)-1}{r} \int_0^z x^c \exp\left\{-(r+1)(\beta x)^c\right\} dx. \end{aligned}$$

The last integral is given by the incomplete gamma function and then the mean deviations for the GW distribution can be calculated immediately from

$$m_1(z) = \beta^{-1} \sum_{k,r=0}^{\infty} \frac{(-1)^r\,(a+k)\binom{a+k-1}{r} b_k}{(r+1)^{1+1/c}} \gamma\left(1+c^{-1}, (r+1)(\beta z)^c\right).$$

A second general formula for $m_1(z)$ can be derived by setting $u = G(x)$ in (12)

$$m_1(z) = \sum_{k=0}^{\infty}(a+k)\, b_k\, T_k(z), \tag{32}$$

where

$$T_k(z) = \int_0^{G(z)} Q_G(u)\, u^{a+k-1} du \tag{33}$$

is a simple integral defined from the baseline quantile function $Q_G(u) = G^{-1}(u)$.

In a similar way, the mean deviations of any gamma-G distribution can be computed from equations (32)-(33). For example, the mean deviations of the gamma-exponential (with parameter λ), gamma-standard logistic and gamma-Pareto (with parameter $\nu > 0$) are calculated immediately (by using the generalized binomial expansion) from the functions

$$T_k(z) = \lambda^{-1}\, \Gamma(a+k-1) \sum_{j=0}^{\infty} \frac{(-1)^j\, \left\{1-\exp\left(-j\lambda z\right)\right\}}{\Gamma(a+k-1-j)\,(j+1)!},$$

$$T_k(z) = \frac{1}{\Gamma(k)} \sum_{j=0}^{\infty} \frac{(-1)^j\, \Gamma(a+k+j)\, \left\{1-\exp(-jz)\right\}}{(j+1)!}$$

and

$$T_k(z) = \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(-1)^j \binom{a+k}{j} \binom{j}{r}}{(1 - r\nu)} z^{1-r\nu},$$

respectively.

Applications of these equations can be given to obtain Bonferroni and Lorenz curves defined for a given probability π by

$$B(\pi) = \frac{T(q)}{\pi \mu_1'} \qquad \text{and} \qquad L(\pi) = \frac{T(q)}{\mu_1'},$$

respectively, where $\mu_1' = E(X)$ and $q = G^{-1}\{1 - \exp[-Q^{-1}(a, 1-\pi)]\}$ is the quantile gamma-G function at π (see Section 5).

9 Extreme values

If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the mean of a random sample from (1), then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \to \infty} \frac{1 - G(t + xh(t))}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using (15), we note that

$$\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(t + xh(t))}{1 - F(t)} &= \lim_{t \to \infty} \frac{[1 - G(t + xh(t))] \{-\log[1 - G(t + xh(t))]\}^{a-1}}{[1 - G(t)] \{-\log[1 - G(t)]\}^{a-1}} \\
&= \lim_{t \to \infty} \frac{1 - G(t + xh(t))}{1 - G(t)} \left\{ \frac{1 - G(t)}{1 - G(t + xh(t))} \frac{(d/dt)[1 - G(t + xh(t))]}{(d/dt)[1 - G(t)]} \right\}^{a-1} \\
&= \lim_{t \to \infty} \frac{1 - G(t + xh(t))}{1 - G(t)} \\
&= \exp(-x)
\end{aligned}$$

for every $x \in (-\infty, \infty)$. So, it follows by Leadbetter *et al.* (1987, Chapter 1) that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} \Pr\{a_n (M_n - b_n) \le x\} = \exp\{-\exp(-x)\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a $\beta > 0$, such that

$$\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{\beta}$$

for every $x > 0$. But, using (15), we note that

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} &= \lim_{t \to \infty} \frac{[1 - G(tx)] \{- \log [1 - G(tx)]\}^{a-1}}{[1 - G(t)] \{- \log [1 - G(t)]\}^{a-1}} \\
&= \lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} \left\{ \frac{1 - G(t)}{1 - G(tx)} \frac{(d/dt) [1 - G(tx)]}{(d/dt) [1 - G(t)]} \right\}^{a-1} \\
&= \lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} \\
&= x^{\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows by Leadbetter *et al.* (1987, Chapter 1) that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n \to \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp \left(-x^{\beta}\right)$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a $\alpha > 0$, such that

$$\lim_{t \to 0} \frac{G(tx)}{G(t)} = x^{\alpha}$$

for every $x < 0$. But, using (14), we note that

$$
\begin{aligned}
\lim_{t \to 0} \frac{F(tx)}{F(t)} &= \lim_{t \to 0} \frac{[1 - G(tx)] \{- \log [1 - G(tx)]\}^{a-1}}{[1 - G(t)] \{- \log [1 - G(t)]\}^{a-1}} \\
&= \lim_{t \to 0} \left\{ \frac{1 - G(t)}{1 - G(tx)} \frac{(d/dt) G(tx)}{(d/dt) G(t)} \right\}^{a-1} \\
&= \lim_{t \to 0} \left\{ \frac{G(tx)}{G(t)} \right\}^{a-1} \\
&= x^{(a-1)\beta}.
\end{aligned}
$$

So, it follows by Leadbetter *et al.* (1987, Chapter 1) that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} \Pr \{a_n (M_n - b_n) \leq x\} = \exp\{-(-x)^{(a-1)\alpha}\}$$

for some suitable norming constants $a_n > 0$ and b_n.

The same argument applies to min domains of attraction. That is, F belongs to the same min domain of attraction as that of G.

10 Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies (Shannon, 1951; Rényi, 1961). The Rényi entropy of a random variable with pdf $f(\cdot)$ is defined as

$$I_R(\gamma) = \frac{1}{1 - \gamma} \log \int_0^{\infty} f^{\gamma}(x) dx$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy of a random variable X is defined by $E[- \log f(X)]$. It is the particular case of the Rényi entropy for $\gamma \uparrow 1$.

Here, we derive expressions for the Rényi and Shannon entropies when X is a gamma-G random variable. By using (10), we can write

$$\{-\log[1-G(x)]\}^{\gamma a-\gamma} = (\gamma a-\gamma)\sum_{k=0}^{\infty}\binom{k-\gamma a+\gamma}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\,p_{j,k}}{[\gamma(a-1)-j]}\,G(x)^{[\gamma(a-1)+k]}.$$

So,

$$\begin{aligned}\int_0^\infty g^\gamma(x)dx &= \int_0^\infty \frac{1}{\Gamma^\gamma(a)}\,\{-\log[1-G(x)]\}^{\gamma a-\gamma}\,g^\gamma(x)dx\\ &= \frac{1}{\Gamma^\gamma(a)}\sum_{k=0}^{\infty}\binom{k-\gamma a+\gamma}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\,p_{j,k}}{[\gamma(a-1)-j]}\,I_k\end{aligned}$$

where I_k comes from the baseline distribution as

$$I_k=\int_0^\infty G(x)^{[\gamma(a-1)+k]}\,g^\gamma(x)dx.$$

Hence, the Rényi entropy of X is given by

$$I_R(\gamma)=-\frac{\gamma\log\Gamma(a)}{1-\gamma}+\frac{1}{1-\gamma}\log\left\{\sum_{k=0}^{\infty}\binom{k-\gamma a+\gamma}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\,p_{j,k}}{[\gamma(a-1)-j]}\,I_k\right\}. \tag{34}$$

The Shannon entropy can be obtained by limiting $\gamma\uparrow 1$ in (34). However, it is easier to derive an expression for it from first principles. Using the series expansion for $\log(1-z)$, we can write

$$\begin{aligned}E\left[-\log f(X)\right] &= \log\Gamma(a)+(1-a)E\left[\log\{-\log[1-G(X)]\}\right]-E\left[\log g(X)\right]\\ &= \log\Gamma(a)+(1-a)E\left[\log\left\{\sum_{r=1}^{\infty}\frac{G^r(X)}{r}\right\}\right]-E\left[\log g(X)\right]\\ &= \log\Gamma(a)+(1-a)E\left[\log\left\{1+\sum_{r=2}^{\infty}\frac{G^{r-1}(X)}{r}\right\}\right]+E\left[\log G(X)\right]-E\left[\log g(X)\right]\\ &= \log\Gamma(a)+(1-a)\left[\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}E\left\{G(X)\sum_{r=0}^{\infty}\frac{G^r(X)}{(r+2)}\right\}^j\right]\\ &\quad +E\left[\log G(X)\right]-E\left[\log g(X)\right].\end{aligned}$$

From equation (18), we have

$$\left\{G(X)\sum_{r=0}^{\infty}\frac{G^r(X)}{(r+2)}\right\}^j=\sum_{k=0}^{\infty}e_{j,r}\,G^{r+j}(X),$$

where

$$e_{j,r}=2r^{-1}\sum_{m=1}^{r}\frac{[m(j+1)-r]}{m+2}\,e_{r,r-m}$$

for $r=1,2,\ldots$ and $e_{j,0}=2^{-j}$. Then,

$$\begin{aligned}E\left[-\log f(X)\right] &= \log\Gamma(a)+(1-a)\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\sum_{r=0}^{\infty}e_{j,r}\,E\left[G^{r+j}(X)\right]\\ &\quad +E\left[\log G(X)\right]-E\left[\log g(X)\right].\end{aligned} \tag{35}$$

The three expectations in (35) can be easily evaluated numerically for a given $G(\cdot)$ and $g(\cdot)$. Using (12), they can also be represented as

$$\begin{aligned} E\left[G^{r+j}(X)\right] &= \sum_{k=0}^{\infty}(a+k)\,b_k \int_0^{\infty} G^{a+r+j+k-1}(x)\,g(x)dx \\ &= \sum_{k=0}^{\infty} \frac{(a+k)\,b_k}{(a+r+j+k)}, \end{aligned}$$

$$\begin{aligned} E\left[\log G(X)\right] &= \sum_{k=0}^{\infty}(a+k)\,b_k \int_0^{\infty} \log[G(x)]\,G^{a+k-1}(x)g(x)dx \\ &= -\sum_{k=0}^{\infty} \frac{b_k}{a+k}, \end{aligned}$$

and

$$E\left[\log g(X)\right] = \sum_{k=0}^{\infty}(a+k)\,b_k \int_0^{\infty} \log[g(x)]\,G^{a+k-1}(x)g(x)dx,$$

respectively. The last of these representations can also be expressed in terms of the baseline quantile function, $Q_G(u) = G^{-1}(u)$, as

$$E\left[\log g(X)\right] = \sum_{k=0}^{\infty}(a+k)\,b_k \int_0^{1} \log\left[g\left(Q_G(u)\right)\right]\,u^{a+k-1}du.$$

The last integral can be calculated for most baseline distributions using a power series expansion for $Q_G(u)$.

11 Reliability

Here, we derive the reliability, $R = \Pr(X_2 < X_1)$, when $X_1 \sim$ gamma-$G(a_1)$ and $X_2 \sim$ gamma-$G(a_2)$ are independent random variables. Probabilities of this form have many applications especially in engineering concepts. Let f_i denote the pdf of X_i and F_i denote the cdf of X_i. By using the representations, (12) and (13), we can write

$$R = \sum_{j,k=0}^{\infty} c_{jk} \int_0^{\infty} H_{a_2+j}(x)\,h_{a_1+k}(x)dx = \sum_{j,k=0}^{\infty} c_{jk}\,R_{jk}, \tag{36}$$

where

$$c_{jk} = \frac{\binom{j+1-a_2}{k}}{(a_2+j)\,\Gamma\left(a_2-1\right)}\,\frac{\binom{k+1-a_1}{k}}{(a_1+k)\,\Gamma\left(a_1-1\right)}\left[\sum_{i=0}^{j}\frac{(-1)^{i+j}\binom{j}{i}\,p_{i,j}}{(a_2-1-i)}\right]\left[\sum_{i=0}^{k}\frac{(-1)^{i+k}\binom{k}{i}\,p_{i,k}}{(a_1-1-i)}\right],$$

and $R_{jk} = \Pr(Y_j < Y_k)$ is the reliability between the independent random variables $Y_j \sim$ exp-$G(a_2+j)$ and $Y_k \sim$ exp-$G(a_1+k)$. Hence, the reliability for gamma-G random variables is a linear combination of those for exp-G random variables. In the particular case $a_1 = a_2$, (36) reduces to $R = 1/2$.

12 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the gamma-G distribution. Let $X_{i:n}$ denote the ith order statistic. From (12) and (13), it is clear that the pdf of $X_{i:n}$ is

$$\begin{aligned}
f_{i:n}(x) &= K\, f(x)\, F^{i-1}(x)\, \{1 - F(x)\}^{n-i} \\
&= K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F^{j+i-1}(x) \\
&= K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\sum_{r=0}^{\infty} b_r\, (a+r)\, G(x)^{a+r-1}\, g(x)\right] \left[\sum_{k=0}^{\infty} b_k\, G(x)^{a+k}\right]^{j+i-1},
\end{aligned}$$

where $K = n!/[(i-1)!\,(n-i)!]$. Using (18) and (19), we can write

$$\left[\sum_{k=0}^{\infty} b_k\, G(x)^{a+k}\right]^{j+i-1} = \sum_{k=0}^{\infty} f_{j+i-1,k}\; G(x)^{a(j+i-1)+k},$$

where $f_{j+i-1,0} = b_0^{j+i-1}$ and

$$f_{j+i-1,k} = (k\, b_0)^{-1} \sum_{m=1}^{k} [m(j+i) - k]\, b_m\, f_{j+i-1,k-m}.$$

Hence,

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{r,k=0}^{\infty} m_{j,r,k}\, h_{a(j+i)+r+k}(x), \tag{37}$$

where

$$m_{j,r,k} = \frac{(-1)^j\, n!}{(i-1)!\,(n-i-j)!\,j!}\; \frac{(a+r)\, b_r\, f_{j+i-1,k}}{[a(j+i)+r+k]}.$$

Equation (37) is the main result of this section. It reveals that the pdf of the gamma-G order statistics is a triple linear combination of exp-G density functions. So, several mathematical quantities of the gamma-G order statistics like ordinary, incomplete and factorial moments, mgf, mean deviations and several others can be obtained from those quantities of gamma-G distributions. Clearly, the cdf of $X_{i:n}$ can be expressed as

$$F_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{r,k=0}^{\infty} m_{j,r,k}\, H_{a(j+i)+r+k}(x).$$

13 Maximum likelihood estimation

Here, we consider estimation of the unknown parameters of the gamma-G distribution by the method of maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from (1). Let $\boldsymbol{\Theta}$ be a q-vector parameter vector specifying $G(\cdot)$. The log-likelihood function $\log L = \log L\,(a, \boldsymbol{\Theta})$ is

$$\log L = \sum_{i=1}^{n} \log g\,(x_i) + (a-1) \sum_{i=1}^{n} \log\{-\log[1 - G\,(x_i)]\} - n \log \Gamma(a). \tag{38}$$

The first derivatives of $\log L$ with respect to the parameters a and $\boldsymbol{\Theta}$ are:

$$\frac{\partial \log L}{\partial a} = \sum_{i=1}^{n} \log \left\{-\log \left[1 - G\left(x_i\right)\right]\right\} - n\psi(a), \tag{39}$$

$$\frac{\partial \log L}{\partial \boldsymbol{\Theta}} = \sum_{i=1}^{n} \frac{\partial g\left(x_i\right)/\partial \theta}{g\left(x_i\right)} + (a-1)\sum_{i=1}^{n} \frac{\partial G\left(x_i\right)/\partial \theta}{\log\left\{1 - G\left(x_i\right)\right\}\left\{1 - G\left(x_i\right)\right\}}, \tag{40}$$

where $\psi(a) = d\log\Gamma(a)/da$ is the digamma function. The maximum likelihood estimates (MLEs) of $(a, \boldsymbol{\Theta})$, say $(\widehat{a}, \widehat{\boldsymbol{\Theta}})$, are the simultaneous solutions of the equations $\partial \log L/\partial a = 0$ and $\partial \log L/\partial \boldsymbol{\Theta} = \mathbf{0}$.

Maximization of (38) can be performed by using well established routines like nlm or optimize in the R statistical package. Our numerical calculations showed that the surface of (38) was smooth for given smooth functions $g(\cdot)$ and $G(\cdot)$. The routines were able to locate the maximum in all cases and for different starting values. However, to easy the computations it is useful to have reasonable starting values. These can be obtained, for example, by the method of moments. For $r = 1, \ldots, q+1$, let $m_r = n^{-1}\sum_{i=1}^{n} x_i^r$ denote the first $q+1$ sample moments. Equating these moments with the theoretical versions given in Section 6, we have $m_r = E(X^r)$, for $r = 1, \ldots, q+1$. These equations can be solved simultaneously to obtain the moments estimates.

For interval estimation of $(a, \boldsymbol{\Theta})$ and tests of hypothesis, one requires the Fisher information matrix. We can express the observed Fisher information matrix of $(\widehat{a}, \widehat{\boldsymbol{\Theta}})$ as

$$\mathbf{I} = \begin{pmatrix} I_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{12} & \mathbf{I}_{22} \end{pmatrix},$$

where

$$\begin{aligned}
I_{11} &= \frac{\partial^2 \log L}{\partial \widehat{a}^2} = n\psi^{\prime}\left(\widehat{a}\right), \\
\mathbf{I}_{12} &= \frac{\partial^2 \log L}{\partial \widehat{a}\partial \widehat{\boldsymbol{\Theta}}} = -\sum_{i=1}^{n} \frac{\partial G\left(x_i\right)/\partial \widehat{\boldsymbol{\Theta}}}{\log\left\{1 - G\left(x_i\right)\right\}\left\{1 - G\left(x_i\right)\right\}}, \\
\mathbf{I}_{22} &= \frac{\partial^2 \log L}{\partial \widehat{\boldsymbol{\Theta}}^2} = -\sum_{i=1}^{n} \frac{\partial^2 g\left(x_i\right)/\partial \widehat{\boldsymbol{\Theta}}^2}{g\left(x_i\right)} + \sum_{i=1}^{n} \left[\frac{\partial g\left(x_i\right)/\partial \widehat{\boldsymbol{\Theta}}}{g\left(x_i\right)}\right]^2 \\
&\quad +(1-a)\sum_{i=1}^{n} \frac{\partial^2 G\left(x_i\right)/\partial \widehat{\boldsymbol{\Theta}}^2}{\log\left\{1 - G\left(x_i\right)\right\}\left\{1 - G\left(x_i\right)\right\}} \\
&\quad +(1-a)\sum_{i=1}^{n} \frac{\left[\partial G\left(x_i\right)/\partial \widehat{\boldsymbol{\Theta}}\right]^2\left[1 + \log\left\{1 - G\left(x_i\right)\right\}\right]}{\log^2\left\{1 - G\left(x_i\right)\right\}\left\{1 - G\left(x_i\right)\right\}^2}.
\end{aligned}$$

For large n, the distribution of $\sqrt{n}(\widehat{a} - a, \widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})$ approximates to a $(q+1)$ variate normal distribution with zero means and variance-covariance matrix $\mathbf{I}^{-1}$. The properties of $(\widehat{a}, \widehat{\boldsymbol{\Theta}})$ can be derived based on this normal approximation.

Often with lifetime data, one encounters censoring. There are different forms of censoring: type I censoring, type II censoring, etc. Here, we consider the general case of multicensored data: there are n subjects of which

- n_0 are known to have the values $t_1, \ldots, t_{n_0}$.
- n_1 are known to belong to the interval $[s_{i-1}, s_i]$, $i = 1, \ldots, n_1$.
- n_2 are known to have exceeded r_i, $i = 1, \ldots, n_2$ but not observed any longer.

Note that $n = n_0 + n_1 + n_2$. Note too that type I censoring and type II censoring are contained as particular cases of multicensoring.

In the case of multicensoring, the log-likelihood function is:

$$\log L\left(a, \mathbf{\Theta}\right) = \sum_{i=1}^{n_0} \log f\left(t_i\right) + \sum_{i=1}^{n_1} \log\left[F\left(s_i\right) - F\left(s_{i-1}\right)\right] + \sum_{i=1}^{n_2} \log\left[1 - F\left(r_i\right)\right], \tag{41}$$

where $f(\cdot)$ and $F(\cdot)$ are given by (1) and (2), respectively. The first derivatives of the log-likelihood function with respect to the parameters a and $\mathbf{\Theta}$ are:

$$\begin{aligned}\frac{\partial \log L}{\partial a} &= \sum_{i=1}^{n_0} \frac{1}{f\left(t_i\right)} \frac{\partial f\left(t_i\right)}{\partial a} + \sum_{i=1}^{n_1} \frac{1}{F\left(s_i\right) - F\left(s_{i-1}\right)} \left[\frac{\partial F\left(s_i\right)}{\partial a} - \frac{\partial F\left(s_{i-1}\right)}{\partial a}\right] \\ &\quad - \sum_{i=1}^{n_2} \frac{1}{1 - F\left(r_i\right)} \frac{\partial F\left(r_i\right)}{\partial a},\end{aligned} \tag{42}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \mathbf{\Theta}} &= \sum_{i=1}^{n_0} \frac{1}{f\left(t_i\right)} \frac{\partial f\left(t_i\right)}{\partial \mathbf{\Theta}} + \sum_{i=1}^{n_1} \frac{1}{F\left(s_i\right) - F\left(s_{i-1}\right)} \left[\frac{\partial F\left(s_i\right)}{\partial \mathbf{\Theta}} - \frac{\partial F\left(s_{i-1}\right)}{\partial \mathbf{\Theta}}\right] \\ &\quad - \sum_{i=1}^{n_2} \frac{1}{1 - F\left(r_i\right)} \frac{\partial F\left(r_i\right)}{\partial \mathbf{\Theta}}.\end{aligned} \tag{43}$$

The first term in (41) is the same as (38) with (x_i, n) replaced by (t_i, n_0). Also the first terms in (42)-(43) are the same as (39)-(40) with (x_i, n) replaced by (t_i, n_0). So, it is sufficient to find explicit expressions for the partial derivatives in (42)-(43). They are

$$\frac{\partial F(x)}{\partial a} = \frac{1}{\Gamma^2(a)} \frac{\partial}{\partial a} \gamma\left(a, -\log\left[1 - G(x)\right]\right) - \frac{\Gamma^{'}(a)}{\Gamma^2(a)} \gamma\left(a, -\log\left[1 - G(x)\right]\right) \tag{44}$$

and

$$\frac{\partial F(x)}{\partial \mathbf{\Theta}} = \frac{1}{\Gamma(a)} \left\{-\log\left[1 - G(x)\right]\right\}^{a-1} \frac{\partial G(x)}{\partial \mathbf{\Theta}}. \tag{45}$$

The partial derivative of the incomplete gamma function is given by

$$\frac{\partial \gamma(a, x)}{\partial a} = \Gamma'(a) - \Gamma^2(a)\, x^a \,{}_2F_2\left(a, a; a+1, a+1; -x\right) + \gamma(a, x) \log x - \Gamma(a)\psi(a),$$

see http://functions.wolfram.com/GammaBetaErf/Gamma2/20/01/01/0002/, where ${}_2F_2$ (a, b; c, d) is a generalized hypergeometric function.

The MLEs of $(a, \mathbf{\Theta})$, say $(\widehat{a}, \widehat{\mathbf{\Theta}})$, are the simultaneous solutions of the equations $\partial \log L / \partial a = 0$ and $\partial \log L / \partial \mathbf{\Theta} = \mathbf{0}$.

14 Application: Actuarial data

Here, we use a real data set to compare the fits of some gamma-G distributions and those of other sub-models, i.e., the GW, Weibull, GLN, LN, GLL and LL distributions. In each case, the parameters are estimated by maximum likelihood (Section 13) using the subroutine NLMixed in SAS. The data correspond to age of death (in years) of retired women with temporary disabilities, which are incorporated in the Mexican insurance public system and who died during 2004, are given in Balakrishnan *et al.* (2009). The data was provided by the Mexican Institute of Social Security (IMSS) to study the distributional behavior of the mortality of retired people on disability. Table 1 lists the MLEs of the parameters (standard errors between parentheses) and

Table 1: MLEs of the model parameters for the mechanical components data, the corresponding SEs (given in parentheses) and the AIC, CAIC and BIC statistics for actuarial data.

Model	a	α	β	AIC	CAIC	BIC
GW model	2.8675	2.8180	0.0292	2102.3	2102.4	2113.2
	(1.3892)	(0.7496)	(0.0083)			
Weibull model	1	5.0088	0.0192	2108.8	2108.9	2116.0
	-	(0.2215)	(0.0002)			
GLL model	0.3432	57.8222	13.2615	2100.1	2100.2	2111.0
	(0.0683)	(1.3525)	(1.8236)			
LL model	1	47.4080	7.6472	2121.3	2121.4	2128.6
	-	(0.6523)	(0.37777)			
	a	μ	σ			
GLN model	0.1784	4.1464	0.1300	2101.6	2101.7	2112.5
	(0.0223)	(0.0391)	(0.0034)			
LN model	1	3.8430	0.2284	2116.1	2116.2	2123.4
	-	(0.0137)	(0.0097)			

the values of the statistics: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). From the values of these statistics, we note that the GLL model provides a better fit to these data. Further, the GW, GLL and GLN models are much better than the Weibull, LN and LL models respectively.

More information is provided by a visual comparison of the fitted density functions and the histogram of the data. The plots of the fitted GW, GLL, GLN, Weibull, LL and LN density functions and estimated cumulative functions are given in Figure 6. These plots show that the new distributions provide adequate fits.

(a) (b) (c)

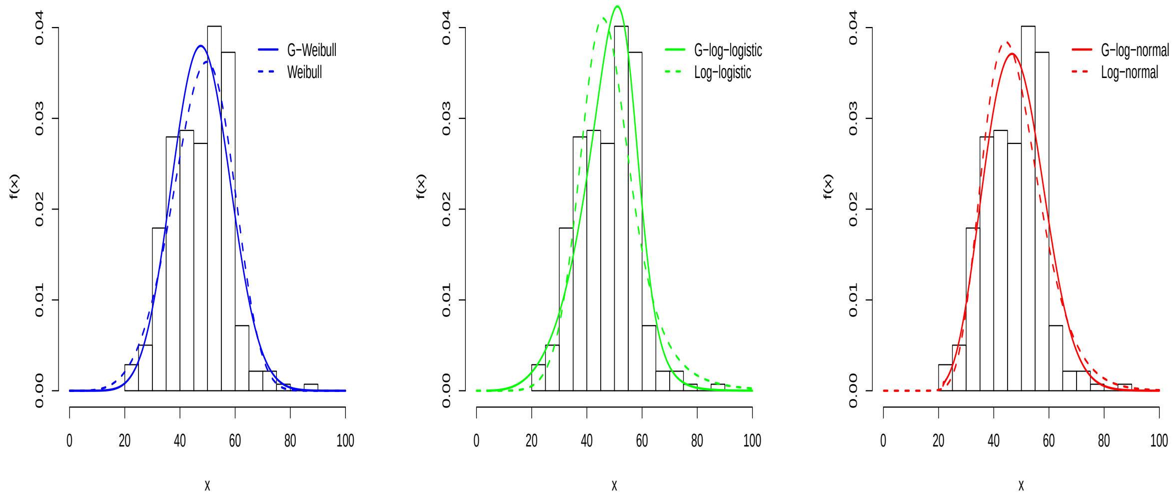

Figure 6: Estimated densities for actuarial data: (a) GW versus Weibull densities; (b) LG versus GLG densities; (c) LN versus GLN densities.

Next, we perform the LR tests, described in Section 13, for a formal test of the need for a third skewness parameter.

Table 2: LR tests.

Models	Hypotheses	Statistic w	p-value
GW vs Weibull	$H_0 : a = 1$ vs $H_1 : H_0$ is false	8.5	0.0036
GLL vs LL	$H_0 : a = 1$ vs $H_1 : H_0$ is false	23.2	<0.0001
GLN vs LN	$H_0 : a = 1$ vs $H_1 : H_0$ is false	16.8	0.000042

A formal test of the need for the third skewness parameter in G-family distributions is based on the LR statistics described in Section 13. Applying these to our three data sets, the results are shown in Table 2. However, for the actuarial data, we reject the null hypotheses of all three LR tests in favor of the GW, GLL and GLN distributions. This gives clear evidence of the potential need for one skewness parameter when modelling real data.

To better visualize the behavior of new models proposed in Figure 7, we can observe it a good fit of the new models.

15 Bivariate generalizations

An immediate bivariate generalization of (2) has the joint cdf specified by

$$F(x, y) = \frac{\gamma\left(a, -\log\left[1 - G(x, y)\right]\right)}{\Gamma(a)} \tag{46}$$

for $x > 0$, $y > 0$ and $a > 0$. The joint pdf is:

$$f(x, y) = \frac{1}{\Gamma(a)} \left\{-\log\left[1 - G(x, y)\right]\right\}^{a-2} A(x, y) \tag{47}$$

for $x > 0$, $y > 0$ and $a > 0$, where

$$A(x, y) = \frac{a-1}{1 - G(x, y)} \frac{\partial G(x, y)}{\partial x} \frac{\partial G(x, y)}{\partial y} - \log\left[1 - G(x, y)\right] \frac{\partial^2 G(x, y)}{\partial x \partial y}.$$

The marginal cdfs are:

$$F(x) = \frac{\gamma\left(a, -\log\left[1 - G(x)\right]\right)}{\Gamma(a)}$$

and

$$F(y) = \frac{\gamma\left(a, -\log\left[1 - G(y)\right]\right)}{\Gamma(a)}$$

for $x > 0$, $y > 0$ and $a > 0$. The marginal pdfs are:

$$f(x) = \frac{1}{\Gamma(a)} \left\{-\log[1 - G(x)]\right\}^{a-1} g(x)$$

and

$$f(y) = \frac{1}{\Gamma(a)} \left\{-\log[1 - G(y)]\right\}^{a-1} g(y)$$

for $x > 0$, $y > 0$ and $a > 0$. The conditional cdfs are:

$$F(x|y) = \frac{\gamma\left(a, -\log\left[1 - G(x, y)\right]\right)}{\gamma\left(a, -\log\left[1 - G(y)\right]\right)}$$

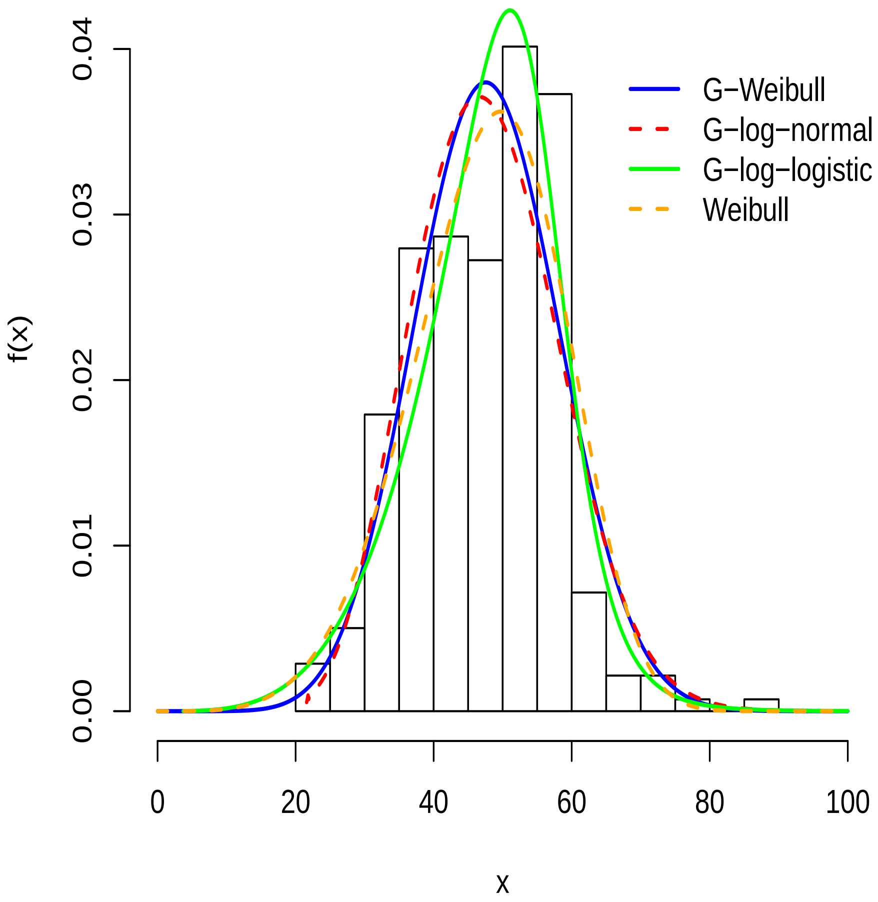

Figure 7: Estimated GW, GLN, GLL and Weibull densities for actuarial data.

and

$$F(y|x) = \frac{\gamma\left(a, -\log\left[1 - G(x, y)\right]\right)}{\gamma\left(a, -\log\left[1 - G(x)\right]\right)}$$

for $x > 0$, $y > 0$ and $a > 0$. The conditional pdfs are:

$$g(x|y) = \frac{\left\{-\log\left[1 - G(x, y)\right]\right\}^{a-2} A(x, y)}{\left\{-\log[1 - G(y)]\right\}^{a-1} g(y)}$$

and

$$g(y|x) = \frac{\left\{-\log\left[1 - G(x, y)\right]\right\}^{a-2} A(x, y)}{\left\{-\log[1 - G(x)]\right\}^{a-1} g(x)}$$

for $x > 0$, $y > 0$ and $a > 0$. Also

$$\frac{\partial F(x, y)}{\partial x} = \frac{1}{\Gamma(a)} \left\{-\log\left[1 - G(x, y)\right]\right\}^{a-1} \frac{\partial G(x, y)}{\partial x}$$

and

$$\frac{\partial F(x,y)}{\partial y} = \frac{1}{\Gamma(a)} \left\{ -\log\left[1 - G(x,y)\right] \right\}^{a-1} \frac{\partial G(x,y)}{\partial y}$$

for $x > 0$, $y > 0$ and $a > 0$.

As in Section 2, it is useful to have series expansions for (46) and (47). First, take bivariate generalizations of (8) and (9) to be given by

$$h_a(x,y) = a\, G^{a-1}(x,y) \frac{\partial^2 G(x,y)}{\partial x \partial y} + a\,(a-1)\, G^{a-2}(x,y) \frac{\partial G(x,y)}{\partial x} \frac{\partial G(x,y)}{\partial y}$$

and

$$H_a(x,y) = G^a(x,y),$$

respectively. Using the Taylor series expansion for the incomplete gamma function (see http:// functions.wolfram.com/ GammaBetaErf/ Gamma2/ 06/ 01/ 04/ 01/ 01/ 0003/) and (10), we can express (46) as:

$$F(x,y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (a+k)\, \omega_{j,k}\, H_{a+j+k}(x,y),$$

where

$$\omega_{j,k} = \binom{j-k-a}{j} \sum_{i=0}^{j} \frac{(-1)^{i+j} \binom{j}{i} p_{i,j}}{(a+k-i)},$$

where $p_{i,j}$ are given by (11). Correspondingly, we can express (47) as:

$$f(x,y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (a+k)\, \omega_{j,k}\, h_{a+j+k}(x,y).$$

These representations can be used to derive expressions for joint moment generating function, joint characteristic function, joint cumulant generating function, product moments and others corresponding to (46).

We now consider estimation of the unknown parameters of (46) by the method of maximum likelihood. Suppose $G(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are parameterized by $\mathbf{\Theta}$. Let (x_1,y_1), (x_2,y_2), ..., (x_n,y_n) be a random sample from (46). Then the log-likelihood function is

$$\log L\,(a, \mathbf{\Theta}) = \sum_{i=1}^{n} \log A\,(x_i, y_i) + (a-2) \sum_{i=1}^{n} \log\left\{ -\log\left[1 - G\,(x_i,y_i)\right] \right\} - n \log \Gamma(a). \qquad (48)$$

The first derivatives with respect to the parameters a and $\mathbf{\Theta}$ are:

$$\frac{\partial \log L}{\partial a} = \sum_{i=1}^{n} \log\left\{ -\log\left[1 - G\,(x_i,y_i)\right] \right\} - n\psi(a) + \sum_{i=1}^{n} \frac{\partial A\,(x_i,y_i)\,/\partial a}{A\,(x_i,y_i)}, \qquad (49)$$

and

$$\frac{\partial \log L}{\partial \mathbf{\Theta}} = (a-2) \sum_{i=1}^{n} \frac{\partial G\,(x_i,y_i)\,/\partial \mathbf{\Theta}}{\log\left\{1 - G\,(x_i,y_i)\right\}\left\{1 - G\,(x_i,y_i)\right\}} + \sum_{i=1}^{n} \frac{\partial A\,(x_i,y_i)\,/\partial \mathbf{\Theta}}{A\,(x_i,y_i)}, \qquad (50)$$

where

$$\frac{\partial A(x,y)}{\partial a} = \frac{1}{\left[1 - G(x,y)\right]} \frac{\partial G(x,y)}{\partial x} \frac{\partial G(x,y)}{\partial y},$$

and

$$\frac{\partial A(x,y)}{\partial \mathbf{\Theta}} = \frac{(a-1)}{[1-G(x,y)]}\left\{\frac{1}{[1-G(x,y)]}\frac{\partial G(x,y)}{\partial \mathbf{\Theta}}\frac{\partial G(x,y)}{\partial x}\frac{\partial G(x,y)}{\partial y} + \frac{\partial G(x,y)}{\partial x \partial \mathbf{\Theta}}\frac{\partial G(x,y)}{\partial y}\right.$$
$$\left. + \frac{\partial G(x,y)}{\partial y \partial \mathbf{\Theta}}\frac{\partial G(x,y)}{\partial x}\right\} + \frac{1}{[1-G(x,y)]}\frac{\partial G(x,y)}{\partial \mathbf{\Theta}}\frac{\partial^2 G(x,y)}{\partial x \partial y}$$
$$- \log\left[1-G(x,y)\right] \frac{\partial^3 G(x,y)}{\partial x \partial y \partial \mathbf{\Theta}}.$$

The MLEs of $(a, \mathbf{\Theta})$, say $(\widehat{a}, \widehat{\mathbf{\Theta}})$, are the simultaneous solutions of the equations $\partial \log L/\partial a = 0$ and $\partial \log L/\partial \mathbf{\Theta} = \mathbf{0}$.

For interval estimation of $(a, \mathbf{\Theta})$ and tests of hypothesis, one requires the Fisher information matrix. We can express the observed Fisher information matrix of $(\widehat{a}, \widehat{\mathbf{\Theta}})$ as

$$\mathbf{I} = \begin{pmatrix} I_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{12} & \mathbf{I}_{22} \end{pmatrix},$$

where

$$I_{11} = -\frac{\partial^2 \log L}{\partial \widehat{a}^2} = n\psi'\left(\widehat{a}\right) - \sum_{i=1}^{n} \frac{1}{\partial A\left(x_i,y_i\right)}\frac{\partial^2 A\left(x_i,y_i\right)}{\partial \widehat{a}^2} + \sum_{i=1}^{n}\left[\frac{1}{\partial A\left(x_i,y_i\right)}\frac{\partial A\left(x_i,y_i\right)}{\partial \widehat{a}}\right]^2,$$

$$\mathbf{I}_{12} = \frac{\partial \log L}{\partial \widehat{a} \partial \widehat{\mathbf{\Theta}}} = -\sum_{i=1}^{n} \frac{\partial G\left(x_i,y_i\right)/\partial \widehat{\mathbf{\Theta}}}{\log\left\{1-G\left(x_i,y_i\right)\right\}\left\{1-G\left(x_i,y_i\right)\right\}} - \sum_{i=1}^{n} \frac{\partial^2 A\left(x_i,y_i\right)/\partial \widehat{a}\partial \widehat{\mathbf{\Theta}}}{A\left(x_i,y_i\right)}$$
$$+ \sum_{i=1}^{n} \frac{\partial A\left(x_i,y_i\right)/\partial \widehat{a} \partial A\left(x_i,y_i\right)/\partial \widehat{\mathbf{\Theta}}}{A^2\left(x_i,y_i\right)},$$

and

$$\mathbf{I}_{22} = \frac{\partial^2 \log L}{\partial \widehat{a} \partial \widehat{\mathbf{\Theta}}^2} = (2-\widehat{a})\sum_{i=1}^{n} \frac{\partial^2 G\left(x_i,y_i\right)/\partial \widehat{\mathbf{\Theta}}^2}{\log\left\{1-G\left(x_i,y_i\right)\right\}\left\{1-G\left(x_i,y_i\right)\right\}}$$
$$+ (2-\widehat{a})\sum_{i=1}^{n} \frac{\left[\partial G\left(x_i,y_i\right)/\partial \widehat{\mathbf{\Theta}}\right]^2\left[1+\log\left\{1-G\left(x_i,y_i\right)\right\}\right]}{\log^2\left\{1-G\left(x_i,y_i\right)\right\}\left\{1-G\left(x_i,y_i\right)\right\}^2}$$
$$- \sum_{i=1}^{n} \frac{\partial^2 A\left(x_i,y_i\right)/\partial \widehat{\mathbf{\Theta}}^2}{A\left(x_i,y_i\right)} + \sum_{i=1}^{n}\left[\frac{1}{\partial A\left(x_i,y_i\right)}\frac{\partial A\left(x_i,y_i\right)}{\partial \widehat{\mathbf{\Theta}}}\right]^2,$$

where

$$\frac{\partial^2 A(x,y)}{\partial a^2} = 0,$$
$$\frac{\partial^2 A(x,y)}{\partial a \partial \mathbf{\Theta}} = \frac{1}{\{1-G(x,y)\}^2}\frac{\partial G(x,y)}{\partial \mathbf{\Theta}}\frac{\partial G(x,y)}{\partial x}\frac{\partial G(x,y)}{\partial y} + \frac{1}{\{1-G(x,y)\}}\frac{\partial 2G(x,y)}{\partial x \partial \mathbf{\Theta}}\frac{\partial G(x,y)}{\partial y},$$

and

$$
\begin{aligned}
\frac{\partial^2 A(x,y)}{\partial \boldsymbol{\Theta}^2} = \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} & \frac{(a-1)}{\{1-G(x,y)\}^2} \left\{ \frac{1}{[1-G(x,y)]} \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial x} \frac{\partial G(x,y)}{\partial y} \right. \\
& \left. + \frac{\partial G(x,y)}{\partial x \partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial y} + \frac{\partial G(x,y)}{\partial y \partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial x} \right\} \\
& + \frac{(a-1)}{[1-G(x,y)]} \left\{ \frac{1}{[1-G(x,y)]^2} \left[\frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \right]^2 \frac{\partial G(x,y)}{\partial x} \frac{\partial G(x,y)}{\partial y} \right. \\
& + \frac{1}{[1-G(x,y)]} \frac{\partial^2 G(x,y)}{\partial \boldsymbol{\Theta}^2} \frac{\partial G(x,y)}{\partial x} \frac{\partial G(x,y)}{\partial y} \\
& + \frac{1}{[1-G(x,y)]} \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \frac{\partial^2 G(x,y)}{\partial x \partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial y} \\
& + \frac{1}{[1-G(x,y)]} \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial x} \frac{\partial^2 G(x,y)}{\partial y \partial \boldsymbol{\Theta}} \\
& + \frac{1}{[1-G(x,y)]^2} \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \frac{\partial^2 G(x,y)}{\partial x \partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial y} \\
& + \frac{1}{[1-G(x,y)]} \frac{\partial^3 G(x,y)}{\partial x \partial \boldsymbol{\Theta}^2} \frac{\partial G(x,y)}{\partial y} \\
& + \frac{1}{[1-G(x,y)]} \frac{\partial^2 G(x,y)}{\partial x \partial \boldsymbol{\Theta}} \frac{\partial^2 G(x,y)}{\partial y \partial \boldsymbol{\Theta}} \\
& + \frac{1}{[1-G(x,y)]^2} \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \frac{\partial G(x,y)}{\partial x} \frac{\partial^2 G(x,y)}{\partial y \partial \boldsymbol{\Theta}} \\
& + \frac{1}{[1-G(x,y)]} \frac{\partial^2 G(x,y)}{\partial x \partial \boldsymbol{\Theta}} \frac{\partial^2 G(x,y)}{\partial y \partial \boldsymbol{\Theta}} \\
& \left. + \frac{1}{[1-G(x,y)]} \frac{\partial G(x,y)}{\partial x} \frac{\partial^3 G(x,y)}{\partial y \partial \boldsymbol{\Theta}^2} \right\} \\
& + \frac{1}{\{1-G(x,y)\}^2} \left[\frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \right]^2 \frac{\partial^2 G(x,y)}{\partial x \partial y} + \frac{1}{[1-G(x,y)]} \frac{\partial^2 G(x,y)}{\partial \boldsymbol{\Theta}^2} \frac{\partial^2 G(x,y)}{\partial x \partial y} \\
& + \frac{2}{[1-G(x,y)]} \frac{\partial G(x,y)}{\partial \boldsymbol{\Theta}} \frac{\partial^3 G(x,y)}{\partial x \partial y \partial \boldsymbol{\Theta}} - \log\left[1-G(x,y)\right] \frac{\partial^4 G(x,y)}{\partial x \partial y \partial \boldsymbol{\Theta}^2}.
\end{aligned}
$$

For large n, the distribution of $\sqrt{n}(\widehat{a}-a, \widehat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})$ approximates to a $(q+1)$ variate normal distribution with zero means and variance-covariance matrix $\mathbf{I}^{-1}$. The properties of $(\widehat{a}, \widehat{\boldsymbol{\Theta}})$ can be derived based on this normal approximation.

Finally, we consider the case of multicensoring for bivariate data. We suppose that there are n bivariate failure times of which

- n_0 are known to occur at $(x_1^{(0)}, y_1^{(0)}), \ldots, (x_{n_0}^{(0)}, y_{n_0}^{(0)})$.
- n_1 are known to have x components occurring at $x_1^{(1)}, \ldots, x_{n_1}^{(1)}$ and y components exceeding $y_i^{(1)}$, $i = 1, \ldots, n_1$ but not observed any longer.
- n_2 are known to have x components occurring at $x_1^{(2)}, \ldots, x_{n_2}^{(2)}$ and y components belonging to the interval $[y_{i-1}^{(2)}, y_i^{(2)}]$, $i = 1, \ldots, n_2$.
- n_3 are known to have y components occurring at $y_1^{(3)}, \ldots, y_{n_3}^{(3)}$ and x components exceeding $x_i^{(3)}$, $i = 1, \ldots, n_3$ but not observed any longer.

- n_4 are known to have y components occurring at $y_1^{(4)}, \ldots, y_{n_4}^{(4)}$ and x components belonging to the interval $[x_{i-1}^{(4)}, x_i^{(4)}]$, $i = 1, \ldots, n_4$.
- n_5 are known to have y components belonging to the interval $[y_{i-1}^{(5)}, y_i^{(5)}]$, $i = 1, \ldots, n_5$ and x components belonging to the interval $[x_{i-1}^{(5)}, x_i^{(5)}]$, $i = 1, \ldots, n_5$.
- n_6 are known to have y components belonging to the interval $[y_{i-1}^{(6)}, y_i^{(6)}]$, $i = 1, \ldots, n_6$ and x components exceeding $x_i^{(6)}$, $i = 1, \ldots, n_6$ but not observed any longer.
- n_7 are known to have x components belonging to the interval $[x_{i-1}^{(7)}, x_i^{(7)}]$, $i = 1, \ldots, n_7$ and y components exceeding $y_i^{(7)}$, $i = 1, \ldots, n_7$ but not observed any longer.
- n_8 are known to have x components exceeding $x_i^{(8)}$, $i = 1, \ldots, n_8$ but not observed any longer and y components exceeding $y_i^{(8)}$, $i = 1, \ldots, n_8$ but not observed any longer.

Note that $n = n_0 + n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8$.

In the multicensoring scheme described, the log-likelihood function, (48), becomes

$$
\begin{aligned}
\log L(a, \boldsymbol{\Theta}) \quad = \quad & \sum_{i=1}^{n_0} \log f\left(x_i^{(0)}, y_i^{(0)}\right) + \sum_{i=1}^{n_1} \log \left[f\left(x_i^{(1)}\right) - \frac{\partial F\left(x_i^{(1)}, y_i^{(1)}\right)}{\partial x_i^{(1)}} \right] \\
& + \sum_{i=1}^{n_2} \log \left[\frac{\partial F\left(x_i^{(2)}, y_i^{(2)}\right)}{\partial x_i^{(2)}} - \frac{\partial F\left(x_i^{(2)}, y_{i-1}^{(2)}\right)}{\partial x_i^{(2)}} \right] \\
& + \sum_{i=1}^{n_3} \log \left[f\left(y_i^{(3)}\right) - \frac{\partial F\left(x_i^{(3)}, y_i^{(3)}\right)}{\partial y_i^{(3)}} \right] \\
& + \sum_{i=1}^{n_4} \log \left[\frac{\partial F\left(x_i^{(4)}, y_i^{(4)}\right)}{\partial y_i^{(4)}} - \frac{\partial F\left(x_{i-1}^{(4)}, y_i^{(4)}\right)}{\partial x_i^{(4)}} \right] \\
& + \sum_{i=1}^{n_5} \log \left[F\left(x_i^{(5)}, y_i^{(5)}\right) - F\left(x_{i-1}^{(5)}, y_i^{(5)}\right) - F\left(x_i^{(5)}, y_{i-1}^{(5)}\right) + F\left(x_{i-1}^{(5)}, y_{i-1}^{(5)}\right) \right] \\
& + \sum_{i=1}^{n_6} \log \left[F\left(y_i^{(6)}\right) - F\left(x_i^{(6)}, y_i^{(6)}\right) - F\left(y_{i-1}^{(6)}\right) + F\left(x_i^{(6)}, y_{i-1}^{(6)}\right) \right] \\
& + \sum_{i=1}^{n_7} \log \left[F\left(x_i^{(7)}\right) - F\left(x_{i-1}^{(7)}\right) - F\left(x_i^{(7)}, y_i^{(7)}\right) + F\left(x_{i-1}^{(7)}, y_i^{(7)}\right) \right] \\
& + \sum_{i=1}^{n_8} \log \left[1 - F\left(x_i^{(8)}\right) - F\left(y_i^{(8)}\right) + F\left(x_i^{(8)}, y_i^{(8)}\right) \right]. \qquad (51)
\end{aligned}
$$

The derivatives of (51) with respect to a and $\mathbf{\Theta}$ are:

$$
\begin{aligned}
\frac{\partial \log L(a, \mathbf{\Theta})}{\partial a} = & \sum_{i=1}^{n_0} \frac{\partial f\left(x_i^{(0)}, y_i^{(0)}\right)/\partial a}{\partial f\left(x_i^{(0)}, y_i^{(0)}\right)} + \sum_{i=1}^{n_1} \frac{\dfrac{\partial f\left(x_i^{(1)}\right)}{\partial a} - \dfrac{\partial^2 F\left(x_i^{(1)}, y_i^{(1)}\right)}{\partial a \partial x_i^{(1)}}}{f\left(x_i^{(1)}\right) - \dfrac{\partial F\left(x_i^{(1)}, y_i^{(1)}\right)}{\partial x_i^{(1)}}} \\
& + \sum_{i=1}^{n_2} \frac{\dfrac{\partial^2 F\left(x_i^{(2)}, y_i^{(2)}\right)}{\partial a \partial x_i^{(2)}} - \dfrac{\partial^2 F\left(x_i^{(2)}, y_{i-1}^{(2)}\right)}{\partial a \partial x_i^{(2)}}}{\dfrac{\partial F\left(x_i^{(2)}, y_i^{(2)}\right)}{x_i^{(2)}} - \dfrac{\partial F\left(x_i^{(2)}, y_{i-1}^{(2)}\right)}{x_i^{(2)}}} \\
& + \sum_{i=1}^{n_3} \frac{\dfrac{\partial f\left(y_i^{(3)}\right)}{\partial a} - \dfrac{\partial^2 F\left(x_i^{(3)}, y_i^{(3)}\right)}{\partial a \partial y_i^{(3)}}}{f\left(y_i^{(3)}\right) - \dfrac{\partial F\left(x_i^{(3)}, y_i^{(3)}\right)}{\partial y_i^{(3)}}} \\
& + \sum_{i=1}^{n_4} \frac{\dfrac{\partial^2 F\left(x_i^{(4)}, y_i^{(4)}\right)}{\partial a \partial y_i^{(4)}} - \dfrac{\partial^2 F\left(x_{i-1}^{(4)}, y_i^{(4)}\right)}{\partial a \partial x_i^{(4)}}}{\dfrac{\partial F\left(x_i^{(4)}, y_i^{(4)}\right)}{\partial y_i^{(4)}} - \dfrac{\partial F\left(x_{i-1}^{(4)}, y_i^{(4)}\right)}{\partial x_i^{(4)}}} \\
& + \sum_{i=1}^{n_5} \frac{\dfrac{\partial F\left(x_i^{(5)}, y_i^{(5)}\right)}{\partial a} - \dfrac{\partial F\left(x_{i-1}^{(5)}, y_i^{(5)}\right)}{\partial a} - \dfrac{\partial F\left(x_i^{(5)}, y_{i-1}^{(5)}\right)}{\partial a} + \dfrac{\partial F\left(x_{i-1}^{(5)}, y_{i-1}^{(5)}\right)}{\partial a}}{F\left(x_i^{(5)}, y_i^{(5)}\right) - F\left(x_{i-1}^{(5)}, y_i^{(5)}\right) - F\left(x_i^{(5)}, y_{i-1}^{(5)}\right) + F\left(x_{i-1}^{(5)}, y_{i-1}^{(5)}\right)} \\
& + \sum_{i=1}^{n_6} \frac{\dfrac{\partial F\left(y_i^{(6)}\right)}{\partial a} - \dfrac{\partial F\left(x_i^{(6)}, y_i^{(6)}\right)}{\partial a} - \dfrac{\partial F\left(y_{i-1}^{(6)}\right)}{\partial a} + \dfrac{\partial F\left(x_i^{(6)}, y_{i-1}^{(6)}\right)}{\partial a}}{F\left(y_i^{(6)}\right) - F\left(x_i^{(6)}, y_i^{(6)}\right) - F\left(y_{i-1}^{(6)}\right) + F\left(x_i^{(6)}, y_{i-1}^{(6)}\right)} \\
& + \sum_{i=1}^{n_7} \frac{\dfrac{\partial F\left(x_i^{(7)}\right)}{\partial a} - \dfrac{\partial F\left(x_{i-1}^{(7)}\right)}{\partial a} - \dfrac{\partial F\left(x_i^{(7)}, y_i^{(7)}\right)}{\partial a} + \dfrac{\partial F\left(x_{i-1}^{(7)}, y_i^{(7)}\right)}{\partial a}}{F\left(x_i^{(7)}\right) - F\left(x_{i-1}^{(7)}\right) - F\left(x_i^{(7)}, y_i^{(7)}\right) + F\left(x_{i-1}^{(7)}, y_i^{(7)}\right)} \\
& + \sum_{i=1}^{n_8} \frac{-\dfrac{\partial F\left(x_i^{(8)}\right)}{\partial a} - \dfrac{\partial F\left(y_i^{(8)}\right)}{\partial a} + \dfrac{\partial F\left(x_i^{(8)}, y_i^{(8)}\right)}{\partial a}}{1 - F\left(x_i^{(8)}\right) - F\left(y_i^{(8)}\right) + F\left(x_i^{(8)}, y_i^{(8)}\right)},
\end{aligned} \tag{52}
$$

and

$$
\begin{aligned}
\frac{\partial \log L\left(a,\mathbf{\Theta}\right)}{\partial \mathbf{\Theta}} \quad = \quad & \sum_{i=1}^{n_0} \frac{\partial f\left(x_i^{(0)}, y_i^{(0)}\right)/\partial \mathbf{\Theta}}{\partial f\left(x_i^{(0)}, y_i^{(0)}\right)} + \sum_{i=1}^{n_1} \frac{\dfrac{\partial f\left(x_i^{(1)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial^2 F\left(x_i^{(1)}, y_i^{(1)}\right)}{\partial \mathbf{\Theta} \partial x_i^{(1)}}}{f\left(x_i^{(1)}\right) - \dfrac{\partial F\left(x_i^{(1)}, y_i^{(1)}\right)}{\partial x_i^{(1)}}} \\
& + \sum_{i=1}^{n_2} \frac{\dfrac{\partial^2 F\left(x_i^{(2)}, y_i^{(2)}\right)}{\partial \mathbf{\Theta} \partial x_i^{(2)}} - \dfrac{\partial^2 F\left(x_i^{(2)}, y_{i-1}^{(2)}\right)}{\partial \mathbf{\Theta} \partial x_i^{(2)}}}{\dfrac{\partial F\left(x_i^{(2)}, y_i^{(2)}\right)}{x_i^{(2)}} - \dfrac{\partial F\left(x_i^{(2)}, y_{i-1}^{(2)}\right)}{x_i^{(2)}}} \\
& + \sum_{i=1}^{n_3} \frac{\dfrac{\partial f\left(y_i^{(3)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial^2 F\left(x_i^{(3)}, y_i^{(3)}\right)}{\partial \mathbf{\Theta} \partial y_i^{(3)}}}{f\left(y_i^{(3)}\right) - \dfrac{\partial F\left(x_i^{(3)}, y_i^{(3)}\right)}{\partial y_i^{(3)}}} \\
& + \sum_{i=1}^{n_4} \frac{\dfrac{\partial^2 F\left(x_i^{(4)}, y_i^{(4)}\right)}{\partial \mathbf{\Theta} \partial y_i^{(4)}} - \dfrac{\partial^2 F\left(x_{i-1}^{(4)}, y_i^{(4)}\right)}{\partial \mathbf{\Theta} \partial x_i^{(4)}}}{\dfrac{\partial F\left(x_i^{(4)}, y_i^{(4)}\right)}{\partial y_i^{(4)}} - \dfrac{\partial F\left(x_{i-1}^{(4)}, y_i^{(4)}\right)}{\partial x_i^{(4)}}} \\
& + \sum_{i=1}^{n_5} \frac{\dfrac{\partial F\left(x_i^{(5)}, y_i^{(5)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(x_{i-1}^{(5)}, y_i^{(5)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(x_i^{(5)}, y_{i-1}^{(5)}\right)}{\partial \mathbf{\Theta}} + \dfrac{\partial F\left(x_{i-1}^{(5)}, y_{i-1}^{(5)}\right)}{\partial \mathbf{\Theta}}}{F\left(x_i^{(5)}, y_i^{(5)}\right) - F\left(x_{i-1}^{(5)}, y_i^{(5)}\right) - F\left(x_i^{(5)}, y_{i-1}^{(5)}\right) + F\left(x_{i-1}^{(5)}, y_{i-1}^{(5)}\right)} \\
& + \sum_{i=1}^{n_6} \frac{\dfrac{\partial F\left(y_i^{(6)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(x_i^{(6)}, y_i^{(6)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(y_{i-1}^{(6)}\right)}{\partial \mathbf{\Theta}} + \dfrac{\partial F\left(x_i^{(6)}, y_{i-1}^{(6)}\right)}{\partial \mathbf{\Theta}}}{F\left(y_i^{(6)}\right) - F\left(x_i^{(6)}, y_i^{(6)}\right) - F\left(y_{i-1}^{(6)}\right) + F\left(x_i^{(6)}, y_{i-1}^{(6)}\right)} \\
& + \sum_{i=1}^{n_7} \frac{\dfrac{\partial F\left(x_i^{(7)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(x_{i-1}^{(7)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(x_i^{(7)}, y_i^{(7)}\right)}{\partial \mathbf{\Theta}} + \dfrac{\partial F\left(x_{i-1}^{(7)}, y_i^{(7)}\right)}{\partial \mathbf{\Theta}}}{F\left(x_i^{(7)}\right) - F\left(x_{i-1}^{(7)}\right) - F\left(x_i^{(7)}, y_i^{(7)}\right) + F\left(x_{i-1}^{(7)}, y_i^{(7)}\right)} \\
& + \sum_{i=1}^{n_8} \frac{-\dfrac{\partial F\left(x_i^{(8)}\right)}{\partial \mathbf{\Theta}} - \dfrac{\partial F\left(y_i^{(8)}\right)}{\partial \mathbf{\Theta}} + \dfrac{\partial F\left(x_i^{(8)}, y_i^{(8)}\right)}{\partial \mathbf{\Theta}}}{1 - F\left(x_i^{(8)}\right) - F\left(y_i^{(8)}\right) + F\left(x_i^{(8)}, y_i^{(8)}\right)},
\end{aligned} \tag{53}
$$

where

$$
\frac{\partial F(x,y)}{\partial a} = -\frac{\Gamma^{'}(a)}{\Gamma^2(a)}\gamma\left(a, -\log\left[1-G(x,y)\right]\right) + \frac{1}{\Gamma(a)}\frac{\partial}{\partial a}\gamma\left(a, -\log\left[1-G(x,y)\right]\right),
$$

$$
\frac{\partial F(x,y)}{\partial \mathbf{\Theta}} = -\frac{1}{\Gamma(a)}\left\{-\log\left[1-G(x,y)\right]\right\} a - 1\frac{\partial G(x,y)}{\partial \mathbf{\Theta}},
$$

$$
\frac{\partial^2 F(x,y)}{\partial x \partial a} = \frac{1}{\Gamma(a)}\left\{-\log\left[1-G(x,y)\right]\right\}^{a-1}\frac{\partial G(x,y)}{\partial x}\left[-\frac{\Gamma^{'}(a)}{\Gamma(a)} + \log\left\{-\log\left[1-G(x,y)\right]\right\}\right],
$$

and

$$\frac{\partial^2 F(x,y)}{\partial x \partial \mathbf{\Theta}} = \frac{1}{\Gamma(a)} \{-\log[1-G(x,y)]\}^{a-2} \left\{(a-1)\frac{\partial G(x,y)}{\partial x}\frac{\partial G(x,y)}{\partial \mathbf{\Theta}} - \log[1-G(x,y)]\frac{\partial^2 G(x,y)}{\partial x \partial \mathbf{\Theta}}\right\}.$$

The remaining partial derivatives required for (52)-(53) follow from previous results: $\partial f(x,y)/\partial a$ and $\partial f(x,y)/\partial \mathbf{\Theta}$ follow from (49)-(50); $\partial f(x)/\partial a$ and $\partial f(x)/\partial \mathbf{\Theta}$ follow from (39)-(40); and, $\partial F(x)/\partial a$ and $\partial F(x)/\partial \mathbf{\Theta}$ follow from (44)-(45).

The MLEs of $(a, \mathbf{\Theta})$, say $(\widehat{a}, \widehat{\mathbf{\Theta}})$, are the simultaneous solutions of the equations $\partial \log L/\partial a = 0$ and $\partial \log L/\partial \mathbf{\Theta} = \mathbf{0}$.

16 Conclusions and Future work

We propose a new class of gamma-G distributions which can include as special cases all classical continuous distributions. For any parent continuous distribution G, we can define the corresponding gamma-G distribution with an extra positive parameter. So, the new class extends several common distributions such as normal, log-normal, Weibull, Gumbel and log-logistic distributions. The mathematical properties of the new class such as ordinary, incomplete and factorial moments, cumulants, generating and quantile functions, mean deviations, Bonferroni and Lorenz curves, asymptotic distribution of the extreme values, Shannon entropy, Rényi entropy, reliability and order statistics are obtained for any gamma-G distribution. The model parameters are estimated by maximum likelihood. An example to real data illustrates the importance and potentiality of the new class.

Properties and extensions of (1) not considered in this paper are: stochastic orderings, cumulative residual entropy, Song's measure, acceptance sampling plans, goodness of fit tests, tolerance intervals, distribution of the sum of gamma-G random variables, distribution of the product of gamma-G random variables, distribution of the ratio of gamma-G random variables, multivariate generalizations of the gamma-G distribution, Bayesian estimation, empirical Bayes estimation, uniform minimum variance unbiased estimation, estimation using weighted least squares, estimation using bootstrap, estimation using quantiles, estimation using order statistics, estimation using L moments, and estimation using record statistics. We hope to address some of these in a future paper.

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