Vehicle Routing with Stochastic Demands: Properties and Solution Frameworks

MOSHE DROR[1,2]

Ben Gurion University of the Negev, Beer Sheva 84 105, Israel, and Université de Montréal, Montréal, Canada

GILBERT LAPORTE[2] and PIERRE TRUDEAU[2]

Université de Montréal, Montréal, Canada

This paper considers the vehicle routing problem with stochastic demands. The objective is to provide an overview of this problem, and to examine a variety of solution methodologies. The concepts and the main issues are reviewed along with some properties of optimal solutions. The existing stochastic mathematical programming formulations are presented and compared and a new formulation is proposed. A new solution framework for the problem using Markovian decision processes is then presented.

INTRODUCTION

The classical *vehicle routing problem* (VRP) can be defined as follows. Consider a set $N = \{0, 1, \ldots, n\}$ of cities or customers, a depot located at city 0 and a distance matrix $C = (c_{ij})$ defined on N^2. The c_{ij}'s can also be interpreted as costs or travel times. Non-negative weights (or single commodity demands) ξ_i are associated with the cities of $N\backslash\{0\}$. In addition, there is a fleet of identical vehicles of capacity Q at the depot which provide service to the set $N\backslash\{0\}$. The VRP consists of determining vehicle routes in such a way that (i) all routes start and end at the depot; (ii) each city other than the depot is visited exactly once; (iii) the total demand of any given route does not exceed Q; (iv) the total distance travelled by all vehicles is minimized. The VRP is an important problem in the area of distribution–logistics, but only small problems can be solved optimally. There exists an extensive literature on the VRP and on several of its variants. Comprehensive recent surveys can be found in CHRISTOFIDES,[6] LAPORTE and NOBERT[25] and GOLDEN and ASSAD.[16]

The classical VRP statement does not capture an important aspect of real-life distribution problems, namely that several of the problem parameters (time-distances, city locations, demands, etc.) are often *stochastic*. Moreover in practice, several operating and service policies and a number of objective criteria can be considered. The number of variants becomes even larger in stochastic contexts. Until now much less effort has been devoted to the study of *stochastic vehicle routing problems* (SVRPs) as opposed to their deterministic counterparts. This is due, we believe, to the fact that when stochastic elements are introduced into this difficult combinatorial problem, it becomes in general even less tractable for analytical treatment.

Documented real-world applications of the SVRP in the context examined in this paper include among others the planning of cash collection from bank branches (LAPORTE et al.[24] and BERTSIMAS[4]). In this case, the maximal amount of cash carried in a vehicle is limited by a ceiling set by an insurance company and the amount collected at each branch is a random variable. Other examples are sludge disposal (LARSON[27]) where sludge accumulation in a plant is a random process, and the delivery of home heating oil (DROR et al.[11]) where daily customer consumption is also random.

The objective of this paper is to survey the work done so far on the VRP with stochastic demands, while clarifying some of the concepts and issues involved. As we survey past work, we offer a more focused classification of solution methodologies. We also propose a new integer programming model as well as a new solution framework in the form of a Markov

[1] Industrial Engineering and Management, Ben Gurion University of the Negev, P.O. Box 653, Beer Sheva 84 105, Israel.

[2] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, Succursale A, Montréal, Canada H3C 3J7.

Transportation Science
Vol. 23, No. 3, August 1989

0041-1655/89/2303-0166 $01.25

decison process model. The paper is organized as follows. In Section 1, we describe problem characteristics, and examine problem parameters, operating and service policies. Solution properties are discussed in Section 2. In Section 3, we present stochastic programming models for the SVRP. In Section 4, we describe the new Markov decision process model. Section 5 contains the conclusions.

1. PROBLEM CHARACTERISTICS

1.1. Problem Scope

We assume that customers have known and fixed locations, that the problem is defined on a complete graph, and that the distance matrix C satisfies the triangular inequality ($c_{ik} + c_{kj} \geqslant c_{ij}$ for all i, j, k). The assumption of a complete graph eliminates from the discussion a number of important studies of routing and location restricted to special graph structures such as trees. We also assume that *all* customers have to be visited and fully serviced. This excludes from consideration the class of *probabilistic traveling salesman problems* studied by DERMAN and KLEIN,[10] JAILLET[19,20] as well as JAILLET and ODONI.[21] The problem examined by Derman and Klein was that of determining an optimal inspection sequence for n components. It was characterized by the cost of inspecting component j after component i and by the required inspection frequency for each component. The objective was to minimize the expected inspection cost per unit time. Derman and Klein called this problem the *stochastic traveling salesman problem.* The name *probabilistic traveling salesman problem* was used by Jaillet for problems in which only a random subset of customers have to be visited. The subset of customers is determined according to a known probability distribution and these are always visited in the same sequence. The problem consists of determining a visiting sequence for all customers which minimizes expected travel distance. Other problems belonging to this class are VRPs with stochastic customers having equal demands (Bertsimas[4]) and a family of stochastic location-routing problems on trees (BERMAN and SIMCHI-LEVI[1] and BERMAN et al.[2]).

Also excluded are problems in which the c_{ij}'s are random variables. Such problems occur, for example, in the context where the c_{ij}'s are travel times which may vary according to traffic conditions. For the study of the estimated value for the expected length of an optimal traveling salesman problem (TSP) tour, see LEIPÄLÄ.[28] For the problem of determining the least expected travel time path between two customers, see HALL.[17]

1.2. Operating Policies

We consider problems with a single given depot, a homogeneous fleet and where customer demand is the only source of uncertainty. Our problem is therefore a particular case of the SVRP, although we will continue to refer to it by the SVRP abbreviation throughout the paper. Demands ξ_i are discrete or continuous nonnegative independent random variables with finite means μ_i and finite variances σ_i^2, and the variables $(\xi_i - \mu_i)/\sigma_i$ are also identically distributed. The fact that demand is not deterministic introduces a major structural change into the problem. It is no longer possible to assume, as was the case in the deterministic VRP, that vehicle routes can be followed as planned. Indeed a *failure* may occur on a planned route if, at some point (customer) along that route, customer demand cannot be met by the vehicle. In such a case, some *recourse* or corrective action which may generate extra cost must be taken. For example, the vehicle may have to return to the depot, reload and go back to the customer where failure occurred, or a new routing sequence may have to be planned for the remaining customers (route reoptimization). Another possibility would be to break the planned vehicle routes at various points in order to reduce the expected cost of corrective actions based on the anticipated customer demand. A *break* implies that at some point the vehicle returns to the depot for replenishment before it resumes its deliveries. These route breaks do not have to coincide with route failures. The range of recourse options is clearly wide and is dependent upon the acceptable service policy and the amount of available information on customer demand (on this subject, see LAPORTE and LOUVEAUX[23]). These options will be examined in the next subsection.

The SVRP consists of determining a first stage solution, i.e., planned vehicle routes, taking into account the recourse actions. SVRPs are sometimes viewed as two-stage problems; however, information on customer demands is often revealed gradually as vehicles follow their planned routes, so that these problems are more appropriately defined as multistage problems. The number of stages is sometimes equal to the number of customers, but as will be seen later, it can also exceed that number and cannot always be known a priori. This will be clarified when we examine service policies for the SVRP in the next subsection. The characterization of the SVRP as a multistage problem brings up the issue of reoptimization. Since new information is learned at each stage, one may want to benefit from this by reconsidering the customer sequence designed in the first stage.

A large range of reoptimization actions can be considered as valid operating policies. One important distinction is whether route reoptimization is allowed or not. At one extreme, the original path must be followed. If a route failure occurs, the vehicle goes to the depot for replenishment and resumes its journey at the failure point. The case where reoptimization is not allowed leads to a stochastic programming formulation. In less restrictive operating policies, reoptimizations are possible but only upon route failures. At the other end of the range, one may reoptimize at any location, using the newly obtained data about customer demand. This leads to a formulation of the problem as a Markov decision process model. Breaks are a form of reoptimization. They can be planned a priori in the first stage, or they can be decided on the basis that when facing an imminent route failure, it may be better to go back to the depot for replenishment before visiting the next customer. Finally, note that the first stage assignment need not be simply a sequence of customers but may also include reoptimization instructions in the form of a decison tree for each stop on the route.

1.3. Service Policies

In the VRP case, we distinguish between two major service policies: (i) *full delivery*: whenever a vehicle arrives at a customer, it delivers all that customer's demand. Hence, it is necessary in this case, to make the additional assumption that $\xi_i \leqslant Q$ $(i = 1, \ldots, n)$, for otherwise, the problem is infeasible. This service policy is the most widely used in the VRP literature (see, for example, Laporte and Nobert[25]); (ii) *split delivery*; here, customer demand can be supplied by several vehicles or split between several visits of the same vehicle: the assumption that a customer is visited exactly once (by exactly one vehicle) no longer holds and it is not necessary to assume that $\xi_i \leqslant Q$ $(i = 1, \ldots, n)$. This policy constitutes in fact a relaxation of the first case. Another possibility belonging to this category is when demand can be split, but only in prespecified fractions. The split delivery VRP has been less widely studied. For a recent paper on this subject, see DROR and TRUDEAU.[13]

In order to examine the major service policies in the case of the SVRP, an important attribute is the time at which customer demand becomes known. There are two extreme cases: (i) *full advance information*: all demands are known before vehicle routes are planned, which leads to the VRP; (ii) *late information:* customer demand becomes known only after the vehicle arrives at the customer's location, which leads to the SVRP. This case can be further split into two: the demand is known immediately before delivery or immediately after full service is completed (for example, in fuel delivery). Between these two extremes, there is a whole spectrum of possibilities (i.e., demand becomes known one, two, three steps ahead, etc.).

For the SVRP, the *full delivery* service policy implicitly assumes that the demand becomes known after the vehicle arrives at the customer's location but before delivery. The *split delivery* service policy for the SVRP is more complicated. If no service preemption is allowed, then in the case of late information, the amount delivered is determined by the demand and the quantity left on the vehicle at arrival (i.e., if the delivery has started, it will end when either the full demand is delivered or the vehicle becomes empty). If service preemption is allowed, then new decision variables have to be added. These determine the maximum level of replenishment on each visit to a customer.

Service policy and availability of information determine the number of stages in the SVRP. For example, in full advance information problems, there is only one stage; in late information and full delivery problems, there are n stages; in the case of late information and split deliveries, there can be more than n stages.

2. SOLUTION PROPERTIES

IT HAS already been observed by Jaillet[19,20] and Jaillet and Odoni[21] that a number of properties of an optimal TSP solution do not translate to the probabilistic TSP. In this section, we reexamine these properties in the context of the SVRP.

PROPERTY 1. *For the Euclidean TSP, the shortest closed path through n given points never intersects itself (i.e., it is a simple polygon unless all the points are collinear).*

For the SVRP the expected minimal cost vehicle tour does not, in general, correspond to a simple polygon. This is illustrated in Figure 1. The coordinates for the depot and customers 1, 2, and 3 are (0, 0), (0, 10), (−1, 1), and (2, 4) respectively. Distances between the points are computed using the Euclidean norm and there is only one vehicle of capacity 10. Customer 1 has a deterministic demand of 2 units. Customer 2 has a demand of 1 unit with probability 0.8 and demand of 8.2 units with probability of 0.2. Customer 3 has a demand of 2 units with probability of 0.8 and demand of 8.1 units with probability of 0.2. Assume that once the service sequence is set, no reoptimization is allowed. When a vehicle arrives at a customer, it delivers all of that customer's demand, if possible. Otherwise, it goes back to the depot upon failure and returns to the same customer to resume

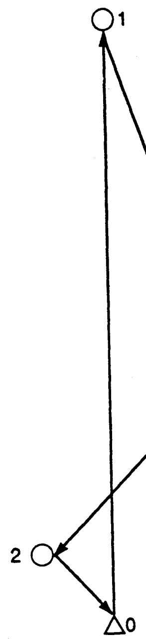

Fig. 1. Two basic geometric TSP properties do not translate to the SVRP.

delivery. In this case, it is simple to enumerate the routes and compute their expected value.

Route	*Expected Distance*
(0, 1, 3, 2, 0)	24.22
(0, 2, 3, 1, 0)	26.97
(0, 1, 2, 3, 0)	29.77
(0, 3, 2, 1, 0)	31.54
(0, 2, 1, 3, 0)	26.70
(0, 3, 1, 2, 0)	25.72.

The routing sequence (0, 1, 3, 2, 0) with the minimal expected value of 24.22 (= $c_{0,1} + c_{1,3} + c_{3,2} + c_{2,0} + 2c_{0,3}P(\xi_3 = 8.1) + 2c_{0,2}P(\xi_3 = 2)P(\xi_2 = 8.2)$) obviously does not possess the nonintersection property. Note that route direction has an impact on expected distance.

PROPERTY 2. *Every Euclidean TSP has an optimal solution in which customers on the boundary of the convex hull of all the customers (including the depot) are visited in the same order in which they appear on the boundary of the convex hull* (FLOOD[14]).

Using the same example, it is clear from Figure 1 that this property does not hold for the SVRP. The next observation has more serious repercussions on the design of SVRP algorithms.

PROPERTY 3. *In the case of the optimal TSP tour, the principle of optimality (usually stated in the context of dynamic programming) can easily be verified. It implies that any segment of an optimal TSP tour is optimal (i.e., if the sequence $i_0 = 0, i_1, \ldots, i_n, 0$ corresponds to the shortest simple cycle through the $n + 1$ points, then any subsequence, say $i_{p+1}, \ldots, i_{p+q}$, through $q < n + 1$ points also corresponds to the shortest simple path through these q points).*

The states in the dynamic programming version of the TSP correspond to the cities already visited and to the current position of the vehicle. In the case of the SVRP, the optimal vehicle tour has to account for all possible demand realizations which, under a certain service policy, could require redesigning a remaining segment of planned city visits. Thus, any segment of the initially designed optimal route would not necessarily be optimal when examined in isolation.

We illustrate this important distinction between the TSP and the SVRP in a special restricted case of tour construction. The restriction states that once a tour is selected, the sequence of the cities serviced remains fixed (i.e., no reoptimization of the tour occurs). Figure 2 shows that in this case, a SVRP tour can be optimal in the sense that it minimizes the expected cost, but the segments of such a tour are not necessarily optimal when examined separately. Indeed, assume there are three customers 1, 2, and 3, and a depot at 0 located at (−5, 5), (0, 1), (5, 5), and (0, 0) respectively. The distances between the points are again Euclidean. Only customer 1 has a stochastic demand (1 unit with probability 0.5 and 9 units with probability 0.5). Customers 2 and 3 each have deterministic demands of 2 units each. The vehicle capacity is 10 units. Computing the expected distances for all tour configurations gives the following values:

Route	*Expected Distance*
(0, 1, 2, 3, 0)	27.948
(0, 3, 2, 1, 0)	34.019
(0, 1, 3, 2, 0)	31.545
(0, 2, 3, 1, 0)	31.545
(0, 2, 1, 3, 0)	31.545
(0, 3, 1, 2, 0)	31.545.

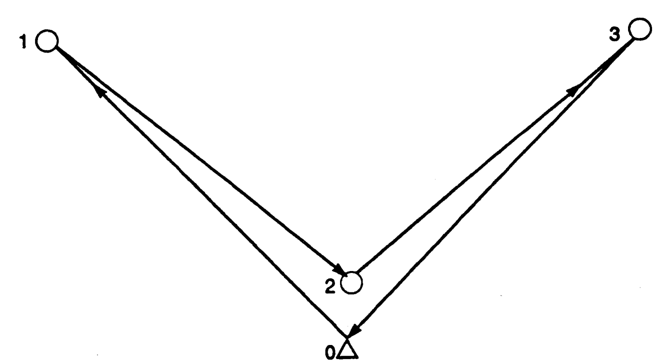

Fig. 2. Segments of an optimal tour are not optimal for the SVRP.

The optimal tour in terms of expected distance is (0, 1, 2, 3, 0). Its cost is computed as $c_{0,1} + c_{1,2} + c_{2,3} + c_{3,0} + 2c_{0,2}\mathrm{P}(\xi_1 = 9) = 27.948$. The problem of segments not being optimal occurs once we reach customer 1. If the demand at 1 is one unit, then the subsequent best decision is to continue to customer 3 followed by 2 and then the depot, with an expected added distance of 17.403 versus the original tour continuation which adds 19.877 to the distance. On the other hand if the demand at 1 is 9 units, then the best solution is to return to the depot and from there to go to customer 2 followed by 3, with an expected added distance of 21.545. If we attempt to follow the original tour, then the vehicle can only partially replenish customer 2, return to the depot then go back to 2 followed by 3, with an expected distance of 21.877. The third option of continuing from 1 to 3, with route failure at 3, return to the depot and then to 3 followed by 2 has an expected distance of 31.545. The optimal tour (0, 1, 2, 3, 0) would never be followed if route re-optimization was allowed.

3. STOCHASTIC PROGRAMMING MODELS

As mentioned earlier, two solution frameworks for SVRPs are stochastic programming and Markov decision processes. We examine them in this and the next section, respectively. For a general overview of stochastic programming, see Dempster,[9] Kall[22] and Wets.[31] For a description of Markov decision processes, the reader is referred to Heyman and Sobel[18] and to White and White.[32]

3.1. Stochastic Programming

By a mathematical programming problem we usually understand a problem of the type

MP $\quad \min f(x)$

subject to: $\quad g_i(x) \leqslant 0 \quad (i = 1, \ldots, m)$
$\quad x \in S \subseteq R^n,$

where the real functions f, $g_i (i = 1, \ldots, m)$ are assumed to be deterministic and known.

The stochastic counterpart can be described as

SP $\quad$ "min" $f(x, \xi)$

subject to: $\quad g_i(x, \xi) \leqslant 0 \quad (i = 1, \ldots, m)$
$\quad x \in S \subseteq R^n,$

where ξ is a vector of random variables defined on a probability space $(\Omega, \Sigma, \mathrm{P})$, hence $(f(x, \xi), g_i(x, \xi);$ $i = 1, \ldots, m)$ is a random vector.

This stochastic programming problem is not well-defined in general since the requirement of choosing $x \in S$ is not clearly specified in SP. Thus, we have to determine the *solution concept* (Kall[22]) appropriate for a particular problem. Stochastic programming terminology distinguishes between "wait and see" and "here and now" situations. The first leads to so called *distribution problems* by computing the decision variables x (planned vehicle routes) after observing the realization of ξ (customer demands). The second, "here and now," is the situation where the computation of x has to precede the observation of ξ. In the *distribution problem*, given a realization ξ of customer demands, we would have to solve a version of the *split delivery problem* in the case where split delivery is allowed (see Dror and Trudeau[13]) or the generic VRP if only full delivery is allowed. In essence, if $z(\xi)$ denotes the cost value of the optimal solution as a function of the demand vector ξ, then we may be interested in the distribution of $z(\xi)$ or at least, in some of its moments.

The SVRP more naturally belongs to the "here and now" category since usually, routes have to be planned in anticipation of customer demands which become known only in the process of executing the planned routing sequence. Again, the problems in this category are usually modeled in two ways: so-called *chance constrainted stochastic programing* and *stochastic programming with recourse.* We restate the general chance constrained formulation and the stochastic programming with recourse formulation as presented in Kall.[22]

(i) Chance constrained programming

CCP $\quad \min E_\xi[f(x, \xi)]$

subject to:

$$x \in S$$
$$P(\{\xi \mid g_i(x, \xi) \leqslant 0,\ i = 1, \ldots, m\}) \geqslant 1 - \alpha$$
or $$P(\{\xi \mid g_i(x, \xi) \leqslant 0\}) \geqslant 1 - \alpha_i \quad (i = 1, \ldots, m).$$

(ii) Stochastic programming with recourse

RM $\quad \min E_\xi[f(x, \xi) + \phi(x, \xi)]$

subject to: $\quad x \in S,$

where $\phi(x, \xi)$ is some non-negative real function $H(g_i(x, \xi),\ i = 1, \ldots, m)$ taking the value 0 if $g_i(x, \xi) \leqslant 0$ $(i = 1, \ldots, m)$. This expresses the fact that if all the constraints are satisfied, then we are concerned with the value of the objective function f only, but when a constraint violation occurs, we have to add a penalty to the value of f. In the SVRP, $\phi(x, \xi)$ corresponds to the cost of corrective or recourse actions.

3.2. Chance Constrained Models

Both the chance constrained formulation and the stochastic programming with recourse formulation have been used for the SVRP. STEWART and GOLDEN[29] propose the following chance constrained formulation which contains no expectation of the total distance in its objective and implicitly assumes no reoptimization. Let x_{ijk} be a binary variable equal to 1 if vehicle k travels from i to j, and equal to 0 otherwise. Also let m be the number of vehicles available, T_m the set of all feasible solutions to the m-TSP and α, the maximum allowable probability of route failure. The model is then:

CCP1 $$\min z = \textstyle\sum_k \sum_{i,j} c_{ij} x_{ijk} \quad (1)$$

subject to:

$$P(\textstyle\sum_{i,j} \xi_i x_{ijk} \leq Q) \geq 1 - \alpha \quad (k = 1, \ldots, m) \quad (2)$$

$$x = (x_{ijk}) \in T_m. \quad (3)$$

In this model, probabilistic constraints (2) can be replaced by the following nonlinear deterministic constraints:

$$\textstyle\sum_{i,j} \mu_i x_{ijk} + \tau(\sum_{i,j} \sigma_i^2 x_{ijk}^2)^{1/2} \leq Q \quad (k = 1, \ldots, m), \quad (4)$$

where $\mu_i = E[\xi_i]$, $\sigma_i^2 = \mathrm{Var}[\xi_i]$. In this expression, τ is a constant such that

$$P[(\textstyle\sum_{i,j} \xi_i x_{ijk} - M_k)/S_k \leq \tau] = 1 - \alpha,$$

where $M_k = \sum_{i,j} \mu_i x_{ijk}$ and $S_k = (\sum_{i,j} \sigma_i^2 x_{ijk}^2)^{1/2}$. Stewart and Golden show that provided (i) the ξ_i's are independent random variables, (ii) $(\sum_{i,j} \xi_i x_{ijk} - M_k)/S_k$ and $(\xi_i - \mu_i)/\sigma_i$ have the same distribution for all i, and (iii) σ_i^2/μ_i is equal to the same constant θ for all i, then constraints (2) or (4) are equivalent to

$$\textstyle\sum_{i,j} \mu_i x_{ijk} \leq \bar{Q} \quad (k = 1, \ldots, m) \quad (5)$$

where $\bar{Q} = [2Q + \tau^2\theta - (\tau^4\theta^2 + 4Q\tau^2\theta)^{1/2}]/2$. In other words, the SVRP then reduces to a deterministic VRP with demands μ_i and vehicle capacity equal to $\bar{Q}$.

Laporte et al.[24] obtain a similar result by making less restrictive assumptions and by using fewer variables. They define two-index variables x_{ij} instead of three-index variables x_{ijk}, but their model cannot take into account heterogeneous vehicle capacities, as was the case in **CCP1**. Let x_{ij} be a binary variable equal to 1 if and only if a vehicle travels on arc (i, j). Then the Laporte et al. model, adapted to the case of general distance matrices, can be written as:

CCP2 $$\min z = \textstyle\sum_{i,j} c_{ij} x_{ij} \quad (6)$$

subject to:

$$\textstyle\sum_{i \in S, j \notin S} x_{ij} \geq V_\alpha(S) \quad (S \subseteq \{1, \ldots, n\}; |S| \geq 2) \quad (7)$$

$$x = (x_{ij}) \in T_m. \quad (8)$$

Here $V_\alpha(S)$ is the minimum number of vehicles required to serve all customers of S so that the probability of route failure in S does not exceed α, i.e., $V_\alpha(S)$ is the smallest integer such that $P(\sum_{i \in S} \xi_i > QV_\alpha(S)) \leq \alpha$. The value of $V_\alpha(S)$ is easily determined provided the probability distribution of $\sum_{i \in S} \xi_i$ is known or even approximated for any non-empty subset S of $\{1, \ldots, n\}$. This model and that of the corresponding deterministic VRP (LAPORTE et al.[26]) are equivalent.

Both **CCP1** and **CCP2** minimize distance traveled while controlling the probability of route failure; however, the cost of such failures is not taken into account. They are not equivalent to minimizing the expected distance traveled while controlling the probability of route failure. Two routes with the same probability of failure and the same distance might have quite different expected distance values since the potential location of a failure has a significant impact on the expected route length. Under appropriate assumptions, solving **CCP1** or **CCP2** involves the same level of difficulty as solving a deterministic VRP with the same parameters.

3.3. Recourse Models

When modeling the SVRP as a stochastic programming problem with recourse, it is important to recognize that the problem is not a two-stage program as is sometimes suggested, but a more general multistage stochastic program. Whenever exact demands become known (e.g., upon arrival of a vehicle at a customer site), a new set of decision variables can in principle be generated reflecting the objective of minimizing the expected cost. The number of such stages is at least equal to the number of customers in the SVRP, but it may be even larger depending on the service policy assumed. For instance, if the vehicle cannot fully supply a customer, it may lead to a number of stages in excess of the number of customers. Moreover, for a general routing problem with stochastic demands, it is difficult to express the objective of minimizing the expected distance value in a closed tractable form. To our knowledge, this type of a general multistage stochastic programming problem has received little attention so far (on this subject, see BIRGE[5]) and is usually considered computationally intractable.

For the stochastic model with recourse, STEWART and GOLDEN[29] propose the following formulation:

RM1 $$\min z = \textstyle\sum_k \sum_{i,j} c_{ij} x_{ijk} + \sum_k \lambda_k E[l_k] \quad (9)$$

subject to: $$x \in T_m, \quad (10)$$

where λ_k is the penalty per unit of demand in excess of Q on route k and $E[l_k]$ is the expected number of units by which demand will exceed the vehicle capacity on route k. In **RM1** the terms $\lambda_k \mathrm{E}[l_k]$ do not reflect the true cost of an extra trip to the depot in case of route failure, since the location of failure is not taken into account.

In the stochastic models with recourse proposed by Laporte and Louveaux,[23] full service (i.e., no split deliveries) is assumed and two cases are considered: (i) in the SVRP with "early information," all customer demands are known after the route is constructed but before the vehicle leaves the depot and route breaks are planned in advance so as to minimize their expected cost; (ii) in the SVRP with "late information," breaks and failures coincide. In both cases, it is assumed that the planned routing sequence is unaltered by failures or breaks. The two models differ only in their recourse function. For a more detailed analysis we refer the reader to the original paper since the explicit recourse representation is quite complicated. The general model can be written as

$$\textbf{RM2} \qquad \min z = cx + \mathrm{E}_\xi[\phi(x, \xi)] \tag{11}$$

$$\text{subject to:} \qquad x \in T_m, \tag{12}$$

where $\phi(x, \xi)$ is the recourse function. This function is determined as the solution of a large scale mixed integer linear program inspired by the commodity flow formulation of Gavish and Graves[15] for the deterministic VRP. For problems of realistic dimensions, the size of this formulation precludes solution by exact methods.

A different approach to the SVRP as a stochastic recourse problem was presented in Dror and Trudeau.[12] Denote that approach as **RM3**. The recourse function $\phi(x, \xi)$ in the event of route failure is assumed to take the value of back and forth trips from the depot to each of the remaining customers on the initial route. Note that this type of recourse function eliminates the issue of reoptimizing the remaining route segments after a route failure and in fact, it penalizes heavily the occurrence of failure on a planned route. The objective criterion is to minimize the expected value of distance plus the recourse value, subject to the routing constraints.

The actual solution procedure was based on heuristic route construction adapted from the deterministic VRP heuristic introduced by Clarke and Wright,[8] in which the decision whether to combine two routes with customers i and j respectively into a single route depended on the savings value $s_{ij} = c_{0i} + c_{0j} - c_{ij}$ and on the feasibility of such an operation. In **RM3**, the savings term is replaced by $\max\{c_i' + c_j' - c_{ij}',\ c_i' + c_j' - c_{ji}'\}$, where c_i' is the expected cost of the current route including i, and c_{ij}' is the expected cost of the combined route where i immediately precedes j. This kind of heuristic route construction accounts for the location of potential route failures, the direction of the route, and the customers already supplied on that route. The computational experiments reported in Dror and Trudeau[12] suggest that their approach is successful in constructing vehicle routes with considerably lower expected routing cost than either **CCP1** or **RM1**.

3.4. A New Recourse Model

The three recourse models presented so far address similar SVRP settings. In **RM1** and **RM2**, no route reoptimization is permitted, only full deliveries are allowed and the demands ξ_i are assumed to be independent random variables. In **RM1**, the objective does not represent the exact expected cost of the solution, and the location at which route failure occurs is not accounted for (i.e., the back and forth distance which the vehicle has to travel in order to resume its route). In **RM2**, the location of route failure (route break) is accounted for in the recourse penalty function. The problem is that this function (expressed by $\phi(x, \xi)$) is implicitly defined and its evaluation may require the solution of a very large number of optimization problems, especially in the case of continuous demand distributions. In **RM3**, the recourse penalty eliminates the reoptimization option (limiting the number of route failures to at most one per route), forcing the vehicle to empty its load at the point of failure.

In this subsection we present a mathematical programming formulation of the SVRP under the same service policy as **RM3**, without reoptimization. This formulation has the following properties:

(i) the expected cost of the solution accounts for the location of route failures;
(ii) the number of variables is independent of the number of demand realizations;
(iii) in the event that a vehicle cannot fully replenish a customer, it empties its load and travels back and forth to the depot before resuming its route which might include a number of additional failures.

In this model, we assume for the sake of simplicity that the cumulative demand distributions are characterized by two parameters. We require that these two parameters be obtained additively from the parameters of the individual demand distributions of ξ_i. For example, when the variables ξ_i are normally distributed with mean μ_i and variance σ_i^2, then their cumulative demand distribution is also normal and has two parameters, each obtained additively from those of the individual demand distributions, i.e., $\sum_{i=1}^n \xi_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$. Similarly, if $\xi_i \sim \Gamma(\alpha_i, \beta)$ then $\sum_{i=1}^n \xi_i \sim \Gamma(\sum_{i=1}^n \alpha_i, \beta)$. In case of the gamma distribution, only the first parameter is additive. When ξ_i

can take negative values (e.g., if ξ_i is normally distributed), it is assumed that the distribution parameters are such that $P[\xi_i < 0] \simeq 0$. If no split deliveries are allowed, it is also assumed that these parameters are such that $P[\xi_i \leq Q] \simeq 1$. The model is of course usable if the demand distribution has only one parameter (e.g., the Poisson distribution) and is easy to expand if it has more than two parameters.

We will first examine the recourse function in this problem setting. Assume a planned route is denoted by $x = (0, i_1, \ldots, i_n, 0)$. Further, to simplify the presentation, we assume continuous demand distributions so that the probability of a vehicle becoming empty simultaneously with service completion is zero. If this case was not excluded, the recourse cost computation would have to be modified. The expected recourse cost now has the following value (TRUDEAU and DROR[30]):

$$
\begin{aligned}
&E_\xi[\phi(x, \xi)]\\
&= \sum_{k=1}^{n} (c_{0i_k} + c_{i_k0}) \sum_{j=1}^{k} P[\textstyle\sum_{l=1}^{k-1} \xi_{i_l} \leq jQ < \sum_{l=1}^{k} \xi_{i_l}]\\
&= \sum_{k=1}^{n} (c_{0i_k} + c_{i_k0}) \sum_{j=1}^{k} (P[\textstyle\sum_{l=1}^{k-1} \xi_{i_l} \leq jQ] \qquad (13)\\
&\quad - P[\textstyle\sum_{l=1}^{k} \xi_{i_l} \leq jQ])\\
&= \sum_{k=1}^{n} (c_{0i_k} + c_{i_k0}) \sum_{j=1}^{k} (F_{k-1}(j) - F_k(j)),
\end{aligned}
$$

where

$$
F_k(t) = \begin{cases} P[\sum_{l=1}^{k} \xi_{i_l} \leq tQ] & \text{if } k \in \{1, \ldots, n\} \\ 1 & \text{otherwise.} \end{cases}
$$

Note that index k in the first sum of (13) refers to customer i_k; index j corresponds to the jth route failure (occurring at customer i_k).

In developing (13), we assumed that the routing sequence of customers is known and therefore the parameters of the cumulative demand distribution up to and including customer i can be easily calculated. In general, the optimal sequence is unknown and in order to obtain the parameter values for the cumulative demand distribution up to node i, we introduce a commodity flow variable for each parameter. On arc (i, j) the flow variable u_{ij} is used to accumulate the first parameter of the cumulative demand and v_{ij} serves the same purpose for the second parameter of this cumulative demand.

In order to simplify the presentation of the objective function, we introduce four variables y_i^1, y_i^2, r_i^1, r_i^2, defined as follows:

$$
y_i^1 \equiv \sum_{k=0}^{n} u_{ki}; \quad y_i^2 \equiv \sum_{k=0}^{n} u_{ik};
$$
$$
r_i^1 \equiv \sum_{k=0}^{n} v_{ki}; \quad r_i^2 \equiv \sum_{k=0}^{n} v_{ik} \quad (i = 1, \ldots, n).
$$

Let η_i^1 and η_i^2 denote the first and the second parameters of the demand distribution at i and define $F(j, y, r)$ as the value of the cumulative demand distribution evaluated at jQ with parameter values y and r. The problem is then:

$$
\textbf{RM4} \quad \min z = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij} + \sum_{i=1}^{n} (c_{0i} + c_{i0}) \sum_{j=1}^{n} [F(j, y_i^1, r_i^1) - F(j, y_i^2, r_i^2)] \tag{14}
$$

subject to:

$$\sum_{i=1}^{n} x_{0i} = \sum_{j=1}^{n} x_{j0} \leq m \tag{15}$$
$$\sum_{i=0}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n) \tag{16}$$
$$\sum_{j=0}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n) \tag{17}$$
$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \quad (S \subseteq N; |S| \geq 2) \tag{18}$$
$$x_{ij} \in \{0, 1\} \quad (i, j = 0, \ldots, n) \tag{19}$$
$$\sum_{k=0}^{n} u_{jk} = \sum_{i=0}^{n} u_{ij} + \eta_j^1 \quad (j = 1, \ldots, n) \tag{20}$$
$$-Mx_{ij} \leq u_{ij} \leq Mx_{ij} \quad (i = 1, \ldots, n; j = 0, \ldots, n) \tag{21}$$
$$\sum_{k=0}^{n} v_{jk} = \sum_{i=0}^{n} v_{ij} + \eta_j^2 \quad (j = 1, \ldots, n) \tag{22}$$
$$-Mx_{ij} \leq v_{ij} \leq Mx_{ij} \quad (i = 1, \ldots, n; j = 0, \ldots, n) \tag{23}$$
$$u_{0j} = v_{0j} = 0 \quad (j = 1, \ldots, n). \tag{24}$$

The first term of (14) corresponds to the a priori planned tour. Because demand distribution is continuous, and since no reoptimization is allowed, the costs associated with such a tour will be incurred, independently of the number of route failures that may occur. The second term is the recourse function which was obtained in (13). Note that in the second summation, index j goes up to n (instead of i) since the sequence of customers up to i in the optimal solution is not known in advance. Constraints (15)–(19) ensure connectivity and flow conservation. Note that these constraints are identical to $x \in T_m$. Constraints (20), (21), (24) and [(22)–(24)], respectively describe the commodity flow variables u_{ij} and v_{ij} related to the two parameters of the cumulative demand distribution function F. M is a large positive constant.

We see that (21) impose that commodity flows are non-zero only on planned routes. By (20), the flow leaving customer j is equal to the flow entering customer j plus the value of the first parameter of the demand distribution of customer i (recall the additivity hypothesis). Therefore, y_i^1 and y_i^2 will be the accumulated values of the first parameter on the route up to node i, respectively, excluding and including the value of the first parameter for customer i.

Note that constraints (18) may often not be required. Constraints (20) by themselves could sometimes prevent the formation of subtours. This would be the case for instance, if variables η_j^1 (or η_j^2) were of the same sign for all j.

The new model presented in this subsection is of general applicability whenever the operating policy is such that no reoptimization is allowed. It should be noted however that its objective function may not have a closed form representation and is in general nonlinear. For this reason, a heuristic approach derived from this model would appear appropriate.

4. A MARKOV DECISION PROCESS MODEL

We now show how the SVRP can be modeled as a Markov decision process. The model involves a very large number of states and its solution would require, in all likelihood, some form of relaxation (not unlike the methodology proposed by Christofides et al.[7] for the VRP or some of the aggregation techniques described by White and White[32]). Meanwhile, our discussion remains descriptive and will hopefully provide new insight into the problem.

Consider a SVRP with a single vehicle of fixed capacity Q located at the depot and assume that no customer demand exceeds Q. In addition, assume that the exact demand becomes known immediately upon arrival of the vehicle at the customer location (i.e., before replenishment begins). Once the vehicle has arrived at a customer location, two possible decisions–actions can be taken prior to the start of service:

(i) not to replenish the customer and then move to another location (which could be another customer or the depot),
(ii) to replenish the customer and then move to another location.

We assume a nonpreemptive service policy which implies that if customer service (replenishment) has started, it is not interrupted until either the customer is fully replenished or the vehicle is empty, in which case the only option is to return to the depot. When there are no more customers to be replenished, then again the only option is to return to the depot. This is the final state of the system whereas in the initial state, there is a full vehicle at the depot with all n customers to be serviced.

The system is observed each time the vehicle arrives at one of the locations of $\{0, 1, \ldots, n\}$. Let $\tau_0 = 0$, τ_1, $\tau_2, \ldots$ ($\tau_i \leq \tau_{i+1}$; $i = 0, 1, 2, \ldots$) be the times (cumulative distance traveled by the vehicle) at which these events occur. Note that τ_i corresponds to the time of the ith arrival which is not necessarily an arrival time at location i. These are called *transition times* and correspond to the times at which decisions are taken. The *state* of the system at a transition time τ_k is described by a vector $s = (r, l, x_1, \ldots, x_n)$, where $r \in \{0, 1, \ldots, n\}$ denotes the position of the vehicle and $l \in [0, Q]$ describes the stock level in the vehicle (if $r = 0$ then $l = Q$ i.e., the vehicle is full). If customer i has been visited, then the exact demand is known and x_i represents the remaining demand ($x_i \geq 0$). If customer i has not yet been visited, then the demand is unknown and is denoted by $x_i = -1$. The *state space* is a subset S of $\{0, 1, \ldots, n\} \times [0, Q] \times (\{-1\} \cup [0, Q])^n$ which satisfies the above conditions.

At each transition time, a decision d is selected from the *decision space* $D = \{d_1, d_2\} \times \{0, 1, \ldots, n\}$, where $d = (d_1, i)$, means that the vehicle goes to customer i from its present location, say j, without servicing j and $d = (d_2, i)$ corresponds to the case where the vehicle first replenishes j before going to i. The decision $d = (\cdot, i)$, ($i \neq 0$) is admissible even if $l = 0$, which represents the value of collecting the information on the demand at i without being able to satisfy any part of that demand. If $x_i = 0$ for all i, then the only admissible action is $d = (d_1, 0)$ i.e., returning to the depot. For each $s \in S$, let $D(s) \subset D$ denote the set of admissible decisions when the system is in state s, and $\Gamma = \{(s, d) \mid d \in D(s)\}$ the set of admissible state-decision pairs.

At transition time τ_k, the system is in some state $s_k \in S$ and a decision $d_k \in D(s_k)$ is taken. The time τ_{k+1} of the next transition is deterministic (specified by the distance matrix C), and the next state s_{k+1} is generated according to the probability distribution which governs the demand variables. Suppose $s_k = (r_{\tau_k}, l, x_1, \ldots, x_n,) \in S$ is the observed state at the current transition time τ_k, and $d = (d_*, i) \in D(s)$ is the decision taken, where $d_* \in \{d_1, d_2\}$. Then the time (distance) until the next event is simply the difference $\tau_{k+1} - \tau_k = c_{r_{\tau_k}, i}$ where r_{τ_k} is the location of the vehicle at transition time τ_k (we implicitly assume that service time is zero). Let $p(\cdot \mid s, d)$ be the transition law, i.e., for every Borel subset $\bar{S}$ of S, $p(\bar{S} \mid s, d)$ is the probability that the next state belongs to $\bar{S}$, given s and d.

A control policy is a function μ which associates with each state $s \in S$ a decision $d = \mu(s) \in D(s)$. The aim of the decision maker is to find a policy which minimizes the expected cost of a decision sequence, starting at $s_0 = (0, Q, -1, \ldots, -1)$ and terminating at $s^* = (0, Q, 0, \ldots, 0)$. More precisely, one seeks a policy μ^* which minimizes $\mathrm{E}_\mu[\sum_{k=0}^{T} c_{r_{\tau_k}, r_{\tau_{k+1}}}]$ where T is the random number of transitions.

The problem can be characterized as a sequential decision process under uncertainty. Appropriate solution methodologies, both exact and approximate, for this class of problems can be found in Bertsekas[3] and in White and White.[32].

In the case where a preemptive service policy is allowed (i.e., the replenishment of a customer can be stopped at any point of the service), then the decision space of the above model becomes much larger. The new decision space is $D = \{0, 1, \ldots, n\} \times [0, Q]$, where a decision d taken at point j has the form $d = (i, t_j)$ $(0 \leq t_j \leq x_j)$. The decision variables t_j determine the replenishment level at j (note that the previous decision variables d_1 and d_2 are no longer required).

5. CONCLUSIONS

IN THIS paper, we have considered a class of SVRPs where all customers have to be visited and where customer demand is the only source of uncertainty. Because of the large number of potential operating and service policies, and the possible variations in the availability of demand information, it becomes clear that the variety of problems in this setting is much larger than in the deterministic case. Furthermore, some elementary properties of deterministic VRPs no longer hold in the stochastic case.

The present status of research into problems in the SVRP category is not very impressive. Only a limited number of mathematical programming formulations and exact algorithms have been proposed for VRPs with stochastic demands and these only apply to a small number of problem settings. As such, versions of chance constrained programming formulations which minimize planned route costs while controlling the probability of failure can be shown to be equivalent to deterministic VRP formulations. In recourse programming formulations, on the other hand, the number of integer variables necessary to account for the possible demand realizations increases problem size and difficulty substantially. So far, such formulations have been proposed for very particular problem settings and have not produced exact algorithms.

In addition to providing an overview of the SVRP, this paper proposes two new solution frameworks. The first one is an integer programming formulation where the number of variables is independent of the possible demand realizations and can be used to model problems with relatively general recourse functions. Its major disadvantage is the non-linearity of the objective function and the restrictive operating and service policies it assumes. The second solution framework is a Markov decision process. Here, the large size of the model indicates that a state space relaxation approach could be appropriate. At present, the value of these two solution frameworks is mainly descriptive. Further research is required to explore the two new modeling avenues presented here.

ACKNOWLEDGMENTS

THE AUTHORS wish to thank Professor Pierre L'Écuyer from Laval University for his comments on the Markov decision process model and René Séguin from the Centre de recherche sur les transports for his helpful suggestions. The first two authors acknowledge the support of the Canadian Natural Sciences and Engineering Research Council (grants OGPIN 020 and A4747).

REFERENCES

1. O. BERMAN AND D. SIMCHI-LEVI, "Minisum Location of a Traveling Salesman," *Networks* **16,** 239–254 (1986).
2. O. BERMAN, D. SIMCHI-LEVI AND A. TAMIR, "The Minimax Multistop Location Problem on a Tree," *Networks* **17,** 39–49 (1988).
3. D. P. BERTSEKAS, *Dynamic Programming: Deterministic and Stochastic Models*, Prentice-Hall, Englewood Cliffs, N.J., 1987.
4. D. J. BERTSIMAS, "Probabilistic Combinatorial Optimization Problems," Ph.D. thesis, Massachusetts Institute of Technology, 1988.
5. J. R. BIRGE, "Decomposition and Partitioning Methods for Multistage Stochastic Linear Programs," *Opns. Res.* **33,** 989–1007 (1985).
6. N. CHRISTOFIDES, "Vehicle Routing," in *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, pp. 431–448, E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys (eds.). Wiley, Chichester, 1985.
7. N. CHRISTOFIDES, A. MINGOZZI AND P. TOTH, "State Space Relaxation Procedures for the Computation of Bounds to Routing Problems," *Networks* **11,** 145–164 (1981).
8. C. CLARKE AND J. W. WRIGHT, "Scheduling of Vehicles from a Central Depot to a Number of Delivery Points," *Opns. Res.* **12,** 568–581 (1964).
9. M. A. H. DEMPSTER (ed.), *Stochastic Programming*, Academic Press, London, 1980.
10. C. DERMAN AND M. KLEIN, "Surveillance of Multicomponent Systems: A Stochastic Traveling Salesman's Problem," *Naval Res. Logist. Quart.* **13,** 103–111 (1966).
11. M. DROR, M. BALL AND B. L. GOLDEN, "Computational Comparison of Algorithms for Inventory Routing," *Ann. Opns. Res.* **4,** 3–23 (1985/6).
12. M. DROR AND P. TRUDEAU, "Stochastic Vehicle Routing with Modified Savings Algorithm," *Eur. J. Opnl. Res.* **23,** 228–235 (1986).
13. M. DROR AND P. TRUDEAU, "Savings with Split Delivery Routing," *Trans. Sci.* **23,** 141–148 (1989).
14. M. M. FLOOD, "The Traveling-Salesman Problem," *Opns. Res.* **4,** 61–75 (1956).
15. B. GAVISH AND S. C. GRAVES, "Scheduling and Routing in Transportation and Distribution Systems: Formulations and New Relaxations," Working Paper, Graduate School of Management, University of Rochester, 1982.
16. B. L. GOLDEN AND A. A. ASSAD, "Vehicle Routing: Methods and Studies," *Studies in Management Science and Systems* **16,** North-Holland Publishing, Amsterdam, 1988.
17. R. W. HALL, "The Fastest Path through a Network with Random Time-Dependent Travel Times," *Trans. Sci.* **20,** 182–188 (1986).

18. D. P. HEYMAN AND M. J. SOBEL, *Stochastic Models in Operations Research*, Vol. II, McGraw-Hill, New York, 1984.
19. P. JAILLET, "Probabilistic Traveling Salesman Problem," Ph.D. thesis, Massachusetts Institute of Technology, 1985.
20. P. JAILLET, "A Priori Solution of Traveling Salesman Problem in Which a Random Subset of Customers Are Visited," *Opns. Res.* **36,** 929–936 (1988).
21. P. JAILLET AND A. R. ODONI, "The Probabilistic Vehicle Routing Problem," in *Vehicle Routing: Methods and Studies*, pp. 293–318, B. L. Golden and A. A. Assad (eds.), *Studies in Management Science and Systems* **16,** North-Holland Publishing, Amsterdam, 1988.
22. P. KALL, "Stochastic Programming," *Eur. J. Opnl. Res.* **10,** 125–130 (1982).
23. G. LAPORTE AND F. LOUVEAUX, "Formulations and Bounds for the Stochastic Capacitated Vehicle Routing Problem with Uncertain Supplies," Cahier du GERAD G-87-23, École des Hautes Études Commerciales de Montréal, 1987 (revised 1989).
24. G. LAPORTE, F. LOUVEAUX AND H. MERCURE, "Models and Exact Solutions for a Class of Stochastic Location-Routing Problems," *Eur. J. Opnl. Res.* **39,** 71–78 (1989).
25. G. LAPORTE AND Y. NOBERT, "Exact Algorithms for the Vehicle Routing Problem," in *Surveys in Combinatorial Optimization*, pp. 147–184, S. Martello, G. Laporte, M. Minoux and C. Ribeiro (eds.). *Annals of Discrete Mathematics* **31,** North-Holland Publishing, Amsterdam, 1987.
26. G. LAPORTE, Y. NOBERT AND M. DESROCHERS, "Optimal Routing under Capacity and Distance Restrictions," *Opns. Res.* **33,** 1050–1073 (1985).
27. R. C. LARSON, "Transportation of Sludge to the 106 Mile Site: An Inventory Routing Algorithm for Fleet Sizing and Logistic System Design," *Trans. Sci.* **22,** 186–198 (1988).
28. T. LEIPÄLÄ, "On the Solutions of Stochastic Traveling Salesman Problems," *Eur. J. Opnl. Res.* **2,** 291–297 (1978).
29. W. R. STEWART, JR., AND B. L. GOLDEN, "Stochastic Vehicle Routing: A Comprehensive Approach," *Eur. J. Opnl. Res.* **14,** 371–385 (1983).
30. P. TRUDEAU AND M. DROR, "Stochastic Inventory Routing: Stockouts and Route Failure," (submitted for publication) (1987).
31. R. WETS, "Stochastic Programming: Solution Techniques and Approximation Schemes," in *Mathematical Programming: State of the Art 1982*, pp. 566–603, A. Bachem, M. Grötschel, and B. Korte (eds.). Springer-Verlag, Berlin, 1983.
32. C. C. WHITE AND D. J. WHITE, "Markov Decision Processes," *Eur. J. Opnl. Res.* **39,** 1–16 (1989).

(Received, June 1988; revision received February 1989; accepted March 1989)