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Bootstrap control charts in monitoring value at risk in insurance

Babak Abbasi [a], Montserrat Guillen [b,*,1]

[a] School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia
[b] Department of Econometrics, Riskcenter-IREA, University of Barcelona, Barcelona, Spain

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ABSTRACT

A risk measure is a mapping from the random variables representing the risks to a number. It is estimated using historical data and utilized in making decisions such as allocating capital to each business line or deposit insurance pricing. Once a risk measure is obtained, an efficient monitoring system is required to quickly detect any drifts in the risk measure. This paper investigates the problem of detecting a shift in value at risk as the most widely used risk measure in insurance companies. The probabilistic C control chart and the parametric bootstrap method are employed to establish a risk monitoring scheme in insurance companies. Since the number of claims in a period is a random variable, the proposed method is a variable sample size scheme. Monte Carlo simulations for Weibull, Burr XII, Birnbaum–Saunders and Pareto distributions are carried out to investigate the behavior and performance of the proposed scheme. In addition, a real example from an insurance company is presented to demonstrate the applicability of the proposed method.

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1. Introduction

A risk measure attempts to assign a numerical value to a random functional loss. Risk measurement is used as an input in many decisions such as the amount of holding capital for an insurance company or prices of different types of insurance services. There are several risk measures in the literature, however, the most widely used risk measure in insurance is value at risk (VaR). Wang et al. (2005) present the applications of VaR in insurance companies and discuss on optimal insurance contracts. VaR_p (also denoted by ξ_p) is the pth quantile of the loss distribution, or it can be defined as the size of loss for which there is a small probability $1-p$ of exceeding. For risk managers it is very important to detect (quickly) in which areas of their insurance business there is a "deviation" from what could be considered as a "normal" activity. If VaR is at 99%, then 1 out of 100 losses is expected to be larger than VaR. However, in day-to-day operations, we may observe less or more than 1 loss out of 100 losses, then it is vital to know whether the observed deviation is a random error or it is due to an assignable cause which necessitates a change in decisions. The control chart is a useful aid, frequently used in the process control, to discriminate the effects of assignable causes versus the effects of chance causes. In insurance companies, a control chart can be used to alarm when too many and so the risk manager should do something to control the claims.

In mathematical form $\mathrm{VaR}_p(X)$ is:

$$\mathrm{VaR}_p(X) \equiv \inf\{x|Pr(X > x) \leqslant 1-p\} \tag{1}$$

where p is called confidence level and often is selected 0.95 or 0.99 in practice. X denotes the random variable that refers to the loss size. We will drop sub-index p reference to X for an easier notation. Although VaR has been criticized for not being a coherent risk measure as introduced by Artzner et al. (1999) since it is not sub-additive in general but it is classified into natural risk statistics introduced by Heyde et al. (2007), it is known as a robust risk measure with respect to modeling assumptions. For a justification of the concept and a comprehensive study of VaR we refer to Jia and Dyer (1996) and Krause (2003). Ma and Wong (2010) establish some behavior foundations for various types of VaR models and discuss several alternative risk measures for investors. It is well known that loss data usually have right-skewed distributions (Bolancé et al., 2003, 2008; Buch-Larsen et al., 2005; Lane, 2000). Fig. 1 shows a histogram of third party claims severity data borrowed from an insurance company which exhibits a typical right-skewed behavior with many small claims and only a few large claims. There are two approaches to estimate the distribution of loss data. In the first approach, the data are fitted to a distribution with high flexibility in shape such as Weibull, Gamma, Pareto or Burr XII distributions; this approach is called the parametric approach. Another approach is called the non-parametric approach which uses empirical distribution or kernel function to estimate the probability density function

* Corresponding author. Address: Av. Diagonal, 690, E-08034 Barcelona, Spain. Tel.: +34 934037039; fax: +34 934021821.

E-mail addresses: babak.abbasi@rmit.edu.au (B. Abbasi), mguillen@ub.edu (M. Guillen).

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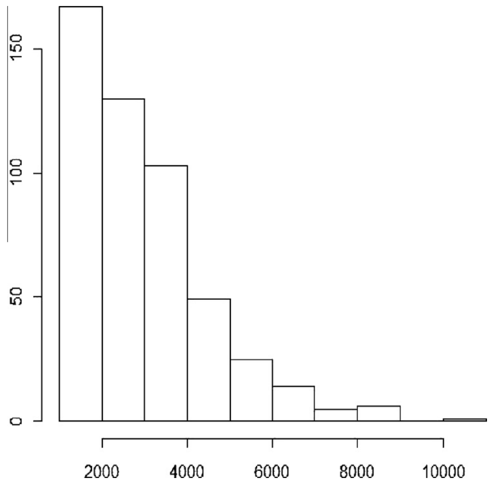

Fig. 1. Histogram of the third party claims ($n = 500$).

of data. The parametric approach in loss distribution analysis is widely used in practice.

After estimating VaR, it is important to monitor and detect any shifts in the risk.[2] In statistical quality control terminology, estimating VaR using historical data can be interpreted as phase I of statistical process control whereas we estimate the parameters of population distribution and establish the control limits. Mihailescu (2004) applies an Exponentially Weighted Moving Average (EWMA) control chart in monitoring VaR when the loss random variable follows a normal distribution. However, normality assumption is not reliable for the loss function in the most of the cases (Bolancé et al., 2003). Detecting shifts enables insurance companies to revisit their policies and decisions to prevent from enormous losses. In this article we design bootstrap control charts with variable sample size to monitor VaR in insurance companies. Moreover, we employ a probabilistic C control chart in monitoring the number of claims in a specific horizon of time. Hence, In phase I we establish two control charts: (1) a probabilistic C chart for monitoring the number of claims (C) in each period and (2) a control chart to monitor VaR based on claim values (X) in each period. To construct a control chart for a parameter, the distribution of the parameter estimator is required otherwise the bootstrapping should be used. In monitoring VaR, since the distribution of the VaR estimator is unknown we implement bootstrap control charts. Moreover, since the sample size is the number of claims in each period, which is a random variable, the designed monitoring scheme is a variable sample size bootstrap scheme.

The remainder of this paper is organized as follows. Section 2 gives an overview on the methods to estimate VaR when data are fitted to one of the following distributions: Weibull, Burr XII, Birnbaum–Saunders or Pareto distributions. Section 3 describes bootstrap methods to establish control limits for quantiles. The monitoring scheme for the number of claims and the severity of claims is explained in Section 4. The performance evaluation using extensive simulation studies considering in-control average run length (ARL) and out-of-control ARL is discussed in Section 5. A real example from an insurance company is illustrated in Section 6. Finally, Section 7 offers conclusions and final remarks.

[2] Risk monitoring and customer auditing is essential to the guarantee solvency in the insurance industry (see recent contributions by Guelman, 2012; Kaishev et al., 2013; Koyuncugil and Ozgulbas, 2012; Shin et al., 2012; Thuring et al., 2012).

2. Estimating value at risk

The estimation of VaR is equal to estimating a quantile of a population, and the estimation is subject to errors. Stephens (1983) shows the nonparametric estimation of VaR is biased. For the parametric approach, Kupiec (1995) in a simulation study using return distributions that are normal or Student-t, indicates that the estimation of VaR is subject to both high variation and bias. In order to take the bias into consideration, there are some bias corrected methods. For instance Efron (1982) estimates bias, b, by $\hat{b} = \Phi^{-1}(\hat{\xi}_{p,boot})/\hat{\sigma}_{\xi_p,boot}$ where $\hat{\xi}_{p,boot}$ and $\hat{\sigma}_{\xi_p,boot}$ are bootstrap estimates of pth quantile and its standard error respectively. The standard error of VaR_p estimator typically increases when p increases and the behavior of a quantile estimate is critically altered by the tail of the loss distribution. Despite of the large standard error of VaR, it is known as a robust risk measure. Kendall and Stuart (1972) derive formula for the asymptotic variance of quantile estimator: for ξ_p (pth quantile) of X with density $f_X(x)$, this variance is:

$$\sigma^2_{\xi_p,n} = n^{-1}p(1-p)f_X(\xi_p)^{-2} + O(n^{-3/2}) \tag{2}$$

where n is the sample size. There are two principle approaches in estimating VaR from the historical data. The first approach is fitting a parametric distribution to the data, in this approach a distribution that takes a variety of shapes such as Weibull or Burr XII distribution is fitted to the data and the goodness-of-fit test is employed to check the model fit. Then VaR as a quantile of the distribution is obtained from the best fitted distribution. For an instance of this approach we refer to Sun and Hong (2010), who apply importance sampling in the parametric method to estimate VaR. The second approach is using the non-parametric density estimation techniques that fits a non-parametric distribution based on the empirical distribution or the kernel function to the historical data; one can then estimate VaR from non-parametric density function. Chang et al. (2003) and Jeong and Kang (2009) use the non-parametric method to estimate VaR. Zmeskal (2005) proposes a method to estimate VaR in fuzzy environments. If the empirical distribution function is used in a non-parametric approach the VaR_p is obtained by one of the following formulas:

$$\hat{\xi}_p^{p,method1} = X_{(\lfloor (n-1)p\rfloor+1)} \tag{3}$$

$$\hat{\xi}_{p,method2} = X_{(\lfloor (n+1)p\rfloor)} \tag{4}$$

$$\hat{\xi}_{p,method3} = X_{(\lfloor (n-1)p\rfloor+1)} + ((n-1)p - \lfloor (n-1)p\rfloor)(X_{(\lfloor (n-1)p\rfloor+2)} - X_{(\lfloor (n-1)p\rfloor+1)}) \tag{5}$$

$$\hat{\xi}_{p,method4} = X_{(\lfloor (n+1)p\rfloor)} + ((n+1)p - \lfloor (n+1)p\rfloor)(X_{(\lfloor (n+1)p\rfloor+1)} - X_{(\lfloor (n+1)p\rfloor)}) \tag{6}$$

where $\lfloor . \rfloor$ is the floor function and $X_{(i)}$ is the ith order statistic of X in the sample. Another method to estimate pth quantile is the jackknife method. The delete-one jackknife estimate of ξ_p is obtained by:

$$\hat{\xi}_{p,jackknife} = (\lfloor (n-1)p\rfloor + 1)n^{-1}X_{(\lfloor (n-1)p\rfloor+2)} + (1 - (\lfloor (n-1)p\rfloor + 1)n^{-1})X_{(\lfloor (n-1)p\rfloor+1)} \tag{7}$$

and the jackknife estimate of the variance of $\hat{\xi}_{p,jackknife}$ is given by (see Martin, 1990; Miller, 1974):

$$\hat{\sigma}_{\xi_{p,jackknife}} = (n-1)(n - \lfloor (n-1)p\rfloor + 1)(\lfloor (n-1)p\rfloor + 1)n^{-2} (X_{(\lfloor (n-1)p\rfloor+2)} - X_{(\lfloor (n-1)p\rfloor+1)})^2 \tag{8}$$

In the appendix we explore the biases of the above estimations. Martin (1990) shows that the variance of the jackknife estimator of sample quartile is inconsistent. Shao (1987) obtains a consistent estimator of the variance of sample quartile by using a delete-d jackknife. To a review on the jackknife methods we refer to Shao and Tu (1996). Even by having a consistent estimator of variance of sample quantile, to construct a closed formula for the confidence interval, the distribution of the estimation is required. Kaigh (1983) introduces another estimator of pth quantile which is called generalized quantile sample and denoted by $\widehat{Q}_U(p,k_n,n)$. Kaigh demonstrates asymptotic equivalence of $\widehat{Q}_U(p,k_n,n)$ and $X_{\lfloor (n+1)p \rfloor}$ if $\lim\inf_{n\to\infty}\inf(k_n/n) > 0$, where k is the sub-sample of n in generalized quantile sample method. Kaigh (1983) obtains the variance of $\widehat{Q}_U(p,k_n,n)$ and proves that by satisfying some conditions, the asymptotic distribution of $\widehat{Q}_U(p,k_n,n)$ is normal. Then Kaigh (1983) constructs a symmetric confidence interval for the pth quantile as:

$$\widehat{Q}_U(p,k_n,n) \pm \Phi^{-1}(1-\alpha/2)S(p,k_n,n)/n^{1/2} \tag{9}$$

where $S(p,k_n,n)$ is the standard deviation of $\widehat{Q}_U(p,k_n,n)$, $\Phi^{-1}(.)$ is the inverse of the standard normal cumulative distribution function and α is the risk level for the confidence interval. Using this method requires a large sample size ($n \to \infty$), therefore, we are not going to use this method to construct control limits for VaR. Moreover, in simulation studies in this paper we show that the confidence interval (control limits) for pth quantile is asymmetric while Kaigh's formula provides a symmetric confidence interval. Likewise, there are other methods to construct a confidence interval for pth quantile where $p \to 0$, $n \to \infty$ and $np \to \infty$, for details we refer to Dekkers and de Haan (1989), Dekkers et al. (1989) and Peng and Yang (2009). However, according to their assumptions these methods are not applicable to construct confidence intervals or control limits for VaR in insurance. Wang et al. (2010) perform a comparison study between the generalized bootstrap (a parametric method) and the non-parametric bootstrap methods to estimate high quantiles and remark that the generalized bootstrap has overall better performance. Some more recent approaches include mixtures of distributions are presented in Peters et al. (2011).

In insurance companies VaR is estimated from a relatively large sample size for example $n = 300$. Therefore, in this paper we apply the parametric approach to estimate VaR by using the three-parameter Weibull, Burr XII Birnbaum–Saunders and Pareto distributions. In the following we explain briefly the parametric distributions that we use in this research.

2.1. Weibull distribution

The Weibull distribution is often used in modeling failure rates and risks. The probability density function of the Weibull distribution is written as

$$f_X(x) = \frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} e^{\left(-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right)}, \quad \alpha,\beta > 0,\ \gamma \geqslant 0,\ x \geqslant \gamma \tag{10}$$

where α, β and γ are scale, shape and location parameters respectively. Since the Weibull density function takes a variety of shapes, it is broadly used in non-normal data analysis. However, the successful application of the Weibull distribution depends on having acceptable statistical estimates of its three parameters. Estimating the parameters of the three-parameter Weibull distribution is intrinsically a difficult task. Ross (1994) presents a graphical method to estimate the three parameters of the Weibull distributions. Johnson et al. (1994) propose a moment method to estimate the three parameter of the Weibull distribution; they present Eqs. (11)–(13) for estimating the unknown parameters.

$$\widehat{Sk} = \frac{\Gamma((3/\widehat{\beta})+1) - 3\Gamma((2/\widehat{\beta})+1)\Gamma((1/\widehat{\beta})+1) + 2\Gamma((1/\widehat{\beta})+1)^3}{[\Gamma((2/\widehat{\beta})+1) - \Gamma((1/\widehat{\beta})+1)^2]^{3/2}} \tag{11}$$

$$\widehat{\alpha} = \left(\frac{n^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}{\Gamma((2/\widehat{\beta})+1) - \Gamma((1/\widehat{\beta})+1)^2}\right)^{1/(2\widehat{\beta^2})} \tag{12}$$

$$\widehat{\gamma} = \bar{x} - \widehat{\alpha}\Gamma((1/\widehat{\beta})+1) \tag{13}$$

where $\bar{x}$ and $\widehat{Sk}$ are sample mean and sample skewness respectively. Note that Eq. (11) should be solved numerically. Abbasi et al. (2008) present a neural network based model to estimate the three parameters of the Weibull distribution, using a simulation study they remark that the trained neural network model and Johnson et al.'s method both perform well and relatively comparable. In this paper we use Eqs. (11)–(13) to estimate the Weibull parameters. The pth quantile of the Weibull distribution is obtained by

$$\xi_{Weibull,p} = \gamma + \alpha(-\ln(1-p))^{1/\beta} \tag{14}$$

2.2. Pareto distribution

The Pareto distribution is often used in loss data analysis. Also, it is extensively applied in the estimation of percentiles and extreme values. It well describes the random behavior of large losses. The Pareto density function (see Arnold, 1983) can be written as

$$f_X(x) = \alpha\beta^{\alpha}x^{-(\alpha+1)}, \quad \alpha,\beta > 0,\ x \geqslant \beta \tag{15}$$

where α and β are the shape and the scale parameters respectively. The maximum likelihood estimators of α and β are

$$\widehat{\alpha} = n/\sum_{i=1}^{n}\ln\left(\frac{x_i}{\widehat{\beta}}\right) \tag{16}$$

$$\widehat{\beta} = \underset{i}{Min}(x_i). \tag{17}$$

The pth quantile of the Pareto distribution is obtained by

$$\xi_{Pareto,p} = \frac{\beta}{(1-p)^{1/\alpha}} \tag{18}$$

2.3. Burr XII distribution

The Burr XII distribution has a wide range of shape and scale parameters, and it has an important role in reliability modeling and risk analysis. However, estimating the Burr XII parameters is not a straightforward task. The Burr XII density function is defined as:

$$f_X(x) = kcx^{c-1}(1+x^c)^{-(k+1)}, \quad x \geqslant 0,\ c,k > 0 \tag{19}$$

There is a standardized transformation between a Burr variable (say Z) and another random variable (say X). This transformation may be expressed as $(Z-\mu_z)/\sigma_z = (X-\mu_x)/\sigma_x$, where μ_z and σ_z are the mean and standard deviation of Z that are given in Burr tables and μ_x and σ_x are the mean and standard deviation of X. Abbasi et al. (2010) develop a Perceptron neural network to estimate the Burr XII parameters. In this paper, we use Abbasi et al.'s method to estimate the Burr XII parameters. The pth quantile of the Burr XII distribution is expressed by

$$\xi_{Burr,p} = ((1-p)^{-1/k} - 1)^{1/c} \tag{20}$$

Note that this distribution is closely related to the Champernowne distribution used in Buch-Larsen et al. (2005).

2.4. Birnbaum–Saunders distribution

The two-parameter Birnbaum–Saunders distribution is widely used in failure analysis and risk analysis (Chang and Tang, 1994). Given the shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, the probability density function of the Birnbaum–Saunders distribution is given by

$$f_X(x) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\sqrt{\frac{\beta}{x}} + \left(\frac{\beta}{x}\right)^{3/2}\right] e^{\left[\frac{-1}{2\alpha^2}\left(\frac{x}{\beta}-2+\frac{\beta}{x}\right)\right]}, \quad x, \alpha, \beta > 0 \tag{21}$$

Ng et al. (2003) propose the modified moment estimators, $\widehat{\alpha}$ and $\widehat{\beta}$, for α and β as

$$\widehat{\alpha} = \left(2\sqrt{\frac{s}{r}} - 2\right)^{1/2} \tag{22}$$

and

$$\widehat{\beta} = \sqrt{sr} \tag{23}$$

where $s = \sum_{i=1}^{n} x_i/n$ and $r = \left(\sum_{i=1}^{n}(1/x_i)/n\right)^{-1}$. We use these estimators for the Birnbaum–Saunders parameters in this paper. The pth quantile of the Birnbaum–Saunders distribution is

$$\xi_{Birnbaum-Saunders.p} = \frac{\beta}{4}\left(\alpha(\Phi^{-1}(p)) + \left(\alpha^2(\Phi^{-1}(p))^2 + 4\right)^{1/2}\right)^2 \tag{24}$$

where $\Phi^{-1}(.)$ is the inverse of the standard normal cumulative distribution function.

3. Bootstrap control charts

The bootstrap control charts have been studied by many authors such as Jones and Woodall (1998). When the data present a non-normal distribution, bootstrapping is an efficient way to construct the control limits. In the risk measure estimation literature Kim and Hardy (2007) propose the exact bootstrap to estimate certain risk measures including VaR and the tail conditional expectation (TCE). They show that the bias can be corrected for the TCE using the bootstrap. Kim (2011) uses bootstrap in capital allocation problem as a useful nonparametric tool (see also Kim (2010)). Bajgier (1992) developed a bootstrap control chart for process mean regardless of the distribution of the process. Liu and Tang (1996) apply the bootstrap control charts to both dependent and independent observations. Niaki and Abbasi (2007) use bootstrapping to design the control limits for multi-attribute processes. In constructing a confidence interval for pth quantile, Hall and Martin (1989) show that conventional percentile bootstrap confidence interval has poor coverages and gives a coverage error of order $O(n^{-1/2})$, which cannot be improved by nominal coverage calibration using iterated bootstrap in general. Ho and Lee (2005) show that by selecting an appropriate order of bandwidths iterating the bootstrap percentile method gives a coverage of order $O(n^{-5/6})$. Also the coverage error of the smoothed bootstrap-t method in constructing a confidence interval for pth quantile is $O(n^{-2/3})$ in general (Falk and Janas, 1992) and by selecting an appropriate order of bandwidths Ho and Lee (2005) remark that it can be reduced to $O(n^{-58/57})$. However, the bootstrap-t method is hardly an efficacious alternative because it requires a consistent estimate for the variance of pth quantile. In designing control charts for quantiles; Park (2009) applies the bootstrap methods to design a median control chart. Nichols and Padgett (2006) develop a parametric bootstrap control chart for the two-parameter Weibull percentiles to monitor the process of producing carbon fibers and compare it with the Shewhart control charts for percentiles of the two-parameter Weibull distribution, concluding bootstrapping approach has the better performance. Lio and Park (2008) use the bootstrapping technique to design the control limits in detecting shifts in the percentile of the Birnbaum–Saunders distribution. They show that the bootstrap control chart outperforms the Shewhart control chart in detecting shifts in percentiles. In both of the above research studies, sample size is constant. Lio and Park (2010) apply the bootstrap method to construct the control limits for the inverse Gaussian percentiles with constant sample sizes. In this paper we use the bootstrap percentile method to construct the control limits for VaR with the variable sample size. In general the steps to construct a bootstrap control chart for a quantile using the bootstrap percentile method when the distribution of data is $f_X(x)$[3] are as follows:

1. Take a relatively large sample from the random variable $(X_1, \ldots, X_m)$ (sample size = m (e.g. m = 300)).
2. Find the best parametric distribution for the data;
   (a) Plot the histogram of the data.
   (b) Estimate the parameters of each distribution.
   (c) Apply the goodness-of-fit test to select the best distribution among the above distributions that you have estimated their parameters in substep b.
3. Generate a bootstrap sample $(X_1^*, \ldots, X_n^*)$ from the best fitted distribution and estimate the parameters of the selected distribution using the generated sample. Then estimate the pth quantile of the distribution with estimated parameters. n is equal to the sample size in the phase II of control chart (Phase II is when we use control charts in monitoring).
4. Repeat Steps 3 a large number of times, B times, obtaining B bootstrap samples. For each bootstrap sample, estimate the parameters of the distribution obtained in step 2-c. Then calculate the pth quantile from the estimated distribution. They are denoted by $\left(\widehat{\xi}_{p,1}^*, \widehat{\xi}_{p,2}^*, \ldots, \widehat{\xi}_{p,B}^*\right)$ where $\widehat{\xi}_{p,i}^*$ is the pth quantile of the ith bootstrap sample.[4]
5. The confidence interval for pth quantile (ξ_p) with $(1-\alpha)100\%$ confidence (L_p, U_p)[5] is obtained by

$$\begin{aligned} L_p &= \widehat{\xi}_{p,(\lfloor (B-1)\alpha/2\rfloor+1)} + (B-1)\alpha/2 - (\lfloor (B-1)\alpha/2\rfloor) \\ &\quad \times \left(\widehat{\xi}_{p,(\lfloor (B-1)\alpha/2\rfloor+2)} - \widehat{\xi}_{p,(\lfloor (B-1)\alpha/2\rfloor+1)}\right) \\ U_p &= \widehat{\xi}_{p,(\lfloor (B-1)(1-\alpha/2)\rfloor+1)} + (B-1)(1-\alpha/2) - (\lfloor (B-1)(1-\alpha/2)\rfloor) \\ &\quad \times \left(\widehat{\xi}_{p,(\lfloor (B-1)(1-\alpha/2)\rfloor+2)} - \widehat{\xi}_{p,(\lfloor (B-1)(1-\alpha/2)\rfloor+1)}\right) \end{aligned} \tag{25}$$

6. Use the upper band of confidence interval as the upper control limit (UCL) and the lower band of confidence interval as the lower control limit (LCL) ($LCL = L_p$ and $UCL = U_p$) whereas α is set according to type I error in control charts.

[3] It can be estimated by the parametric distributions as explained in Section 2 or by the non-parametric functions. In the non-parametric approach one can use the empirical distribution function F_n defined as $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leqslant x)$), where $I(.)$ is the indicator function (see Wang et al. (2010)) or $\widehat{f}_{n,\eta}(t) = (n\eta)^{-1}\sum_{i=1}^{n} k((t - X_i)/\eta)$ where $k(.)$ is a kernel function and $\eta > 0$ is the bandwidth (see Ho and Lee, 2005).

[4] It can be proved that $Pr\left(\underset{i(i=1,\ldots,B)}{Min}\left(\widehat{\xi}_{p,i}^*\right) < \xi_p < \underset{i\,(i=1,\ldots,B)}{Max}\left(\widehat{\xi}_{p,i}^*\right)\right) = 1 - p^B - (1-p)^B$, Strelen (2005) uses this to construct a confidence interval for ξ_p, where B random samples are available instead of bootstrapping. However, the confidence level always is $(1 - p^B - (1-p)^B)100\%$ and we are not able to construct a confidence interval for a given confidence level.

[5] Detail explanation of Step 5: if $G_{\xi_p}(x_p) = Pr(\xi_p \leqslant x_p)$ then the bootstrap estimate $G_{\xi_p,boot}(x_p)$ of $G_{\xi_p}(x_p)$ can be approximated by $G_{\xi_p,boot}(x_p) = \frac{1}{B}\sum_{i=1}^{B} I\left(\widehat{\xi}_{p,i}^* \leqslant x_p\right)$ where $I(.)$ is the indicator function and $\widehat{\xi}_{p,i}^*$ is the estimate of $\xi_{p,i}$ from the ith bootstrap sample. Then the $(1-\alpha)100\%$ bootstrap confidence interval for ξ_p is $\left(G^{-1}_{\xi_p,boot}\left(\frac{\alpha}{2}\right),\ G^{-1}_{\xi_p,boot}\left(1-\frac{\alpha}{2}\right)\right)$.

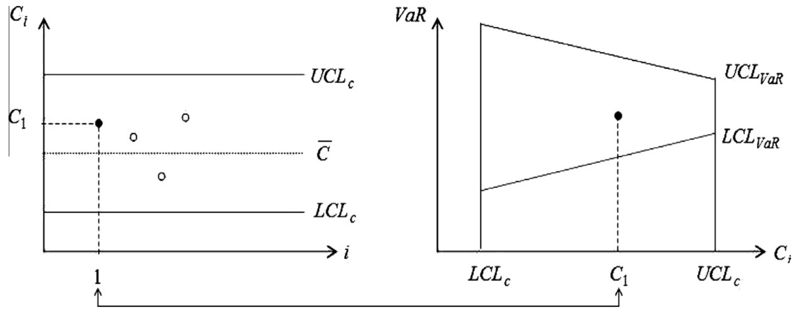

Fig. 2. Monitoring scheme for VaR.

Using the above steps we can construct the bootstrap control charts. However, in monitoring the risk measure the number of claims in a specific period is not constant. Therefore we should find the confidence intervals for different sample sizes.

4. Monitoring scheme for VaR

For an insurance company, monitoring both the number of claims over a specific horizon and VaR are important. Here, we use two control charts simultaneously for this purpose. First, we use a probabilistic C control chart to monitor the number of claims in a specific period (for example in a business day) and second we apply a bootstrap control chart to monitor VaR based on steps explained in the previous section. The steps of the monitoring scheme are as follows:

1. Using the historical data estimate the parameter of the Poisson distribution for the number of claims in a period (e.g. a week or a day).
2. Use the following equations to find the control limits for the number of claims in the defined period (α is the type I error and is set to 0.02[6]

$$LCL_c = \inf\{x|Pr(X < x) \leqslant \alpha/2\} \tag{26}$$

$$UCL_c = \inf\{x|Pr(X > x) \leqslant \alpha/2\} \tag{27}$$

3. For different sample sizes (n) varying from LCL_c to UCL_c use the bootstrapping procedure explained in Section 3 to construct control limits for VaR at different sample sizes (in step 5 of bootstrapping procedure α is type I error).
4. Afterward, the number of claims (C_i) for the ith period is plotted against probabilistic C control chart, if it falls in probabilistic C control chart, the parameters of distributions which used in constructing bootstrap control limits are estimated using claim values in the ith period, then VaR is calculated and compared with the bootstrap control limits with the sample size C_i. For example, if the Weibull distribution is used in constructing the bootstrap control charts, based on data from ith period the three parameters of Weibull distribution are estimated and VaR is calculated from Eq. (14) and is checked whether the estimated VaR falls inside the control limits or not. Fig. 2 depicts the proposed monitoring scheme. The ARL_0 (in-control average run length) is calculated by $1/\alpha$ and is defined as the average number of subgroup samples between false alarms (i.e. the average number of samples to observe an out-of-control signal when the data are not shifted). The ARL_1 (out-of- control average run length) is defined as the average number of subgroup samples falls in control region when data are out of control. The ARL_1 is related to the type II error and depends on the magnitude of shifts in the data.

Lemma 1. *The ARL_0 for the bootstrap control charts obtained from step 3 is $1/\alpha$.*

Proof. The probabilistic C control chart is in control, then $\sum_{c=LCL_C}^{UCL_C} Pr(C=c) = 1$ and

$$Pr(L_p < \hat{\xi}_{p,i} < U_p) = \sum_{c=LCL_C}^{UCL_C} Pr(L_{p,c} < \hat{\xi}_{p,i} < U_{p,c}|C)Pr(C=c)$$
$$= \sum_{c=LCL_C}^{UCL_C} (1-\alpha)Pr(C=c) = (1-\alpha)\sum_{c=LCL_C}^{UCL_C} Pr(C=c) = (1-\alpha)$$

Then $ARL_0 = 1/(1-(1-\alpha)) = 1/\alpha$. □

Lemma 2. *If α_1 is the type I error of the probabilistic C chart and α_2 is the type I error of the bootstrap control chart. The ARL_0 for the monitoring scheme is $1/(\alpha_1 + (1-\alpha_1)\alpha_2)$.*

Proof of Lemma 2 is straightforward and similar to the proof of Lemma 1, hence we omit the proof.

5. Simulation study and performance analysis

In this section, using simulation studies we examine the performance of the proposed monitoring scheme when data follow different distributions such as the Weibull, Pareto, Burr XII and Birnbaum–Saunders distributions. To the best of our knowledge there is no other method in the literature to do a comparison study. We quantify the performance of the proposed method by ARL_1 (which is $1/\beta$ where β denotes the type II error). The ARL is broadly used in literature to evaluate the performance of a monitoring technique (refer to Jones and Woodall, 1998; Nichols and Padgett, 2006). We assume that the number of claims per business day follows a Poisson distribution with the average 10 claims per business day. The control limits by setting α = 0.02 (i.e. ARL_0 = 50 as it means 50 days in insurance companies) for monitoring the number of claims per day using Eqs. (26) and (27) would be $LCL_c = 4$ and $UCL_c = 18$. Using the obtained control limits, the nominal in-control ARL becomes $1/(Pr(4 < X < 18) \simeq 57$. Table 1 shows the ARL_1 values for different shifts in the number of claims by using 1000 replications. Table 1 indicates that the probabilistic C control chart

[6] There is a compromise between the type I error (α) and type II error (β). Selecting a small value for α will increase the type II error and vice versa. It means if we want to reduce the number of error signals (the situations that control chart triggers an alarm while there is no actual shift) we have to bear an increase in time between when the control chart detects a shift and when the data are actually shifted. For insurance companies that monitor the risk on a daily basis, α = 0.02 means that in average every 50 days the control chart alerts a shift when there is no actual shift. Hence, α = 0.02 is practical for insurance companies. However, it can be set to any values by risk analysts.

Table 1
In-control/out-of-control ARLs for the number of claims ($\lambda = 10$).

λ	7	8	9	10	11	12	13	14	15	16
ARL	12.00	22.40	44.89	56.41	44.83	26.37	13.23	8.88	5.71	3.59
Std (ARL)	11.58	22.18	42.86	55.12	42.47	25.45	12.72	8.14	5.41	2.95

Table 2
The control limits for VaR_p (claim values follow *Weibull* (1,1,10)).

$p = 0.99$			$p = 0.95$		
Nominal in-control ARL = 100					
$ARL_0 = 100.94$			$ARL_0 = 105.28$		
$Std\ (ARL_0) = 108.07$			$Std\ (ARL_0) = 104.38$		
n	*LCL*	*UCL*	*n*	*LCL*	*UCL*
4	10.44	20.07	4	10.33	17.23
5	10.55	20.06	5	10.45	17.03
6	10.65	20.02	6	10.57	16.81
7	10.83	19.78	7	10.68	16.66
8	10.85	19.71	8	10.75	16.34
9	11.03	19.69	9	10.84	16.29
10	11.07	19.52	10	10.91	16.15
11	11.23	19.46	11	10.94	15.95
12	11.23	19.36	12	10.94	15.82
13	11.31	19.18	13	11.00	15.72
14	11.41	19.12	14	11.05	15.66
15	11.45	19.09	15	11.07	15.60
16	11.51	19.01	16	11.13	15.45
17	11.54	18.79	17	11.24	15.35
18	11.57	18.58	18	11.24	15.31
Nominal in-control ARL = 50					
$ARL_0 = 50.10$			$ARL_0 = 53.71$		
$Std\ (ARL_0) = 49.74$			$Std\ (ARL_0) = 51.34$		
n	*LCL*	*UCL*	*n*	*LCL*	*UCL*
4	10.47	19.08	4	10.44	16.44
5	10.66	19.03	5	10.53	16.32
6	10.78	18.94	6	10.63	16.20
7	10.95	18.81	7	10.75	16.05
8	11.03	18.80	8	10.81	15.85
9	11.11	18.70	9	10.93	15.72
10	11.22	18.67	10	10.95	15.57
11	11.27	18.66	11	11.02	15.54
12	11.37	18.40	12	11.09	15.39
13	11.45	18.36	13	11.14	15.29
14	11.49	18.24	14	11.20	15.29
15	11.60	18.23	15	11.24	15.24
16	11.66	18.13	16	11.27	15.15
17	11.70	18.06	17	11.31	15.05
18	11.77	18.06	18	11.34	14.87

performs well in detecting shifts in the number of claims. For example when the average number of claims jumps to 16, the control chart detects it in 3.5 days in average. We then use the procedure explained in Section 4 to construct the bootstrap control charts in monitoring the claim values. For all simulation studies we round the bootstrap control limits by using $LCL_{boot} = \lfloor 100L_{\alpha/2} \rfloor /100$ and $UCL_{boot} = \lceil 100U_{1-\alpha/2} \rceil /100$.

Table 2 shows the bootstrap control limits for $VaR_{0.99}$ and $VaR_{0.95}$ when the distribution of loss follows the three-parameter Weibull distribution with $\alpha = 1$, $\beta = 1$ and $\gamma = 10$. In Table 2 ARL_0 is calculated using 1000 replications where the nominal *ARL* values are set to 100 and 50.

Table 3 shows the out-of-control ARLs for detecting shifts in $\mathrm{VaR}_{0.95}$ and $\mathrm{VaR}_{0.99}$ when the scale parameter of the Weibull distribution is shifted. In addition, VaRs for the shifted scenarios have been presented in Table 3. Tables 4 and 5 report the out-of-control ARL for $\mathrm{VaR}_{0.95}$ and $\mathrm{VaR}_{0.99}$ when the shape and location parameters of the Weibull distribution are shifted. Considering the results in Tables 3–5 we remark for the same positive shift in VaR the bootstrap control chart is more sensitive to shift in the shape parameter. Also results show that the bootstrap control chart is more sensitive to shift in the scale parameter than shift in the location parameter. Furthermore, for the same negative shifts in VaR the control charts is more sensitive to shift in the location parameter. Also, for the same negative shift in VaR when the shift is triggered from the scale parameter it is more quickly detected than when the shape parameter is shifted.

Table 6 presents the control limits obtained by the bootstrap method for the Pareto distribution with $\alpha = 30$ and $\beta = 10$, where the nominal value for ARL_0 is 50. Tables 7 and 8 show the out-of-control ARLs for the different shifts in α and β. They indicate that for the same positive shift in VaR the bootstrap control chart is more sensitive to shift in the scale parameter than shift in the shape parameter. While for the same negative shift in VaR the control chart is more sensitive to shift in the shape parameter than shift in the scale parameter.

Table 9 presents the control limits obtained by the bootstrap method for the Burr XII distribution with the parameters $c = 3$ and $k = 1$, where the nominal value for ARL_0 is 50. Tables 10 and 11 display the out-of-control ARL values for the different shifts in

Table 3
In-control/Out-of-control ARLs for the shifts in the scale parameter of Weibull $(\alpha, 1, 10)$.

Scale parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
2.5	17.48	1.67	1.09	21.51	2.07	1.47
2	15.99	2.67	1.99	19.21	3.70	3.00
1.8	15.39	3.64	3.08	18.28	5.30	4.71
1.5	14.49	7.52	6.83	16.90	10.23	9.30
1.3	13.89	16.49	15.72	15.98	20.75	19.91
1.1	13.29	38.86	38.31	15.06	41.65	42.41
1 (No shift)	12.99	53.71	51.34	14.60	50.10	49.74
0.85	12.54	40.46	40.37	13.91	38.33	37.94
0.7	12.09	15.91	15.34	13.22	16.56	16.41
0.6	11.79	8.26	8.29	12.76	8.41	8.22
0.5	11.49	4.12	3.70	12.30	4.65	4.05
0.4	11.19	2.35	1.82	11.84	2.52	1.94
0.3	10.89	1.42	0.79	11.38	1.53	0.90
0.2	10.59	1.11	0.33	10.92	1.14	0.40

Table 4
In-control/Out-of-control ARLs for the shifts in the shape parameter of Weibull $(1, \beta, 10)$.

Shape parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
0.5	18.97	1.83	1.28	31.20	1.84	1.26
0.7	14.79	4.69	4.07	18.86	4.65	3.99
0.8	13.94	9.29	8.60	16.74	9.39	9.10
0.9	13.38	21.88	20.35	15.45	21.15	21.75
0.95	13.17	33.95	32.41	14.99	34.27	32.25
0.98	13.06	43.97	43.13	14.75	42.61	40.39
1 (No shift)	12.99	53.71	51.34	14.60	50.10	49.74
3	11.44	46.53	46.18	11.66	9.61	9.41
4	11.31	20.39	19.74	11.46	5.07	4.78
5	11.24	11.99	10.45	11.35	3.42	2.86
10	11.11	4.84	4.07	11.16	1.88	1.30
15	11.07	3.74	3.36	11.10	1.59	0.92
20	11.05	3.38	2.73	11.07	1.50	0.90

Table 5
In-control/Out-of-control ARLs for the shifts in the location parameter of Weibull $(1, 1, \gamma)$.

Location parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
6.5	9.49	1.07	0.27	11.10	1.28	0.60
7	9.99	1.12	0.34	11.60	1.41	0.80
7.5	10.49	1.24	0.54	12.10	1.64	1.02
8	10.99	1.51	0.89	12.60	2.13	1.50
8.5	11.49	2.09	1.52	13.10	3.35	2.77
9	11.99	3.75	3.12	13.60	6.21	5.69
9.5	12.49	11.75	10.91	14.10	15.55	15.99
10 (No shift)	12.99	53.71	51.34	14.60	50.10	49.74
10.5	13.49	48.87	46.05	15.10	46.33	46.95
11	13.99	23.85	23.20	15.60	44.38	45.04
11.5	14.49	11.21	11.03	16.10	29.67	29.46
12	14.99	6.71	6.20	16.60	20.86	20.45
12.5	15.49	3.63	2.93	17.10	13.27	12.69
13	15.99	2.20	1.59	17.60	9.38	8.49

c and k. They reveal that for the same positive shift in VaR the bootstrap control chart has the same performance, with either the first parameter or the second parameter shifts. However for the same negative shift in VaR the control chart is more sensitive to shift in k than shift in c.

Table 6
The control limits for VaR_p (claim size has *Pareto* $(30, 10)$ & nominal $ARL_0 = 50$).

$p = 0.99$			$p = 0.95$		
$ARL_0 = 50.63$ $Std\ (ARL_0) = 50.56$			$ARL_0 = 50.82$ $Std\ (ARL_0) = 48.42$		
n	LCL	UCL	n	LCL	UCL
4	32.11	81.25	4	31.41	57.94
5	32.89	76.74	5	31.96	55.79
6	33.59	73.06	6	32.37	53.69
7	34.21	72.63	7	32.70	53.90
8	34.67	69.93	8	32.93	52.22
9	35.15	69.21	9	33.25	52.17
10	35.53	67.28	10	33.63	51.48
11	35.87	67.22	11	33.73	50.52
12	36.29	65.74	12	34.06	49.79
13	36.40	64.63	13	34.16	49.60
14	37.00	64.57	14	34.46	49.16
15	37.10	62.82	15	34.57	49.13
16	37.61	62.36	16	34.67	48.63
17	37.66	62.13	17	34.87	48.21
18	37.88	61.98	18	34.93	47.83

Table 7
In-control/Out-of-control ARLs for the shifts in the shape parameter of Pareto $(\alpha, 10)$.

Shape parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
20	26.98	1.01	0.13	31.69	1.15	0.40
22	29.68	1.07	0.27	34.86	1.44	0.77
25	33.73	1.63	0.93	39.62	3.02	2.47
27	36.43	3.58	3.02	42.79	7.48	6.63
29	39.12	20.07	19.42	45.96	26.65	27.25
30 (No shift)	40.47	50.82	48.42	47.58	50.63	50.56
31	41.82	49.96	49.70	49.13	48.07	46.31
33	44.52	18.20	17.00	52.30	30.07	29.84
35	47.22	7.10	6.62	55.47	16.17	14.82
38	51.27	2.51	2.08	60.22	7.16	6.42
40	53.97	1.70	1.06	63.39	4.30	3.82

Table 12 presents the control limits obtained by the bootstrap method for the Birnbaum–Saunders distribution with $\alpha = 2$ and $\beta = 2$, where the nominal value for ARL_0 is set to 50. Tables 13 and 14 show the out-of-control ARLs for the different shifts in α and β. They point out that for the same positive shift in VaR the bootstrap control chart is more sensitive to shift in the scale parameter than shift in the shape parameter. Also it is sensitive to detect negative shift in VaR when the scale parameter shifts but it is unable to detect negative shifts when the shape parameter shifts, Table 13 shows that out-of-control ARLs for negative shifts in the shape parameter are sometimes higher than in-control ARL. To investigate this matter Fig. 3 shows the Birnbaum–Saunders distributions for the different values of α when $\beta = 2$. Fig. 3 shows when α decreases, VaR decreases and also the variance of VaR dwindles; therefore the control chart is impotent to detect such shifts. This is similar to monitor mean by Shewhart control chart, when μ decreases and at the same time the standard deviation of the process declines considerably, in this situation most of the points remains in the in-control region, so control chart is unable to detect any shifts. To tackle this phenomenon the runs rules techniques are developed and they can be embedded in the bootstrap control charts in a further research study.

All simulation studies show that the control limits obtained by using bootstrapping perform well in monitoring VaR and are capable to quickly detect shifts in VaR. Some observations based on simulation studies are:

Remark 1. Obtained control limits are not symmetric $(UCL - VaR_p \neq VaR_p - LCL)$

Table 8
In-control/Out-of-control ARLs for the shifts in the scale parameter of Pareto $(30,\beta)$.

Scale parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
5	45.61	2.01	1.39	75.35	1.91	1.25
6	49.42	3.32	2.69	64.63	3.54	2.91
7	46.02	6.52	6.22	57.92	6.37	5.70
8	43.62	13.91	14.40	53.34	14.60	14.70
9	41.84	31.60	32.61	50.04	30.59	31.19
10 (No shift)	40.47	50.82	48.42	47.58	50.63	50.56
11	39.39	47.13	47.05	45.59	46.00	46.37
12	38.50	32.99	32.20	44.03	33.52	33.53
13	37.77	21.39	19.33	42.75	24.22	22.58
14	37.15	14.38	14.74	41.68	15.16	14.31
15	36.63	11.05	10.12	40.78	11.02	10.48
17	35.78	5.98	5.74	39.33	6.02	5.41
19	35.12	3.96	3.36	38.22	4.04	3.49
20	34.84	3.39	2.89	37.86	3.35	2.65

Table 9
The control limits for VaR_p (claim size has *BurXII* (3,2) & nominal $ARL_0 = 50$).

$p = 0.99$			$p = 0.95$		
$ARL_0 = 50.52$; $Std\ (ARL_0) = 51.28$			$ARL_0 = 50.54$; $Std\ (ARL_0) = 49.02$		
n	*LCL*	*UCL*	*n*	*LCL*	*UCL*
4	1.72	160.55	4	1.32	27.82
5	1.76	135.38	5	1.32	24.64
6	1.83	115.40	6	1.39	22.19
7	1.89	102.24	7	1.42	21.18
8	1.89	101.06	8	1.43	18.99
9	2.04	93.63	9	1.45	17.97
10	2.09	84.51	10	1.52	16.83
11	2.14	81.08	11	1.58	16.78
12	2.21	75.50	12	1.59	15.93
13	2.22	74.31	13	1.66	15.41
14	2.36	72.88	14	1.73	15.40
15	2.41	69.74	15	1.76	14.87
16	2.61	68.75	16	1.83	14.66
17	2.69	65.30	17	1.86	14.49
18	2.80	64.56	18	1.88	14.00

Table 10
In-control/Out-of-control ARLs for the shifts in the first parameter (c) of Burr XII $(c,1)$.

c	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
1	19.00	2.16	1.59	99.00	2.05	1.41
1.5	7.12	4.33	3.88	21.40	4.56	4.00
2	4.35	11.07	9.88	9.49	11.27	11.58
2.5	3.24	25.79	25.92	6.28	26.36	24.48
3 (No shift)	2.66	50.54	49.02	4.62	50.52	51.26
4	2.08	46.41	45.55	3.15	46.61	42.68
5	1.80	31.34	30.00	2.50	27.86	27.03
6	1.63	21.94	20.71	2.15	18.49	18.43
7	1.52	17.20	16.47	1.92	15.11	14.96
8	1.44	15.32	13.89	1.77	12.96	12.75
9	1.38	12.57	12.82	1.66	10.91	10.89
10	1.34	11.11	10.59	1.58	10.19	10.39

Remark 2. The bootstrap control limits for $VaR_{0.99}$ are wider than the bootstrap control limits for $VaR_{0.95}$.

Remark 3. In-control ARLs are very close to their nominal values for all distributions in our simulation studies.

Remark 4. For all distributions in the simulation studies the larger positive shifts in VaR are detected more quickly.

Table 11
In-control/Out-of-control ARLs for the shifts in the second parameter (k) of Burr XII $(3,k)$.

k	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
0.5	7.36	2.11	1.46	21.54	2.11	1.47
0.6	5.27	3.63	3.06	12.91	3.68	3.20
0.7	4.14	7.46	7.41	8.95	7.15	6.62
0.8	3.45	15.38	14.40	6.80	14.82	14.13
1 (No shift)	2.66	50.54	49.02	4.62	50.52	51.26
1.2	2.23	43.30	43.58	3.56	38.59	38.45
1.5	1.85	19.22	18.57	2.73	21.81	21.96
2	1.51	7.78	7.15	2.08	9.50	9.07
2.5	1.32	4.38	3.94	1.74	5.66	4.94

Table 12
The control limits for VaR_p (claim size has *Birnbaum–Saunders* (2,2) & nominal $ARL_0 = 50$).

$p = 0.99$			$p = 0.95$		
$ARL_0 = 51.49$; $Std\ (ARL_0) = 51.56$			$ARL_0 = 51.35$; $Std\ (ARL_0) = 50.40$		
n	*LCL*	*UCL*	*n*	*LCL*	*UCL*
4	1.89	96.21	4	1.23	56.74
5	2.61	82.76	5	1.69	49.48
6	3.46	77.89	6	2.11	44.55
7	4.40	74.50	7	2.51	42.66
8	4.73	68.97	8	3.08	40.95
9	5.44	66.11	9	3.32	38.42
10	6.03	63.28	10	3.59	36.14
11	6.69	61.24	11	3.84	34.44
12	7.16	57.99	12	4.21	33.68
13	7.54	56.85	13	4.67	33.31
14	8.03	54.65	14	4.75	31.95
15	8.20	54.38	15	5.00	31.16
16	8.27	52.79	16	5.14	29.95
17	9.01	51.75	17	5.46	29.52
18	9.27	50.65	18	5.53	28.82

Table 13
In-control/Out-of-control ARLs for the shifts in the shape parameter of Birnbaum–Saunders $(\alpha,2)$.

Shape parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
0.5	4.45	26.39	25.94	6.04	7.78	7.40
1	8.96	107.27	102.52	14.54	60.51	62.22
1.5	15.92	111.43	112.30	28.21	96.00	98.55
2 (No shift)	25.48	51.35	50.40	47.21	51.49	51.56
2.2	30.05	32.37	31.97	56.31	30.45	29.32
2.5	37.71	18.38	18.60	71.59	17.42	16.68
2.7	43.35	13.30	12.46	82.85	12.24	11.58
3	52.62	8.82	8.60	101.37	8.24	8.27
4	90.53	4.19	3.63	177.15	3.55	2.77
5	139.24	2.62	2.11	274.58	2.38	1.89

Remark 5. For all distributions except the Birnbaum–Saunders distribution in the simulation studies larger negative shifts in VaR are detected more quickly.

Remark 6. Since the standard error of $VaR_{0.99}$ is higher than the standard error of $VaR_{0.95}$, for the same amount of positive shift (Υ) in VaR, the control chart for $VaR_{0.95}$ detects shifts quicker than the control chart for $VaR_{0.99}$. In the other word a shift Υ in VaR is different in $VaR_{0.95}$ and $\mathrm{VaR}_{0.99}$ in terms of the factor k where $\Upsilon = k\ \sigma_{VaR_p}$.

Table 14
In-control/Out-of-control ARL for the shifts in the scale parameter of Birnbaum–Saunders $(2, \beta)$.

Shape parameter	$p = 0.95$			$p = 0.99$		
	VaR	ARL	Std (ARL)	VaR	ARL	Std (ARL)
1	12.74	8.44	7.49	23.60	8.89	8.32
1.3	16.56	19.22	18.55	30.68	21.48	19.76
1.5	19.11	31.80	30.73	35.40	33.29	31.47
1.8	22.93	49.73	50.05	42.48	48.52	48.65
1.9	24.21	51.02	50.38	44.84	48.89	48.77
2 (No shift)	25.48	51.35	50.40	47.21	51.49	51.56
2.2	31.85	23.11	22.28	59.01	22.78	21.68
2.5	31.84	22.57	22.24	59.01	21.53	20.59
2.7	34.40	15.88	15.16	63.73	14.69	14.96
3	38.23	9.98	9.16	70.81	9.48	9.05
4	50.97	3.63	3.16	94.42	3.55	3.04
5	63.71	2.24	1.62	118.02	2.21	1.72

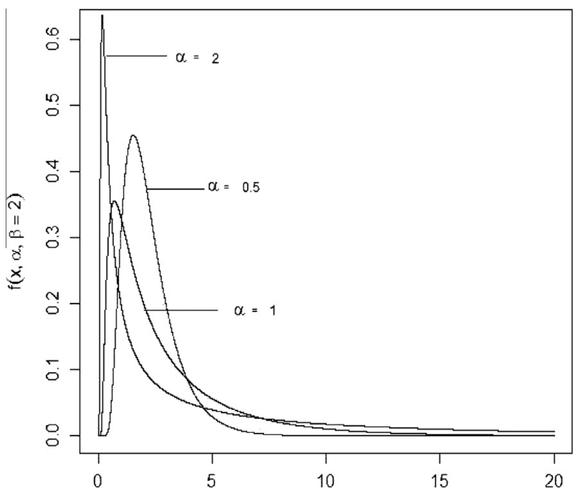

Fig. 3. The Birnbaum–Saunders distributions with shifts in the shape parameter.

Remark 7. The ARL_1 values of the proposed method are very small, showing that the designed control charts are able to detect the shifts very quickly.

6. An empirical example

In this section, we apply the proposed monitoring scheme to real data borrowed from an insurance company. Claims from the last two months are used to establish the control limits. There are five hundred and five data points available to estimate VaR and construct the control limits. The data are well fitted to a three-parameter Weibull distribution with $\widehat{\alpha} = 119.4321$, $\widehat{\beta} = 2.0878$ and $\widehat{\gamma} = 30.5175$. The p-value of Kolmogorov–Smirnov goodness-of-fit test is greater than 0.1, which shows that the Weibull distribution fits well. Fig. 4 shows the Weibull distribution with the estimated parameters. The control limits for the probabilistic C chart are obtained as $LCL_c = 11$ and $UCL_c = 38$. The bootstrap control charts for $VaR_{0.95}$ when type I error is considered as 0.02 is displayed in Fig. 5. Fig. 5 shows the data points (the estimated VaRs) for the past fifty-five days.

7. Conclusion

In this paper we present a method for monitoring VaR as the most widely used risk measure in insurance companies. The parametric bootstrap method using the three-parameter Weibull, Burr XII, Pareto and Birnbaum–Saunders distributions was developed to construct control charts for monitoring VaR. A probabilistic C chart

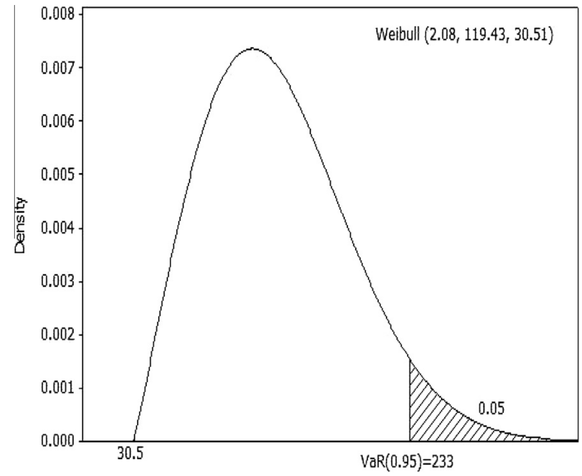

Fig. 4. The parametric distribution of the claim size.

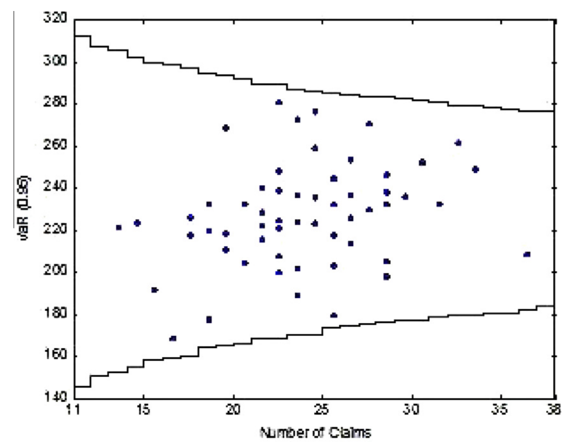

Fig. 5. The bootstrap control chart for the real example.

was appended to the bootstrap control chart to monitor the number of the claims in insurance companies. Simulation studies for three-parameter Weibull, Burr XII, Pareto and Birnbaum–Saunders distributions verified the successful performance of the proposed scheme. Moreover, a real case study was presented to illustrate the proposed scheme. Monitoring VaR in multivariate cases when a company has several lines of business is worth pursuing in future research. Additionally, future research will involve designing monitoring scheme for other risk measures such as tail conditional expectation and applying non-parametric bootstrap techniques to construct control charts for risk measures.

Appendix A

Proposition A.1. *For $p \geqslant 0.5$, $\widehat{\xi}_{p,method2} \geqslant \widehat{\xi}_{p,method1}$ and for any p,*

$\widehat{\xi}_{p,method4} \geqslant \widehat{\xi}_{p,method2}$ and $\widehat{\xi}_{p,method3} \geqslant \widehat{\xi}_{p,method1}$

and when np is integer $\widehat{\xi}_{p,method3} \geqslant \widehat{\xi}_{p,method2}$

- Proof of $\widehat{\xi}_{p,method2} \geqslant \widehat{\xi}_{p,method1}$

  As $p > 0.5$, always $\lfloor np + p \rfloor \geqslant \lfloor np - p + 1 \rfloor$.
  Thus

$\widehat{\xi}_{p,method2} \geqslant \widehat{\xi}_{p,method1}$.

- Proof of $\widehat{\xi}_{p,method4} \geqslant \widehat{\xi}_{p,method2}$

According to the definitions

$$\widehat{\xi}_{p,method4} = \widehat{\xi}_{p,method2} + ((n+1)p - \lfloor (n+1)p \rfloor)(X_{(\lfloor (n+1)p \rfloor+1)} - X_{(\lfloor (n+1)p \rfloor)}).$$

However $((n+1)p - \lfloor (n+1)p \rfloor)(X_{(\lfloor (n+1)p \rfloor+1)} - X_{(\lfloor (n+1)p \rfloor)})$ is always non-negative as

$(n+1)p \geqslant \lfloor (n+1)p \rfloor$ and $X_{(\lfloor (n+1)p \rfloor+1)} \geqslant X_{(\lfloor (n+1)p \rfloor)}$

- Proof of $\widehat{\xi}_{p,method3} \geqslant \widehat{\xi}_{p,method1}$

The proof is similar to the previous part.

- Proof of $\widehat{\xi}_{p,method3} \geqslant \widehat{\xi}_{p,method2}$ when np is integer.

When np is integer, $\widehat{\xi}_{p,method2} = X_{(np)}$ and

$\widehat{\xi}_{p,method3} = pX_{(np)} + (1-p)X_{(np+1)}$.

Since $X_{(np+1)} \geqslant X_{(np)}$, $\widehat{\xi}_{p,method3} \geqslant \widehat{\xi}_{p,method2}$.

Proposition A.2. *$\widehat{\xi}_{p,jackknife} = \widehat{\xi}_{p,method4}$ when np is integer. In addition, For $p \geqslant 0.5$, $\widehat{\xi}_{p,jackknife} \geqslant \widehat{\xi}_{p,method3}$ when np is integer.*

- Proof. When np is integer

$\widehat{\xi}_{p,jackknife} = pX_{(np+1)} + (1-p)X_{(np)}$

and $\widehat{\xi}_{p,method4} = pX_{(np+1)} + (1-p)X_{(np)}$. It proves that when np is integer

$\widehat{\xi}_{p,jackknife} = \widehat{\xi}_{p,method4}$.

when np is integer

$\widehat{\xi}_{p,method3} = (1-p)X_{(np+1)} + pX_{(np)}$.

For $p \geqslant 0.5$
since $X_{(np+1)} \geqslant X_{(np)}$ it is proved that

$\widehat{\xi}_{p,jackknife} \geqslant \widehat{\xi}_{p,method3}$.

Proposition A.3. *$\widehat{\xi}_{p,jackknife} \geqslant \widehat{\xi}_{p,method3}$ under $n \geqslant \frac{n^2p-np-1}{\lfloor np-p \rfloor} - 1$ and $\lfloor (n-1)p \rfloor \geqslant 1$ conditions.*

- Proof $\widehat{\xi}_{p,method3}$ and $\widehat{\xi}_{p,jackknife}$ both are linear combinations of $X_{(\lfloor (n-1)p \rfloor+2)}$ and $X_{(\lfloor (n-1)p \rfloor+1)}$ with sum of the coefficient equals to one.

Since $X_{(\lfloor (n-1)p \rfloor+2)} \geqslant X_{(\lfloor (n-1)p \rfloor+1)}$, if we show that the coefficient of $X_{(\lfloor (n-1)p \rfloor+2)}$ in $\widehat{\xi}_{p,jackknife}$ is higher than its coefficient in $\widehat{\xi}_{p,method3}$ we reach the proof.

Hence, we need to show that

$$(\lfloor (n-1)p \rfloor + 1)n^{-1} \geqslant (n-1)p - \lfloor (n-1)p \rfloor$$

with some simplifications, the conditions for the above statement are obtained as

$$n \geqslant \frac{n^2p - np - 1}{\lfloor np - p \rfloor} - 1 \quad \text{and} \quad \lfloor (n-1)p \rfloor \geqslant 1.$$

Proposition A.4. *$\widehat{\xi}_{p,method1}$ is a biased estimator for ξ_p and when $1 - np + \lfloor (n-1)p \rfloor \geqslant 0$ and $p \geqslant 0.5$,*

$\widehat{\xi}_{p,method1}$, $\widehat{\xi}_{p,method2}$, $\widehat{\xi}_{p,method3}$ and $\widehat{\xi}_{p,method4}$

are upward biased estimations of ξ_p.

- Proof. Stephens (1983) shows

$$E(X_{(\lfloor np+1 \rfloor)}) = \xi_p + \frac{1-p-np+\lfloor np \rfloor}{n+1}\frac{\partial \xi_p}{\partial p} + \frac{p(1-p)}{2n+4}\frac{\partial^2 \xi_p}{\partial p^2}$$

It can be shown that

$$E(X_{(\lfloor (n-1)p \rfloor+1)}) = \xi_p + \frac{1-np+\lfloor (n-1)p \rfloor}{n}\frac{\partial \xi_p}{\partial p} + \frac{p(1-p)}{2n+2}\frac{\partial^2 \xi_p}{\partial p^2}$$

It indicates that $\widehat{\xi}_{p,method1}$ is biased. However, always $\frac{\partial \xi_p}{\partial p} > 0$ (as ξ_p increases by p) and for most of the distributions (such as Weibull, Burr XII and Pareto distributions) $\frac{\partial^2 \xi_p}{\partial p^2} > 0$.

$\frac{p(1-p)}{2n+2}$ is always positive but $\frac{1-np+\lfloor (n-1)p \rfloor}{n}$ can be negative or positive.

Therefore, when $1 - np + \lfloor (n-1)p \rfloor \geqslant 0$, $\widehat{\xi}_{p,method1}$ is always a upward biased estimator for ξ_p. In A1 we showed $\widehat{\xi}_{p,method1}$ is less than (or equal to) other estimators at least for $p \geqslant 0.5$. Therefore, under $1 - np + \lfloor (n-1)p \rfloor \geqslant 0$ and $p \geqslant 0.5$ conditions, $\widehat{\xi}_{p,method2}$, $\widehat{\xi}_{p,method3}$ and $\widehat{\xi}_{p,method4}$ are upward estimators for ξ_p with more bias than $\widehat{\xi}_{p,method1}$.

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