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Research

A Bootstrap Control Chart for Weibull Percentiles

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The problem of detecting a shift of a percentile of a Weibull population in a process monitoring situation is considered. The parametric bootstrap method is used to establish lower and upper control limits for monitoring percentiles when process measurements have a Weibull distribution. Small percentiles are of importance when observing tensile strength and it is desirable to detect their downward shift. The performance of the proposed bootstrap percentile charts is considered based on computer simulations, and some comparisons are made with an existing Weibull percentile chart. The new bootstrap chart indicates a shift in the process percentile substantially quicker than the previously existing chart, while maintaining comparable average run lengths when the process is in control. An illustrative example concerning the tensile strength of carbon fibers is presented. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: percentiles; Weibull distribution; average run length; statistical process control; tensile strength; control chart limits; Shewhart chart

1. INTRODUCTION

The well-known Shewhart-type control charts are commonly used to monitor and detect shifts in the mean and/or variance of a quality characteristic of interest in a process. The usual Shewhart $\bar{X}$ and R control charts assume that the observed process data come from a near-normal distribution. However, when the distribution of the process under observation is unknown or non-normal, the sampling distribution of a parameter estimator may not be available theoretically. In this case, computational methods, such as parametric or non-parametric bootstrap methods, can be employed to set control limits for an appropriate control chart. See Efron and Tibshirani[1], Gunter[2,3] and Young[4] for a basic discussion of bootstrap techniques. An advantage of the bootstrap methods is that they are not restricted by assumptions on the distribution of the process measurements. The computation time the method often requires is perhaps a perceived disadvantage, but actually is not, given the current computing power available and considering the possibility of using an inappropriate control chart. The bootstrap method uses only the sample data to estimate the sampling distribution of the parameter estimator, and then to determine appropriate control limits. Only the usual assumptions (for Phase II of the statistical process control paradigm) that the process is stable and subgroup observations are independent (and identically distributed) are required.

Here, we use the parametric bootstrap method to construct control chart limits for monitoring a specified percentile of the distribution of the process characteristic of interest, e.g. the tensile strength of a material. Although the method applies more generally, we will be concerned specifically with small percentiles of the

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Weibull distribution. The probability density function of the Weibull distribution is typically written in the form

$$f(w) = \frac{\delta}{\beta}\left(\frac{w}{\beta}\right)^{\delta-1} \exp\left[-\left(\frac{w}{\beta}\right)^{\delta}\right], \quad w > 0\ (\delta, \beta > 0) \tag{1}$$

where δ is the shape parameter and β is the scale parameter. The Weibull density can take a variety of shapes and the exponential distribution is a special case when the shape parameter is one. The $100p$th percentile of the Weibull distribution is $W_p = \beta[-\ln(1-p)]^{1/\delta}$ for $0 < p < 1$.

The Weibull distribution is often used to model the tensile strength of brittle materials, such as carbon and boron, among other phenomena, for example. A minimum tensile strength of such materials is usually required for engineering design considerations and small percentiles are of interest. A downward shift in a lower percentile of the strength distribution indicates a decrease in the tensile strength. Since the Weibull distribution is asymmetric, when the quality characteristic of interest is a lower percentile whose estimator has a non-normal distribution, the standard $\bar{X}$ and R Shewhart control charts may fail to detect important shifts in the specific lower percentile in question. This was discussed in detail by Padgett and Spurrier[5]. They developed Shewhart-type control charts for percentiles of Weibull or log-normal distributions for the typical values of α, the type I error rate, used in statistical process control. The Weibull distribution was transformed to the extreme value distribution, with parameters η and λ, by taking the logarithm of the Weibull data, and best linear invariant estimators (BLIEs) of the parameters, $\tilde{\eta}$ and $\tilde{\lambda}$, were obtained. The BLIEs were given by

$$\tilde{\eta} = \sum_{i=1}^{n} A_{n,n,i} X_i \quad \text{and} \quad \tilde{\lambda} = \sum_{i=1}^{n} C_{n,n,i} X_i$$

where $A_{n,n,i}$ and $C_{n,n,i}$ are found in Table 5.3 of Mann *et al.*[6], and the $100p$th percentile of the extreme-value distribution, $x_p = \eta + \lambda \ln[-\ln(1-p)]$, was estimated by $\tilde{x}_p = \tilde{\eta} + \tilde{\lambda} \ln[-\ln(1-p)]$. The estimate of the $100p$th percentile of the Weibull was then given by $\tilde{W}_p = \exp(\tilde{x}_p)$.

In this paper, the parametric bootstrap method is used to develop control charts for small percentiles of the Weibull distribution. The method works for any percentile, but as mentioned earlier, lower percentiles are of particular interest for tensile strength considerations. The existing Weibull percentile control chart of Padgett and Spurrier[5] will be briefly described in the next section, and the proposed bootstrap-type control chart is presented in Section 3. An advantage of the bootstrap chart for Weibull percentiles is that the tabled coefficients required for the BLIEs used by Padgett and Spurrier[5] are not needed. The behavior of the bootstrap Weibull percentile control limits, the performance of the proposed bootstrap Weibull percentile chart, and the comparison with the existing control chart will be investigated briefly by computer simulations in Section 4. An example concerning the tensile strength of carbon fibers will be given in Section 5 for illustration.

The concern, both here and in Padgett and Spurrier's[5] paper, is with process monitoring after the process has been stabilized and it has been justified that the observed values from the process are from a Weibull population, at least approximately. Weibull probability plots of observed sample data from the process can be used for the latter.

2. *THE PARAMETRIC WEIBULL PERCENTILE CONTROL CHART*

Monte Carlo simulations were used by Padgett and Spurrier[5] to find the extreme percent points of the probability distribution of the statistic $V^{*}_{1-p} = (\tilde{\eta} - x_{po})/\tilde{\lambda}$ when x_{po} is the true percentile. The simulations were generated with sample sizes (or 'subgroup' sizes) typically used in statistical process control to define the limits of the Shewhart-type control chart. In situations without a specified value of x_{po}, it was replaced in V^{*}_{1-p} by the estimator $\hat{\mu}_{\tilde{x}}$, the average of k estimates of x_p from the last k samples or subgroups. The statistic then becomes $V^{**}_{1-p} = (\tilde{\eta} - \hat{\mu}_{\tilde{x}})/\tilde{\lambda}$ for the subsequent samples. The control limits for V^{**}_{1-p} were also obtained by Padgett and Spurrier via simulation under the 'in-control' assumption. If the value of the statistic is extreme in either direction, the process is signaling out of control.

The 'in-control' average run lengths (ARLs) for both cases were also obtained by Padgett and Spurrier via Monte Carlo simulation. The ARLs based on each of the statistics, V^{*}_{1-p} and V^{**}_{1-p}, were estimated for control limits corresponding to the 0.001 35 and 0.998 65 percent points. For ARLs when the process shifted to an 'out-of-control' percentile, the Weibull distribution was shifted out of control by changing the shape parameter δ. Then samples taken *after* the shift were generated until an out-of-control signal was received. The number of samples was counted including the first sample that signaled. This entire process was replicated 1000 times, yielding 1000 simulated run lengths, and the ARL and standard error (SE) were calculated from those 1000 values.

It is unclear if the standard $\bar{X}$ and R charts will have the correct error probabilities or if they will correctly signal a shift in the $100p$th percentile, W_p, of the Weibull distribution. In both the in-control and out-of-control cases, the $\bar{X}$ and R charts can signal out of control much too soon or not soon enough. This was also illustrated by Padgett and Spurrier for Weibull-distributed data.

3. BOOTSTRAP CONTROL CHARTS

Several authors have investigated the application of bootstrap methods to statistical quality control charts. The non-parametric bootstrap method can be applied to control charts, thereby eliminating the necessity of traditional parametric assumptions. For example, the 'bootstrap $\bar{X}$ chart' for monitoring the process mean requires the process to be in control, but not the assumption of normally distributed process data. Bootstrap methods can also be used when the distribution of the statistic used to monitor the process is not available.

Bajgier[7] developed a bootstrap control chart for the process mean, which is a competitor to Shewhart's $\bar{X}$ chart, and is appropriate when the process data may not be normal. Bajgier's bootstrap control chart assumes only that the process is stable and in control when the control limits are computed. If this assumption is not satisfied when the control limits are computed, the limits will be too wide, regardless of the distribution of the process variable. In an attempt to eliminate the assumption required by Bajgier's method (that the process was stable and in control when the control limits were computed), Seppala *et al.*[8] proposed the subgroup bootstrap method, which uses observations from k independent subgroups of size n from the process, and calculates the within subgroup residuals by subtracting the mean of each subgroup from each observation within the subgroup. Then a bootstrap sample of size n is taken from these residuals, a correction factor is used to adjust the variance of these resampled subgroups, and the mean of the subgroup residuals is added to the grand mean. The bootstrap estimates are then sorted and the appropriate upper control limit (UCL) and lower control limit (LCL) are found. Liu and Tang[9] extended $\bar{X}$ bootstrap control charts to both dependent and independent observations. Jones and Woodall[10] evaluated the performance of these three proposed bootstrap charts and concluded that in general they did not perform *substantially* better than the standard method in terms of the in-control ARL, noting that the work of Seppala *et al.*[8] was flawed. However, they indicated that for skewed distributions, bootstrap techniques seemed to yield better estimates of the true percentile values on average than Shewhart methods. The problem we address in the present paper falls into the latter category using the parametric bootstrap approach. This was motivated in part by the results of Padgett and Tomlinson[11] that parametric bootstrap approaches produce good lower confidence bounds on Weibull percentiles.

Our interest in this paper is the Weibull distribution. It has been used to model the tensile strength of brittle materials, among other types of failures (see, for examples, Padgett and Spurrier[5], Mann *et al.*[6] and Padgett *et al.*[12]). Knowledge of the strength distribution is desired for structural design purposes, and in particular, lower percentiles of the strength distribution are important, since a downward shift in a lower percentile indicates a decrease in the material's tensile strength. Since the Weibull distribution is asymmetric, for a quality characteristic of interest such as the breaking strength of brittle materials, the standard $\bar{X}$ and R Shewhart control charts too often may fail to detect important shifts in a specific lower percentile of interest. We are interested in using the parametric bootstrap (percentile) method to construct control charts for percentiles of the Weibull distribution and then comparing the performance of the resulting bootstrap control chart to the Padgett–Spurrier Shewhart-type charts for Weibull percentiles. Some graphs of Weibull density functions used in the simulations in Section 4 are shown in Figure 1.

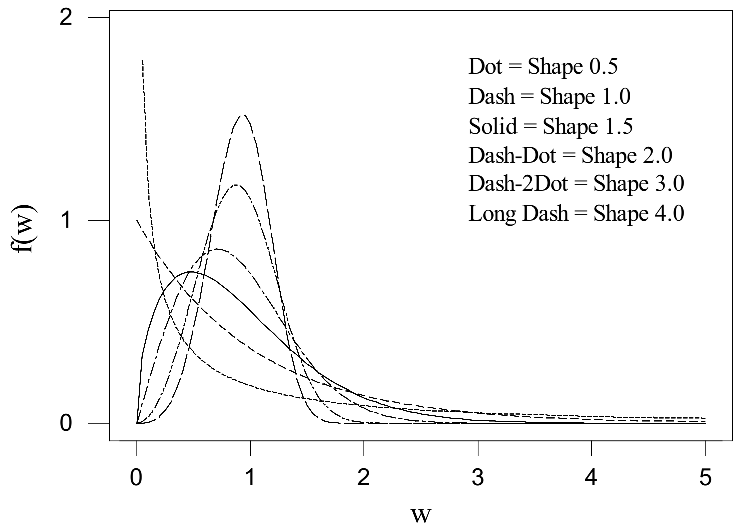

Figure 1. Weibull density functions (scale = 1)

The following steps are used to construct the bootstrap Weibull percentile control chart.

1. From an in-control, stable process, observe $n \times k$ observations assuming they come from a Weibull distribution with unknown scale and shape parameters, β and δ, respectively. A Weibull probability plot should be constructed to check this assumption. The observations are denoted by x_{ij}, $i = 1, \ldots, n$, and $j = 1, \ldots, k$, and are assumed to come from k independent subgroups of size n.
2. Find $\hat{\beta}$ and $\hat{\delta}$, the maximum likelihood estimators (MLEs) of the unknown parameters based on all $n \times k$ observations, which solve the following two equations[13,14]:

$$\delta = \left[\frac{\sum_{j=1}^{k} \sum_{i=1}^{n} x_{ij}^{\delta} \log x_i}{\sum_{j=1}^{k} \sum_{i=1}^{n} x_{ij}^{\delta}} - \frac{\sum_{j=1}^{k} \sum_{i=1}^{n} \log x_{ij}}{nk} \right]^{-1} \quad \text{and} \quad \beta = \left(\frac{\sum_{j=1}^{k} \sum_{i=1}^{n} x_{ij}^{\delta}}{nk} \right)^{1/\delta} \qquad (2)$$

   Here, the Newton–Raphson method is used to find $\hat{\delta}$ from (2) numerically.
3. Generate a bootstrap subgroup of size n, $x_1^*, x_2^*, \ldots, x_n^*$, from the Weibull distribution using the MLEs, $\hat{\beta}$ and $\hat{\delta}$, as the estimated parameters (i.e. a parametric bootstrap sample). (Subgroup sizes will be taken as $n = 3$, 4, 5, or 6, as typical in control charts.)
4. Find the parameter MLEs from the bootstrap subgroup and denote these as $\hat{\beta}^*$ and $\hat{\delta}^*$.
5. For the bootstrap subgroup, find $W_p^* = \hat{\beta}^*[-\ln(1-p)]^{1/\hat{\delta}^*}$, the bootstrap estimate of the $100p$th percentile of interest, W_p.
6. Repeat steps 3–5 a large number of times, B, obtaining B bootstrap estimates of W_p, denoted by $W_{p1}^*, W_{p2}^*, \ldots, W_{pB}^*$.
7. Order the B bootstrap estimates W_{pi}^*, from smallest to largest. The LCL is the value of the smallest ordered W_{pi}^*, such that $(\alpha/2)B$ values are below it. Here, α is the probability that an observation is considered out of control when the process is actually in control. (A typical value of α is 0.0027 for Shewhart-type charts[15].) The UCL is the value of the smallest ordered W_{pi}^*, such that $(\alpha/2)B$ values are above it.

Once the control limits have been computed, future subgroup samples of size n are taken from the process at regular time intervals, and W_p is estimated for each of the new subgroups by the MLE as indicated in step 5 above. If the estimate, $\hat{W}_p$, falls between the UCL and LCL calculated in step 7, the process is assumed to be in control. Values of $\hat{W}_p$ that are below the LCL or above the UCL signal that the process may be out of control.

Hence, once the bootstrap control limits are found, the process is monitored using the statistic $\hat{W}_p$ in the usual way.

In the next section, the performance of the proposed bootstrap Weibull percentile control chart will be considered by performing some computer simulations. In particular, the statistical behavior of the bootstrap Weibull percentile control limits is investigated. Further simulations are used to check the performance of the bootstrap percentile control chart with respect to ARLs, and then some comparisons with the Padgett–Spurrier chart are discussed.

4. PERFORMANCE OF THE BOOTSTRAP WEIBULL PERCENTILE CHART

In this section, computer simulations to check the performance of the bootstrap Weibull percentile control chart are presented. The behavior of the bootstrap control limits is investigated by calculating the average UCL and LCL and their associated variances from simulations. Further simulations examine the ARL while the process is in control and the variance associated with each of the ARLs. The simulations are run with varying sample sizes, different percentiles of interest and different α levels. The ARL when the process is out of control is computed, as well as its variance. These simulations are also run with varying sample sizes, different percentiles of interest and different α values, and comparison with the Padgett–Spurrier charts is discussed.

The average UCL and LCL and their associated variances were computed in the following way: $n \times k$ observations were generated from a Weibull distribution with scale and shape parameters, β and δ, respectively, as described in step 1 of Section 3. Steps 2–7 of Section 3 were then carried out. The value of B was 10 000 here. This *entire* process (steps 1–7) was repeated 500 times and the average LCL and the average UCL were computed from the 500 generated values of each. The variances of the control limits were also computed from the 500 values. This simulation procedure was done for varying sample sizes ($n = 4, 5, 6$), different percentiles ($p = 0.01, 0.10, 0.50$), and different α values ($\alpha = 0.0027, 0.002, 0.01$). The simulations were coded in Fortran, and some of the results are given in Tables I–VI. As one would expect, as the subgroup size is increased from four to five to six, the control limits get closer together, and as the percentile is increased from 0.01 to 0.10 to 0.50, the limits become farther apart.

The in-control ARL was obtained by generating $k = 20$ subgroups of size 5 and determining the corresponding bootstrap UCL and LCL by steps 2–7 in Section 3. Once the control limits were computed, future subgroup samples were simulated from the Weibull with the same shape and scale parameters. For each of the new subgroups, W_p was estimated by the MLE. The number of times that the estimate, $\hat{W}_p$, was within the limits was counted until an estimate fell outside of the control limits. The run length was the number of times that the estimate remained within the limits plus the first subgroup to signal out of control. This entire process of obtaining the run length was replicated 1000 times and the average of the resulting 1000 run lengths (ARL) and their standard errors were calculated. These simulations for the in-control ARL were run using different values of the shape parameter ($\delta = 0.5$, 1, 2, and 4), for 'J-shape', exponential, and unimodal Weibull distributions. (The scale parameter β remained unchanged at one since scale changes do not affect the results.) The simulations were also done for $p = 0.01$, 0.10, and 0.50. The value of $\alpha = 0.0027$ was used for all runs. Some results are listed in Table VII. These results are comparable to those of Padgett and Spurrier[5] for their Shewhart-type charts based on the BLIEs of the parameters. Based on standard theory, we expect the inverse of the α value to be the theoretical ARL. Since $\alpha = 0.0027$, the in-control ARLs for these simulations should be $1/\alpha$, or about 370. Smaller ARLs indicate that the control limits computed may typically be too narrow, and ARLs larger than 370 indicate that the control limits computed may be too wide, or that the bootstrap control charts give fewer false signals. Seven of the ARLs among the 12 cases given in Table VII were higher than 370, and three were slightly less than 370, while two were very close to 370. Overall, the in-control ARLs for the bootstrap Weibull percentile control chart seemed to be somewhat closer to 370 than Padgett and Spurrier's values.

The out-of-control ARLs were obtained by generating $k = 20$ samples of size 5 from a Weibull (δ, $\beta = 1$) population and determining the corresponding UCL and LCL in the same manner as for the in-control ARLs. Future subgroup samples, each of size 5, were simulated from a Weibull population with a different value of δ.

Table I. Average control limits ($\alpha = 0.0027$)

Subgroup size	Percentile		
	$p = 0.01$	$p = 0.10$	$p = 0.50$
Average LCL			
$n = 4$	0.007 590 2965	0.063 058 2996	0.278 253 6345
$n = 5$	0.007 826 0913	0.067 662 9219	0.316 678 3523
$n = 6$	0.008 007 3509	0.071 014 9708	0.344 198 7628
Average UCL			
$n = 4$	0.874 023 2393	1.193 373 8261	1.666 454 1705
$n = 5$	0.795 401 6348	1.100 814 2779	1.573 748 7164
$n = 6$	0.726 805 1954	1.026 886 2114	1.507 090 2970

Table II. Variance of control limits ($\alpha = 0.0027$)

Subgroup size	Percentile		
	$p = 0.01$	$p = 0.10$	$p = 0.50$
Variance of LCLs			
$n = 4$	0.000 009 3924	0.000 234 1094	0.001 286 1540
$n = 5$	0.000 009 0684	0.000 217 7837	0.001 255 4688
$n = 6$	0.000 007 0679	0.000 181 0093	0.001 018 2064
Variance of UCLs			
$n = 4$	0.002 249 2447	0.001 369 3515	0.008 864 1743
$n = 5$	0.001 643 4531	0.003 928 9051	0.005 543 4459
$n = 6$	0.001 369 3515	0.002 060 8460	0.004 342 5351

Table III. Average control limits ($\alpha = 0.002$)

Subgroup size	Percentile		
	$p = 0.01$	$p = 0.10$	$p = 0.50$
Average LCL			
$n = 4$	0.007 195 7289	0.060 349 5533	0.267 667 8666
$n = 5$	0.007 413 1990	0.064 688 0668	0.306 384 8419
$n = 6$	0.007 532 5467	0.067 938 0003	0.334 680 2883
Average UCL			
$n = 4$	0.893 921 9303	1.215 325 3864	1.691 873 0974
$n = 5$	0.814 945 2686	1.121 749 8399	1.594 634 1249
$n = 6$	0.746 226 8821	1.046 230 1603	1.527 266 7992

Table IV. Variance of control limits ($\alpha = 0.002$)

Subgroup size	Percentile		
	$p = 0.01$	$p = 0.10$	$p = 0.50$
Variance of LCLs			
$n = 4$	0.000 008 8284	0.000 226 2332	0.001 309 3472
$n = 5$	0.000 008 3544	0.000 208 0062	0.001 256 6324
$n = 6$	0.000 006 4581	0.000 172 9006	0.001 021 9681
Variance of UCLs			
$n = 4$	0.002 477 4847	0.004 254 7691	0.009 345 7203
$n = 5$	0.001 772 9311	0.002 663 8370	0.005 917 3197
$n = 6$	0.001 483 5733	0.002 281 0137	0.004 619 5795

Table V. Average control limits ($\alpha = 0.01$)

	Percentile		
Subgroup size	$p = 0.01$	$p = 0.10$	$p = 0.50$
Average LCL			
$n = 4$	0.010 546 5468	0.081 321 6038	0.339 768 5946
$n = 5$	0.011 069 1247	0.086 289 3502	0.375 275 1299
$n = 6$	0.011 673 4463	0.090 737 4238	0.400 728 0374
Average UCL			
$n = 4$	0.766 731 5668	1.071 318 9099	1.536 492 4810
$n = 5$	0.690 591 4923	0.987 747 3526	1.476 114 725
$n = 6$	0.625 569 9364	0.921 530 3994	1.401 075 8959

Table VI. Variance of control limits ($\alpha = 0.01$)

	Percentile		
Subgroup size	$p = 0.01$	$p = 0.10$	$p = 0.50$
Variance of LCLs			
$n = 4$	0.000 015 7687	0.000 302 9814	0.001 388 6344
$n = 5$	0.000 015 4292	0.000 299 9994	0.001 360 7116
$n = 6$	0.000 012 4640	0.000 239 1656	0.001 042 6552
Variance of UCLs			
$n = 4$	0.001 591 9692	0.002 841 1518	0.006 529 3817
$n = 5$	0.001 358 9843	0.002 074 5440	0.004 334 0427
$n = 6$	0.001 147 6204	0.001 551 7426	0.003 232 5051

Table VII. In-control ARL and standard error of in-control ARL ($\alpha = 0.0027$)

	Percentile of interest		
Shape parameters	$p = 0.01$	$p = 0.10$	$p = 0.50$
In-control ARL			
$\delta = 0.5$	370.685	370.678	402.628
$\delta = 1.0$	366.254	409.990	388.049
$\delta = 2.0$	362.219	349.962	377.839
$\delta = 4.0$	426.133	432.658	418.988
Standard error			
$\delta = 0.5$	15.861 842	17.964 126	17.629 979
$\delta = 1.0$	15.768 186	20.627 973	17.484 485
$\delta = 2.0$	14.681 990	15.513 048	17.492 663
$\delta = 4.0$	22.980 813	19.133 650	17.653 334

This shift in the shape parameter simulates a process percentile that has shifted and is to be detected as an 'out-of-control' condition. For each subgroup of size 5, the percentile estimate, $\hat{W}_p$, was computed. The number of subgroups for which $\hat{W}_p$ falls within the control limits was counted until an estimate falls outside of the limits. The run length was again the number of times that the estimates remained within the limits plus the first subgroup to signal out of control. Again, this entire process of obtaining the run length was replicated 1000 times, and the ARLs and standard errors were calculated from the 1000 generated run lengths. These out-of-control ARL simulations were run using a value of the shape parameter (shape 1) of $\delta = 1$, 1.5, or 3 (with $\beta = 1$), $\alpha = 0.0027$ to determine the control limits. Then, the shape parameter was shifted from the original δ to a different value (shape 2), one of the values 1.0, 1.5, or 2.0. The simulations were repeated for $p = 0.01$, 0.10, and 0.50, and some results are shown in Table VIII. The bootstrap Weibull percentile control chart signals the shift to an out-of-control condition from a low of 13 to a high of 73 ARL, much smaller than the in-control ARLs, as

Table VIII. Out-of-control ARLs and standard errors ($\alpha = 0.0027$)

Percentile	Shape 1	Shift to shape 2	ARL	ARL (SE)
0.10	1.0	1.5	73.557	3.268 6908
0.01	1.5	1.0	13.415	0.478 9363
0.10	3.0	2.0	16.939	0.616 0113
0.01	3.0	2.0	13.644	0.465 6608
0.50	1.5	1.0	25.286	0.826 0508

Table IX. Comparison of out-of-control ARLs ($\alpha = 0.0027$)

Percentile	Shape 1	Shift to shape 2	Booststrap ARL	P-S ARL
0.10	3.0	2.0	16.939	84.82
0.01	1.5	1.0	13.415	42.04
0.01	1.0	1.5	73.557	205.66

Table X. First 10 subgroups—in control. Shape parameter 4.8 and scale parameter 3.2 ($W_{0.01} = 1.227$)

Subgroup	Breaking stress of carbon fibers (GPa)				
1	3.70	2.74	2.73	2.50	3.60
2	3.11	3.27	2.87	1.47	3.11
3	4.42	2.41	3.19	3.22	1.69
4	3.28	3.09	1.87	3.15	4.90
5	3.75	2.43	2.95	2.97	3.39
6	2.96	2.53	2.67	2.93	3.22
7	3.39	2.81	4.20	3.33	2.55
8	3.31	3.31	2.85	2.56	3.56
9	3.15	2.35	2.55	2.59	2.38
10	2.81	2.77	2.17	2.83	1.92

it should be. These results for the out-of-control ARLs were substantially lower than the results reported by Padgett and Spurrier (P-S ARLs) as shown in Table IX.

5. *ILLUSTRATIVE EXAMPLES*

Suppose a process is producing carbon fibers to be used in constructing fibrous composite materials. Also, suppose it is desired that 99% of the fibers exceed a minimum tensile strength, i.e. the parameter of interest in this example is the first percentile of the tensile strength distribution. A decrease in the first percentile would indicate a decrease in tensile strength, and a control chart can be used to detect such changes. As discussed earlier, the Weibull distribution is a reasonable model for the tensile strength of such material.

Carbon fibers of 50 mm in length were sampled from the process, tested, and their tensile strength observed. The UCL and LCLs were computed from k subgroups of size n, assuming the process is stable. Data were simulated using a Weibull distribution with a shape parameter of 4.8 and a scale parameter of 3.2 (and thus, the first percentile $W_{0.01} = 1.227$), which are typical values for Weibull fits to carbon fiber tensile strength data[16]. These data are given in Table X. Future subgroups were simulated from a different Weibull distribution with a shape parameter of 2.0 and a scale parameter of 2.6 ($W_{0.01} = 0.261$), representing a shift to an 'out-of-control' condition to a smaller first percentile. Figure 2 shows the two Weibull density functions. Once the control limits have been set using the first set of data (the $k \times n$ pooled observations), the second set of data will be used to simulate a process that is no longer in control.

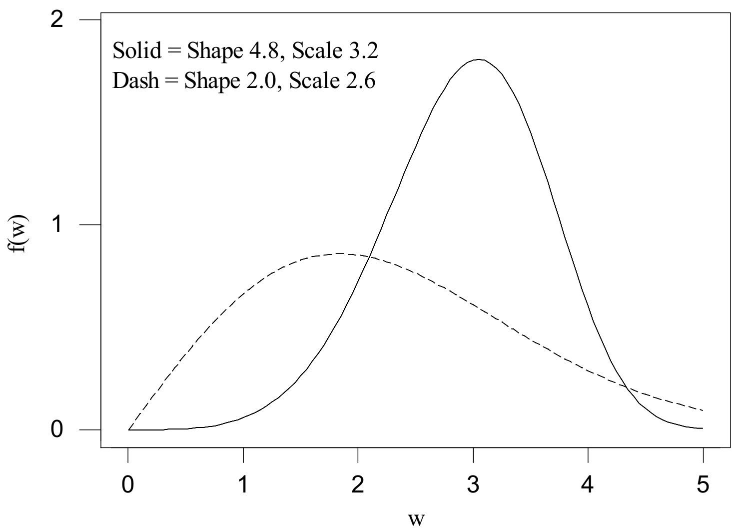

Figure 2. Weibull density functions for carbon fiber strength

Table XI. Subgroups 11–20 after process shift. Shape parameter 2.0 and scale parameter 2.6 ($W_{0.01} = 0.261$)

Subgroup	Breaking stress of carbon fibers (GPa)				
11	1.41	3.68	2.97	1.36	0.98
12	2.76	4.91	3.68	1.84	1.59
13	3.19	1.57	0.81	5.56	1.73
14	1.59	2.00	1.22	1.12	1.71
15	2.17	1.17	5.08	2.48	1.18
16	3.51	2.17	1.69	1.25	4.38
17	1.84	0.39	3.68	2.48	0.85
18	1.61	2.79	4.70	2.03	1.80
19	1.57	1.08	2.03	1.61	2.12
20	1.89	2.88	2.82	2.05	3.65

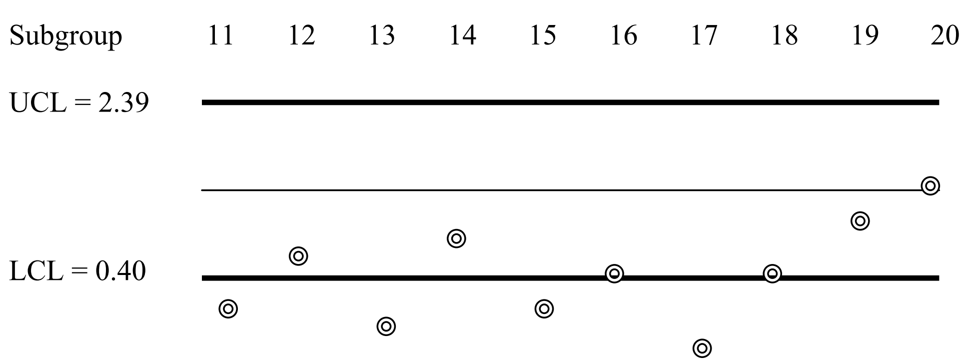

Figure 3. Control chart based on 10 subgroups

Using $\alpha = 0.0027$, the upper and lower bootstrap Weibull percentile control chart limits were calculated from the 10 subgroups in Table X. First, the MLEs were calculated to be $\hat{\delta} = 4.78$ and $\hat{\beta} = 3.20$, very close to the true parameter values. Next, $B = 10\,000$ bootstrap percentile estimates, $\hat{W}^*_{0.01}$, were generated from subgroups of size $n = 5$ from the Weibull distribution with $\hat{\delta} = 4.78$ and $\hat{\beta} = 3.20$, and the bootstrap control limits were found as described in Section 3. The UCL was 2.39 and the LCL was 0.40.

Subgroup samples of size 5 from the shifted process are shown in Table XI, and the MLE, $\hat{W}_{0.01}$, is found for each of the subgroups. For the first subgroup, the estimate, $\hat{W}_{0.01} = 0.28$, falls below the LCL calculated and the process has immediately signaled out of control. The estimate for the second subgroup, $\hat{W}_{0.01} = 0.59$, is within the limits. The results for all subgroups are given in Figure 3. The subgroups that signal out of control are the same as those in the Shewhart-type Weibull percentile chart for this example given by Padgett and Spurrier[5].

The use of parametric bootstrap control charts clearly has advantages when process data come from skewed distributions and/or atypical parameters such as percentiles are to be monitored. Construction of such charts should not be problematic with current computing power available, and should be used especially when standard charts, such as $\bar{X}$ or R charts, are known to not be applicable in these situations.

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