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Median control charts for monitoring asymmetric quality characteristics double bounded

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Abstract
In many practical situations, the quality characteristics of interest assume values in the range (0,1), like rates and proportions (but they are not results from Bernoulli experiments). Most control charts built for these quality characteristics rely on monitoring parameters of their probability distribution functions or on their averages after some reparameterization of their density probability function. However, for highly asymmetric distributions, the median is a more appropriate location parameter than the average. In this paper, we propose Shewhart-type control charts for monitoring the median of observations taken from quality characteristics double bounded after reparameterization of two probability density functions: Kumaraswamy and unit Weibull. The performance of the control charts is evaluated and compared in terms of run length (RL) analysis considering three estimators for the median. Finally, we also carry out two applications to demonstrate the applicability of these control charts.

KEYWORDS
average run length, double-bounded processes, Kumaraswamy distribution, reparameterization, unit-Weibull distribution

1 | INTRODUCTION

Attribute control charts are important tools in statistical process control to monitor processes with discrete data.[1] In particular, there is interest in monitoring/controlling rates and proportions of the quality characteristics. When the quality characteristics are results of Bernoulli experiments, the *np* or *p* control charts have been used for this aim.[2]

Alternative control charts have been proposed by many contributors for situations where the rates and proportions are not results of Bernoulli experiments although they assume values in the standard unit interval. For example, Sant'Anna and ten Caten[1] presented the beta control chart, where the quality characteristic follows a beta probability distribution.[3] In situations where the quality characteristic is affected by external covariates, Bayer et al.[4] suggested the beta regression control chart considering the beta regression model varying dispersion. Lima-Filho et al.[5] proposed a control chart for monitoring double-bounded processes in the intervals (0,1], [0,1), or [0,1], based on the inflated beta probability distribution. For quality characteristics following simplex and unit-gamma distributions, Ho et al.[6] have studied the consequences in terms of speed to sign shifts on the rates when control charts built for beta distribution are equivocally used to monitor simplex (or unit-gamma) quality characteristics (and vice versa). These contributions[4-6] were considered monitoring the average rate or proportion after a reparameterization of the probability density functions.

However, in this paper, we are concerned with situations when the mean may not be a good location measure to represent a set of data, particularly in data sets with strong asymmetry. In these cases, the median appears to be more adequate as it is a robust measure of location parameters mainly in the presence of outliers.

The contribution of this paper is twofold. First, we presented a median-dispersion reparameterization for unit-Weibull distribution.[7] Second, Shewhart control charts are proposed for monitoring the stability of their medians in two cases: when only individual observations are available and for sample size $n>1$, three estimators for the median are considered. The performance of the control charts is made in terms of the usual metrics, to know average (ARL), median (MRL), and standard deviation (SDRL) of the run length (RL).

Kumaraswamy and unit-Weibull distributions are chosen in this study as they exhibit suitable features as closed expressions for their cumulative distribution functions as also their inverse function. Thus, the median-dispersion reparameterizations in these distributions can be achieved. To our knowledge, there is a gap in the current literature on control charts for these distributions. Only few contributions are found in literature. For example, Lima-Filho and Bayer[8] proposed a control chart for monitoring the shape parameter of the Kumaraswamy distribution but not for the median and applied it in an environmental data set.

One may claim that the reason such median-dispersion reparameterizations were not tried for other distributions such as beta, simplex, and unit-gamma density functions is because they also assume values in the standard unit. As the cumulative distribution functions of these distributions do not present closed expressions, such median-reparameterizations are not feasible. On the other hand, the mean reparameterizations are allowable for the beta, simplex, and unit-gamma distributions but unfeasible for the Kumaraswamy and unit-Weibull distributions. For example, the average and variance of unit-Weibull distribution do not have closed expressions.

The paper proceeds as follows. Section 2 covers a brief review of the Kumaraswamy and unit-Weibull distributions and their respective median reparameterizations for modeling rates and proportions in the intervals (0,1). The Shewhart-type control charts for individual observations from the Kumaraswamy and unit-Weibull distributions are presented in Section 3, and their performances are evaluated in terms of RL. In Section 4, control charts for sample size $n>1$ are developed considering three estimators for the median: the sample median (SM), the estimator proposed by Hodges and Lehmann (HL),[9] and the maximum likelihood (ML). The performance of these control charts is compared in terms of ARL. Applications using two real data sets are illustrated in Section 5. Finally, concluding remarks are outlined in Section 6.

2 | DISTRIBUTIONS TO MODEL DOUBLE-BOUNDED PROCESSES

In this section, we briefly present the Kumaraswamy and unit-Weibull distributions reparameterized in terms of their medians in order to be used to model data that assume values in the unit interval.

2.1 | Kumaraswamy distribution

Suppose that Y_K follows the Kumaraswamy distribution with shape parameters $\delta_K > 0$ and $\gamma_K > 0$, denoted by $Y_K \sim$ Kuma (δ_K, γ_K). The probability density function (pdf) is defined by[10]

$$f(y_K;\delta_K,\gamma_K) = \delta_K\,\gamma_K\,y_K^{\delta_K-1}\left(1-y_K^{\delta_K}\right)^{\gamma_K-1},\ 0<y_K<1.$$

The mean and variance of $Y_K \sim$ Kuma(δ_K, γ_K) are given, respectively, by

$$\mathrm{E}(Y_K) = \gamma_K B\left(1+\frac{1}{\delta_K},\gamma_K\right),$$

and

$$\mathrm{Var}(Y_K)=\gamma_K B\left(1+\frac{2}{\delta_K},\gamma_K\right)-\left\{\gamma_K B\left(1+\frac{1}{\delta_K},\gamma_K\right)\right\}^2,$$

where $B(a,b)=\int_0^1 s^{a-1}(1-s)^{b-1}\mathrm{d}s=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, where $\Gamma(\nu)=\int_0^\infty t^{\nu-1}\mathrm{e}^{-t}\mathrm{d}t$ is the gamma function.

The cumulative distribution and quantile functions are defined, respectively, by

$$\begin{aligned} F(y_K;\delta_K,\gamma_K)&=P(Y\le y_K)=\int_0^{y_K} f(t;\delta_K,\gamma_K)dt=1-(1-y_K^{\delta_K})^{\gamma_K},\ 0<y_K<1,\\ \psi(u;\delta_K,\gamma_K)&=F(u;\delta_K,\gamma_K)^{-1}=\{1-(1-u)^{1/\gamma_K}\}^{1/\delta_K},\ 0<u<1. \end{aligned} \tag{1}$$

Using (1), an expression for the median of the distribution can be obtained,[11] which is given by

$$\mathrm{md}(Y_K)=\tilde{\mu}_K=(1-0.5^{1/\gamma_K})^{1/\delta_K}.$$

Thus, making $\varphi_K=\delta_K$ (dispersion parameter), the parameter γ_K can be expressed by

$$\gamma_K=\frac{\ln(0.5)}{\ln(1-\tilde{\mu}_K^{\varphi_K})}.$$

For the Kumaraswamy distribution, we adopted the following notation $Y_K\sim\mathrm{Kuma}(\tilde{\mu}_K,\varphi_K)$. Considering this reparameterization, the probability density, cumulative distribution, and quantile functions can be re-expressed as

$$f(y_K;\tilde{\mu}_K,\varphi_K)=\frac{\varphi_K\ln(0.5)}{\ln(1-\tilde{\mu}_K^{\varphi_K})}y_K^{\varphi_K-1}(1-y_K^{\varphi_K})^{\frac{\ln(0.5)}{\ln(1-\tilde{\mu}_K^{\varphi_K})}-1}, \tag{2}$$

$$F(y_K;\tilde{\mu}_K,\varphi_K)=1-(1-y_K^{\varphi_K})^{\frac{\ln(0.5)}{\ln(1-\tilde{\mu}_K^{\varphi_K})}}, \tag{3}$$

$$\psi(u;\tilde{\mu}_K,\varphi_K)=F(u;\tilde{\mu}_K,\varphi_K)^{-1}=\left[1-(1-u)^{\frac{\ln(1-\tilde{\mu}_K^{\varphi_K})}{\ln(0.5)}}\right]^{1/\varphi_K},$$

for $0<y_K<1$, $\tilde{\mu}_K$ being the median of the y_K, $\varphi_K>0$, and $0<u<1$. Notice that φ_K can be interpreted as a precision parameter, in the sense that for fixed γ_K, the response variance decreases as φ_K increases. The mean and variance under this reparameterization are given, respectively, by

$$\mathrm{E}(Y_K)=\gamma_K B\left(1+\frac{1}{\varphi_K},\gamma_K\right),$$

and

$$\mathrm{Var}(Y_K)=\gamma_K B\left(1+\frac{2}{\varphi_K},\gamma_K\right)-\left\{\gamma_K B\left(1+\frac{1}{\varphi_K},\gamma_K\right)\right\}^2.$$

2.2 | Unit-Weibull distribution

A random variable Y_U follows a unit-Weibull distribution[7] with shape parameters $\delta_U>0$ and $\gamma_U>0$, denoted by $Y_U\sim\mathrm{UW}(\delta_U,\gamma_U)$ if its pdf is given by

$$f(y_U;\delta_U,\gamma_U)=\frac{1}{y_U}\delta_U\,\gamma_W[-\ln(y_U)^{\gamma_U-1}\exp\{-\delta_U[-\ln(y_U)]^{\gamma_U}\},\ 0<y_U<1.$$

The cumulative distribution function and quantile function are defined, respectively, by

$$\begin{aligned}F(y_U;\delta_U,\gamma_U)\ &=P(Y_U\leq y_U)=\textstyle\int_0^{y_U}f(t;\delta_U,\gamma_U)dt=\exp[-\delta_U(-\ln y_U)^{\gamma_U}],\ 0<y_U<1,\\ \psi(u;\delta_U,\gamma_U)\ &=F(u;\delta_U,\gamma_U)^{-1}=\exp\left[-\left(-\tfrac{\ln u}{\delta_U}\right)^{1/\gamma_U}\right],\ 0<u<1.\end{aligned}\tag{4}$$

By Equation (4), an expression for the median can be obtained as

$$\mathrm{md}(Y_U)=\tilde{\mu}_U=\exp\left[-\left(-\frac{\ln(0.5)}{\delta_U}\right)^{1/\gamma_U}\right].$$

Thus, making $\varphi_U=\gamma_U$ (dispersion parameter), the parameter δ_U can be expressed by

$$\delta_U=-\frac{\ln(0.5)}{[-\ln(\tilde{\mu}_U)]^{\varphi_U}}.$$

For the unit-Weibull distribution, we adopted the notation $Y_U\sim UW(\tilde{\mu}_U,\,\varphi_U)$. Considering this reparameterization, the probability density and the cumulative distribution can be expressed as

$$f(y_U;\tilde{\mu}_U,\varphi_U)=\frac{\varphi_U}{y_U}\left[\frac{\ln(0.5)}{\ln(\tilde{\mu}_U)}\right]\left[\frac{\ln(y_U)}{\ln(\tilde{\mu}_U)}\right]^{\varphi_U-1}0.5^{(\ln(y_U)/\ln(\tilde{\mu}_U))^{\varphi_U}},\tag{5}$$

and

$$F(y_U;\tilde{\mu}_U,\varphi_U)=0.5^{[\ln(y_U)/\ln(\tilde{\mu}_U)]^{\varphi_U}},\tag{6}$$

for $0<y_U<1$, $\tilde{\mu}_U$ being the median of the y_U and $\varphi_U>0$. It is worth mentioning that the mean and variance of unit-Weibull distribution do not present closed expressions.

3 | MONITORING INDIVIDUAL OBSERVATIONS OF KUMARASWAMY AND UNIT-WEIBULL DISTRIBUTIONS

Let Y_K be the random variable that follows a Kumaraswamy distribution. Considering an in-control process with parameters $\tilde{\mu}_{0K}$ and φ_{0K} and a Type I error α, the lower and upper probability control limits LCL_K and UCL_K for a Shewhart-type control chart are determined such that

$$\begin{aligned}P(Y_K\ &<LCL_K|\tilde{\mu}_{0K},\varphi_{0K})=P(Y_K>UCL_K|\tilde{\mu}_{0K},\varphi_{0K})=\alpha/2,\\ &\textstyle\int_0^{LCL_K}f(y_K;\tilde{\mu}_{0K},\varphi_{0K})dy_K=\int_{UCL_K}^{1}f(y_K;\tilde{\mu}_{0K},\varphi_{0K})dy_K=\alpha/2,\end{aligned}$$

where $f(y_K;\,\tilde{\mu}_{0K},\,\varphi_{0K})$ is expressed in (2). Using (3), the probability control limits LCL_K and UCL_K are easily achieved.

If Y_U comes from unit-Weibull distribution, the probability control limits LCL_U and UCL_U can be similarly established as

$$P(Y_U < LCL_U|\tilde{\mu}_{0U},\varphi_{0U}) = P(Y_U > UCL_U|\tilde{\mu}_{0U},\varphi_{0U}) = \alpha/2,$$
$$\int_0^{LCL_U} f(y_U;\tilde{\mu}_{0U},\varphi_{0U})dy_U = \int_{UCL_U}^1 f(y_U;\tilde{\mu}_{0U},\varphi_{0U})dy_U = \alpha/2,$$

with $f(y_U; \tilde{\mu}_{0U}, \varphi_{0U})$ expressed in (5). Using (6), LCL_U and UCL_U are easily determined. In both control charts, the central line (CL) is the median $(\tilde{\mu}_j)$, $j = K, U$. Thus, whenever $Y_j < LCL_j$ or $Y_j > UCL_j$ $j = K, U$, the process is declared out of control, and the search for special causes begins. The power $(1-\beta)$ of these control charts can be calculated as

$$1-\beta = P(Y_j < LCL_j|\Theta_{1j}) + P(Y_j > UCL_j|\Theta_{1j}),$$

$j = K,U$, with $\Theta_{1j} = \{(\tilde{\mu}_{1K},\varphi_{1K}),(\tilde{\mu}_{1U},\varphi_{1U})\}$, respectively, for the out-of-control parameters of the Kumaraswamy and unit-Weibull parameters. Depending on the context of interest, the out-of-control median proportion $\tilde{\mu}_{1j}$ can be expressed by $\tilde{\mu}_{1j} = \tilde{\mu}_{0j} + k\sqrt{\mathrm{Var}(Y_j)}$, $\tilde{\mu}_{1j} = \tilde{\mu}_{0j} + \Delta_j$ or $\tilde{\mu}_{1j} = \tilde{\mu}_{0j}\Delta_j$, $j = K,U$. In this paper, we consider the third approach. In addition, we consider shifts only on the median proportion and no shifts on the dispersion parameters.

The performance of these control charts for individual observations is usually measured in terms of RL analysis. The RL distribution follows a geometric distribution with parameter p.[6] For this aim, the average RL (ARL) is one of the most commonly used metrics.[2] For the in-control process, the ARL_0 can be written in terms of the probability of a Type I error (α), that is, $ARL_0=1/\alpha$. In an out-of-control process, the ARL_1 can be written in terms of the probability of a Type II error (β), that is, $ARL_1=1/(1-\beta)$. Thus, when the process is in control, $p=\alpha$, and when the process is out of control, $p=1-\beta$. Additional analysis of the control chart performance can be carried out with the median $(\mathrm{MRL} = \ln(0.5)/\ln(1-p))$ and standard deviation $\left(\mathrm{SDRL} = \sqrt{(1-p)/p^2}\right)$ of the RL distribution.[6,8]

In this section, we evaluate and compare the performance of the control charts in terms of ARL, SDRL, and MRL. These measures are evaluated in two situations: considering the process in control and the process out of control. All results are performed using the statistical computing environment R.[12] We wrote R codes to determine the limits of the proposed control charts, which are available upon request.

Table 1 presents the values of the parameters that will compose the different cases evaluated in this study. These scenarios are determined considering different degrees of variability, asymmetry, and kurtosis. As noted in Figure 1, Cases 1 and 4 have lower and higher variances, respectively.

All results presented in this section are obtained considering as in-control median proportion $\tilde{\mu}_{0K} = \tilde{\mu}_{0_U} = 0.1, 0.3, 0.5, 0.7$ and out-of-control $\tilde{\mu}_{1_K} = \tilde{\mu}_{1_U} = \tilde{\mu}(\cdot)\Delta$, Δ={0.6,0.7,0.8,0.9,1.0,1.1,1.2,1.3,1.4}. When Δ=1.0, the process is in control. In addition, a Type I error (α) equal to 0.0027 was used to evaluate the performance of the control charts.

Furthermore, it would be interesting to measure the impact on the speed to detect shifts in the median proportion if a set of control limits determined under distribution "A" is equivocally used to monitor proportions under distribution "B," or vice versa.

Tables 2–4 present the performance for the Shewhart control chart. Table 2 considers the case where the median proportion shifts in two directions (decreases and increases). Tables 3 and 4 are built to describe the results related to the shifts of the median proportion in a single direction, respectively, considering only decreases or increases.

The results in terms of ARL, SDRL, and MRL are presented in Tables 2 to 4. In this study, two approaches are considered: the correct control limits are used (Kumaraswamy—true or unit-Weibull—true), and the incorrect limits are used (unit-Weibull control chart when the data follow Kumaraswamy distribution or vice versa).

According to Table 2, when the correct control limits are used, both control charts have obtained good results. However, due to the asymmetry of probability distributions, the impact on the detection of change in the process occurs differently. For example, considering Case 1, the unit Weibull to be true, the control chart takes approximately five

TABLE 1 Cases considered in the study

Case	$\tilde{\mu}$	φ_K	φ_U
1	0.1	10	14.73
2	0.3	8	6.84
3	0.5	6	3.43
4	0.7	4	1.57

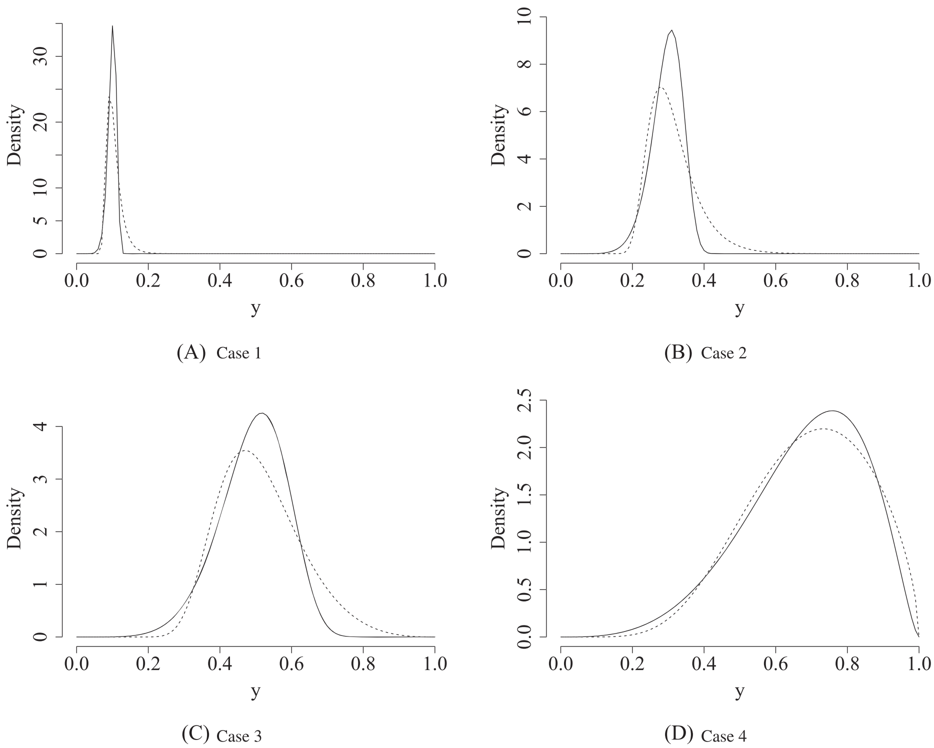

FIGURE 1 Kumaraswamy (solid line) and unit-Weibull (dashed line) probability density function (pdf) of cases

samples to detect a 20% decrease in the median and 220 samples to detect an increase in the median of the same magnitude. The use of equivocal control limits produces a great variety of impacts. For example, in Case 2, the Kumaraswamy control chart wrongly used results in an ARL of approximately 87 (previous false alarms), when Δ=1.0, and an ARL of approximately 960 (no detection) when Δ=0.6. In general, the cases with lower variability can detect changes in the process more quickly.

Tables 3 and 4 present the results of unilateral control charts. Like bilateral control charts, when the correct control limits are used, both control charts have obtained good results. However, the use of equivocal control limits produces poor results. This result is expected if we analyze the density function shown in Figure 1. The tail densities (left and right) of the two distributions are vastly different, which explains the different performances observed in the four cases. Note that more impact is observed in cases of unilateral shifts (decreases or increases in the median proportion).

The results shown in Tables 2–4 highlight the importance of considering adequate data distribution to reduce false alarms and ensuring the control chart power to detect changes in the process. When this information is unknown, assuming that the process is in control state at Phase I, a general framework with the following steps can be used:

1 Select the possible distributions.
2 Estimate the parameters of the candidate distributions.
3 Use one or more criteria for the selection of the best distribution, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), or some adherence tests: Kolmogorov-Smirnov, Anderson-Darling, and Cramér von Mises.
4 Check the adjustment of the selected distribution. Considering that the choice was adequate, determine the control limits and start monitoring the process; otherwise, go back to Step 1.

4 | CONTROL CHARTS TO MONITOR THE MEDIAN PARAMETER OF THE KUMARASWAMY AND UNIT-WEIBULL DISTRIBUTIONS BUILT WITH SEVERAL ESTIMATORS FOR THE MEDIAN

In this section, we proposed control charts to monitor the median proportion of the Kumaraswamy and unit-Weibull distributions considering three estimators for this parameter. The first one is the SM. Let $y_1,y_2,...,y_n$, be a random sample

TABLE 2 Comparing the performances of the control charts

		Kumaraswamy—true			Unit Weibull			Kumaraswamy			Unit Weibull—true		
	Δ	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**
Case 1:	0.6	4.99	4.46	3.10	1.18	0.46	0.37	1630.68	1630.18	1129.95	1.41	0.76	0.56
$\varphi_K = 10.00$	0.7	21.39	20.89	14.48	3.03	2.48	1.73	711.41	710.91	492.76	2.21	1.63	1.15
$\varphi_U = 14.73$	0.8	79.90	79.40	55.04	10.00	9.49	6.58	333.38	332.88	230.74	5.42	4.89	3.40
	0.9	258.35	257.85	178.73	31.33	30.83	21.37	165.26	164.75	114.20	29.88	29.37	20.36
	1	370.00	369.50	256.12	88.93	88.43	61.29	85.74	85.24	59.09	370.00	369.50	256.12
	1.1	12.69	12.18	8.44	229.85	229.35	158.98	46.24	45.74	31.70	396.57	396.07	274.53
	1.2	2.90	2.35	1.64	548.01	547.51	379.51	25.80	25.29	17.53	219.93	219.43	152.10
	1.3	1.61	1.00	0.72	1219.53	1219.03	844.97	14.86	14.35	9.95	125.00	124.50	86.30
	1.4	1.26	0.57	0.44	2558.27	2557.77	1772.91	8.83	8.32	5.77	72.70	72.19	50.04
	LCL	0.0536			0.0663			0.0249			0.0683		
	UCL	0.1253			0.2177			0.1746			0.2215		
Case 2:	0.6	12.93	12.42	8.61	2.02	1.44	1.02	960.31	959.81	665.29	1.80	1.20	0.85
$\varphi_K = 8.00$	0.7	43.13	42.63	29.55	5.55	5.03	3.49	514.36	513.86	356.18	3.06	2.51	1.75
$\varphi_U = 6.84$	0.8	124.56	124.06	85.99	15.17	14.66	10.16	279.20	278.70	193.18	7.86	7.34	5.09
	0.9	318.80	318.30	220.63	38.12	37.61	26.07	155.12	154.62	107.17	40.08	39.58	27.44
	1	370.00	369.50	256.12	87.88	87.38	60.57	87.61	87.10	60.38	370.00	369.50	256.12
	1.1	21.50	21.00	14.56	187.81	187.31	129.84	50.05	49.55	34.35	419.57	419.07	290.48
	1.2	4.64	4.11	2.86	376.26	375.76	260.46	28.83	28.33	19.64	241.01	240.51	166.71
	1.3	2.25	1.67	1.18	713.45	712.95	494.18	16.72	16.21	11.24	138.15	137.65	95.41
	1.4	1.56	0.94	0.68	1288.69	1288.19	892.91	9.76	9.25	6.41	79.07	78.57	54.46
	LCL	0.1375			0.1796			0.0771			0.1875		
	UCL	0.3976			0.6090			0.5163			0.6166		

(Continues)

TABLE 2 (Continued)

	Δ	Kumaraswamy—true ARL	Kumaraswamy—true SDRL	Kumaraswamy—true MRL	Unit Weibull ARL	Unit Weibull SDRL	Unit Weibull MRL	Kumaraswamy ARL	Kumaraswamy SDRL	Kumaraswamy MRL	Unit Weibull—true ARL	Unit Weibull—true SDRL	Unit Weibull—true MRL
Case 3:	0.6	34.75	34.24	23.74	5.13	4.60	3.20	73.95	73.44	50.91	2.70	2.15	1.50
$\varphi_K = 6.00$	0.7	86.90	86.40	59.89	12.15	11.64	8.07	328.45	327.95	227.31	4.91	4.38	3.04
$\varphi_U = 3.43$	0.8	193.23	192.73	133.59	26.47	25.96	18.00	286.87	286.37	198.49	12.57	12.06	8.36
	0.9	391.47	390.97	271.00	53.25	52.75	36.57	179.85	179.35	124.32	55.70	55.19	38.26
	1	370.00	369.50	256.12	100.13	99.63	69.06	110.92	110.42	76.54	370.00	369.50	256.12
	1.1	39.49	38.99	27.02	178.09	177.59	123.09	67.00	66.50	46.09	442.46	441.96	306.34
	1.2	8.79	8.28	5.74	298.01	297.51	206.22	39.28	38.78	26.88	260.24	259.74	180.04
	1.3	3.77	3.23	2.25	222.72	222.22	154.03	22.13	21.62	14.99	145.31	144.81	100.38
	1.4	2.30	1.72	1.21	40.72	40.22	27.88	11.83	11.32	7.85	76.31	75.81	52.55
	LCL	0.1770			0.2472			0.1268			0.2624		
	UCL	0.7201			0.8887			0.8224			0.8938		
Case 4:	0.6	85.64	85.14	59.01	25.25	24.75	17.15	11.03	10.52	7.29	5.16	4.63	3.21
$\varphi_K = 4.00$	0.7	160.42	159.92	110.85	46.99	46.48	32.22	25.61	25.11	17.40	9.27	8.75	6.07
$\varphi_U = 1.57$	0.8	279.32	278.82	193.26	81.55	81.05	56.18	80.15	79.64	55.21	21.60	21.09	14.62
	0.9	456.83	456.32	316.30	134.69	134.19	93.01	250.16	249.66	173.05	76.21	75.71	52.48
	1	370.00	369.50	256.12	210.29	209.78	145.41	268.47	267.97	185.74	370.00	369.50	256.12
	1.1	62.37	61.87	42.88	163.62	163.12	113.07	167.91	167.41	116.04	450.50	450.00	311.92
	1.2	13.81	13.30	9.22	34.87	34.37	23.82	89.29	88.79	61.55	241.50	241.00	167.05
	1.3	4.79	4.26	2.96	8.53	8.02	5.56	34.41	33.91	23.50	92.54	92.04	63.80
	1.4	2.03	1.45	1.02	2.65	2.09	1.46	3.56	3.02	2.10	8.75	8.23	5.71
	LCL	0.1521			0.2071			0.1463			0.2221		
	UCL	0.9812			0.9931			0.9875			0.9934		

TABLE 3 Comparing the performance of the control charts: Decreases in median proportion

		Kumaraswamy—true			Unit Weibull			Kumaraswamy			Unit Weibull—true		
	Δ	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**
Case 1:	0.6	2.77	2.22	1.55	1.11	0.35	0.30	3.84×10^{8}	3.84×10^{8}	2.66×10^{8}	1.36	0.70	0.52
$\varphi_K = 10.00$	0.7	10.95	10.43	7.23	2.58	2.02	1.41	4.85×10^{19}	4.85×10^{19}	Inf	2.03	1.45	1.02
$\varphi_U = 14.73$	0.8	40.18	39.67	27.50	8.26	7.74	5.37	1.10×10^{42}	1.10×10^{42}	Inf	4.55	4.02	2.79
	0.9	129.34	128.84	89.30	25.66	25.15	17.44	8.79×10^{84}	8.79×10^{84}	Inf	21.32	20.81	14.43
	1	370.00	369.50	256.12	72.65	72.14	50.01	1.50×10^{164}	Inf	Inf	370.00	369.50	256.12
	LCL	0.0574			0.0676			0.0292			0.0697		
Case 2:	0.6	6.72	6.20	4.30	1.75	1.14	0.82	1142.15	1141.65	791.33	1.69	1.08	0.78
$\varphi_K = 8.00$	0.7	21.80	21.30	14.76	4.56	4.03	2.80	6.67×10^{5}	6.67×10^{5}	4.62×10^{5}	2.72	2.17	1.52
$\varphi_U = 6.84$	0.8	62.49	61.99	42.97	12.26	11.75	8.15	5.47×10^{10}	5.47×10^{10}	3.79×10^{10}	6.35	5.83	4.04
	0.9	159.56	159.05	110.25	30.66	30.16	20.91	2.27×10^{19}	2.27×10^{19}	Inf	27.98	27.47	19.04
	1	370.00	369.50	256.12	70.57	70.07	48.57	2.28×10^{34}	2.28×10^{34}	Inf	370.00	369.50	256.12
	LCL	0.1500			0.1846			0.0901			0.1926		
Case 3:	0.6	17.62	17.11	11.86	4.06	3.52	2.45	28.77	28.26	19.59	2.44	1.87	1.31
$\varphi_K = 6.00$	0.7	43.67	43.17	29.92	9.43	8.91	6.18	215.50	215.00	149.03	4.16	3.62	2.52
$\varphi_U = 3.43$	0.8	96.80	96.30	66.75	20.40	19.89	13.79	5247.80	5247.30	3637.15	9.69	9.18	6.37
	0.9	196.14	195.64	135.61	40.92	40.42	28.02	9.16×10^{5}	9.16×10^{5}	6.35×10^{5}	38.12	37.61	26.07
	1	370.00	369.50	256.12	76.84	76.34	52.92	4.83×10^{9}	4.83×10^{9}	3.35×10^{9}	370.00	369.50	256.12
	LCL	0.1987			0.2584			0.1484			0.2738		
Case 4:	0.6	43.04	42.54	29.49	16.59	16.09	11.15	7.89	7.37	5.11	4.35	3.81	2.65
$\varphi_K = 4.00$	0.7	80.41	79.91	55.39	30.72	30.22	20.95	16.54	16.04	11.12	7.36	6.85	4.75
$\varphi_U = 1.57$	0.8	139.82	139.32	96.57	53.19	52.68	36.52	48.53	48.03	33.29	15.84	15.33	10.63
	0.9	231.18	230.68	159.90	87.74	87.24	60.47	254.85	254.35	176.30	51.55	51.04	35.38
	1	370.00	369.50	256.12	140.23	139.73	96.86	4067.13	4066.63	2818.77	370.00	369.50	256.12
	LCL	0.1809			0.2307			0.1752			0.2462		

TABLE 4 Comparing the performance of the control charts: Increases in median proportion

		Kumaraswamy—true			Unit Weibull			Kumaraswamy			Unit Weibull—true		
	Δ	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**	**ARL**	**SDRL**	**MRL**
Case 1:	1	370.00	369.50	256.12	Inf	Inf	Inf	69.42	68.92	47.77	370.00	369.50	256.12
$\varphi_K = 10.00$	1.1	9.78	9.26	6.42	Inf	Inf	Inf	37.48	36.98	25.63	198.76	198.25	137.42
$\varphi_U = 14.73$	1.2	2.60	2.04	1.43	Inf	Inf	Inf	20.95	20.45	14.17	110.14	109.64	76.00
	1.3	1.54	0.91	0.66	Inf	Inf	Inf	12.11	11.60	8.04	62.71	62.21	43.12
	1.4	1.23	0.53	0.41	8.50×10^{11}	8.50×10^{11}	5.89×10^{11}	7.24	6.72	4.66	36.58	36.07	25.00
	UCL	0.1239			0.2023			0.1702			0.2059		
Case 2:	1	370.00	369.50	256.12	Inf	Inf	Inf	68.58	68.08	47.19	370.00	369.50	256.12
$\varphi_K = 8.00$	1.1	15.78	15.27	10.59	Inf	Inf	Inf	39.23	38.72	26.84	210.69	210.19	145.69
$\varphi_U = 6.84$	1.2	3.96	3.42	2.38	1.97×10^{13}	1.97×10^{13}	1.37×10^{13}	22.64	22.14	15.35	120.68	120.17	83.30
	1.3	2.06	1.48	1.05	1.02×10^{7}	1.02×10^{7}	7.05×10^{6}	13.18	12.67	8.78	69.28	68.78	47.67
	1.4	1.49	0.86	0.63	7455.55	7452.66	5167.45	7.74	7.23	5.01	39.76	39.26	27.21
	UCL	0.3922			0.5776			0.5039			0.5855		
Case 3:	1	370.00	369.50	256.12	2.71×10^{10}	2.70×10^{10}	1.88×10^{10}	80.60	80.10	55.52	370.00	369.50	256.12
$\varphi_K = 6.00$	1.1	27.59	27.09	18.78	7.14×10^{5}	7.11×10^{5}	4.95×10^{5}	48.74	48.24	33.44	223.03	222.53	154.24
$\varphi_U = 3.43$	1.2	7.02	6.50	4.51	2747.60	2742.99	1904.15	28.64	28.13	19.50	130.28	129.78	89.96
	1.3	3.28	2.73	1.91	124.57	124.29	86.00	16.19	15.68	10.87	72.86	72.36	50.15
	1.4	2.10	1.52	1.08	20.55	20.29	13.89	8.72	8.21	5.69	38.38	37.88	26.26
	UCL	0.7078			0.8656			0.8067			0.8715		
Case 4:	1	370.00	369.50	256.12	2921.60	2914.64	2024.75	162.99	162.49	112.63	370.00	369.50	256.12
$\varphi_K = 4.00$	1.1	42.46	41.96	29.08	157.35	156.87	108.72	100.34	99.84	69.21	227.54	227.04	157.37
$\varphi_U = 1.57$	1.2	10.56	10.05	6.97	24.06	23.71	16.33	53.46	52.95	36.71	120.92	120.42	83.47
	1.3	4.07	3.53	2.46	6.64	6.24	4.25	20.73	20.22	14.02	46.49	45.99	31.88
	1.4	1.89	1.29	0.92	2.36	1.86	1.26	2.36	1.79	1.25	4.64	4.11	2.85
	UCL	0.9751			0.9892			0.9827			0.9897		

from Y. Because the parameter $\tilde{\mu}$ is the median of the distribution of interest, the sample median (SM) seems to be a natural choice as an estimator of $\tilde{\mu}$. We denote the SM estimator of $\tilde{\mu}$ by

$$\widehat{\tilde{\mu}}_{\text{SM}} = \text{median}\{y_1, y_2, \ldots, y_n\},$$

which can be defined as follows:

$$\widehat{\tilde{\mu}}_{\text{SM}} = \begin{pmatrix} Y_{(k+1)}, & \text{if } n = 2k+1, \\ (Y_{(k)} + Y_{(k+1)})/2, & \text{if } n = 2k, \end{pmatrix}$$

where $Y_{(k)}$ is the kth-order statistic from $y_1, y_2, \ldots, y_n$.

The second is related to the Hodges and Lehmann's proposal[9] and introduces the point estimator currently known as the HL estimator of the median parameter. The HL estimator of $\tilde{\mu}$ is the median of all pairwise means of those observations, that is,

$$\widehat{\tilde{\mu}}_{\text{HL}} = \text{median}\left[\frac{Y_i + Y_j}{2}, \ 1 \le i \le j \le n\right].$$

Finally, the third estimator is the one with ML, denoted by $\widehat{\tilde{\mu}}_{ML}$ with

$$\widehat{\tilde{\mu}}_{ML} = arg\,max \prod\nolimits_{i=1}^{n} f(\tilde{\mu}, \tilde{\varphi} | y_i),$$

where $f(\tilde{\mu}|y_i)$ is the probability density function of Y (see expressions 2 and 5, respectively, for the Kumaraswamy and unit-Weibull distributions). ML estimators will be obtained using numerical methods, as equating the first-order log-likelihood derivatives to 0 leads us to a complicated system of nonlinear equations.

To compare the performance of these three estimators in terms of speed to sign a shift in the median proportion, metrics like ARL (or MRL and SDRL) are considered. Due to challenges in obtaining closed expressions for distributions of the three median estimators, we use the Monte Carlo simulation approach in two procedures. In the first procedure, empirical probability control limits are stated as follows:

- Step 1: Choose the in-control parameters $\tilde{\mu}_0$ and φ of the random variable Y.
- Step 2: Take n (in-control) values of random variable Y.
- Step 3: Obtain the statistics $\widehat{\tilde{\mu}}_{(j)}$ $j = SM, HL, ML$.
- Step 4: Repeat Steps 2 and 3 t times (such as 10 000) to obtain the empirical distribution of $\widehat{\tilde{\mu}}_{(j)}$ $j = SM, HL, ML$.
- Step 5: Obtain the quantiles $q_{\alpha/2, \hat{\tilde{\mu}}_{(j)}}$, $q_{1-\alpha/2, \hat{\tilde{\mu}}_{(j)}}$, $q_{\alpha, \hat{\tilde{\mu}}_{(j)}}$, $q_{1-\alpha, \hat{\tilde{\mu}}_{(j)}}$, $j = SM, HL, ML$ where α is the Type I error. The first two quantiles are, respectively, LCL_j and UCL_j, the probability control limits for bidirectional shifts, and the last two are, respectively, LCL_j and UCL_j,the probability control limits for unidirectional shifts (respectively, for decreases or increases in the median proportion).

In the second procedure, metrics such as out-of-control ARL (ARL_1) (or other metrics such as out-of-control MRL [MRL_1] and out-of-control SDRL [SDRL_1]) proceed as follows:

- Step 1: Use LCL_j and/or UCL_j from Step 5 of the first procedure; make k=0; and choose the out-of-control parameters $\tilde{\mu}_1$ and φ of the random variable Y.
- Step 2: Take n (out-of-control) values of random variable Y.
- Step 3: Obtain the statistics $\widehat{\tilde{\mu}}_{(j)}$, $j = SM, HL, ML$.
- Step 4: If $\widehat{\tilde{\mu}}_{(j)} < LCL_j$ or $\widehat{\tilde{\mu}}_{(j)} > UCL_j$, then $k = k$+1.
- Step 5: Repeat Steps 2–4 t times (like 10 000).
- Step 6: $p = k/t$ is the power of the control chart and its $\text{ARL}_1 = 1/p$. Other metrics like MRL_1 or SDRL_1 can be easily derived from p.

TABLE 5 Kumaraswamy distribution: Comparing the performance of SM, HL, and ML estimators for the median in terms of ARL (bilateral shifts)

		n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
$\tilde{\mu}_0; \varphi$	**Δ**	**SM**	**HL**	**ML**	**SM**	**HL**	**ML**	**SM**	**HL**	**ML**	**SM**	**HL**	**ML**
0.1; 10	0.6	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	1.558	**1.163**	**1.036**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	8.772	5.244	**3.220**	**1.068**	**1.010**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	81.967	58.480	**37.594**	9.615	5.618	**3.774**	**1.250**	**1.091**	**1.050**	**1.000**	**1.000**	**1.000**
	1.0	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	] 357.143	357.143	357.143	357.143
	1.1	8.110	6.452	**5.537**	2.759	**2.174**	**1.822**	**1.185**	**1.072**	**1.060**	**1.000**	**1.000**	**1.000**
	1.2	**1.865**	**1.576**	**1.493**	**1.065**	**1.026**	**1.016**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.3	**1.185**	**1.097**	**1.080**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.046**	**1.021**	**1.013**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.0719	0.0745	0.0768	0.0849	0.0861	0.0879	0.0918	0.0921	0.0931	0.0954	0.0956	0.0961
	UCL	0.1190	0.1163	0.1159	0.1115	0.1097	0.1097	0.1071	0.1056	0.1061	0.1041	0.1029	0.1033
0.3; 8	0.6	**1.205**	**1.010**	**1.005**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	4.120	**1.911**	**1.735**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	22.472	10.460	**9.823**	1.502	**1.154**	**1.080**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	153.846	**86.207**	91.743	19.342	10.870	**7.686**	1.910	**1.442**	**1.381**	**1.002**	**1.000**	**1.000**
	1.0	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	13.459	14.409	**10.593**	5.013	**3.611**	**3.698**	**1.714**	**1.294**	**1.266**	**1.012**	**1.001**	**1.001**
	1.2	2.722	**2.527**	**2.212**	**1.268**	**1.134**	**1.123**	**1.005**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.3	**1.479**	**1.347**	**1.277**	**1.023**	**1.005**	**1.005**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.154**	**1.107**	**1.080**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.1956	0.2093	0.2133	0.2452	0.2505	0.2555	0.2704	0.2717	0.2742	0.2842	0.2841	0.2869
	UCL	0.3717	0.3651	0.3651	0.3442	0.3370	0.3393	0.3278	0.3208	0.3229	0.3159	0.3109	0.3131

(Continues)

TABLE 5 (Continued)

$\tilde{\mu}_0$; φ	Δ	n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
		SM	HL	ML	SM	HL	ML	SM	HL	ML	SM	HL	ML
0.5; 6	0.6	3.855	**1.727**	**1.532**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	17.007	7.022	**5.491**	**1.119**	**1.010**	**1.004**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	67.568	37.175	**26.042**	4.054	**2.128**	**1.768**	**1.032**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	303.030	181.818	**158.730**	52.632	28.169	**17.182**	5.984	**2.869**	**2.591**	**1.093**	**1.012**	**1.012**
	1.0	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	**22.779**	**22.779**	24.390	11.198	8.598	**5.400**	2.595	**2.063**	**2.190**	**1.121**	**1.026**	**1.023**
	1.2	4.963	**4.203**	**4.290**	2.008	**1.678**	**1.388**	**1.050**	**1.012**	**1.011**	**1.000**	**1.000**	**1.000**
	1.3	**2.216**	**1.947**	**1.905**	**1.162**	**1.086**	**1.041**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.456**	**1.335**	**1.313**	**1.025**	**1.007**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.2748	0.3045	0.3142	0.3796	0.3953	0.4051	0.4306	0.4399	0.4457	0.4644	0.4663	0.4697
	UCL	0.6584	0.6403	0.6481	0.5991	0.5856	0.5794	0.5567	0.5464	0.5521	0.5335	0.5240	0.5281
0.7; 4	0.6	10.638	4.090	3.459	1.060	1.008	1.005	1.000	1.000	1.000	1.000	1.000	1.000
	0.7	36.630	13.947	**11.416**	2.133	**1.526**	**1.365**	**1.003**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	98.039	47.847	**38.610**	9.337	6.658	**5.479**	1.635	**1.171**	**1.087**	**1.000**	**1.000**	**1.000**
	0.9	263.158	166.667	**151.515**	69.444	70.423	**54.645**	16.722	9.328	**6.293**	1.707	**1.221**	**1.178**
	1.0	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	50.505	47.619	**43.478**	20.661	22.173	**10.627**	6.116	4.912	**3.722**	1.621	**1.309**	**1.160**
	1.2	9.416	8.013	**6.887**	3.068	3.216	**1.872**	**1.246**	**1.157**	**1.051**	**1.001**	**1.000**	**1.000**
	1.3	3.106	2.894	**2.259**	**1.291**	**1.335**	**1.082**	**1.005**	**1.003**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.407**	**1.437**	**1.191**	**1.012**	**1.021**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.3115	0.3702	0.3820	0.4870	0.5036	0.5162	0.5703	0.5845	0.5992	0.6339	0.6371	0.6471
	UCL	0.9459	0.9105	0.9199	0.8645	0.8438	0.8370	0.8075	0.7800	0.7889	0.7633	0.7397	0.7508

TABLE 6 Kumaraswamy distribution: Comparing the performance of SM, HL, and ML estimators for the median in terms of ARL (unilateral shifts)—Increases in the median

		n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
$\tilde{\mu}_0$; φ	Δ	SM	HL	ML	SM	HL	ML	SM	HL	ML	SM	HL	ML
0.1; 10	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	5.549	4.857	**4.088**	2.304	**1.911**	**1.665**	**1.149**	**1.052**	**1.034**	**1.000**	**1.000**	**1.000**
	1.2	**1.634**	**1.455**	**1.371**	**1.047**	**1.019**	**1.013**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.3	**1.138**	**1.077**	**1.062**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.035**	**1.016**	**1.009**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.1173	0.1150	0.1145	0.1107	0.1090	0.1091	0.1067	0.1052	0.1055	0.1038	0.1026	0.1031
0.3; 8	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	9.328	9.881	**7.628**	3.831	**2.898**	**2.959**	**1.584**	**1.206**	**1.198**	**1.007**	**1.001**	**1.000**
	1.2	**2.335**	**2.163**	**1.932**	**1.199**	**1.099**	**1.093**	**1.003**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.3	**1.382**	**1.273**	**1.215**	**1.016**	**1.003**	**1.004**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.122**	**1.082**	**1.064**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.3664	0.3600	0.3601	0.3408	0.3340	0.3365	0.3263	0.3190	0.3213	0.3147	0.3100	0.3120
0.5; 6	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	17.301	**15.038**	17.544	7.918	6.039	**4.933**	**2.284**	**1.856**	**1.835**	**1.090**	**1.017**	**1.014**
	1.2	4.274	**3.349**	**3.575**	**1.792**	**1.486**	**1.351**	**1.036**	**1.009**	**1.006**	**1.000**	**1.000**	**1.000**
	1.3	**2.042**	**1.733**	**1.744**	**1.125**	**1.059**	**1.037**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.396**	**1.261**	**1.258**	**1.020**	**1.005**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.6503	0.6291	0.6399	0.5924	0.5786	0.5772	0.5534	0.5436	0.5481	0.5314	0.5220	0.5262
0.7; 4	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	29.412	31.056	**26.316**	13.870	12.970	**8.613**	4.425	3.932	**1.034**	**1.449**	**1.213**	**1.111**
	1.2	6.341	6.386	**5.339**	2.532	2.489	**1.725**	**1.170**	**1.120**	**1.000**	**1.000**	**1.000**	**1.000**
	1.3	**2.489**	2.561	**1.997**	**1.227**	**1.234**	**1.066**	**1.003**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.303**	**1.364**	**1.157**	**1.009**	**1.014**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.9304	0.9011	0.9099	0.8540	0.8312	0.8312	0.7994	0.7748	0.7824	0.7591	0.7358	0.7477

TABLE 7 Kumaraswamy distribution: Comparing the performance of SM, HL, and ML estimators for the median in terms of ARL (unilateral shifts)—Decreases in the median

		n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
$\tilde{\mu}_0$; φ	Δ	SM	HL	ML	SM	HL	ML	SM	HL	ML	SM	HL	ML
0.1; 10	0.6	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	**1.194**	**1.063**	**1.012**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	4.579	3.541	**2.392**	**1.025**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	36.101	34.602	**23.810**	5.834	3.415	**2.586**	**1.158**	**1.038**	**1.028**	**1.000**	**1.000**	**1.000**
	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.0749	0.0763	0.0783	0.0862	0.0875	0.0890	0.0924	0.0929	0.0935	0.0958	0.0959	0.0965
0.3; 8	0.6	**1.040**	**1.002**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	2.218	**1.547**	**1.375**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	9.579	7.072	**5.893**	**1.243**	**1.076**	**1.047**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	60.976	56.497	**44.643**	10.537	6.798	**5.777**	**1.611**	**1.246**	**1.215**	**1.001**	**1.000**	**1.000**
	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.2078	0.2151	0.2202	0.2505	0.2545	0.2580	0.2726	0.2742	0.2766	0.2853	0.2852	0.2878
0.5; 6	0.6	2.236	**1.372**	**1.217**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	8.052	4.365	**3.360**	**1.037**	**1.004**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	30.864	20.833	**13.947**	2.534	**1.746**	**1.496**	**1.009**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	136.986	106.383	**70.922**	23.981	17.331	**11.481**	3.531	**2.323**	**2.319**	**1.056**	**1.005**	**1.005**
	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.2957	0.3182	0.3306	0.3919	0.4020	0.4117	0.4384	0.4436	0.4476	0.4668	0.4692	0.4725
0.7	0.6	6.826	**2.686**	**2.521**	**1.028**	**1.002**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	21.978	8.130	**7.502**	1.706	**1.300**	**1.199**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	56.818	25.445	**23.585**	6.177	4.361	**3.805**	**1.357**	**1.107**	**1.052**	**1.000**	**1.000**	**1.000**
	0.9	153.846	**74.627**	81.967	39.841	36.232	**28.736**	9.625	6.640	**4.581**	**1.500**	**1.168**	**1.113**
	1.0	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.3344	0.3954	0.4029	0.5020	0.5181	0.5301	0.5816	0.5910	0.6065	0.6383	0.6395	0.6511

TABLE 8 Unit-Weibull distribution: Comparing the performance of SM, HL, and ML estimators for the median in terms of ARL (bilateral shifts)

		n=3 Estimator			*n*=10 Estimator			*n*=30 Estimator			*n*=100 Estimator		
$\tilde{\mu}_0; \varphi$	**Δ**	**SM**	**HL**	**ML**	**SM**	**HL**	**ML**	**SM**	**HL**	**ML**	**SM**	**HL**	**ML**
0.1; 14.73	0.6	**1.083**	**1.066**	**1.039**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	**1.403**	**1.348**	**1.243**	**1.019**	**1.009**	**1.003**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	**2.869**	**2.801**	**2.374**	**1.355**	**1.240**	**1.144**	**1.011**	**1.003**	**1.001**	**1.000**	**1.000**	**1.000**
	0.9	15.748	16.502	**13.072**	6.532	5.280	**4.237**	2.293	**1.925**	**1.621**	**1.037**	**1.016**	**1.003**
	1	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	344.828	**123.457**	153.846	54.348	**20.284**	**20.576**	5.855	4.847	**2.711**	**1.154**	**1.051**	**1.009**
	1.2	163.934	45.045	**40.816**	5.942	**2.878**	**2.434**	**1.093**	**1.037**	**1.005**	**1.000**	**1.000**	**1.000**
	1.3	45.249	13.605	**10.977**	1.600	**1.174**	**1.094**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	17.036	5.165	**3.979**	**1.041**	**1.003**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.0760	0.0785	0.0778	0.0841	0.0868	0.0862	0.0894	0.0921	0.0909	0.0940	0.0961	0.0949
	UCL	0.1690	0.1554	0.1487	0.1277	0.1248	0.1214	0.1143	0.1153	0.1112	0.1075	0.1086	0.1057
0.3; 6.84	0.6	**1.237**	**1.183**	**1.127**	**1.004**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	1.846	**1.681**	**1.533**	**1.068**	**1.047**	**1.027**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	4.312	**3.861**	**3.370**	**1.624**	**1.531**	**1.429**	**1.046**	**1.014**	**1.007**	**1.000**	**1.000**	**1.000**
	0.9	25.641	20.661	**18.622**	8.368	8.190	**7.616**	3.234	**2.320**	**2.081**	**1.127**	**1.056**	**1.032**
	1	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	243.902	243.902	**188.679**	51.282	**36.364**	45.872	8.475	4.914	**4.037**	**1.296**	**1.070**	**1.050**
	1.2	91.743	**64.935**	67.568	6.743	**4.333**	**4.376**	**1.213**	**1.048**	**1.018**	**1.000**	**1.000**	**1.000**
	1.3	30.395	18.868	**17.544**	**1.740**	**1.344**	**1.274**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	11.682	6.317	**5.685**	**1.053**	**1.011**	**1.004**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.2146	0.2252	0.2221	0.2465	0.2538	0.2492	0.2636	0.2734	0.2694	0.2793	0.2867	0.2817
	UCL	0.4850	0.4663	0.4557	0.3862	0.3816	0.3748	0.3473	0.3457	0.3376	0.3245	0.3257	0.3196

(Continues)

TABLE 8 (Continued)

		n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
$\tilde{\mu}_0$; φ	Δ	SM	HL	ML	SM	HL	ML	SM	HL	ML	SM	HL	ML
0.5; 3.43	0.6	**1.655**	**1.537**	**1.454**	**1.040**	**1.028**	**1.012**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	**2.827**	**2.480**	**2.358**	**1.221**	**1.199**	**1.120**	**1.005**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	7.077	6.532	**5.977**	**2.265**	**2.282**	**1.916**	**1.165**	**1.063**	**1.051**	**1.000**	**1.000**	**1.000**
	0.9	36.496	43.103	**34.483**	11.834	12.937	**10.672**	4.230	**3.171**	**3.182**	**1.302**	**1.136**	**1.093**
	1	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	294.118	285.714	**166.667**	96.154	65.789	**42.194**	16.920	7.418	**4.843**	**1.531**	**1.164**	**1.118**
	1.2	87.719	57.471	**47.619**	10.549	6.506	**4.433**	**1.483**	**1.094**	**1.044**	**1.000**	**1.000**	**1.000**
	1.3	28.490	15.432	**13.298**	2.132	**1.467**	**1.250**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	9.242	**4.421**	**3.982**	**1.059**	**1.006**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.3214	0.3413	0.3341	0.3943	0.4029	0.4013	0.4312	0.4468	0.4392	0.4610	0.4730	0.4669
	UCL	0.7734	0.7400	0.7316	0.6526	0.6390	0.6241	0.5900	0.5784	0.5672	0.5445	0.5433	0.5362
0.7; 1.57	0.6	2.926	**2.574**	**2.205**	**1.300**	**1.151**	**1.168**	**1.007**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	5.365	4.625	**3.828**	**1.930**	**1.542**	**1.606**	**1.074**	**1.012**	**1.016**	**1.000**	**1.000**	**1.000**
	0.8	12.970	11.587	**8.591**	4.433	**3.182**	**3.547**	**1.635**	**1.250**	**1.269**	**1.009**	**1.000**	**1.000**
	0.9	51.813	46.729	**36.630**	24.938	**16.978**	22.989	7.937	**5.179**	**5.285**	1.890	**1.317**	**1.340**
	1	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143	357.143
	1.1	163.934	**204.082**	227.273	94.340	61.728	**42.373**	13.624	**6.053**	6.658	1.793	**1.143**	**1.189**
	1.2	58.480	**39.216**	51.282	7.138	3.784	**3.194**	**1.216**	**1.013**	**1.021**	**1.000**	**1.000**	**1.000**
	1.3	9.699	**4.498**	6.131	**1.139**	**1.011**	**1.007**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.021**	**1.000**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	LCL	0.3490	0.3939	0.3925	0.4945	0.5248	0.5132	0.5747	0.5946	0.5945	0.6317	0.6403	0.6440
	UCL	0.9557	0.9307	0.9419	0.8806	0.8546	0.8556	0.8165	0.7876	0.7973	0.7673	0.7453	0.7546

TABLE 9 Unit-Weibull distribution: Comparing the performance of SM, HL, and ML estimators for the median in terms of ARL (unilateral shifts)—Increases in the median

		n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
$\tilde{\mu}_0$; φ	Δ	SM	HL	ML	SM	HL	ML	SM	HL	ML	SM	HL	ML
0.1; 14.73	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	149.254	**76.336**	80.000	26.525	13.736	**13.021**	3.903	2.987	**2.205**	**1.085**	**1.027**	**1.005**
	1.2	49.261	24.752	**19.841**	3.586	**2.247**	**1.898**	**1.040**	**1.011**	**1.002**	**1.000**	**1.000**	**1.000**
	1.3	16.835	8.006	**6.064**	**1.296**	**1.101**	**1.046**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	6.916	3.229	**2.506**	**1.014**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.1578	0.1498	0.1430	0.1247	0.1230	0.1195	0.1129	0.1136	0.1104	0.1068	0.1080	0.1054
0.3; 6.84	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	144.928	129.870	**126.582**	31.447	**22.272**	25.773	6.321	3.799	**2.921**	**1.198**	**1.050**	**1.025**
	1.2	50.000	**33.784**	38.760	4.805	**3.163**	**3.094**	**1.139**	**1.029**	**1.009**	**1.000**	**1.000**	**1.000**
	1.3	18.622	**11.198**	**11.338**	**1.484**	**1.203**	**1.155**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	7.369	**3.967**	**3.970**	**1.026**	**1.004**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.4705	0.4518	0.4444	0.3802	0.3755	0.3688	0.3443	0.3431	0.3345	0.3228	0.3246	0.3180
0.5; 3.43	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	125.000	161.290	**95.238**	33.223	**22.222**	**22.124**	8.177	4.936	**3.697**	**1.337**	**1.117**	**1.083**
	1.2	46.729	38.314	**27.778**	5.084	**2.999**	**3.008**	**1.219**	**1.049**	**1.026**	**1.000**	**1.000**	**1.000**
	1.3	15.699	11.025	**8.230**	**1.466**	**1.144**	**1.119**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	5.394	3.342	**2.752**	**1.012**	**1.000**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.7525	0.7292	0.7160	0.6336	0.6195	0.6133	0.5795	0.5727	0.5626	0.5407	0.5414	0.5342
0.7; 1.57	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	1.1	**93.458**	96.154	128.205	47.847	32.787	**22.883**	9.285	**4.177**	4.636	**1.561**	**1.079**	**1.114**
	1.2	32.787	**18.553**	27.397	4.671	**2.755**	**2.293**	**1.130**	**1.005**	**1.009**	**1.000**	**1.000**	**1.000**
	1.3	6.053	**2.691**	3.965	**1.060**	**1.004**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	1.4	**1.004**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	UCL	0.9468	0.9164	0.9320	0.8702	0.8450	0.8447	0.8098	0.7807	0.7906	0.7633	0.7410	0.7504

TABLE 10 Unit-Weibull distribution: Comparing the performance of SM, HL, and ML estimators for the median in terms of ARL (unilateral shifts)—Decrease in the median

		n=3 Estimator			n=10 Estimator			n=30 Estimator			n=100 Estimator		
$\tilde{\mu}_0$; φ	Δ	SM	HL	ML	SM	HL	ML	SM	HL	ML	SM	HL	ML
0.1; 14.73	0.6	**1.069**	**1.052**	**1.035**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	**1.339**	**1.282**	**1.218**	**1.015**	**1.006**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	**2.521**	**2.376**	**2.218**	**1.289**	**1.193**	**1.117**	**1.007**	**1.002**	**1.000**	**1.000**	**1.000**	**1.000**
	0.9	11.641	**11.236**	**11.001**	5.102	4.288	**3.522**	**1.886**	**1.647**	**1.421**	**1.025**	**1.006**	**1.002**
	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.0772	0.0800	0.0785	0.0850	0.0876	0.0869	0.0903	0.0928	0.0917	0.0945	0.0967	0.0953
0.3; 6.84	0.6	**1.202**	**1.155**	**1.107**	**1.004**	**1.001**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	**1.705**	**1.569**	**1.443**	**1.059**	**1.037**	**1.018**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	3.602	**3.286**	**2.897**	**1.538**	**1.430**	**1.308**	**1.030**	**1.009**	**1.004**	**1.000**	**1.000**	**1.000**
	0.9	17.699	14.970	**13.055**	7.097	6.317	**5.441**	2.640	**1.949**	**1.737**	**1.094**	**1.035**	**1.018**
	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.2191	0.2294	0.2261	0.2485	0.2564	0.2531	0.2661	0.2758	0.2719	0.2805	0.2880	0.2835
0.5; 3.43	0.6	**1.539**	**1.432**	**1.384**	**1.032**	**1.022**	**1.010**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	**2.499**	**2.167**	**2.101**	**1.185**	**1.154**	**1.101**	**1.004**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.8	5.718	**5.094**	**4.866**	**2.049**	**2.001**	**1.780**	**1.139**	**1.044**	**1.036**	**1.000**	**1.000**	**1.000**
	0.9	24.631	26.247	**23.641**	9.328	9.416	**8.696**	3.654	**2.554**	**2.596**	**1.240**	**1.092**	**1.068**
	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.3311	0.3515	0.3430	0.3995	0.4090	0.4053	0.4338	0.4515	0.4435	0.4629	0.4755	0.4691
0.7; 1.57	0.6	**2.540**	**2.240**	**2.083**	**1.257**	**1.118**	**1.127**	**1.005**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
	0.7	4.357	**3.690**	**3.482**	**1.794**	**1.437**	**1.470**	**1.054**	**1.007**	**1.010**	**1.000**	**1.000**	**1.000**
	0.8	9.506	8.460	**7.457**	3.882	**2.755**	**2.924**	**1.487**	**1.175**	**1.194**	**1.006**	**1.000**	**1.000**
	0.9	34.247	31.949	**30.303**	18.727	**12.690**	14.970	5.967	**3.885**	**3.940**	1.730	**1.220**	**1.260**
	1	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370	370.370
	LCL	0.3716	0.4146	0.4033	0.5036	0.5346	0.5263	0.5837	0.6024	0.6030	0.6351	0.6445	0.6474

We use the same set of in-control parameters (see Table 1) and the same magnitude of shifts as in the previous Section 3 with the following sample sizes $n = \{3,10,30,100\}$ in Monte Carlo simulations.

Tables 5–7 and 8–10 present results related to the Kumaraswamy and unit-Weibull distributions, respectively, considering the three estimators for the median parameter. In these tables, we assumed similar performances for those ARL_1 values, which differ less than 0.5 from the minimum ARL_1 among the values provided by the estimators SM, HL, and ML for a fixed case, Δ and n. These values are bold in the tables. Results with sample sizes $n > 100$ are obtained but not shown here as most of the cases present $ARL_1 = 1$ even for $n = 30,100$. In general, the control charts present good performance (with $ARL_1 < 2$) even for small samples like $n = 3$ for increases or decreases $\geq 30\%$ (that is, $\Delta = 0.6,0.7,1.3,1.4$) in cases with larger precise parameters ($\varphi \geq 8$). As expected, the ML estimator provides the lowest ARL_1 in most cases. Instabilities, nonconvergence, more process time, and dependence of the initial values are the prices paid for this good performance. However, the HL estimator also presents good performance and some cases with similar performance as the ML. Therefore, it can be viewed as an alternative due to its simplicity to obtain. In some cases, it has better performance than ML.

5 | REAL-DATA APPLICATIONS

In this section, we present two applications to real data to illustrate our methodology. The first is an application of a control chart for individual proportion observations, and the second is for sample sizes $n > 1$. In order to estimate the parameters of these models, we adopt the ML method (as discussed in Section 4) and perform all the computations using the function `optim` of the `R` software.

5.1 | First application: Control chart for individual observations

This application consists of an individual observation of a data set of a study of contaminated peanuts by toxic substances in 34 batches of 120 pounds.[13], p. 63, Exercise K This data set was used in the study of Ho et al.[6] The quality characteristic monitored is the proportion of noncontaminated peanuts (ratio between continuous numbers). For this reason, the use of the Kumaraswamy and unit-Weibull distributions for fitting this data set is well justified.

In this application, we follow the steps described at the end of Section 3. First, the Kumaraswamy and unit-Weibull distributions are selected, and the parameters of these distributions are estimated using the ML method. For this stage (Phase I), the first 20 observations are used. The remaining 14 observations (Phase II) are used to monitor the proportion of noncontaminated peanuts. The empirical median and standard deviation for the first 20 observations are 0.9655 and 0.0345, respectively. The ML estimates of the parameters AIC and BIC for the Kumaraswamy and unit-Weibull models for the first 20 observations are listed in Table 11. In addition, Figure 2 illustrates the histogram and pdf of the Kumaraswamy and unit-Weibull fitted models to these data. According to the criteria AIC, BIC, and graphical analysis, the Kumaraswamy distribution seems to be the most appropriate model.

Figure 3 graphically presents the control limits of the Kumaraswamy control chart for α=0.0027. As it may be of interest to control the quality characteristic in a single direction, we present the bilateral and unilateral control chart. During Phase I, the control charts do not trigger an alarm, so we do not obtain any contradictions against the models. In Phase II, the control chart detected six and seven points out of control considering the bilateral and unilateral control charts, respectively. In the same phase, the control chart triggers a signal, for the first time, at Sample 5.

TABLE 11 Estimates of the parameters and goodness-of-fit statistics for the proportion of noncontaminated peanuts

Model	Parameter	Estimate	AIC	BIC
Kumaraswamy	$\tilde{\mu}$	0.9602	−86.103	−84.111
	φ_K	37.078		
Unit Weibull	$\tilde{\mu}$	0.9589	−83.770	−81.778
	φ_{UW}	1.474		

FIGURE 2 Estimated pdf for the Kumaraswamy (solid line) and unit-Weibull (dashed line) models for the proportion of noncontaminated peanuts

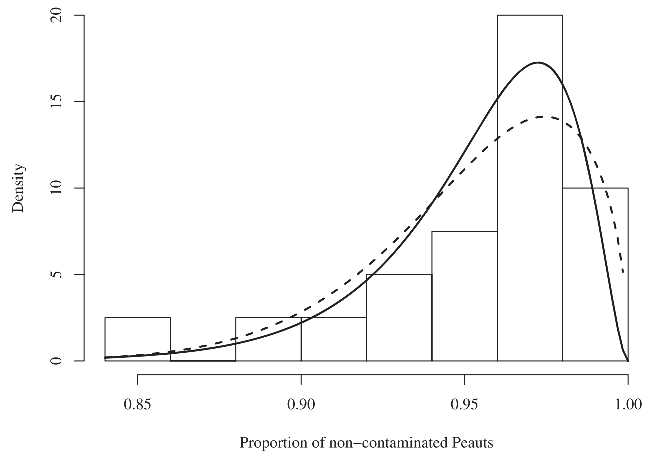

FIGURE 3 Kumaraswamy control chart for the proportion of noncontaminated peanuts (Phase I: 1–20; Phase II: 21–34)

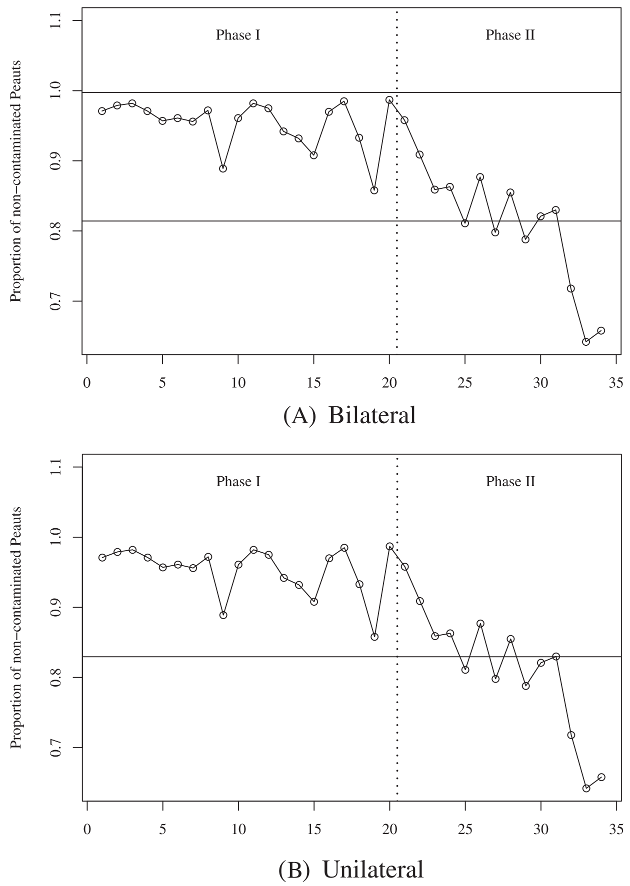

TABLE 12 Estimates of the parameters and goodness-of-fit statistics data set of the relative humidity

Model	Parameter	Estimate	AIC	BIC
Kumaraswamy	$\tilde{\mu}$	0.7240	−2264.443	−2254.080
	φ_K	7.5020		
Unit Weibull	$\tilde{\mu}$	0.7095	−2189.497	−2179.133
	φ_{UW}	2.4060		

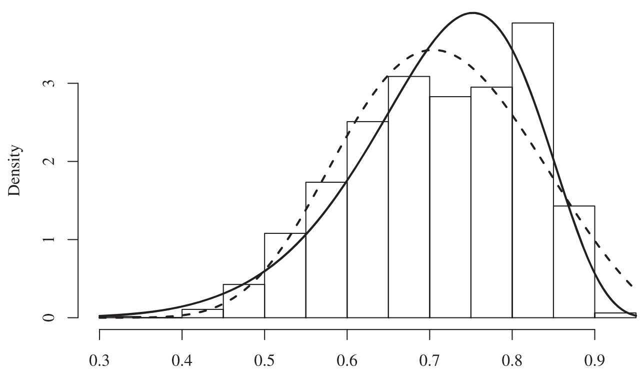

FIGURE 4 Estimated pdf for the Kumaraswamy (solid line) and unit-Weibull (dashed line) models

5.2 | Second application: Control chart for nonindividual observations

This application is related to a data set of relative humidity (RELH). The quality characteristic monitored is given in terms of percentage, being the ratio of the partial pressure of water to the equilibrium vapor pressure of water (ratio between continuous numbers). The quality characteristic is measured several times throughout the day (sample size

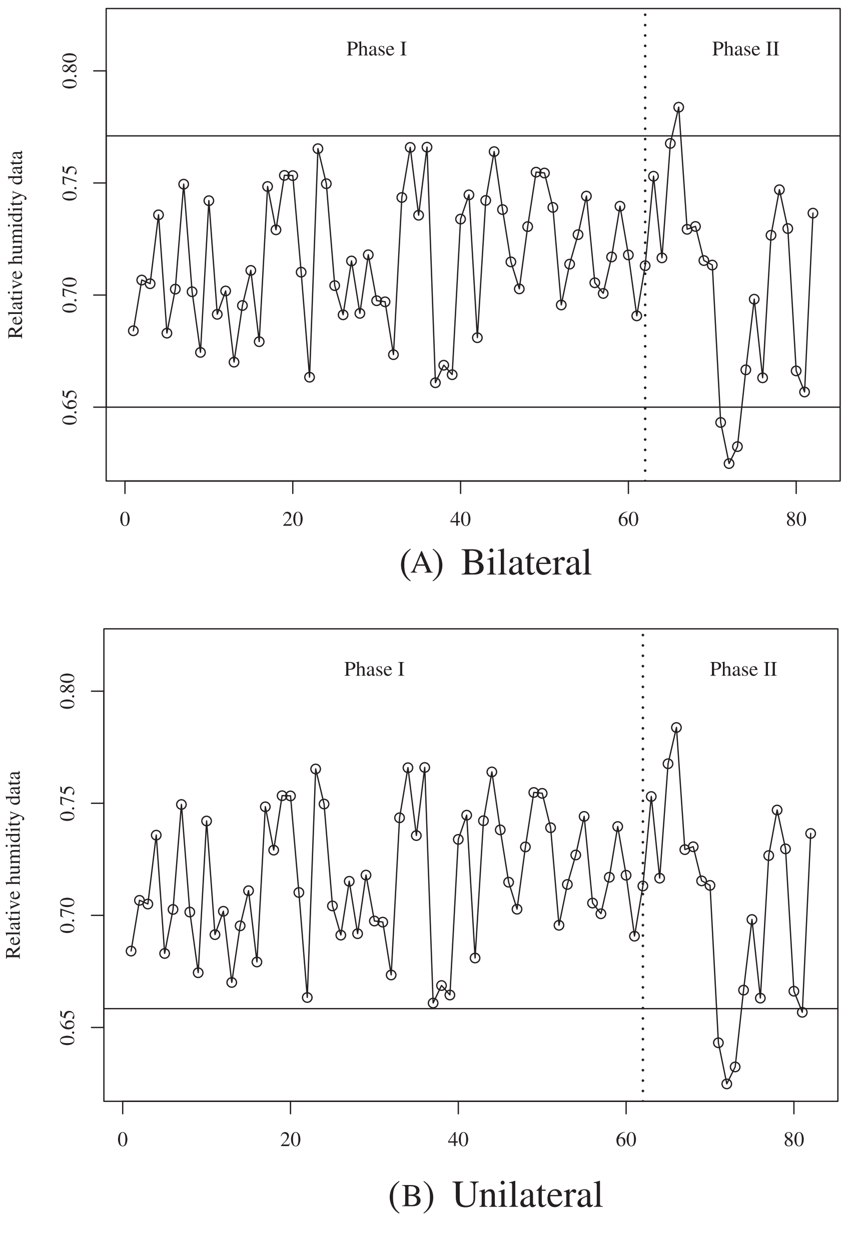

FIGURE 5 Kumaraswamy control chart for the relative humidity data (Phase I: 1–62; Phase II: 63–82)

$n > 1$). This data set was obtained from the database available at the Iowa Environmental Mesonet (IEM) website (http://mesonet.agron.iastate.edu/), from Los Angeles Downtown, California, USA, during 2019, from March 21, 2019, until July 10, 2019. A total of 2718 observations (from March 21 to June 20) were used for Phase I to estimate the parameters of the distributions of interest. Table 12 reports the estimates of the parameters of the two distributions, as well as AIC and BIC values. According to these indexes, the Kumaraswamy distribution is a better fit for the data set. The histogram and probability density functions are drawn together in Figure 4.

With the estimated parameters of the Kumaraswamy distribution, the lower and upper probability control limits 0.650 and 0.771 for the median estimator HL are obtained by the simulated empirical distribution considering that an average of 30 observations of relative humidity are taken daily. The control chart is applied, considering $\alpha = 0.0027$, for the daily HL median estimators of the relative humidity from June 21 to July 10, showing that the relative humidity on June 24 seems higher than usual, as shown in Figure 5. We present the bilateral and unilateral control charts. During Phase I, the control charts do not trigger an alarm. In Phase II, the control charts detected four points out of control. In the same phase, considering bilateral and unilateral control charts, the control charts trigger a signal, for the first time, at Samples 4 and 9, respectively.

6 | CONCLUSION

In this paper, we highlight the fact that most of the control charts proposed to model rates and proportions when they are not results of Bernoulli experiments used mean reparameterized continuous distributions. However, the mean is not always a good measure of a set of data. For this reason, we proposed new control charts for monitoring double-bounded quality characteristic considering median-dispersion reparameterized continuous distributions. The proposed control charts use the Kumaraswamy and unit-Weibull distributions to determine control limits. When the correct limits were used, good results were obtained. However, the use of incorrect control limits provokes, in general, a high number of false alarms, compromising the performance of the control chart. In addition, it was also observed that lower variability can detect changes in the process more quickly. In this way, given the results, the importance of identifying which distribution better fits the data set among many possible distributions is evident.

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