A Parametric Bootstrap Method of Computation of Control Limits of Charts for Skewed Distributions

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Abstract — This article is proposing a new parametric bootstrap method of evaluation of control limits of charts for asymmetrically distributed process measurements. The proposed method modifies the authors' approach to evaluation of control limits based on pseudorandom numbers generation (presented in [1]) by using the unbiased estimator of within-subgroup variation (pooled variance) at the step of evaluation of the distribution parameters hence decreasing the probability of false alarm in a process monitoring phase (Phase II). The use of an average statistic of within-subgroup variation increases the robustness of control limits (to the presence of exceptional variation) that allows applying the proposed method in a retrospective analysis (Phase I). The method does not require nonlinear transformations of data, which may complicate the technical interpretation and application of the results of analysis. The proposed parametric bootstrap method may be used to construct a control chart for any statistic when process measurements are distributed in accordance with any (one- or two-parameter) theoretical law. Quality variables with such distributions are found in many industries: telecommunications, electronics, mechanical engineering, etc. A comparison between the performance of charts for averages and standard deviations of the proposed method and the same charts of alternative methods[1] in Phase II is made in terms of type-I and type-II error rates (using Monte Carlo simulations for process measurements distributed in accordance with the lognormal and the Weibull laws). The type-I error rates of charts for averages and standard deviations of the proposed method are closer to the required value than the type-I error rates of charts for averages and standard deviations of the alternative methods. The type-II error rates of these charts of the proposed method enable them to detect large shifts in the process location and variance very quickly.

Keywords — Statistical process control; control chart; evaluation of control limits; parametric bootstrap method; skewed distribution; type-I and type-II error rates

I. Introduction

Shewhart control charts are powerful tools for identifying a state of a process. Shewhart charts for averages ($\bar{X}$) and standard deviations (S) are the most widespread charts for measures of location and dispersion. Control limits of these charts are calculated under the assumption that process measurements (X) are normally distributed. However, X with such distributions are found in many processes and that assumption will lead to a higher probability of both the type-I and the type-II errors. The examples of X with skewed distributions are: duration of a telephone call, time to failure of a device, deviation of shape from circularity, tensile strength of material, etc.

II. Literature Review

Four general approaches to constructing charts for measures of location and dispersion were proposed to deal with skewed distribution of process measurements.

A. The First Approach

Skewness of a distribution can be disregarded by increasing subgroup sample size (n) so that values of $\bar{X}$ become approximately normally distributed. Then $\bar{X}$ - chart may be used as a chart for measure of location and mR - chart (chart for moving ranges) for averages – as a chart for measure of dispersion. However an increase in n is often expensive and almost impossible in some cases.

B. The Second Approach

Nonparametric (distribution-free) methods may be used to construct a control chart. In [2] the heuristic weighted variance (WV) method based on semivariance approximation [3] was proposed and the formulas for control limits of $\bar{X}$ - and R - (chart for ranges) charts for skewed distributions were presented. In [4] Bai and Choi transformed the WV method and presented the tables of bias correction factors to use them in the formulas for control limits of $\bar{X}$ - and R - charts. In [5] Khoo, Atta and Chen proposed the WV method of computation of control limits of $\bar{X}$ - and S - charts.

Another heuristic method – the weighted standard deviations (WSD) method – was proposed in [6]. It is used to construct the $\bar{X}$ - chart, the cumulative sum (CUSUM) chart and exponentially weighted moving range (EWMA) chart.

In [7] the skewness correction (SC) method of constructing $\bar{X}$ - and R - charts for skewed distributions was proposed. The control limits of the method are based on the degree of skewness estimated from the subgroups.

In [8] Bajgier proposed to compute control limits of $\bar{X}$ - chart via the nonparametric bootstrap method [9]. In [10]

[1] Shewhart charts, weighted variance charts and charts based on Johnson curves.

Jones and Woodall concluded that the bootstrap control charts seem to be adequate for extremely skewed distributions. The bootstrap control chart proposed in [8] assumes only that the process is stable when the control limits are computed. If this assumption is not satisfied when the control limits are computed (for example, in a retrospective analysis (Phase I)), the limits will be too wide, regardless of the distribution of the process variable [11]. Therefore, the method [8] may be used only in process monitoring (Phase II).

C. The Third Approach

When the theoretical distribution of X is unknown, an approach to evaluation of control limits based on Johnson [12] or Pearson curves may be used. For example, a Johnson curve (JC) can be fitted to the first four moments of the observed data to compute the control limits of X- chart [13]. Castagliola and Khoo modified this method to compute the control limits of $\bar{X}$- chart [14]. Their method has the disadvantage: the real process is assumed to be stable with the values of skewness and kurtosis computed from the data of the real process; if the real process contains both chance and assignable causes of variation, the evaluation of the control limits may be incorrect. This laborious method is implement in the versions of the software product STATISTICA [15].

D. The Fourth Approach

In many processes empirical distributions of X can be approximated by some (skewed) theoretical models: a distribution of durations of telephone calls can be approximated by the lognormal distribution, a distribution of values of time to failure of a device – by the Weibull distribution, a distribution of deviations of shape from circularity – by the Rayleigh distribution, etc. In such cases parametric methods may be used to construct control charts for measures of location and dispersion.

Morrison [16] and Nelson [17] proposed the parametric methods of computation of control limits, which are based on the assumption that X follows, respectively, the lognormal and the Weibull distributions. The disadvantage of Morrison's method is the need for exponential transformations of data that may lead to improper evaluation of control limits [18]. Nelson developed constants for the limits of the median, range, scale and location charts (but only for some values of subgroup sample size, shape parameter and alpha level).

Even if the analytical distribution of X is known, the distribution of $\bar{X}$ (and other statistics) is generally unknown (except for some rare cases) [14]. This problem can be solved using parametric bootstrap (PB) methods. PB methods allow estimation of the sampling distribution of almost any statistic (average, median, standard deviation, etc.) by pseudorandom numbers generation of values of X [19].

Several PB methods were proposed to evaluate control limits for monitoring quantiles of some statistic (ξ_p) when process measurements have a theoretical distribution. For example, Nichols and Padgett [11] developed a PB method to establish lower and upper control limits for monitoring percentiles W_p^w when values of X have a Weibull distribution. Lio and Park [20] proposed the same technique to establish control limits for monitoring percentiles W_p^{b-s} when process measurements have a Birnbaum–Saunders distribution. Abbasi and Guillen [21] considered four theoretical distributions of X (Weibull, Burr, Birnbaum–Saunders and Pareto) and for each of them they proposed a PB method to establish lower and upper control limits for quantiles of some ξ_p with variable sample sizes within subgroups. The general scheme of computation of control limits for quantiles of ξ_p using a PB method is proposed in [22]. X_p- chart (chart for quantiles of X) was used in the simulation study of its performance [22]. The main advantage of the PB methods of control limits computation is high accuracy of evaluation of the required value of α (the type-I error rate). But the PB methods of control limits computation proposed in [11], [20], [21] and [22] assume only that the process is stable when the control limits are computed and, therefore, these methods may be used only in Phase II.

The main feature of the PB method of computation of control limits proposed by Lukin and Yaschenko [1] is the use of an average (or a median) of k (k is a number of subgroups) statistics of location and dispersion to evaluate parameters of a chosen theoretical distribution (by means of the analytic connection between the expected value $E[X]$, the variance $Var[X]$ and the distribution parameters $\theta_1,...,\theta_m$) as opposed to the use of estimations of $\theta_1,...,\theta_m$ based on all $n \times k$ observations [11], [20], [21], [22]. The use of an average statistic of within-subgroup variation increases the robustness of control limits (to the presence of exceptional variation) that allows applying the method in Phase I.

The PB method [1] does not require nonlinear transformations of data, which may complicate the technical interpretation and application of the results of analysis. This method may be used to construct a control chart for any statistic when values of X are distributed in accordance with any (one- or two-parameter) theoretical law. In addition, there are no any difficult analytical computational procedures, which could limit the application of the method in practice.

The proposed method modifies the approach presented in [1] by using the unbiased estimator of within-subgroup variation (pooled variance) at the step of evaluation of the distribution parameters in Phase II.

III. Method Selection Criteria

The control charting methods have been selected in accordance with the following criteria:

- an empirical distribution of X can be approximated by some theoretical model (if needed);
- a presence of robustness of control limits;
- an absence of nonlinear transformations of data.

The simulation study is carried out to investigate the performance of the control charts constructed using the selected methods. Some comments are followed to clarify the choice of the criteria.

Since in many processes empirical distributions of measurements (X) can be approximated by some theoretical models, the parametric methods may be used to identify the state of statistical control of these processes. In the simulation study two widespread skewed theoretical distributions (lognormal and Weibull) of X are considered.

Initially in most practical cases it is never known if the process is stable or not until the retrospective analysis is made (in Phase I). Hence the ability of a control charting method to construct robust control limits (to the presence of exceptional variation) is especially important.

Nonlinear transformations of process measurements, used in some control charting methods, complicate the technical interpretation and application of the results of analysis [23]. Hence the use of nonlinear transformations is undesirable [24] and the control charting methods based on nonlinear transformations are not considered in the simulation study.

In telecommunications the compliance of a control charting method with the mentioned criteria is especially critical when analyzing durations of telephone calls. $\bar{X}$ - chart is the most widespread chart for measure of process location (for example, in telecommunications average call duration (ACD) is used as $\bar{X}$ statistic). In the simulation study moderate ($n=10$) and large ($n=20$) sample sizes are applied. For such sample sizes S - chart is preferable to R - chart when monitoring process variability [25].

In accordance with the mentioned criteria the following methods of $\bar{X}$ - and S - charts construction have been chosen for the simulation study:

- Shewhart method [11];
- WV method [5];
- The method based on JC [14], [15];
- PB method by Lukin and Yaschenko.

The algorithms of Shewhart method, WV method and the method based on JC, used in the simulation study, are available from the authors upon request.

IV. The Algorithm of the Proposed Parametric Bootstrap Method

$\bar{X}$ - and S - charts of the proposed PB method are constructed in accordance with the following steps:

1. Draw a sample of $n \times k$ measurements (X_{ij}) from some observed process, where $i=1,...,k$ is a subgroup's sequence number; $j=1,...,n$ is a sequence number of a measurement in a subgroup.

2. Compute the sample average ($\overline{X_i}$), the sample standard deviation (S_i) and the sample variance (S_i^2) for each subgroup as follows:

$$\overline{X_i} = \sum_{j=1}^{n} X_{ij} \Big/ n \tag{1}$$

$$S_i = \sqrt{\sum_{j=1}^{n} \left(X_{ij} - \overline{X_i}\right)^2 \Big/ (n-1)} \tag{2}$$

$$S_i^2 = \sum_{j=1}^{n} \left(X_{ij} - \overline{X_i}\right)^2 \Big/ (n-1) \tag{3}$$

3. Compute the grand average ($\bar{\bar{X}}$), the average standard deviation ($\bar{S}$) and the pooled variance ($\overline{S^2}$) as follows:

$$\bar{\bar{X}} = \sum_{i=1}^{k} \overline{X_i} \Big/ k \tag{4}$$

$$\bar{S} = \sum_{i=1}^{k} (S_i) \Big/ k \tag{5}$$

$$\overline{S^2} = \sum_{i=1}^{k} (S_i^2) \Big/ k \tag{6}$$

4. It is assumed that one- or two- parameter distribution function (F_x) of process measurements (X) is known. If not, F_x of sampled populations (previously taken from stable processes) of X can be determined using a test of goodness-of-fit [1]. The analytic connection between the expected value ($E[X]$), the variance ($Var[X]$) and the distribution parameters ($\theta_1;\theta_2$) of F_x can be presented as follows (for two-parameter distribution):

$$\begin{cases} E[X] = g_1(\theta_1;\theta_2) \\ Var[X] = g_2(\theta_1;\theta_2) \end{cases} \tag{7}$$

Assuming that $\bar{\bar{X}}$ is the estimate of $E[X]$ and $\overline{S^2}$ (or $(\bar{S})^2$) is the estimate of $Var[X]$, the estimations of distribution parameters ($\hat{\theta}_1;\hat{\theta}_2$) of F_x can be computed.

In Phase I it is better to use $(\bar{S})^2$ than $\overline{S^2}$ as the estimator of $Var[X]$, because $(\bar{S})^2$ is more robust to the presence of exceptional variation.

In Phase II it is preferable to use $\overline{S^2}$ than $(\bar{S})^2$, because $\overline{S^2}$ is an unbiased estimate of $Var[X]$, while $(\bar{S})^2$ is biased ($\overline{S^2}$ is a more accurate estimate of $Var[X]$ than $(\bar{S})^2$).

As the performance of $\bar{X}$ - and S - charts of the proposed method is studied in Phase II (chapter V), $\overline{S^2}$ should be used as an estimate of $Var[X]$.

Compute the estimations of parameters ($\hat{\mu};\hat{\sigma}^2$) of lognormal distribution $LogN(\mu;\sigma^2)$ from

$$\begin{cases} \hat{\mu} = \ln(\bar{\bar{X}}) - \dfrac{\hat{\sigma}^2}{2} \\ \hat{\sigma} = \sqrt{\ln(\dfrac{\overline{S^2}}{e^{2\cdot\ln(\bar{\bar{X}})}} + 1)} \end{cases} \tag{8}$$

Compute the estimations of parameters ($\hat{\delta};\hat{\psi}$) of Weibull distribution $W(\delta;\psi)$ from

$$\begin{cases} \hat{\psi} = \dfrac{\overline{\overline{X}}}{\Gamma(1+\frac{1}{\hat{\delta}})} \\ \dfrac{\Gamma(1+\frac{2}{\hat{\delta}})}{\Gamma^2(1+\frac{1}{\hat{\delta}})} = 1 + \dfrac{\overline{S^2}}{\left(\overline{\overline{X}}\right)^2} \end{cases} \tag{9}$$

where $\Gamma(X)$ (the gamma function) is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \cdot e^{-t} dt \tag{10}$$

An estimate ($\hat{\delta}$) of the shape parameter can be computed from (9) numerically by one of the methods available in most mathematical software products.

5. Using one of the methods of pseudorandom numbers generation [26] form a sample of size N. The values (X_{ij}^*) of the sample should be distributed in accordance with the obtained distribution function $F_x(\hat{\theta}_1;\hat{\theta}_2)$. In the simulation study (chapter V) X_{ij}^* (which are distributed in accordance with $LogN(\hat{\mu};\hat{\sigma}^2)$ and $W(\hat{\delta};\hat{\psi})$) are generated using the statistical software product STATGRAPHICS Centurion XV.

6. Arrange X_{ij}^* randomly into $B = N/n$ subgroups (of size n). For each subgroup compute the sample average ($\overline{X_i}^*$) and the sample standard deviation (S_i^*) in accordance with the formulas (1) and (2).

7. Rank B values of $\overline{X_i}^*$ in ascending order, then the value with the sequence number $(\alpha/2)\cdot B$ is the lower control limit ($LCL_{\bar{X}}$) of $\bar{X}$ - chart and the value with the sequence number $\left[1-(\alpha/2)\right]\cdot B$ is the upper control limit ($UCL_{\bar{X}}$) of $\bar{X}$ - chart ($\alpha = 0.0027$ is used in the simulation study).

8. Rank B values of S_i^* in ascending order, then the value with the sequence number $(\alpha/2)\cdot B$ is the lower control limit (LCL_S) of S - chart and the value with the sequence number $\left[1-(\alpha/2)\right]\cdot B$ is the upper control limit (UCL_S) of S - chart.

Steps 6 – 7 may be repeated for any statistic ξ_{pi}^* to compute the control limits of ξ_p - chart.

V. The Simulation Study

In order to study the performance of $\bar{X}$ - and S - charts (constructed using the mentioned methods) in Phase II, the Monte Carlo simulations are carried out. A comparison between the performance of the charts is made in terms of type-I and type-II error rates (α and β probabilities). The process measurements used in the simulation study have been obtained by pseudorandom numbers generation (the samples of X are distributed in accordance with the lognormal and the Weibull laws). Three-sigma limits are used as the control limits of Shewhart and the WV charts; the quantiles $UCL_{99.865\%}$ and $LCL_{0.135\%}$ are used as the control limits of the PB charts and the charts based on JC.

The evaluations of α for $\bar{X}$ - and S - charts are obtained in accordance with the following steps:

1. *Control chart construction*

1.1. Generate n values of X distributed in accordance with $LogN(\mu;\sigma^2)$ or $W(\delta;\psi)$.

1.2. Repeat step 1.1 $k=10$ times to form k subgroups of size n.

1.3. The control limits of $\bar{X}$ - and S - charts are computed in accordance with the algorithms of Shewhart method, the WV method, the PB method and the method based on JC.

1.4. Repeat steps 1.1 – 1.3 10^2 times for each method.

1.5. Repeat steps 1.1 – 1.4 for $n = 10;\ 20$.

1.6. Repeat steps 1.1 – 1.5 for $LogN(0.44;1.32)$, $LogN(1.53;0.52)$, $LogN(1.74;0.1)$, $W(0.75;5)$, $W(1.24;3)$, $W(2.6;3)$.

2. *Control chart operation*

2.1. Generate n values of X distributed in accordance with $LogN(\mu;\sigma^2)$ or $W(\delta;\psi)$ (similar to step 1.1).

2.2. Compute the sample statistics $\overline{X_i}$ and S_i.

2.3. In accordance with n and $LogN(\mu;\sigma^2)$ (or $W(\delta;\psi)$) from step 2.1 choose one of the $\bar{X}$ - charts constructed in steps 1.1 – 1.6 using one of the mentioned methods. $\overline{X_i}$ computed in step 2.2 is compared with $UCL_{\bar{X}}$ and $LCL_{\bar{X}}$ of the chosen $\bar{X}$ - chart in accordance with the formulas:

$$I_{UCL}(\overline{X_i}) = \begin{cases} 1, & если\ \overline{X_i} > UCL_{\bar{X}} \\ 0, & если\ \overline{X_i} \le UCL_{\bar{X}} \end{cases} \tag{11}$$

$$I_{LCL}(\overline{X_i}) = \begin{cases} 1, & если\ \overline{X_i} < LCL_{\bar{X}} \\ 0, & если\ \overline{X_i} \ge LCL_{\bar{X}} \end{cases} \tag{12}$$

The values of $I_{UCL}(\overline{X_i})$ and $I_{LCL}(\overline{X_i})$ are calculated for each of the $\bar{X}$ - charts constructed in steps 1.1 – 1.6 using one of the mentioned methods (for n and $LogN(\mu;\sigma^2)$ (or $W(\delta;\psi)$) from step 2.1).

2.4. In accordance with n and $LogN(\mu;\sigma^2)$ (or $W(\delta;\psi)$) from step 2.1 choose one of the S - charts constructed in steps 1.1 – 1.6 using one of the mentioned methods. S_i computed in step 2.2 is compared with UCL_S and LCL_S of the chosen S - chart in accordance with the formulas:

$$I_{UCL}(S_i) = \begin{cases} 1, & если\ S_i > UCL_S \\ 0, & если\ S_i \le UCL_S \end{cases} \tag{13}$$

$$I_{LCL}(S_i) = \begin{cases} 1, & если\ S_i < LCL_S \\ 0, & если\ S_i \ge LCL_S \end{cases} \tag{14}$$

The values of $I_{UCL}(S_i)$ and $I_{LCL}(S_i)$ are calculated for each of the S - charts constructed in steps 1.1 – 1.6 using one of the mentioned methods (for n and $LogN(\mu;\sigma^2)$ (or $W(\delta;\psi)$) from step 2.1).

2.5. Repeat steps 2.1 – 2.4 10^4 times.

2.6. For the chosen control charting method compute the average of all $I_{UCL}(\overline{X_i})$ (the obtained value is the estimate of $\alpha_{\bar{X}}$ probability that a point falls above $UCL_{\bar{X}}$) and the average of all $I_{LCL}(\overline{X_i})$ (the obtained value is the estimate of $\alpha_{\bar{X}}$ probability that a point falls below $LCL_{\bar{X}}$). For the same method compute the average of all $I_{UCL}(S_i)$ and the average of all $I_{LCL}(S_i)$.

2.7. Repeat steps 2.3 – 2.6 10^4 for each control charting method.

2.8. Repeat steps 2.1 – 2.7 for $n = 10;\ 20$.

2.9. Repeat steps 2.1 – 2.8 for $LogN(0.44;1.32)$, $LogN(1.53;0.52)$, $LogN(1.74;0.1)$, $W(0.75;5)$, $W(1.24;3)$, $W(2.6;3)$.

The estimates of $\alpha_{\bar{X}}$ (type-I error rates) for $\bar{X}$ - charts constructed using the mentioned control charting methods are presented in Table I. The estimates of α_S (type-I error rates) for S - charts constructed using the mentioned methods are presented in Table II.

The proposed PB charts outperform Shewhart charts, WV charts and the charts based on JC in terms of type-I error rate (especially for highly skewed distributions). The estimates of α for the PB charts are the nearest to the required value 0.27%. It is important to notice that the probability that a point falls above UCL and the probability that a point falls below LCL are very similar for the proposed PB charts.

Fig. 1 shows the estimates of $\alpha_{\bar{X}}$ (only LCL is considered) for the studied $\bar{X}$ - charts plotted against the values of P_x (for $LogN(0.44;1.32)$, $LogN(1.53;0.52)$, $LogN(1.74;0.1)$ and $n = 10$), where P_x is the probability that $X \le M[X]$. The probability density plots of X are presented under the corresponding values of P_x. For the mentioned $\bar{X}$ - charts (except the PB chart) the increase in skewness leads to significant deviation of $\alpha_{\bar{X}}$ from the required value 0.135%. Although the use of Shewhart and WV charts for highly skewed distributions leads to the extinction of type-I errors (if only LCL is considered), this fact signals the possible failure of these charts to detect exceptional variation of data (that will be proved further, see Fig. 2).

The evaluates of $(1-\beta)$ for the mentioned $\bar{X}$ - and S - charts are obtained to determine their power in detecting exceptional variation. The most widespread way to simulate the average $M[X]^*$ and the standard deviation $SD[X]^*$ of an unstable process is to use the following formulas [5]:

$$M[X]^* = M[X] + a \cdot SD[X] \quad (15)$$

$$SD[X]^* = b \cdot SD[X] \quad (16)$$

where a and b are the parameters which characterize the magnitudes of shifts in $M[X]$ and $SD[X]$. When the distribution of X (of a stable process) is normal:

- shifts in $M[X]$ and $SD[X]$ do not lead to change in shape of the distribution;
- the distributions shifted by $\pm a \cdot SD[X]$ from $M[X]$ are symmetric.

However, these conditions are not met in case of a skewed distribution that complicates the choice of appropriate values of a and b. That is why in order to model a skewed distribution of an unstable process, we suggest to use the WV

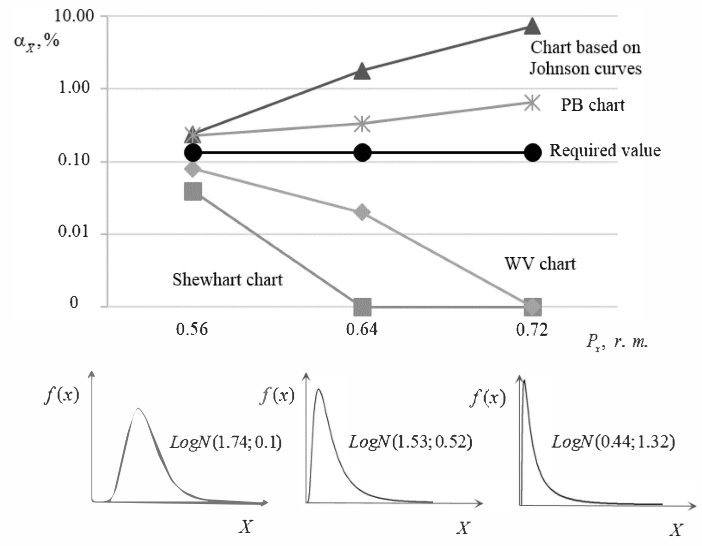

Fig. 1. Graphs of the estimates of $\alpha_{\bar{X}}$ (only LCL is considered) for the studied $\bar{X}$ - charts against the values of P_x.

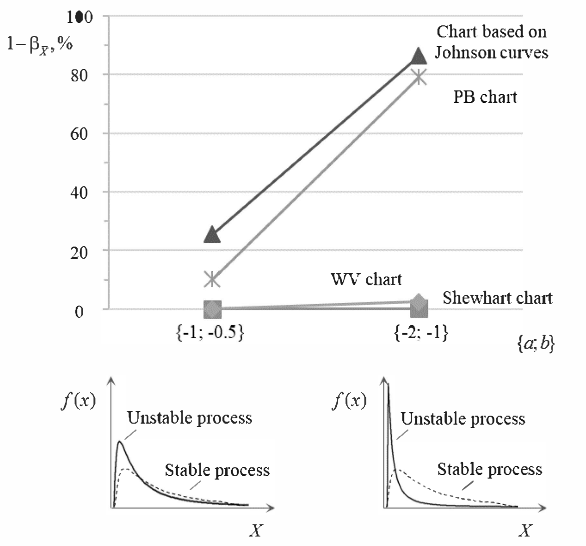

Fig. 2. Graphs of the estimates of $(1-\beta_{\bar{X}})$ (only LCL is considered) for the studied $\bar{X}$ - charts against the values of $\{a;b\}$ (for $n = 10$).

method. The algorithm of modelling a skewed distribution of an unstable process (based on WV method) and the estimates of $(1-\beta)$ for $\bar{X}$- and S- charts constructed using the mentioned control charting methods are available from the authors upon request.

Fig. 2 shows the estimates of $(1-\beta_{\bar{X}})$ (only LCL is considered) for the studied $\bar{X}$- charts plotted against the values of $\{a;b\}$ (for $n=10$). The distribution of X of the stable process is modelled by $LogN(0.44;1.32)$ (with $P_x = 0.72$). The distributions of the unstable processes are modelled by $LogN(-0.12;1.43)$ ($a=-1$; $b=-0.5$) and $LogN(-1.41;1.9)$ ($a=-2$; $b=-1$). The probability density plots of X (of stable and unstable processes) are presented under the corresponding values of $\{a;b\}$. The proposed PB chart identifies large shifts in the process location and variance (e.g. $\{-2;-1\}$) very quickly (the average run length there is 1.25), while Shewhart and WV charts fail to detect exceptional variation of data.

VI. Conclusions

In many processes empirical distributions of X can be approximated by some (skewed) theoretical models. In such cases parametric methods may be used to construct control charts for measures of location and dispersion.

The proposed PB method may be used to construct a control chart for any statistic when values of X are distributed in accordance with any (one- or two-parameter) theoretical law. The main feature of the method is the use of an average of k statistics of location and dispersion to evaluate parameters of a chosen theoretical distribution. The use of an average statistic of within-subgroup variation increases the robustness of control limits that allows applying the method in Phase I. The use of pooled variance statistic at the step of evaluation of the distribution parameters increases the accuracy of control limits in Phase II.

The control charting methods (Shewhart method, the WV method, the method based on JC and the proposed PB method) have been selected in accordance with the following criteria: an empirical distribution of X can be approximated by some theoretical model (if needed), a presence of robustness of control limits, an absence of nonlinear transformations of data. The simulation study has been carried out to investigate the performance of $\bar{X}$- and S- charts (constructed using the selected methods) in terms of type-I and type-II error rates. The proposed PB charts outperform the alternative charts in terms of type-I error rate. The PB charts detect large shifts in the process location and variance very quickly.

The proposed PB control charting method has a successful practical application in monitoring of telecommunicational processes.

References

[1] Lukin, V.; Yaschenko, V. Computation of Control Limits of $\bar{X}$ - Chart Based on Pseudorandom Numbers Generation. *Proceedings of the IEEE Russia. North West Section*, 2014, Vol. 6; pp. 54-58.

[2] Choobineh, F.; Ballard, J. Control-Limits of QC Charts for Skewed Distributions Using Weighted-Variance. *IEEE Transactions on Reliability*, 1987, Vol. 36, No. 4; pp. 473-477.

[3] Choobineh, F.; Branting, D. A Simple Approximation for Semivariance. *European Journal of Operational Research*, 1986, Vol. 27, No 3; pp. 364-370.

[4] Bai, D. S.; and Choi, I. S. $\bar{X}$ and R Control Charts for Skewed Populations. *Journal of Quality Technology*, 1995, Vol. 27, No. 2; pp. 120-131.

[5] Khoo, M. B. C.; Atta, A. M. A.; Chen, C.-H. Proposed $\bar{X}$ and S Control Charts for Skewed Distributions. *Proceedings of the 2009 IEEE IEEM*, 2009; pp. 389-393.

[6] Chang, Y. S.; Bai, D. S. Control Charts for Positively - Skewed Populations with Weighted Standard Deviations. *Quality and Reliability Engineering International*, 2001, Vol. 17, No. 5; pp. 397-406.

[7] Chan, L. K.; Cui, H. J. Skewness Correction $\bar{X}$ and R Charts for Skewed Distributions. *Naval Research Logistics*, 2003, Vol. 50, No. 6; pp. 555-573.

[8] Bajgier, S. M. The Use of Bootstrapping to Constructs Limits on Control Charts. *Proceedings of the Decision Science Institute, San Diego, CA*, 1992; pp. 1611-1613.

[9] Efron, B. Bootstrap Methods: Another Look at the Jackknife. *Annals of Statistics*, 1979, Vol. 7; pp. 1-26.

[10] Jones, L. A.; and Woodall, W. H. The Perfomance of Bootstrap Control Charts. *Journal of Quality Technology,* 1998, Vol. 30, No. 4; pp. 362-375.

[11] Nichols, M. D.; Padgett, W. J. A Bootstrap Control Chart for Weibull Percentiles. *Quality and Reliability Engineering International*, 2005, Vol. 22, No. 2; pp. 141-151.

[12] Johnson, N. Systems of Frequency Curves Generated by Methods of Translation. *Biometrika*, 1949, Vol. 36; pp. 149-176.

[13] Hill, I.; Hill, R.; Holder, R. Fitting Johnson Curves by Moments. *Applied Statistics*, 1976, Vol. 25, No. 2; pp. 180-189.

[14] Castagliola, P.; Khoo, M. B. C. A Synthetic Scaled Weighted Variance Control Chart for Monitoring the Process Mean of Skewed Populations. *Communications in Statistics – Simulation and Computation*, 2009, Vol. 38, No. 8; pp. 1659-1674.

[15] Statistica: Quality Control Charts / StatSoft, URL; http://www.statsoft.com/Products/STATISTICA/Quality-Control/

[16] Morrison, J. The Lognormal Distribution in Quality Control. *Applied Statistics*, 1958, Vol. 7, No. 3; pp. 160-172.

[17] Nelson, R. P. Control Charts for Weibull Processes with Standards Given. *IEEE Transactions on Reliability*, 1979, Vol 28, No. 4; pp. 283-298.

[18] Cheng, S. W.; and Xie, H. Control Charts for Lognormal Data. *Tamkang Journal of Science and Engineering*, 2000, Vol. 3, No. 3; pp. 131-137.

[19] Amiri, S.; von Rosen, D.; Zwanzig, S. On the Comparison of Parametric and Nonparametric Bootstrap. *U.U.D.M. Report 2008:15, Uppsala Universitet*, 2008; pp. 1-13.

[20] Lio, Y. L.; and Park, C. A Bootstrap Control Chart for Birnbaum–Saunders Percentiles. *Quality and Reliability Engineering International*, 2008, Vol. 24; pp. 585-600.

[21] Abbasi, B.; Guillen, M. Bootstrap Control Charts in Monitoring Value at Risk in Insurance. *Expert Systems with Applications*, 2013, Vol. 40; pp. 6125-6135.

[22] Wang, B.; Mishra, S. N.; Mulekar, M. S.; Mishra, N.; Huang, K. Comparison of Bootstrap and Generalized Bootstrap Methods for Estimating High Quantiles. *Journal of Statistical Planning and Inference*, 2010, Vol. 140, No. 10; pp. 2926-2935.

[23] Tippett, H. C. *The Methods of Statistics, 4th edition.* John Wiley and Sons, New York, 1952; pp. 344-345.

[24] Wheeler, D. *Advanced Topics in Statistical Process Control.* SPC Press, 1995.- 470 p.

[25] Woodall, W. H.; Montgomery D. C. Research Issues and Ideas in Statistical Process Control. *Journal of Quality Technology*, 1999, Vol. 31, No. 4; pp. 376-386.

[26] Kroese, D. P.; Taimre, T.; Botev, Z. I. *Handbook of Monte Carlo Methods.* John Wiley and Sons, Hoboken, New Jersey, 2011.

TABLE I. TYPE-I ERROR RATES FOR $\bar{X}$ - CHARTS

Distribution	F_x	$K_3[X]$	$K_4[X]$	$M[X]$	$SD[X]$	n = 10								n = 20							
						Shewhart chart		**WV chart**		**Chart based on Johnson curves**		**PB chart**		**Shewhart chart**		**WV chart**		**Chart based on Johnson curves**		**PB chart**	
						below LCL	*above UCL*	*below LCL*	*above UCL*	*below LCL*	*above UCL*	*below LCL*	*above UCL*	*below LCL*	*above UCL*	*below LCL*	*above UCL*	*below LCL*	*above UCL*	*below LCL*	*above UCL*
Lognormal	0.72	7.76	122.75	3	5	0.00%	4.10%	0.00%	1.74%	7.36%	2.08%	0.65%	0.76%	0.00%	3.34%	0.00%	1.42%	4.74%	1.61%	0.32%	0.59%
						4.10%		1.74%		9.44%		1.41%		3.34%		1.42%		6.35%		0.91%	
	0.64	2.94	16.95	6	5	0.00%	2.22%	0.02%	1.07%	1.77%	1.00%	0.33%	0.45%	0.00%	1.53%	0.03%	0.68%	0.69%	0.60%	0.23%	0.41%
						2.22%		1.09%		2.77%		0.78%		1.53%		0.71%		1.29%		0.64%	
	0.56	1.02	1.90	6	2	0.04%	0.61%	0.08%	0.42%	0.24%	0.31%	0.23%	0.31%	0.07%	0.52%	0.15%	0.35%	0.25%	0.28%	0.17%	0.22%
						0.65%		0.50%		0.55%		0.54%		0.59%		0.50%		0.53%		0.39%	
Weibull	0.68	3.11	15.44	6	8	0.00%	2.72%	0.00%	1.20%	1.11%	1.06%	0.33%	0.47%	0.00%	1.75%	0.03%	0.70%	0.36%	0.66%	0.27%	0.41%
						2.72%		1.20%		2.17%		0.80%		1.75%		0.73%		1.02%		0.68%	
	0.60	1.43	2.77	2.8	2.25	0.02%	1.33%	0.05%	0.75%	0.35%	0.56%	0.21%	0.40%	0.02%	0.68%	0.09%	0.35%	0.18%	0.32%	0.20%	0.24%
						1.35%		0.80%		0.91%		0.61%		0.70%		0.44%		0.50%		0.44%	
	0.52	0.31	-0.20	2.66	1.1	0.13%	0.42%	0.15%	0.36%	0.19%	0.33%	0.28%	0.26%	0.10%	0.27%	0.14%	0.23%	0.14%	0.21%	0.19%	0.18%
						0.55%		0.51%		0.52%		0.54%		0.37%		0.37%		0.35%		0.37%	

TABLE II. TYPE-I ERROR RATES FOR S - CHARTS

Distribution	F_x	$K_3[X]$	$K_4[X]$	$M[X]$	$SD[X]$	n = 10								n = 20							
						Shewhart chart		**WV chart**		**Chart based on Johnson curves**		**PB chart**		**Shewhart chart**		**WV chart**		**Chart based on Johnson curves**		**PB chart**	
						below LCLs	*above UCLs*	*below LCLs*	*above UCLs*	*below LCLs*	*above UCLs*	*below LCLs*	*above UCLs*	*below LCLs*	*above UCLs*	*below LCLs*	*above UCLs*	*below LCLs*	*above UCLs*	*below LCLs*	*above UCLs*
Lognormal	0.72	7.76	122.75	3	5	8.37%	13.96%	0.00%	2.39%	9.80%	6.50%	0.21%	0.70%	17.49%	17.27%	0.00%	2.28%	9.32%	5.27%	0.19%	0.42%
						22.33%		2.39%		16.30%		0.91%		34.76%		2.28%		14.59%		0.61%	
	0.64	2.94	16.95	6	5	0.64%	10.14%	0.00%	2.13%	7.86%	6.46%	0.17%	0.35%	4.09%	10.94%	0.00%	1.69%	5.33%	6.42%	0.20%	0.27%
						10.78%		2.13%		14.32%		0.52%		15.03%		1.69%		11.75%		0.47%	
	0.56	1.02	1.90	6	2	0.05%	2.62%	0.01%	1.20%	6.85%	5.43%	0.22%	0.36%	0.25%	3.09%	0.24%	1.26%	4.38%	4.15%	0.17%	0.25%
						2.67%		1.21%		12.28%		0.58%		3.34%		1.50%		8.53%		0.42%	
Weibull	0.68	3.11	15.44	6	8	2.49%	12.18%	0.00%	2.40%	7.75%	4.91%	0.19%	0.55%	7.22%	12.94%	0.00%	1.54%	5.90%	5.47%	0.22%	0.44%
						14.67%		2.40%		12.66%		0.74%		20.16%		1.54%		11.37%		0.66%	
	0.60	1.43	2.77	2.8	2.25	0.11%	5.25%	0.01%	1.63%	6.10%	7.06%	0.16%	0.41%	0.59%	4.31%	0.05%	0.94%	5.85%	4.41%	0.12%	0.38%
						5.36%		1.64%		13.16%		0.57%		4.90%		0.99%		10.26%		0.50%	
	0.52	0.31	-0.20	2.66	1.1	0.01%	0.54%	0.03%	0.49%	4.24%	3.97%	0.15%	0.33%	0.05%	0.20%	0.28%	0.46%	3.65%	3.48%	0.17%	0.23%
						0.55%		0.52%		8.21%		0.48%		0.25%		0.74%		7.13%		0.40%	