Modelling and Optimization

INF170

#5:LP Models

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Agenda

- Online Advertising Problem
- Portfolio Selection
- Scheduling Postal Workers
- Inventory Management
- Highway Patrol Allocation
- Hospital Location, In Manhattan

Online Advertising Problem

Online Advertising Problem

- The New Bedford Times (NBT) is a small newspaper with a growing web site
- 3 sections: News, Travel, and Sports
- 5 advertisers: Apple Cruises, Bank Boston, CoolTickets, D-Mobile Wireless, eCooking

ONLINE ADVERTISING PROBLEM

- Decision: how do we allocate the ads to the sections of the web site?
  - e.g. How many of D-Mobile's ads do we put on the Sports section?
- Assumption: one ad per page view
- Objective: maximize the total expected number of click-throughs on the NBT web site (in order to improve its attractiveness to potential advertisers)

Index conventions:

- News = 1, Travel = 2, Sports = 3
- Apple Cruises = 1, Bank Boston = 2, CoolTickets = 3, D-Mobile Wireless = 4, eCooking = 5

Data available:

- v_i = forecasted number of page views of section i for this month (i = 1, 2, 3)
- a_j = number of ads sold to advertiser j for this month (j = 1, 2, 3, 4, 5)
- $p_{i,j}$ = projected click-through probability of advertiser j 's ad on section i (i = 1, 2, 3 and j = 1, 2, 3, 4, 5)

"Implicit" constraints:

- Total number of ads assigned to a section is <u>at most</u> the number of ads (page views) available on that section
- Total number of ads delivered for an advertiser is <u>at least</u> the number of ads sold to the advertiser

Additional constraints:

- BankBoston's contract requires that at least 20% of its page views are on the Travel section
- CoolTickets's contract requires that at least 25% of its page views are on the News section
- D-Mobile's contract requires that it receives at least 27,000 expected click-throughs based on the forecast click-through probabilities

ONLINE ADVERTISING PROBLEM

1) Define the decision variables

2) Write the objective function

3) Write the constraints

   - Main constraints

   - Variable-type constraints

Online Advertising Problem

- How do we assign each advertiser's page views to each section of the web site?

- Decision variables:

$x_{i,j}$ = number of advertiser j 's ads assigned to section i

for $i = 1, 2, 3$and $j = 1, 2, 3, 4, 5$

Online Advertising Problem

- Objective is to maximize total expected number of click-throughs
- For example, the expected number of click-throughs for Apple Cruises on the News section = $p_{1,1}x_{1,1}$
- Objective:

$$maximize \quad \sum_{i=1}^{3}\sum_{j=1}^{5} p_{i,j}x_{i,j}$$

Online Advertising Problem

- Total number of ads assigned to a section is at most the number of ads (page views) available on that section
- For section i , the total number of ads assigned is

$$x_{i,1} + x_{i,2} + x_{i,3} + x_{i,4} + x_{i,5} = \sum_{j=1}^{5} x_{i,j}$$

- The number of ads available on section i is v_i
- Constraints:

$$\sum_{j=1}^{5} x_{i,j} \leq v_i \qquad \text{for } i = 1,2,3$$

Online Advertising Problem

- Total number of ads delivered for an advertiser is at least the number of ads sold to the advertiser
- For advertiser j , the number of ads delivered is

$$x_{1,j} + x_{2,j} + x_{3,j} = \sum_{i=1}^{3} x_{i,j}$$

- The number of ads advertiser j ordered is a_j
- Constraints:

$$\sum_{i=1}^{3} x_{i,j} \geq a_j \qquad \text{for } j = 1,2,3,4,5$$

Online Advertising Problem

- BankBoston's contract requires that at least 20% of its page views are on the Travel section

$$x_{2,2} \geq \frac{1}{5} \sum_{i=1}^{3} x_{i,2}$$

- CoolTickets's contract requires that at least 25% of its page views are on the News section

$$x_{1,3} \geq \frac{1}{4} \sum_{i=1}^{3} x_{i,3}$$

- D-Mobile's contract requires that it receives at least 27,000 expected click-throughs based on the forecast click-through probabilities

$$\sum_{i=1}^{3} p_{i,4} x_{i,4} \geq 27000$$

Online Advertising Problem

- Anything missing?
- Nonnegativity:
- $x_{i,j} \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$
- If we consider the integrallity of the variables then we have an integer program

Online Advertising Problem

maximize	$\sum_{i=1}^{3}\sum_{j=1}^{5} p_{i,j} x_{i,j}$		(expected # click-throughs)
subject to	$\sum_{j=1}^{5} x_{i,j} \leq v_i$	for $i = 1,2,3$	(section capacities)
	$\sum_{i=1}^{3} x_{i,j} \geq a_j$	for $j = 1,2,3,4,5$	(advertiser demand)
	$x_{2,2} \geq \frac{1}{5}\sum_{i=1}^{3} x_{i,2}$		(BankBoston's contract)
	$x_{1,3} \geq \frac{1}{4}\sum_{i=1}^{3} x_{i,3}$		(CoolTickets's contract)
	$\sum_{i=1}^{3} p_{i,4} x_{i,4} \geq 27000$		(D-Mobile's contract)
	$x_{i,j} \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$		(nonnegativity)

Portfolio Selection

Portfolio Selection

- A portfolio manager in charge of a bank portfolio has $10 million to invest
- 5 different securities available

Bond name	Bond type	Quality rating	Years to maturity	Yield to maturity
1	Municipal	2	9	4.30%
2	Agency	2	15	2.7
3	Gov't	1	4	2.5
4	Gov't	1	3	2.2
5	Municipal	5	2	4.5

Portfolio Selection

- The bank places the following policy limitations on the portfolio manager's actions
  1) Government and agency bonds must total at least $4 million
  2) The average quality of the portfolio cannot exceed 1.4 (lower quality rating = better)
  3) The average years to maturity of the portfolio must not exceed 5 years
  4) Bonds cannot be "shorted" (cannot buy negative amounts of bonds)
- Objective: maximize earnings
- Decision: how much of each type of bond to purchase?

Portfolio Selection

- Need to determine dollar amount of each security to be purchased
- Decision variables:

x_i = amount to be invested in bond i , in millions

for i = 1, 2, 3, 4, 5

Portfolio Selection

- We want to maximize total earnings

- For one security,

  earnings = (yield to maturity) × (amount invested)

- Objective:

$$maximize\ 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5$$

- Variable-type constraints?
- Cannot short bonds, so amount invested is nonnegative
- Constraints:

$$x_i \geq 0 \qquad \text{for } i = 1, 2, 3, 4, 5$$

PORTFOLIO SELECTION

- Portfolio manager only has a total of 10 million dollars to invest

$$\sum_{i=1}^{5} x_i \leq 10$$

- At least \$4 million must be invested in government and agency bonds

$$x_2 + x_3 + x_4 \geq 4$$

Portfolio Selection

- The average quality of the portfolio must not exceed 1.4

$$\text{average quality of portfolio} = \frac{\text{total quality of portfolio}}{\text{total value of portfolio}}$$

- So,

$$\frac{2x_1 + 2x_2 + x_3 + x_4 + 5x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \leq 1.4$$

- Nonlinear!
- But can be linearized: x_i's are nonnegative, so denominator is nonnegative

$$2x_1 + 2x_2 + x_3 + x_4 + 5x_5 \leq 1.4\,(x_1 + x_2 + x_3 + x_4 + x_5)$$

$$\leftrightarrow 0.6x_1 + 0.6x_2 - 0.4x_3 - 0.4x_4 + 3.6x_5 \leq 0$$

Portfolio Selection

- The average maturity of the portfolio must not exceed 5

$$\text{average maturity of portfolio} = \frac{\text{total maturity of portfolio}}{\text{total value of portfolio}}$$

- So,

$$\frac{9x_1 + 15x_2 + 4x_3 + 3x_4 + 2x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \leq 5$$

- Since x_i's are nonnegative, can be linearized

$$9x_1 + 15x_2 + 4x_3 + 3x_4 + 2x_5 \leq 5\,(x_1 + x_2 + x_3 + x_4 + x_5)$$

$$\leftrightarrow 4x_1 + 10x_2 - x_3 - 2x_4 - 3x_5 \leq 0$$

Portfolio Selection

maximize $\quad 0.043x_1 + 0.027x_2 + 0.025x_3 + 0.022x_4 + 0.045x_5$

(total earnings)

subject to

$\sum_{i=1}^{5} x_i \leq 10$ (cash available)

$x_2 + x_3 + x_4 \geq 4$ (gov't and agency)

$0.6\,x_1 + 0.6\,x_2 - 0.4\,x_3 - 0.4\,x_4 + 3.6\,x_5 \leq 0$ (average quality)

$4\,x_1 + 10\,x_2 - x_3 - 2\,x_4 - 3\,x_5 \leq 0$ (average maturity)

$x_i \geq 0 \quad$ for $i = 1, 2, 3, 4, 5$ (nonnegativity)

Portfolio Selection

- What if we are able to borrow up to \$1 million at a rate of 2.75%?
- In other words, we can increase our cash supply above \$10 million by borrowing at a rate of 2.75%
- How can we change our model to incorporate this?

PORTFOLIO SELECTION

- Define new decision variable

$$y = \text{amount borrowed, in millions}$$

- Limitations on how much can be borrowed
  - Add new constraint:

$$y \leq 1$$

- Cash availability increases
  - Change cash availability constraint:

$$\sum_{i=1}^{5} x_i \leq 10 + y$$

- Define new decision variable

$$y = \text{amount borrowed, in millions}$$

- Borrowed money costs 2.75%
  - Change objective function:

$$0.043x1 + 0.027x2 + 0.025x3 + 0.022x4 + 0.045x5 - 0.0275y$$

- Nonnegativity

$$y \geq 0$$

Portfolio Selection

Maximize $0.043x1 + 0.027x2 + 0.025x3 + 0.022x4 + 0.045x5 - 0.0275y$

(total earnings)

subject to

$y \leq 1$ (borrowing limit)

$\sum_{i=1}^{5} x_i \leq 10 + y$ (cash available)

$x_2 + x_3 + x_4 \geq 4$ (gov't and agency)

$0.6\, x_1 + 0.6 x_2 - 0.4\, x_3 - 0.4\, x_4 + 3.6 x_5 \leq 0$ (average quality)

$4\, x_1 + 10 x_2 - x_3 - 2\, x_4 - 3 x_5 \leq 0$ (average maturity)

$x_i \geq 0$ for $i = 1, 2, 3, 4, 5$, $y \geq 0$ (nonnegativity)

Scheduling

Postal Workers

Scheduling Postal Workers

- Each postal worker works for 5 consecutive days, followed by 2 days off, repeated weekly
- Demand for workers for each day of the week

d_i = demand for workers on day i for $i = 1, \ldots, 7$

➢ Indices: Monday = 1, Tuesday = 2, ... , Sunday = 7

- We want to minimize the number of total workers used
  - For now, let's assume that fractional workers are allowed

Scheduling Postal Workers

- Decision variables

$$y_i = \text{number of workers on day } i \qquad \text{for } i = 1, \ldots, 7$$

- Easy to formulate number of workers on day i is at least d_i

$$y_i \geq d_i \qquad \text{for } i = 1, \ldots, 7$$

- Looks like a "natural" decision variables and easy to formulate some constraints

- How to formulate "each worker works 5 consecutive days followed by 2 consecutive days off"?

Scheduling Postal Workers

- Sometimes the decision variables incorporate constraints of the problem
  - "Art" of optimization
  - Hard to do this well, but worth keeping in mind
  - Good integer programming models often require this

Decision variables

x_i = number of workers who work on days

i , i + 1, i + 2, i + 3, i + 4 for i = 1, … , 7

- For example,

x_1 = number of workers who start work on Monday and work through Friday

x_4 = number of workers who start work on Thursday and work through Monday

x_7 = number of workers who start work on Sunday and work through Thursday

- We want to minimize the total number of workers
- Total number of workers = Add up the number of workers who work in each type of shift

Objective:

$$minimize \sum_{i=1}^{7} x_i$$

Constraints

- Day i needs d_i workers
- Who works on Monday ($i = 1$)?
  - Workers on the Monday-Friday shift (x_1)
  - Workers on the Thursday-Monday shift (x_4)
  - Workers on the Friday-Tuesday shift (x_5)
  - Workers on the Saturday-Wednesday shift (x_6)
  - Workers on the Sunday-Thursday shift (x_7)
- Constraint for meeting demand for workers on Monday:

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1$$

Scheduling Postal Workers

- Similar constraints for each day of the week:

$$
\begin{array}{lr}
x_1 \qquad\qquad\qquad\quad + x_4 + x_5 + x_6 + x_7 \geq d_1 & \text{(Mon)} \\
x_1 + x_2 \qquad\qquad\qquad + x_5 + x_6 + x_7 \geq d_2 & \text{(Tue)} \\
x_1 + x_2 + x_3 \qquad\qquad\qquad + x_6 + x_7 \geq d_3 & \text{(Wed)} \\
x_1 + x_2 + x_3 + x_4 \qquad\qquad\qquad + x_7 \geq d_4 & \text{(Thu)} \\
x_1 + x_2 + x_3 + x_4 + x_5 \qquad\qquad\qquad \geq d_5 & \text{(Fri)} \\
x_2 + x_3 + x_4 + x_5 + x_6 \qquad \geq d_6 & \text{(Sat)} \\
x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 & \text{(Sun)}
\end{array}
$$

Scheduling Postal Workers

$$
\begin{array}{lllll}
\min & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & & \\
s.t. & x_1 \quad\quad\quad\quad\quad\quad + x_4 + x_5 + x_6 + x_7 \geq d_1 & & \text{(Mon)} \\
& x_1 + x_2 \quad\quad\quad\quad\quad\quad + x_5 + x_6 + x_7 \geq d_2 & & \text{(Tue)} \\
& x_1 + x_2 + x_3 \quad\quad\quad\quad\quad\quad + x_6 + x_7 \geq d_3 & & \text{(Wed)} \\
& x_1 + x_2 + x_3 + x_4 \quad\quad\quad\quad\quad\quad + x_7 \geq d_4 & & \text{(Thu)} \\
& x_1 + x_2 + x_3 + x_4 + x_5 \quad\quad\quad\quad\quad\quad \geq d_5 & & \text{(Fri)} \\
& \quad\quad x_2 + x_3 + x_4 + x_5 + x_6 \quad\quad\quad \geq d_6 & & \text{(Sat)} \\
& \quad\quad\quad\quad x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 & & \text{(Sun)} \\
& \quad\quad\quad\quad x_i \geq 0 \quad \text{for } i = 1, \ldots, 7 & &
\end{array}
$$

INVENTORY MANAGEMENT

Inventory Management

- Global Minimum manufactures ski jackets
- Business is highly seasonal
- Demand for quarter *t* is d_t (t = 1, 2, 3, 4)
- Cost of producing 1 jacket in quarter *t* is p_t (t = 1, 2, 3, 4)
- Company can produce up to *C* jackets per quarter
- In order to properly warm up the production line, the total number of jackets produced in quarters 2 and 3 must be at least twice the number of jackets produced in quarter 1.

Inventory Management

- Inventory must be built up to meet larger demands
- Cost of holding 1 jacket from quarter t to quarter $t + 1$ is h_t ($t = 1, 2, 3$)
- No initial inventory
- Jackets in inventory at the end of quarter 4 are salvaged
- Each jacket salvaged <u>reduces</u> total cost by s
- Global wants to meet demand while minimizing production and inventory costs
- Assume volume is large, and you can ignore the integrality <u>for now</u>!

Decision variables

- How much to produce in each quarter?

  x_t = number of jackets produced in quarter t for $t = 1,2,3,4$

- Also want to keep track of how much we have in inventory

  z_t = number of jackets in inventory at the end of quarter t for $t = 1,2,3,4$

- Are these "decisions" that we make?

INVENTORY MANAGEMENT

- The z_t variables do not represent explicit decisions
- Sometimes adding "auxiliary" decision variables helps with modeling
- In this case, we can write a correct model without the z_t variables, but is much more cumbersome
- Sometimes, you might need/want to go back and add decision variables once you start building an optimization model

- We want to minimize production and inventory costs
- Objective:

$$minimize \sum_{t=1}^{4} p_t x_t + \sum_{t=1}^{3} h_t z_t - s z_4$$

Inventory Management

- How are inventory and production levels at different quarters related?

  (Inventory at the end of quarter $t-1$) + (Amount produced in quarter t)

  = (Demand in quarter t) + (Inventory at the end of quarter t)

- Sometimes referred to as *balance constraints*

- Constraints:

$$x_1 = d_1 + z_1 \qquad z_1 + x_2 = d_2 + z_2$$

$$z_2 + x_3 = d_3 + z_3 \qquad z_3 + x_4 = d_4 + z_4$$

Inventory Management

- Company can produce up to C jackets per quarter
- Constraints:

$$x_t \leq C \qquad \text{for } t = 1, 2, 3, 4$$

- Number of jackets produced in quarters 2 and 3 must be at least twice the number of jackets produced in quarter 1
- Constraints:

$$2x_1 \leq x_2 + x_3$$

- Cannot produce a negative jacket
- Constraints:

$$x_t \geq 0 \qquad \text{for } t = 1, 2, 3, 4$$

Inventory Management

minimize $$\sum_{t=1}^{4} p_t x_t + \sum_{t=1}^{3} h_t z_t - s z_4$$

subject to

$$x_1 = d_1 + z_1$$

$$z_1 + x_2 = d_2 + z_2$$

$$z_2 + x_3 = d_3 + z_3$$

$$z_3 + x_4 = d_4 + z_4$$

$x_t \leq C$ for $t = 1, 2, 3, 4$

$$2x_1 \leq x_2 + x_3$$

$x_t \geq 0$ for $i = 1, 2, 3, 4$

$z_t \geq 0$ for $i = 1, 2, 3, 4$

HIGHWAY PATROL ALLOCATION

Highway Patrol Allocation

- The State of Simplex wants to divide the effort of its on-duty officers along 8 highway segments to maximize the reduction of speeding incidents
- 25 officers per week to allocate
- Estimated data from analysts:
  - u_j = maximum number of officers that can be assigned to segment j in one week (for $j = 1, \ldots, 8$)
  - r_j = weekly reduction in speeding incidents for segment j , per officer assigned (for $j = 1, \ldots, 8$)
- Write a linear program that allocates officers to highway segments in a way that maximizes the total reduction in speeding incidents

Highway Patrol Allocation

Decision variables:

x_j = number of officers per week assigned to patrol segment j

for $j = 1, \ldots, 8$

Objective function:

- Reduction in speeding incidents on segment $j = r_j x_j$
- Total reduction in speeding incidents:

$$\sum_{j=1}^{8} r_j x_j$$

Highway Patrol Allocation

Main constraints:

- Number of officers available:

$$\sum_{i=1}^{8} x_j \leq 25$$

- Number of officers allowed on each segment:

$$x_j \leq u_j \qquad \text{for } j = 1, \ldots, 8$$

Variable-type constraints:

- Nonnegativity:

$$x_j \geq 0 \qquad \text{for } j = 1, \ldots, 8$$

- Assume fractional workers are OK (for now...)

Highway Patrol Allocation

Maximize $\sum_{j=1}^{8} r_j x_j$ (max total reduction)

subject to: $\sum_{i=1}^{8} x_j \leq 25$ (officers available)

$x_j \leq u_j$ for $j = 1, \ldots, 8$ (upper bounds)

$x_j \geq 0$ for $j = 1, \ldots, 8$ (nonnegativity)

Highway Patrol Allocation

- Maximizing total reduction may not be the right objective
  - ➢ Optimal solution may assign officers only to one or two highway segments
  - ➢ May result in high reduction in some areas, and no reduction in other areas
- Instead of optimizing a total measure, we might want to focus on optimizing the least satisfactory result
- For example, let's maximize the minimum reduction in speeding incidents among all highway segments
- "Let's make sure the worst we achieve is still pretty good"

Highway Patrol Allocation

- New objective function:
- Want to maximize the minimum reduction in speeding incidents among all highway segments
- Recall: speeding incident reduction on segment $j = r_j x_j$

$$Maximize \quad min\{r_1 x_1, \ldots, r_j x_j\}$$

$$\leftrightarrow Maximize \quad min\{r_j x_j : j = 1, \ldots, 8\}$$

Highway Patrol Allocation

Maximize $\quad min\{rj\, x_j : j = 1, \ldots, 8\}$ (maximin reduction)

subject to: $\quad \sum_{i=1}^{8} x_j \le 25$ (officers available)

$x_j \le u_j \quad$ for $j = 1, \ldots, 8$ (upper bounds)

$x_j \ge 0 \quad$ for $j = 1, \ldots, 8$ (nonnegativity)

- Nonlinear!

Highway Patrol Allocation

- Objective function:

$$Maximize \qquad min\{rj\, x_{j\ :}\, j = 1, \ldots, 8\}$$

- Let f be an auxiliary decision variable that represents the objective function value
- Change objective:

$$maximize\ f$$

- Add constraints

$$f \leq r_j x_j \qquad \text{for } j = 1, \ldots, 8$$

Highway Patrol Allocation

Maximize f (maximin objective)

subject to: $f \leq r_j x_j$ for $j = 1,\ldots, 8$ (maximin constraints)

$\sum_{i=1}^{8} x_j \leq 25$ (officers available)

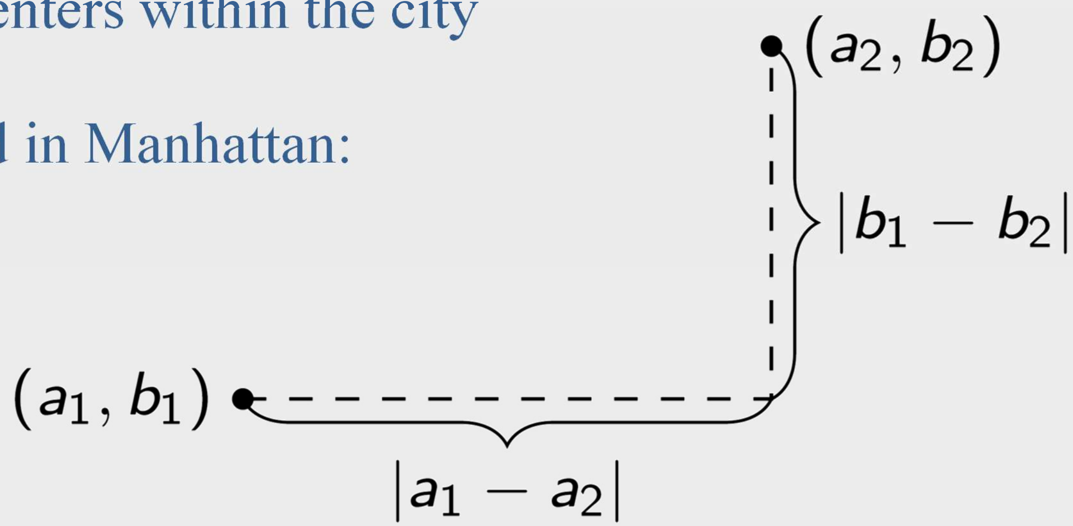
$x_j \leq u_j$ for $j = 1, \ldots, 8$ (upper bounds)

$x_j \geq 0$ for $j = 1, \ldots, 8$ (nonnegativity)

HOSPITAL LOCATION, IN MANHATTAN

Hospital Location, in Manhattan

- Mayor Dantzig formed a task force to determine the location of a new hospital
- The task force decided that the new hospital should be located "close" to 4 major population centers within the city
- Distances measured in Manhattan:



Distance between (a_1, b_1) and $(a_2, b_2) = |a_1 - a_2| + |b_1 - b_2|$

Hospital Location, in Manhattan

- 4 major population centers located at

$$(a_1, b_1)\ ,\ (a_2, b_2)\ ,\ (a_3, b_3)\ ,\ (a_4, b_4)$$

- Want to determine location of new hospital
- Decision variables:

$x = x\_coordinate$ of location of new hospital

$y = y\_coordinate$ of location of new hospital

- Objective:

$$Minimize\ |x - a_1| + |y - b_1| + |x - a_2| + |y - b_2| + |x - a_3| + |y - b_3| + |x - a_4| + |y - b_4|$$

$$\leftrightarrow \quad Minimize \quad \sum_{i=1}^{4} |x - a_i| + \sum_{i=1}^{4} |y - b_i|$$

- Constraints: none

HOSPITAL LOCATION, IN MANHATTAN

$$\textit{Minimize} \qquad \sum_{i=1}^{4} |x - a_i| + \sum_{i=1}^{4} |y - b_i|$$

- Nonlinear!
- Introduce auxiliary decision variable that represents every absolute value term in the objective value

$$s_i \leftrightarrow |x - a_i| \qquad \text{for } i = 1, 2, 3, 4$$

$$t_i \leftrightarrow |y - b_i| \qquad \text{for } i = 1, 2, 3, 4$$

Hospital Location, in Manhattan

- Change objective function

$$\textit{Minimize} \qquad \sum_{i=1}^{4} s_i + \sum_{i=1}^{4} t_i$$

- Add constraints!

$$s_i \geq x - a_i \qquad \text{for } i = 1, 2, 3, 4$$

$$s_i \geq -(x - a_i) \qquad \text{for } i = 1, 2, 3, 4$$

$$t_i \geq y - b_i \qquad \text{for } i = 1, 2, 3, 4$$

$$t_i \geq -(y - b_i) \qquad \text{for } i = 1, 2, 3, 4$$

LECTURE #6:

TRANSPORTATION AND ASSIGNMENT PROBLEM