Control Charts for Dependent and Independent Measurements Based on Bootstrap Methods

Regina Y. LIU and Jen TANG

Shewhart charts are widely accepted as standard tools for monitoring manufacturing processes of univariate, independent, and "nearly" normal measurements. They are not as well developed beyond these types of data. We generalize the idea of Shewhart charts to cover other types of data commonly encountered in practice. More specifically, we develop some valid control charts for dependent data and for independent data that are not necessarily "nearly" normal. We derive the proposed charts from the moving blocks bootstrap and the standard bootstrap methods. Their constructions are completely nonparametric, and no distributional assumptions are required. Some simulated as well as real data examples are included, and they are very supportive of the proposed methods.

KEY WORDS: Dependent processes; Moving blocks bootstrap; Nonparametric process control; Shewhart charts.

1. INTRODUCTION

The control chart, first developed by W. A. Shewhart, is a useful tool in statistical process control. It is an on-line process control technique used to detect the occurrence of any significant process mean change and to call for a corrective action. The construction of a control chart is basically equivalent to the plotting of the acceptance regions of a sequence of hypothesis testing over time. For example, the $\bar{X}$ chart is a control chart used to monitor the process mean μ. It plots the sample means, $\bar{X}$'s, of subgroups of the observations $\{X_1, X_2, \ldots\}$ and is equivalent to testing the hypotheses H_0: $\mu = \mu_0$ versus H_a: $\mu \neq \mu_0$ (for some target value μ_0) conducted over time, using $\bar{X}$ as the test statistic. Here we assume that $\{X_1, X_2, \ldots\}$ are the sample measurements of a particular quality characteristic following the distribution F with mean μ and standard deviation σ. When there is insufficient evidence to reject H_0, the process is said to be in control; otherwise, it is said to be out of control. The decision to accept or to reject H_0 is based on the value of the sample mean $\bar{X}$ observed at each time interval. These decision rules are graphically displayed in the control chart as the *upper control limits* (UCL) and the *lower control limits* (LCL), with the horizontal axis indicating the time order of the observed $\bar{X}$'s. The region between the control limits is the acceptance region of H_0. Hence the process is considered out of control when an observed subgroup sample mean falls outside the limits. When this occurs, it suggests that the process may have been affected by some assignable causes. Investigation of these causes should then be initiated. As in hypothesis testing, to obtain the control limits, we need to find the sampling distribution of $\bar{X} - \mu$ when H_0 is true. More precisely, for a given α, we need to locate two values, U and L, such that under H_0,

$$P(L < \bar{X} - \mu < U) = 1 - \alpha. \tag{1}$$

The probability here is calculated with respect to the underlying distribution of the sample F. Then the LCL and UCL are set at $(\mu_0 + L)$ and $(\mu_0 + U)$, and the level of the test α is now interpreted as the false alarm rate.

If F is normal or the subgroup size n is large, then the standard $\bar{X}$ chart plots the sample mean of each subgroup of n consecutive X_i's with LCL $= \mu_0 - 3\sigma/\sqrt{n}$ and UCL $= \mu_0 + 3\sigma/\sqrt{n}$. In this case α is .27%, and the *center line* is μ_0.

In practice, μ_0 and σ are generally not known or specified and are estimated using some past or training data taken when the process is considered in control. Suppose that the past data consist of k samples and that each sample is of size n. For the ith sample, we denote its sample mean and sample variance by $\bar{X}_i$ and S_i^2. We then estimate μ_0 and σ^2 by

$$\bar{\bar{X}}_N = \sum_{i=1}^{k} \bar{X}_i \quad and \quad S^2 = \sum_{i=1}^{k} S_i^2/k. \tag{2}$$

Note that we use "$=$" and "$-$" on top of X to differentiate the grand mean and the subgroup mean. We also omit the index of $\bar{X}$ when the subgroup mean is viewed as a general statistic. Under the normality assumption of $F, k(n-1)S^2/\sigma^2$ follows a chi-square distribution with $k(n-1)$ degrees of freedom. Consequently, under $H_0, \sqrt{n}(\bar{X} - \mu_0)/S$ follows a t distribution with $k(n-1)$ degrees of freedom. Because $k(n-1)$ is usually large in practice, the t distribution is sufficiently close to the standard normal distribution, and hence the control limits for $\bar{X}$ chart are approximately

$$\text{LCL} = \bar{\bar{X}}_N - z_{(1-\alpha/2)} S/\sqrt{n}$$

and

$$\text{UCL} = \bar{\bar{X}}_N + z_{(1-\alpha/2)} S/\sqrt{n}. \tag{3}$$

Here z_c indicates the c quantile of the standard normal distribution. Note that $\bar{\bar{X}}_N$ is the same as the average of all X_i's, and it serves now as the center line of the chart. An

Regina Y. Liu is Professor, Department of Statistics, Rutgers University, Piscataway, NJ 08855. Jen Tang is Associate Professor, Krannert Graduate School of Management, Purdue University, West Lafayette, IN 47907.
Research of the first author was partially supported by National Science Foundation grants DMS-90-04658 and DMS-90-22126. Research of the second author was partially supported by a grant from the Center for Management of Manufacturing Enterprises (CMME) of Purdue University.

Journal of the American Statistical Association
December 1996, Vol. 91, No. 436, Theory and Methods

alternative approach is to set

$$\text{LCL} = \bar{\bar{X}}_N - z_{(1-\alpha/2)} \bar{S}/(c_4 \sqrt{n})$$

and

$$\text{UCL} = \bar{\bar{X}}_N + z_{(1-\alpha/2)} \bar{S}/(c_4 \sqrt{n}), \tag{4}$$

if σ is estimated by $\bar{S} = \sum_{i=1}^{k} S_i/k$. Here c_4 is the bias-adjusting constant, which is tabulated in most of the standard statistical process control (SPC) textbooks (see, e.g., Montgomery 1991).

Two key assumptions in the foregoing standard methods are the normality of F and the independence between observations. The former may be relaxed as long as F is somewhat symmetric or n is large enough, but the latter is crucial. When the normality or any other parametric assumption of F is in question, we propose in Section 2 to use the bootstrap method to compute control limits. Possible advantages of using the bootstrap method over the normal approximation based on the central limit theorem is also discussed. Outside of the realm of iid (independent and identically distributed) samples, we often encounter observations that are statistically dependent or correlated. Common examples are stock market indices and measurements obtained from an automatically-adjusted manufacturing equipment. When the data are correlated, the true variance of $\bar{X}$ involves covariances between X_i's that are not well estimated by the pooled sample variance S^2 in (2). Thus the $\bar{X}$ chart in (3) or (4) is no longer valid. Although methods exist for dealing with correlated observations, they require normality for the process distribution (see, e.g., Box and Jenkins 1994). Without parametric assumptions, we apply and modify the moving blocks bootstrap method introduced by Künsch (1989) and Liu and Singh (1992) to develop valid control charts for weakly dependent data. These are presented in Section 3. Because the moving blocks bootstrap is a modification of the standard bootstrap, Section 2 may be viewed as a special case of Section 3. Three examples are presented in both Sections 2 and 3 to demonstrate the proposed methods. The results are quite supportive.

Away from the univariate data setting, statistical control of multivariate measurements is another direction that demands much attention today, because well-developed automatic inspection procedures are easily available for measuring multiple quality characteristics of each unit of the products. Several multivariate control methods have been proposed and studied, as surveyed by Alt and Smith (1988). However, most existing methods are suitable only for multivariate normal data. In a recent article of Liu (1995), based on the notions of data depth (see, e.g., Liu 1990 and Liu and Singh 1993), a nonparametric method was introduced to chart independent multivariate measurements. The charts there can be visualized as easily as the standard univariate control charts, and they can simultaneously detect the mean change and scale increase of the process. Perhaps combining this data depth approach and the moving blocks bootstrap ideas discussed in Section 3 may lead to a new nonparametric approach for statistical control of dependent multivariate observations.

2. $\bar{X}$ CHART FOR iid SAMPLES BASED ON BOOTSTRAP METHOD

The bootstrap method, introduced by Efron (1979), is a powerful tool for estimating the sampling distribution of a given statistic. The bootstrap estimate of the sampling distribution is generally better than the normal approximation based on the central limit theorem (cf. Bickel and Freedman 1981 and Singh 1981), even if the statistic is not standardized (cf. Beran 1982 and Liu and Singh 1987). Let $\{X_1, \ldots, X_N\}$ be an iid sample following the distribution F with mean μ and variance σ^2. The standard bootstrap procedure is to draw with replacement a random sample of size N from $\{X_1, \ldots, X_N\}$. Denote the bootstrap sample by $\{X_1^*, \ldots, X_N^*\}$ and denote their mean and standard deviation by $\bar{\bar{X}}_N^*$ and S_N^*. Let F_N indicate the empirical distribution of $\{X_1, \ldots, X_N\}$. The sampling distribution of $(\bar{\bar{X}}_N^* - \bar{\bar{X}}_N)$ under F_N is the bootstrap approximation of the sampling distribution of $(\bar{\bar{X}}_N - \mu)$ under F. Its approximation error is shown to be negligible by the following proposition derived by Bickel and Freedman (1981) and Singh (1981). Let $\|\cdot\|$ stand for the sup norm over $-\infty < x < \infty$.

Proposition 2.1. If the second moment of X's exists, then

$$\|P(\sqrt{N}(\bar{\bar{X}}_N^* - \bar{\bar{X}}_N) \le x | F_N) - P(\sqrt{N}(\bar{\bar{X}}_N - \mu) \le x | F)\| \to 0$$

almost surely as $N \to \infty$.

Hence the sampling distribution of $\sqrt{N}(\bar{\bar{X}}_N - \mu)$ can be approximated by its bootstrap counterpart; that is, $P(\sqrt{N}(\bar{\bar{X}}_N^* - \bar{\bar{X}}_N) \le x | F_N) \approx P(\sqrt{N}(\bar{\bar{X}}_N - \mu) \le x | F)$. In the context of $\bar{X}$ chart construction, N is the total number of observations available from the process. Assume that $N = kn$, with n the subgroup size and k the number of subgroups. Because the $\bar{X}$ chart plots the subgroup sample means, $\bar{X}_n$'s, the control limits should be obtained from the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)$. In fact, this sampling distribution can be approximated by bootstrap following the observation that for any x,

$$P(\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N) \le x | F_N) \approx P(\sqrt{n}(\bar{X}_n - \mu) \le x | F). \tag{5}$$

This observation can be obtained as follows. As n and N tend to ∞, both $\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N)$ and $\sqrt{N}(\bar{\bar{X}}_N^* - \bar{\bar{X}}_N)$ converge weakly to $\mathcal{N}(0, \sigma^2)$, a normal distribution with mean zero and standard deviation σ. Similarly, $\sqrt{n}(\bar{X}_n - \mu)$ and $\sqrt{N}(\bar{\bar{X}}_N - \mu)$ also converge to $\mathcal{N}(0, \sigma^2)$. The observation thus follows using the triangular inequality and Proposition 2.1.

The realization of (5) leads to an alternative approach to constructing an $\bar{X}$ chart for iid observations—to repeat the bootstrap procedure say K times and form a histogram of the resulting K terms of $\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N)$, and then locate the $\alpha/2$ and $(1-\alpha/2)$ quantiles. These are then used as the estimated $\alpha/2$ and $(1-\alpha/2)$ quantiles of $\sqrt{n}(\bar{X}_n - \mu)$. In

Table 1. Comparison Between the Bootstrap and Standard Methods

α	10% LCL	10% UCL	5% LCL	5% UCL	2% LCL	2% UCL
			Normal (0, 1)			
Standard	−.794	.794	−.946	.946	−1.123	1.123
Bootstrap	−.817	.816	−.955	.958	−1.139	1.145
Exact	−.821	.821	−.979	.979	−1.162	1.162
			Exponential (1)			
Standard	.280	1.733	.141	1.872	−.021	2.034
Bootstrap	.333	1.947	.276	2.120	.195	2.328
Exact	.342	1.938	.273	2.192	.206	2.511

other words, using the bootstrap histogram, we obtain $\tau_{\alpha/2}$ such that

$$\alpha/2 = P(\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N) \le \tau_{\alpha/2}|F_N), \quad \text{a.s.}$$

This leads to

$$\alpha/2 \approx P(\bar{X} \le \mu + \tau_{\alpha/2}/\sqrt{n}|F),$$

in view of (5). Finally, we obtain the control limits for the $\bar{X}$ chart as

$$\text{LCL} = \bar{\bar{X}}_N + \tau_{\alpha/2}/\sqrt{n}$$

and

$$\text{UCL} = \bar{\bar{X}}_N + \tau_{(1-\alpha/2)}/\sqrt{n}, \tag{6}$$

using $\bar{\bar{X}}_N$ as an estimate of μ under H_0.

Using Edgeworth expansions, Beran (1982) and Liu and Singh (1987) have shown that when $n = N$ (i.e., $k = 1$), the aforementioned standard bootstrap approximation of the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)$ is superior to that of the normal approximation, in terms of mean squared errors (MSE's). This phenomenon is termed the *partial correction by the bootstrap.* We briefly describe this same phenomenon in our case here for $k \ge 1$, because this allows us to claim that *if F is not normal, then the control limits obtained by the bootstrap in (6) are more accurate than the ones derived from normal approximation in (3) or (4).* We first state the Edgeworth expansions of the relevant sampling distributions and then compare their MSE's.

Consider $N = kn$, where k is a fixed constant. A slight modification of the Edgeworth expansions of Liu and Singh (1987) gives the expressions

$$\begin{aligned} H_n(x) &= P(\sqrt{n}(\bar{X}_n - \mu) \le x|F) \\ &= \Phi\left(\frac{x}{\sigma}\right) + \frac{\nu_F}{6\sigma^3\sqrt{n}}\left(1 - \frac{x^2}{\sigma^2}\right)\phi\left(\frac{x}{\sigma}\right) + o(n^{-1/2}) \end{aligned} \tag{7}$$

and

$$\begin{aligned} H_{n,N}^*(x) &= P(\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N) \le x|F_N) \\ &= \Phi\left(\frac{x}{S_{F_N}}\right) + \frac{\nu_{F_N}}{6S_{F_N}^3\sqrt{n}}\left(1 - \frac{x^2}{S_{F_N}^2}\right) \\ &\quad \times \phi\left(\frac{x}{S_{F_N}}\right) + o(n^{-1/2}), \quad \text{a.s.,} \end{aligned} \tag{8}$$

where $S_{F_N}^2 = \sum_{i=1}^N (X_i - \bar{\bar{X}}_N)^2/(N-1)$ and ν_G denotes the third central moment of the distribution G. Expressions (7) and (8) imply that the approximation of $H_n(x)$ by $H_{n,N}^*(x)$ results in an error of the order of $o(n^{-1/2})$ as n and N tend to ∞. The same order of error also exists if we use the standard normal approximation $\Phi(x/S_{F_N})$ instead. However, if we examine the two approximation errors further, we obtain the following:

Proposition 2.2. Assume that the conditions in Proposition 2.1 hold, and $\sqrt{N}(S_{F_N}^2 - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, a^2)$ for some a as N tends to ∞. Then, as n tends to ∞,

$$\sqrt{n}(H_n(x) - H_{n,N}^*(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, c^2)$$

and

$$\sqrt{n}(H_n(x) - \Phi(x/S_{F_N})) \xrightarrow{\mathcal{L}} \mathcal{N}(b, c^2),$$

where $b = (\nu_F/6\sigma^3)[1 - (x^2/\sigma^2)]\phi(x/\sigma)$ and $c^2 = x^2a^2\phi^2(x/\sigma)/\{k\sigma^6\}$.

In other words, *even though the approximation errors from the bootstrap and the normal approximation are of the same order of magnitude, bootstrap approximation actually eliminates the bias and thus has a smaller asymptotic* MSE. The amount of the reduction in MSE here is b^2. This phenomenon for the standard bootstrap without grouping was first observed by Beran (1982).

1.2 Simulation Example A

Here we provide a simulation comparison of the control limits obtained by the bootstrap in (6) and by the normal approximation in (3) or (4) for two process distributions: $\mathcal{N}(0, 1)$ and the exponential distribution with mean parameter $\lambda = 1$. In each case we draw a random sample of size 100 and form a subgroup for every four consecutive observations; that is, $n = 4$ and $k = 25$. Here $\bar{\bar{X}}$ is the average of $\{X_1, \ldots, X_{100}\}$, and $\bar{X}$ is the average within a subgroup. From $\{X_1, \ldots, X_{100}\}$, we obtain 200 bootstrap samples (each of size 4) and compute the corresponding 200 values of $\sqrt{n}(\bar{X} - \bar{\bar{X}})$'s. Using formula (6), we compute the control limits and list them in Table 1 for $\alpha = 10\%, 5\%$, and 2%. Table 1 also lists the exact control limits and the ones computed from the standard normal approximation formula (3). Note that the exact control limits for the exponential case can be obtained easily through the gamma distribution (with parameter 4) associated with $\bar{X}$. Figures 1 and 2 plot the three sets of control limits with $\alpha = .05$ for $\mathcal{N}(0, 1)$ and exponential(1).

In principle, the standard normal approximation should be expected to outperform the bootstrap for the normal case, because the method is designed specifically under the assumption of normality. However, the results here actually suggest that the bootstrap competes well under the normal model. For the case of exponential distribution, which is highly skewed with a long right tail, the results in Table 1 clearly indicate that the bootstrap is more effective in capturing the skewness of the underlying distribution, if there is any. This supports the assertion that we made in Proposition 2.2, that the bootstrap can provide a better approximation

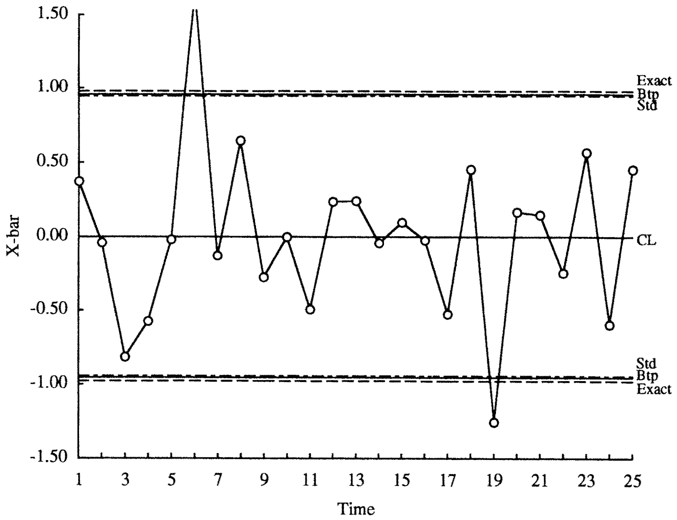

Figure 1. Comparisons Under Standard Normal Distribution (α = 5%).

of the sampling distribution of a statistic. In summary, we conclude that the control limits obtained by bootstrap are superior to those obtained by the normal approximations.

3. CONTROL CHARTS FOR DEPENDENT OBSERVATIONS

Many different types of dependence structure have been studied in the literature, but here we focus only on m-dependence. Here m is a nonnegative integer. A sequence of data $\{X_1, X_2, \ldots\}$ is said to be m-dependent if the data are dependent, unless their time indices are at least m units apart. Generally, m-dependence models quality measurements well, because sample observations from a manufacturing process are generated at equally spaced time intervals where cross correlation is expected to exist between neighboring observations. When observations are dependent, the standard method described in (3) or (4) of Section 1 is not valid, because the S^2/n there generally underestimates (overestimates) the variance of the sample mean if the observations are positively (negatively) correlated. As a matter of fact, this estimation error is higher than that obtained from using S_N^2/n, where $S_N^2 = \sum_{i=1}^N (X_i - \bar{\bar{X}}_N)^2/(N-1)$ is the sample variance based on all $N = nk$ observations without grouping. (See also the discussion in simulation example B in Sec. 3.2.) In addition to m-dependence, we also assume our observations $\{X_1, X_2, \ldots\}$ to be stationary throughout Section 3. This means that the distribution of any subset of observations of X_i's does not change when their indices are shifted by the same number of units. For instance, (X_3, X_5, X_{10}) has the same distribution as (X_6, X_8, X_{13}).

To facilitate the description of our $\bar{X}$ chart for m-dependent data, we begin by reviewing the moving blocks bootstrap, which is a modified bootstrap procedure that can capture the m-dependence structure.

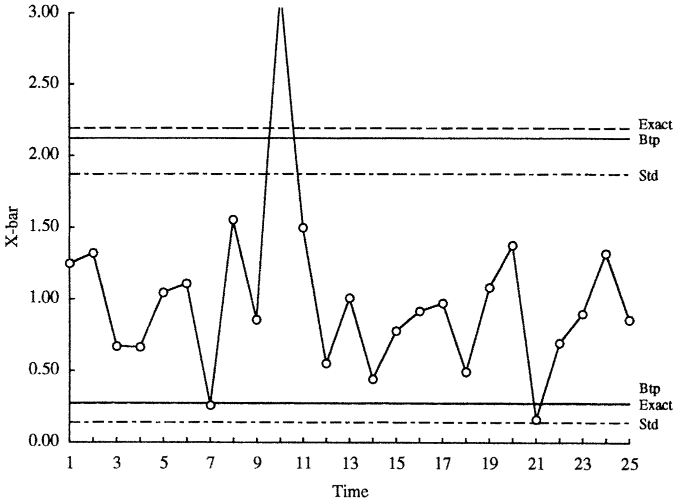

Figure 2. Comparisons Under Exponential Distribution with λ = 1 (α = 5%).

3.1 The Moving Blocks Bootstrap

Let $\{X_1, \ldots X_N\}$ be a stationary m-dependent sample observation, and let B_i's be its moving blocks of size b; namely, $B_i = \{X_i, X_{i+1}, \ldots, X_{i+b-1}\}$ for $i = 1, \ldots, N - b + 1$. The *moving blocks bootstrap* (MBB) method was introduced by Künsch (1989) and Liu and Singh (1992) for bootstrapping weakly dependent data. Under MBB, we sample with replacement from the moving blocks $B_1, \ldots, B_{N-b+1}$, with all the B_i's being equally likely to be drawn. Suppose that k blocks are drawn and are denoted by $\xi_1, \ldots, \xi_k$. Each of ξ_i consists of b elements, which may be expressed as $\xi_i \equiv (\xi_{i1}, \ldots, \xi_{ib})$. Assume that $N = bk$. We refer to $(X_1^*, \ldots, X_N^*) \equiv (\xi_{11}, \ldots \xi_{1b}, \ldots, \xi_{k1}, \ldots \xi_{kb})$ as the bootstrap sample under the MBB. For example, with $b = 2$ and $k = 3$, if $\xi_1 = (9, 2), \xi_2 = (7, 4)$, and $\xi_3 = (4, 6)$, then the bootstrap sample is (9, 2, 7, 4, 4, 6). Again, we use $\bar{\bar{X}}_N^*$ and $\bar{\bar{X}}_N$ to indicate the mean of the bootstrap sample and the mean of the original sample. As in the iid case in Section 2, we need to derive the sampling distribution of $\sqrt{n}(\bar{X}_n - \mu)$ to obtain the control limits for charting $\bar{X}$'s. We show that the MBB procedure provides a valid approximation of the foregoing sampling distribution. To see what the MBB actually achieves in this approximation, we first examine the relevant asymptotic variances involved in different stages of approximation.

Under m dependence, $m \leq n$, it can be shown that the variance of $\sqrt{n}(\bar{X}_n - \mu)$ is $\sigma_n^2 \equiv \sigma^2 + 2\sum_{j=1}^m [1 - (j/n)]\text{cov}(X_1, X_{1+j})$. Clearly, getting a good estimate of σ_n^2 will allow us to obtain proper control limits for charting $\bar{X}$'s. The following two propositions, given by Liu and Singh (1992), show how MBB can provide a good estimate of σ_n^2 and how the sampling distribution of $\sqrt{n}(\bar{X} - \mu)$ can be approximated by its bootstrap counterpart.

Proposition 3.1. Let $\{X_1, X_2, \ldots\}$ be a sequence of stationary m-dependent observations. If the second moment of the X_i's is finite, then

$$\|P(\sqrt{N}(\bar{\bar{X}}_N^* - \bar{\bar{X}}_N) \leq x | F_N) - \Phi(x/\sigma_b)\| \to 0$$

almost surely as $N \to \infty$ for any fixed block size b. Here $\Phi(\cdot)$ denotes the standard normal cumulative distribution.

Proposition 3.2. Let $\{X_1, X_2, \ldots\}$ be a sequence of stationary m-dependent observations. If the $(4 + \delta)$th moment

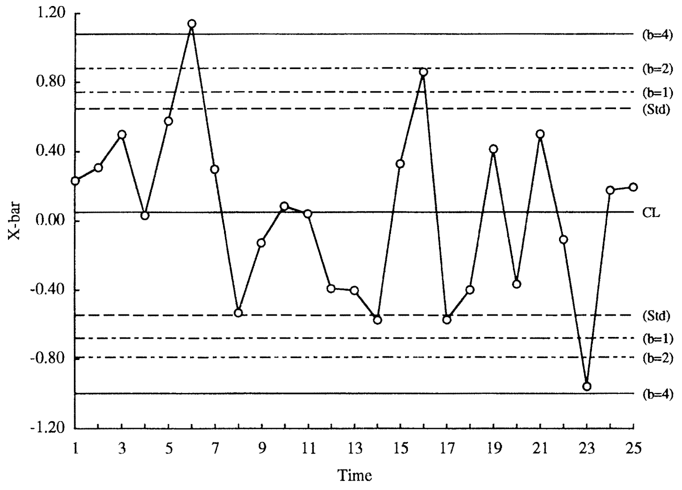

Figure 3. $\bar{X}$-Bar Chart for Dependent Data.

of the X_i's is finite with $\delta > 0$ and if the block size $b \to \infty$ and $b/\sqrt{N} \to 0$ as $N \to \infty$, then

$$\|P(\sqrt{N}(\bar{\bar{X}}_N^* - \bar{\bar{X}}_N) \leq x|F_N) - P(\sqrt{N}(\bar{\bar{X}}_N - \mu) \leq x|F)\| \to 0$$

in probability as $N \to \infty$.

If we let $b = n$ in Proposition 3.1, then we immediately realize that the bootstrap sampling distribution converges to a normal distribution with variance σ_n^2, the true variance of $\sqrt{n}(\bar{X} - \mu)$, which we need for obtaining correct control limits. Using arguments similar to the ones in Section 2 and Proposition 3.2, we obtain

$$P(\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N) \leq x|F_N) \approx P(\sqrt{n}(\bar{X} - \mu) \leq x|F)$$

if n is not too small, which in turn leads to the following two-step computation of the control limits for $\bar{X}$ chart stated in (9). First, we repeat the MBB procedure with $b = n$ say K times to obtain K terms of $\bar{X}_n^*$'s. Locate the $(\alpha/2)$th and $(1 - \alpha/2)$th quantiles from the histogram formed by the K values of $\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N)$, and denote them by $\tau_{\alpha/2}$ and $\tau_{(1-\alpha/2)}$. Then use these two values as the estimates of the $(\alpha/2)$th and $(1 - \alpha/2)$th quantiles of the distribution of $\sqrt{n}(\bar{X} - \mu)$, so that

$$P(\tau_{\alpha/2} \leq \sqrt{n}(\bar{X} - \mu) \leq \tau_{(1-\alpha/2)}) \approx 1 - \alpha.$$

Finally, this suggests that we take

$$\mathrm{LCL} = \bar{\bar{X}}_N + \tau_{\alpha/2}/\sqrt{n}$$

and

$$\mathrm{UCL} = \bar{\bar{X}}_N + \tau_{(1-\alpha/2)}/\sqrt{n}. \tag{9}$$

To illustrate the ideas in the foregoing procedure, we specifically work out an example with $m = 1, n = 4$, and various choices of b. Note that here $\mathrm{var}(\sqrt{n}(\bar{X}_n - \mu)) = \sigma^2 + \frac{3}{2}\,\mathrm{cov}(X_1, X_2)$. In view of Proposition 3.1 and the arguments for (5), the achieved asymptotic variance of $\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N)$, as $N \to \infty$, by MBB under $b = 1, 2, 3$, and 4 are

$$\sigma_1^2 = \sigma^2,$$

$$\sigma_2^2 = \sigma^2 + \mathrm{cov}(X_1, X_2),$$

$$\sigma_3^2 = \sigma^2 + \frac{4}{3}\,\mathrm{cov}(X_1, X_2),$$

and

$$\sigma_4^2 = \sigma^2 + \frac{3}{2}\,\mathrm{cov}(X_1, X_2). \tag{10}$$

Clearly, MBB achieves the correct asymptotic variance by taking the block size b to be 4. With the smaller block sizes, MBB is missing part of the covariance structure but catches up gradually as the block size increases. Note that when $b = 1$, MBB becomes the standard bootstrap for iid observations discussed in Section 2. It should be obvious now that the standard bootstrap is not valid for m-dependent data with $m \geq 1$, because it fails to capture any of the covariance. To further assess this performance of MBB-based control limits, we provide some simulation results in Example B and a detailed analysis of a real data set in Example C in Section 3.3.

3.2 Simulation Example B

First, we obtain a random sample of 101 points from $\mathcal{N}(0,\ 1)$ and denote them by $\{A_1, \ldots, A_{101}\}$. Next, we obtain independently another sample of 100 from $\mathcal{N}(0,\ .01)$ and denote them by $\{y_1, \ldots, y_{100}\}$. Let $X_i = (A_i + A_{i+1})/2 + y_i, i = 1, \ldots, 100$; that is, $N = 100$. Clearly the X_i's are 1-dependent, and X_i follows $\mathcal{N}(0,\ .51)$ with $\mathrm{cov}(X_i, X_{i+1}) = .25$. Applying formula (10), we obtain $\sigma_1^2 = .51, \sigma_2^2 = .76$, and $\sigma_4^2 = .885$. We repeat the MBB procedure with $b = 4 (= n)$ 1,000 times to obtain 1,000 bootstrap sample average $\bar{X}_n^*$'s, and form the histogram of the 1,000 values of $\sqrt{n}(\bar{X}_n^* - \bar{\bar{X}}_N)$. Here $\bar{\bar{X}}_N$ is the average of the original 100 X_i's, which is $-.00499$ in our case. If we assume $\alpha = .05$, then we should locate the 2.5th and the 97.5th percentiles from the histogram, which turn out to be -1.9876 and 2.1674. Finally, the formula in (9) gives LCL $= -.9988$ and UCL $= 1.0787$. These limits seem comparable with the true limits $\pm\ .92193\, (= \pm 1.96\sigma_4/\sqrt{4})$. In Figure 3 we present a $\bar{X}$-chart which plots the 25 subgroup means from a new sample of 100 X_i's. Four sets of control limits are plotted in Figure 3, and their values are reported in Table 2. The center line in Figure 3 is $-.00499$. The most outlying set of limits corresponds to the case of $b = 4$. Moving inward, the next two sets correspond to $b = 2$ and $b = 1$. For comparison, we have computed the standard Shewhart limits, using (4) based on the pooled S. These limits (labeled Std) are obviously worse than all of the aforementioned sets. This is because S^2(pooled) actually estimates

Table 2. UCL's and LCL's Under Different Block Sizes and Standard Method

Block size	LCL	UCL	True var($\bar{X}_n$)	Simulated var($\bar{X}_n$)
$b = 4$	$-.9988$	1.0787	.2130	.2486
$b = 2$	$-.7906$	.8805	.1865	.1776
$b = 1$	$-.6784$	.7407	.1335	.1351
S_N	$-.6662$	.7660		
S(pooled)	$-.5463$	.6462		

Table 3. Reactor Outlet Concentration Data

Subgroup	x_1	x_2	x_3	x_4	x_5	$\bar{x}$	S
1	3.250	3.186	2.865	2.750	2.813	2.973	.229
2	2.970	2.940	3.170	3.000	2.895	2.995	.105
3	2.870	2.750	2.938	3.000	3.060	2.924	.120
4	3.050	2.900	2.980	2.875	2.820	2.925	.091
5	2.875	3.188	3.300	3.200	2.813	3.075	.217
6	2.825	2.900	2.825	2.900	2.825	2.855	.041
7	2.970	2.960	2.900	3.030	3.000	2.972	.049
8	2.830	2.800	2.890	2.850	2.980	2.870	.070
9	2.962	3.050	2.942	3.150	3.188	3.058	.110
10	3.220	3.188	2.875	2.813	2.938	3.007	.186
11	2.955	2.938	2.955	2.938	2.985	2.954	.019
12	3.125	3.280	3.155	3.188	3.090	3.168	.073
13	3.063	3.000	3.030	2.900	2.850	2.969	.090
14	3.188	3.188	3.150	3.175	3.125	3.165	.027
15	3.200	3.100	3.080	2.970	2.940	3.058	.105
16	2.980	2.910	2.940	2.780	2.900	2.902	.075
					Average	2.992	.100

the covariance between X_i and X_{i+1} in the opposite direction, and in this example here it severely *underestimates* the true variance. Table 2 also includes the set of limits derived from $\bar{\bar{X}}_N \pm 1.96 S_N/\sqrt{n}$, which should be appropriate only when the data are iid and normally distributed. This is actually the best that the standard iid bootstrap (or, equivalently, our MBB procedure with $b = 1$) can achieve. Therefore, it is not surprising that this set of limits is quite close to that based on $b = 1$ in MBB. Finally, we point out that the figure shows one out-of-control point according to the limits set by $b = 4$, which confirms our α level. The remaining sets of limits are clearly too narrow, which would have declared the process out of control more often than it should.

We would like to point out that as b decreases in value, their corresponding sets of limits move inward here is due to the positive correlation imposed on X_i's. If X_i's are negatively correlated, then the foregoing sets of limits instead would move outward, leaving out-of-control observations largely undetected.

3.3 Example C: Reactor Outlet Concentration Data

In many industries, automatic process control (APC) is used to identify deviations from the desired operation and make immediate adjustments in a controlled variable. The adjustment is selected to compensate the effects of the disturbance and maintain the controlled variable at its desired value. However, APC does not eliminate the cause of poor operation in terms of variability or disturbance, leaving the process susceptible to future disturbances from the same sources. By contrast, the goal of statistical process control (SPC) is the identification and elimination of disturbances. Its long-term effect is to reduce the variability of process operations and to improve product quality. Because some variability in process operations is unavoidable, SPC alone cannot do the job. However, SPC and APC can complement each other to provide a fuller process control.

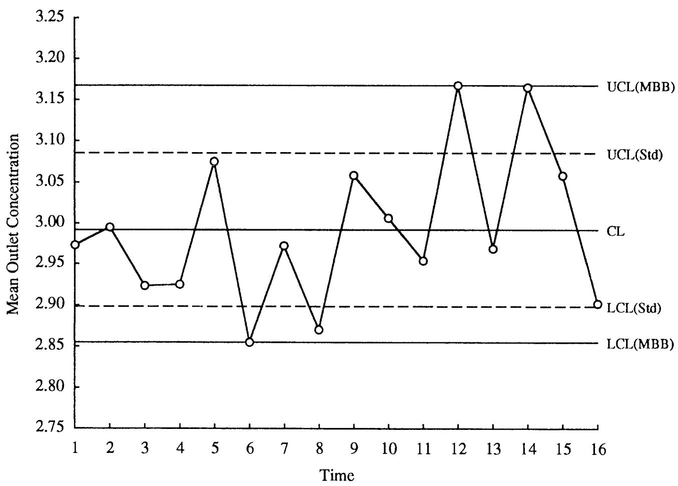

Figure 4. Control Chart for Reactor Outlet Concentration Data.

We consider now the case study given by Marlin (1995, p. 490) for APC using a proportional-integral-derivative (PID) feedback control to regulate the packed-bed reactor of a chemical process. The goal here is to tightly control the exit concentration (in moles/m^3) in the effluent of the reactor. The measurements of the outlet concentration are used for feedback control to adjust the heating medium valve in the reactor preheat exchanger through the PID algorithm. The process flow and feedback loop are described in figures 14.11 and 14.12 of Marlin (1995). There are several possible causes for variability in the outlet concentration, including feed temperature, heating oil pressure, feed flow rate, and feed composition.

Initially, measurements on outlet concentration and valve position (% open) were taken without any feedback action/control and without SPC. The data were shown in figure 26.17 of Marlin (1995). The measurements of outlet concentration are correlated in time, and the feedback dynamics are fast enough so that adjustments in the valve can compensate for slowly changing trends. We use the measurements after the PID control is implemented, starting from the 120th minute to allow PID to take effect so that the process performance is close to the desired value; that is, the process is stationary. The data between $t = 120$ and $t = 179$ contains $5 \times 16 = 80$ observations and are given in Table 3 with subgroup size $n = 5$. We found that this data series X_i's satisfy an auto-regressive model of order 2. That is

$$X_i = 1.579 + .7224X_{i-1} - .2514X_{i-2} + e_i,$$

where e_i is the error variable with mean zero. The p values of the zero coefficient tests here are .00, .00 and .02 respectively. Other higher order auto-regressive explanatory variables all turn out to be insignificant. From p. 60 of Box and Jenkins (1994), we can also show that X_i's follow a stationary process. In other words, the measurements here appear to be m-dependent, with $m = 2$.

For charting the subgroup means listed in Table 3, the MBB procedure give control limits (2.855, 3.168), based on block size $b = 5$ and 4,000 resamplings. Ignoring the dependence, the standard Shewhart control limits are (2.898, 3.085). Both charts have the center line at 2.992. As we have shown in simulation example B, a significant positive correlation between variables will cause the Shewhart control limits to be narrower than they should be. In fact, this happens again in this example, as shown in Figure 4. Furthermore, the original data on the outlet concentration show a right-skewed distribution with the skewness coefficient .374. This skewness is reflected in the bootstrap control limits, as they are asymmetric, whereas the Shewhart charts are still symmetric because they are based on the normal approximation.

If we use Shewhart limits, Figure 4 shows that there are four false alarms in 16 points, giving a Type I risk of 25%. This high false alarm rate will subsequently lead to overadjustments in the implementation of APC. With the MBB control limits, the number of false alarms is reduced to two. Although this rate of $1/8 = 12.5\%$ is still higher than the desired $\alpha = 5\%$, it is clearly due to the relatively small number of subgroups. In summary, the MBB control limits substantially reduce the unnecessarily frequent claims of out-of-control conditions made by the use of Shewhart limits, and avoid/eliminate possible overadjustments altogether.

4. CONCLUSIONS

The main goal of this article is to show how the bootstrap method can be applied in statistical quality control to obtain valid control charts for stationary weakly dependent data, as well as for iid data that are not normally distributed. These bootstrap control limits are new, and they generalize the idea of Shewhart charts beyond the scope of iid normal data. Furthermore, they are easily applicable because they are completely nonparametric. One of the key features of the MBB approach that we introduce in Section 3 is that it applies for any stationary dependent data, *without* the dependence structure being specified. This is in marked contrast to previous existing methods, such as that of Box and Jenkins (1994), where the dependence structure must be specified in addition to the assumption of normality. In fact, *the MBB control limits remain valid even when the underlying distribution of the data is not stationary, as long as the sampling distribution of the statistic for charting remains the same throughout time.*

Although the MBB approach is designed for dependent data, it will still give correct results when the data are independent. By choosing the block size b in MBB to be the same as the subgroup size n, the MBB control limits reflect the true covariance of the data set (cf (10)), which simply vanishes when the data are independent. Thus when one is unsure about whether the data are independent or not, one should use the MBB method.

Finally, we note that, in addition to the $\bar{X}$ charts mentioned earlier, the MBB control limits are also valid for moving average charts if the MBB block size is chosen to be the same as the moving average span in the moving average chart. This is the case for both iid and stationary weakly dependent data.

[Received May 1995. Revised December 1995.]

REFERENCES

Alt, F., and Smith, N. (1988), "Multivariate Process Control," in *Handbook of Statistics*, 7, eds. P. R. Krishnaiah and C. R. Rao, Amsterdam: Elsevier, pp. 333–351.

Beran, R. J. (1982), "Estimated Sampling Distributions: The Bootstrap and Competitors," *The Annals of Statistics*, 10, 212–215.

Bickel, P. J., and Freedman, D. A. (1981), "Some Asymptotic Theory for the Bootstrap," *The Annals of Statistics*, 9, 1196–1217.

Box, G. E. P., and Jenkins, G. M. (1994), *Time Series Analysis, Forecasting, and Control*, (3rd ed.), San Francisco: Prentice-Hall.

Efron, B. (1979), "Bootstrap Methods: Another Look at the Jackknife," *The Annals of Statistics*, 7, 1–26.

Künsch, H. (1989), "The Jackknife and the Bootstrap for General Stationary Observations," *The Annals of Statistics*, 17, 1217–1241.

Liu, R. (1990), "On a Notion of Data Depth Based on Random Simplices," *The Annals of Statistics*, 18, 405–414.

——— (1995), "Control Charts for Multivariate Processes," *Journal of the American Statistical Association*, 90, 1380–1388.

Liu, R., and Singh, K. (1987), "On a Partial Correction by the Bootstrap," *The Annals of Statistics*, 15, 1713–1718.

——— (1992), "Moving Blocks Bootstrap and Jackknife Capture Weak Dependence," in *Exploring the Limits of Bootstrap*, eds. R. LePage and L. Billard, New York: Wiley, pp. 225–248.

——— (1993), "A Quality Index Based on Data Depth and Multivariate Rank Tests," *Journal of the American Statistical Association*, 88, 252–260.

Marlin, T. E. (1995), *Process Control: Designing Processes and Control Systems for Dynamic Performance*, New York: McGraw-Hill.

Montgomery, D. C. (1991), *Introduction to Statistical Quality Control* (2nd ed.), New York: Wiley.

Singh, K. (1981), "On the Asymptotic Accuracy of Efron's Bootstrap," *The Annals of Statistics*, 9, 1187–1195.