Modelling and Optimization

INF170

#2:Unconstrained Optimization

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OPTIMIZATION

Optimization is the science of making the best decision or, more precisely, *making the best possible decision.*

Three Cases

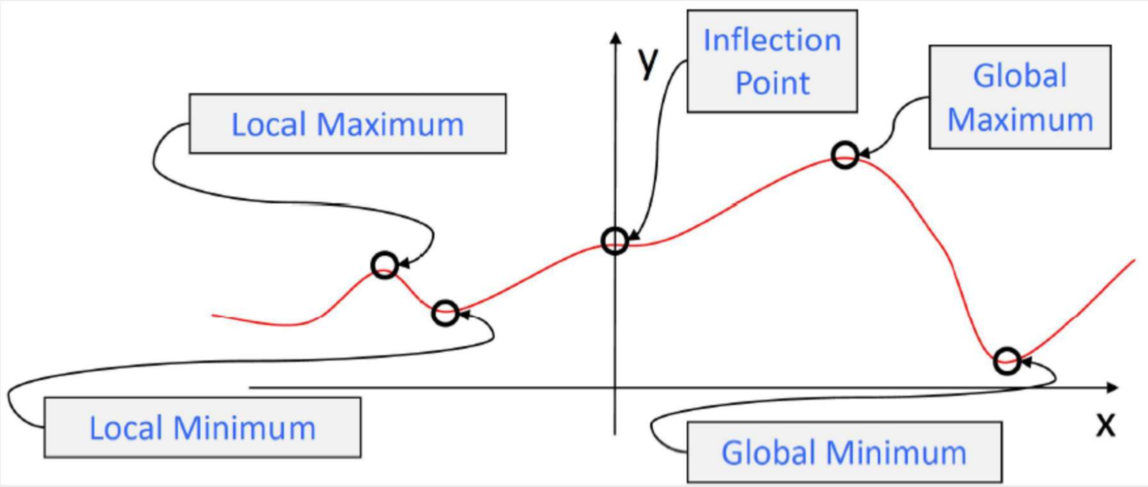
- iWidget – Profit Maximization
  - Given cost and demand functions, find the price for the iWidget that produces maximum profit.
- Gears Unlimited – Inventory Replenishment Policy
  - Given annual demand and transportation and holding costs, calculate the re-order quantity that minimize total cost.
- Boxy.com – Package Optimization
  - Calculate the package dimensions that maximize total usable volume given a specific cardboard sheet.

Common Features

- Each of these problems …
  - Utilizes a math function to make the decision,
  - Looks for an "extreme point" solution, and
  - Are unconstrained in that there is not a resource limit.
- What is an extreme point of a function?
  - The point, or points, where the function takes on an extreme value, either a minimum or a maximum.
  - The point(s) where the slop or "rate of change" of the function is equal to zero.

Extreme Points

- Types of Critical Points
  - Extreme points(Minimum, Maximum), or Inflection Points
  - The minimum and maximum points are either global or local



How do we find these “extreme point” solutions?

We’ll use differential calculus to find where the slope is equal to zero!

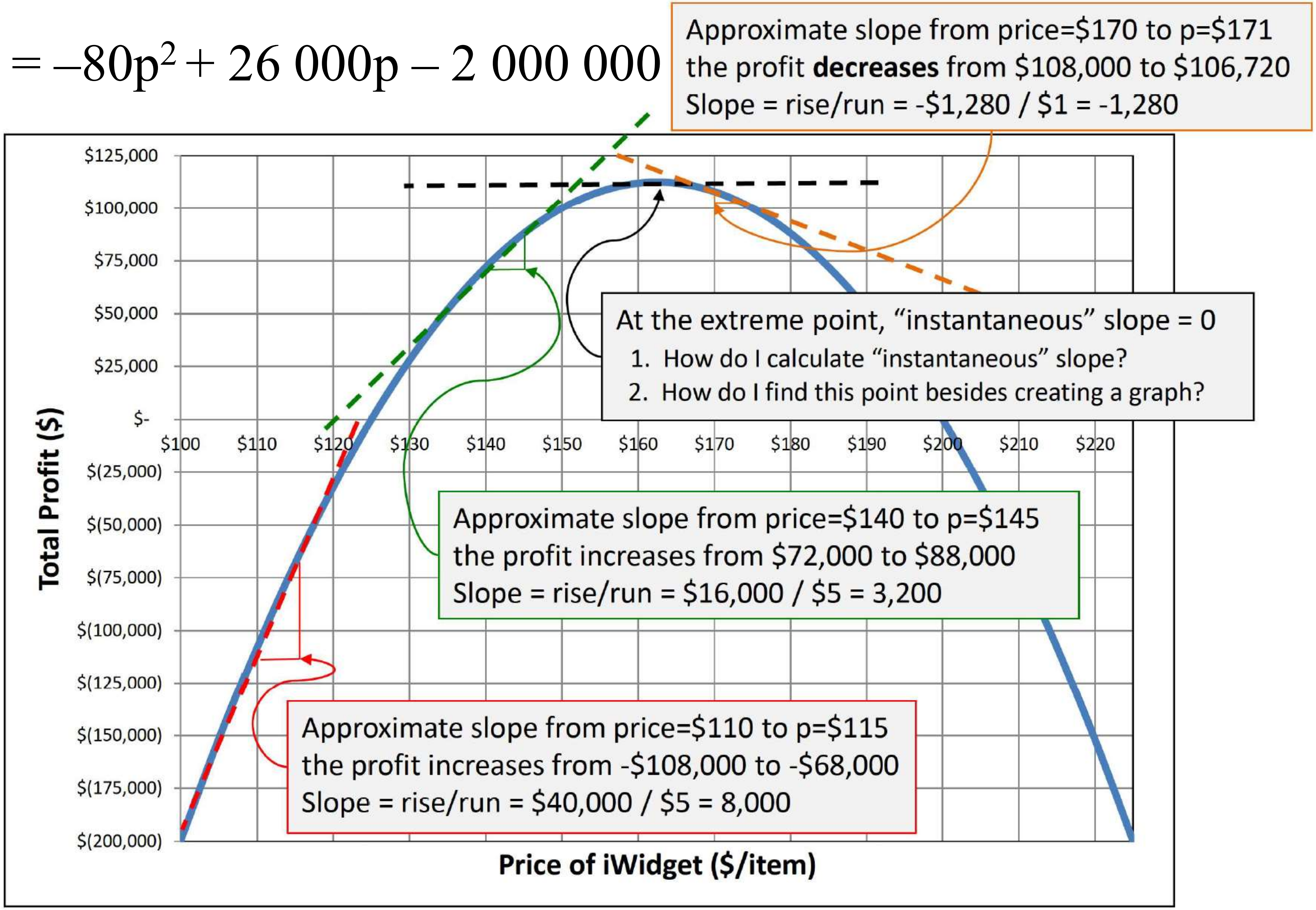
iWidget

- We are manufacturing a product where we know:
  - The cost function = f(#made) = $500\,000 + 75x$
  - The demand function = f(price) = $20\,000 - 80p$
  - And therefore the profit function = $-80p^2 + 26\,000p - 2\,000\,000$

- We want to find the price, p, that maximizes profits.

iWidget

Profit = $-80p^2 + 26\,000p - 2\,000\,000$



iWidget

- Find the price, p, that maximizes the profit function:

$$y = -80p^2 + 26\,000p - 2\,000\,000$$

- Solution
  1. Take the first derivative:

  $$y' = dy/dp = -160p + 26\,000$$

  2. Set the first derivative equal to zero:

  $$-160p + 26\,000 = 0$$

  3. Solve for p^*

  $$p^* = -26\,000 / -160 \rightarrow p^* = 162.50$$

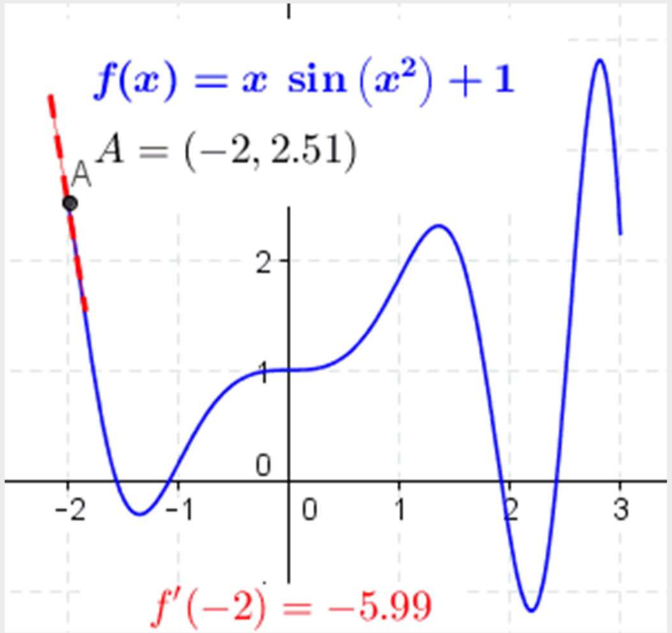
Set price at *\$162.50* in order to maximize profit.

Expected profit will be *\$112 500*.

$$profit = -80(162.50)^2 + 26\,000(162.50) - 2\,000\,000$$

Critical Points

1. How do I know this is a maximum and not a minimum?
2. How do I know whether this is a global or local?



iWidget

1. How do I know this is a maximum and not a minimum?
2. How do I know whether this is a global or local?

Necessary and Sufficient conditions

- In order to determine x^* at the *max/min* of an unconstrained function:
- Necessary condition – the slope has to be zero, that is, $f'(x^*)=0$
- Sufficient condistion – determines whether extreme point is min or max by taking the second derivative, $f''(x)$.
  - If $f''(x) > 0$ then the extreme point is a local minimum
  - If $f''(x) < 0$ then the extreme point is a local maximum
  - If $f''(x) = 0$ then it is inconclusive
- Special cases
  - If $f(x)$ is <u>convex</u> then $f(x^*)$ is a <u>global minimum</u>
  - If $f(x)$ is <u>concave</u> then $f(x^*)$ is a <u>global maximum</u>

Gears Unlimited

Inventory Replenishment Policy

Gears unlimited distributes specialty gears, derailleurs, and brakes for high-end mountain and BMX bikes. One of their most steady selling item is the PK35 derailleur. They sell about 1500 of the PK35's a year. They cost $75 each to procure from a supplier and Gears Unlimited assumes that the cost of capital is 20% a year. It costs about $350 to place and receive an order of the PK35s, regardless of the quantity of the order.

How many PK35s should Gear Unlimited order at a time to minimize the average annual cost in terms of purchase cost, ordering costs and holding cost?

Gears Unlimited

What do we know?

- ✓ D = Demand = 1500 item/year
- ✓ c = Unit cost = 75 \$/item
- ✓ A = Ordering cost = 350 \$/order
- ✓ r = Cost of capital = 0.2 \$/\$/year

What do we want to find?

- ➢ Q = Order Quantity (item/order)
- ➢ Find Q^* that minimizes Total Cost

What is my objective function?

$$\text{TotalCost} = \text{PurchaseCost} + \text{OrderCost} + \text{Holding Cost}$$

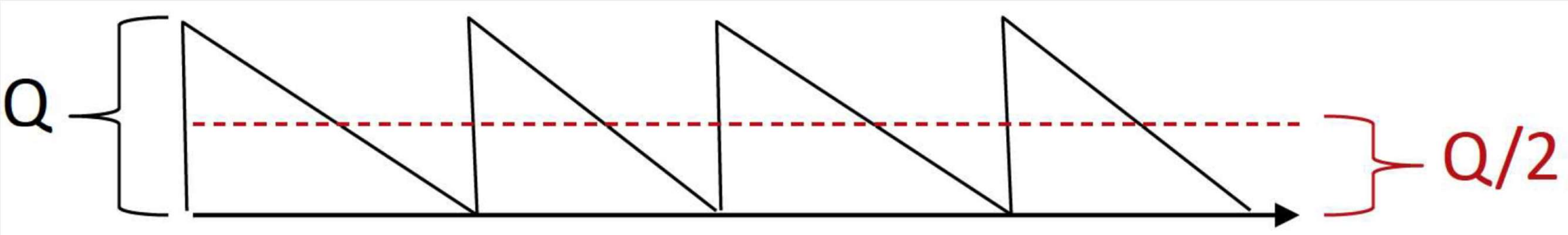
Gears Unlimited

$$\text{TotalCost} = \text{PurchaseCost} + \text{OrderCost} + \text{Holding Cost}$$

$$PurchaseCost = cD = (75)(1500) = 112500\ \$/yr$$

$$OrderCost = A(D/Q) = (350)(1500)/Q = 525000/Q\ \$/yr$$

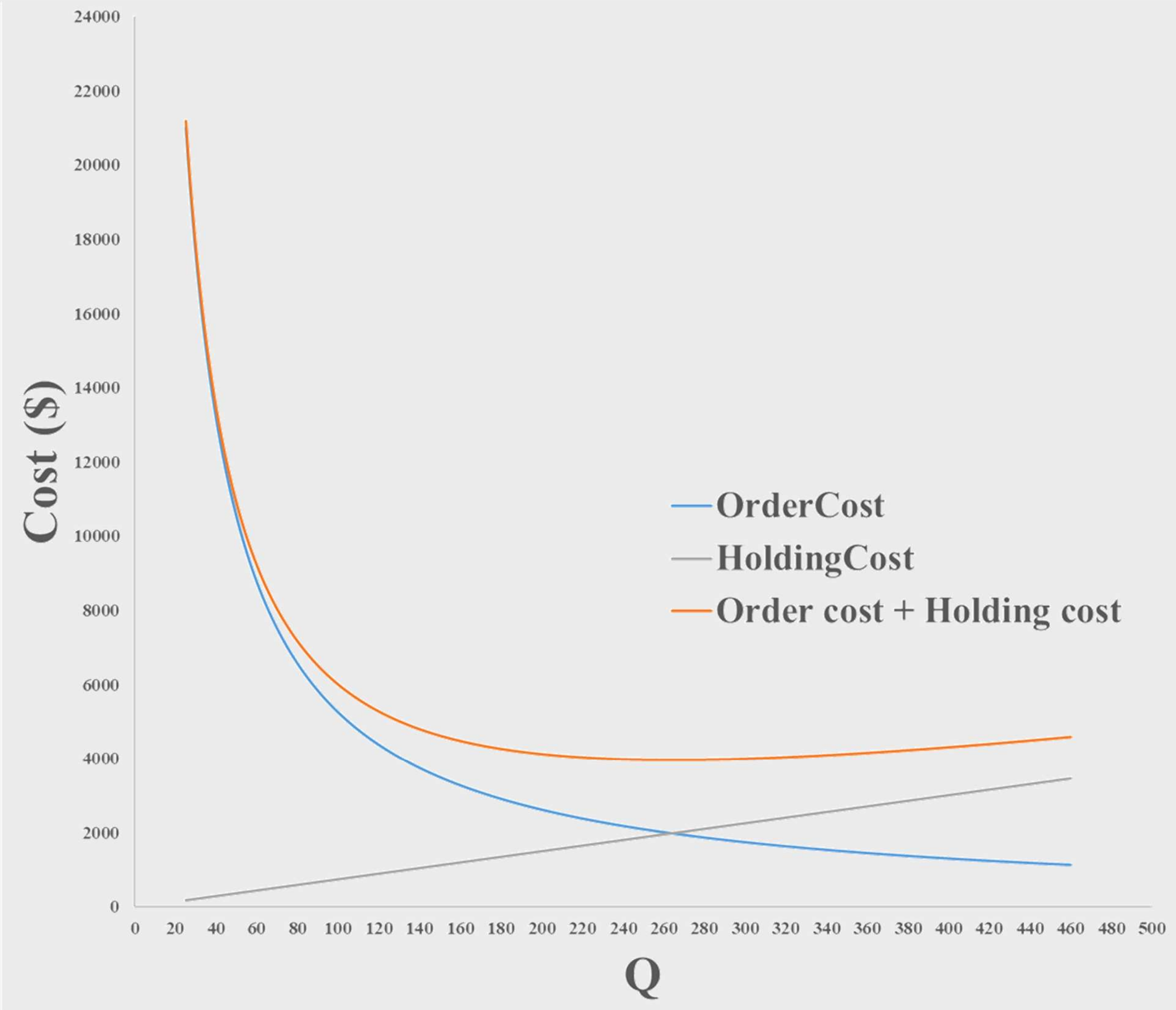
$$HoldingCost = rc(Q/2) = (0.2)(75)(Q/2) = 7.5Q\ \$/yr$$



$$TC(Q) = cD + A(D/Q) + rc(Q/2)$$

$$TC(Q) = 112500 + 525000/Q + 7.5Q$$

Gears Unlimited



Gears Unlimited

1. Determine the Objective Function

$$TC(Q) = cD + A(D/Q) + rc(Q/2)$$

$$= 112500 + 525000/Q + 7.5Q$$

2. Take the first derivative

$$f'(Q) = 0 - 525000/Q^2 + 7.5$$

3. Set the first derivative equal to zero and solve for Q^*

$$f'(Q^*) = -\,525000/Q^{*2} + 7.5 = 0$$

$$Q^{*2} = 70000 \rightarrow Q^* = 264.6 \sim 265 \text{ items/order}$$

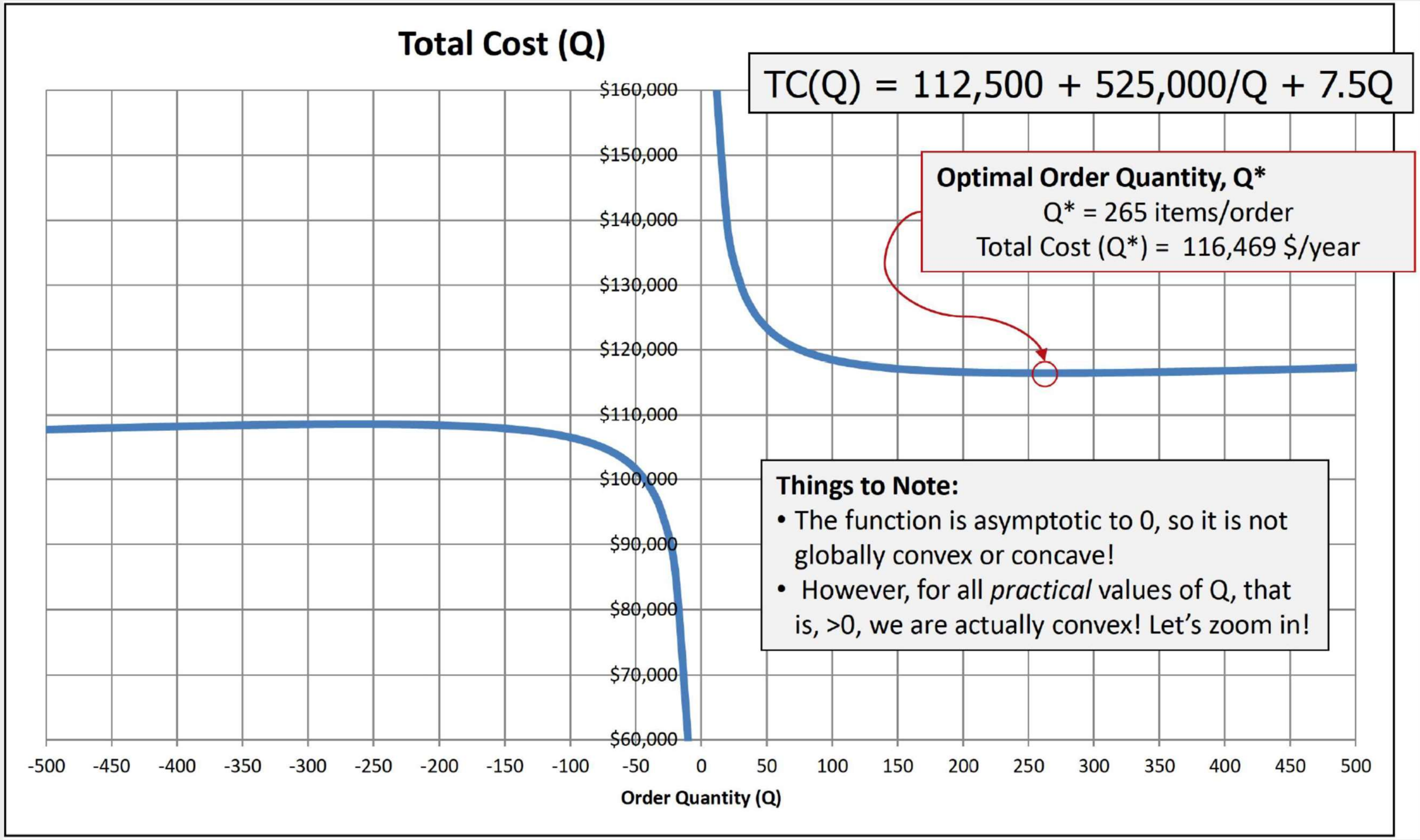
4. Check the second order conditions

$$f''(Q^*) = -\,525000(-\,2)/Q^{*3} = 1050000/\,Q^{*3} > 0$$

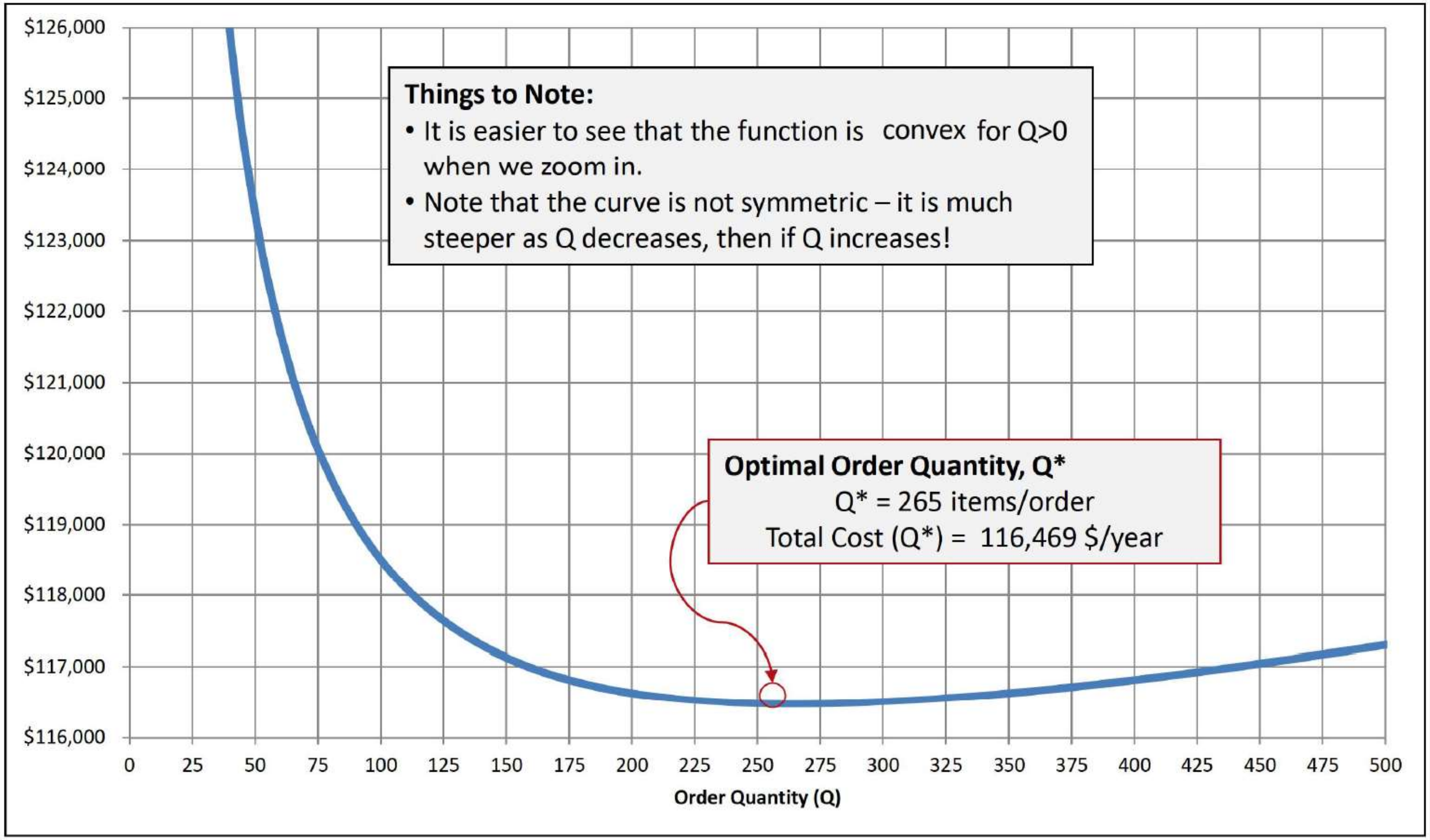
Because Q^* will always be greater than zero,

We know that this Q^* is a local minimum

Gears Unlimited



Gears Unlimited



Optimal Package Design

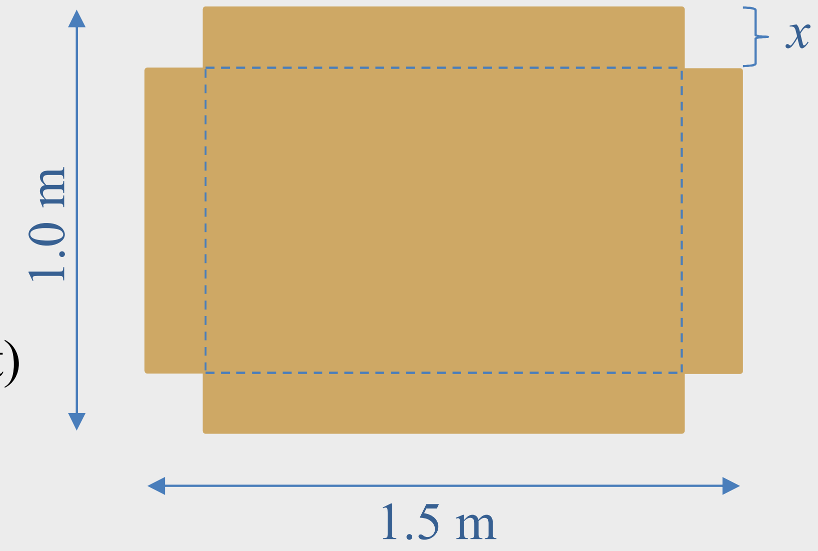
You are consulting with boxy.com, the premier online packaging company. They just received a large quantity of heavy duty cardboard from a third party at an extremely low cost. All of the sheets are 1 meter by 1.5 meters in dimension. You have been asked to come up with the design that maximizes the total volume of the box made from this sheet. The only cutting that can be made, however, are equal-sized squares from each of the four corners. The edges then fold up to form the box.

- How big should the square cut-outs be to maximize the box's volume?

Boxy.com

What do we know?

- ✓ W = Width = 1 m
- ✓ L = Length = 1.5 m
- ✓ x = Height of box (also the amount cut)



What do we want to find?

- ➢ Find x^* that maximizes Volume
- ➢ V = Volume = (Width)(Length)(Height)

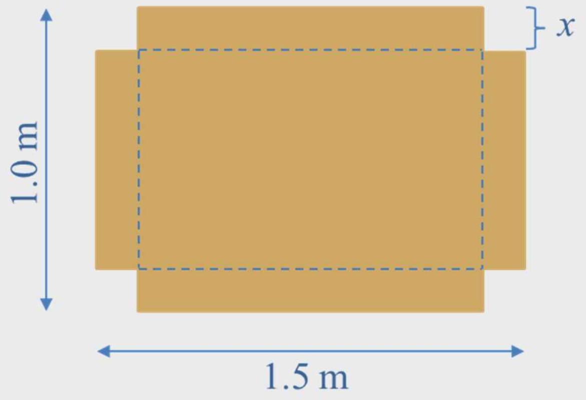
What is my objective function?

$$Max\ V = (W - 2x)(L - 2x)(x)$$

$$= (WL - 2xL - 2Wx + 4x^2)\,x = 4x^3 - 2Wx^2 - 2Lx^2 + WLx$$

$$= 4x^3 - 5x^2 + 1.5x$$

Gears Unlimited



1. Determine the Objective Function

$$V = f(x) = (W - 2x)(L - 2x)(x)$$
$$= 4x^3 - 5x^2 + 1.5x$$

2. Take the first derivative

$$f'(x) = 12x^2 - 10x + 1.5$$

3. Set the first derivative equal to zero and solve for x^*

$$f'(x^*) = 12x^2 - 10x + 1.5 = 0$$

$$r_1, r_2 = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(12)(1.5)}}{2(12)} = \frac{10 \pm \sqrt{28}}{24}$$

4. Check the second order conditions

$$f''(x^*) = 24x^* - 10 < 0$$

The function at $x^* = 0.196$ is a local maximum.

Maximum volume = 0.132 m^3

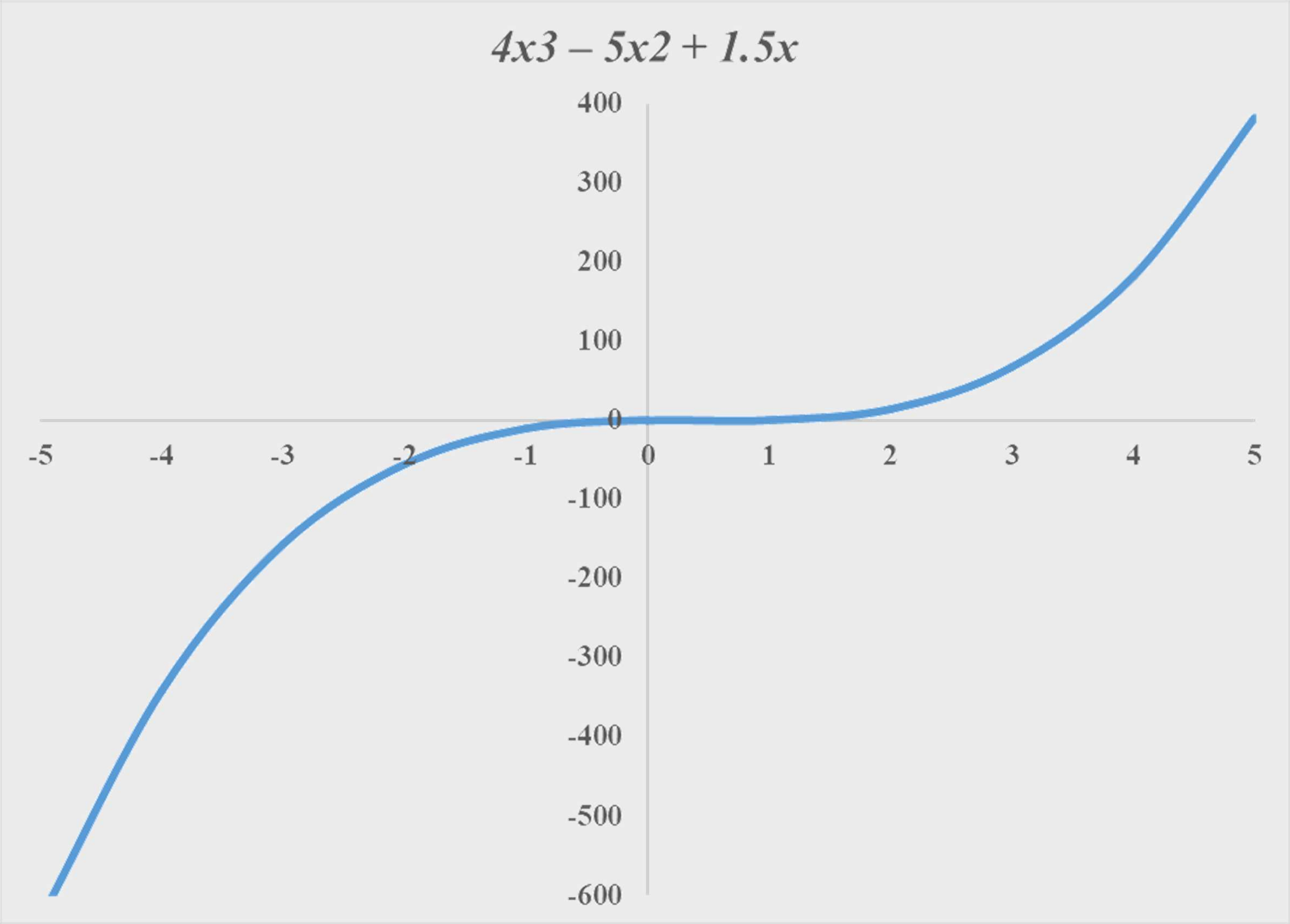
Recall…
$y = ax^2 + bx + c$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = \frac{10 + \sqrt{28}}{24} = 0.637 \text{ ✗}$$

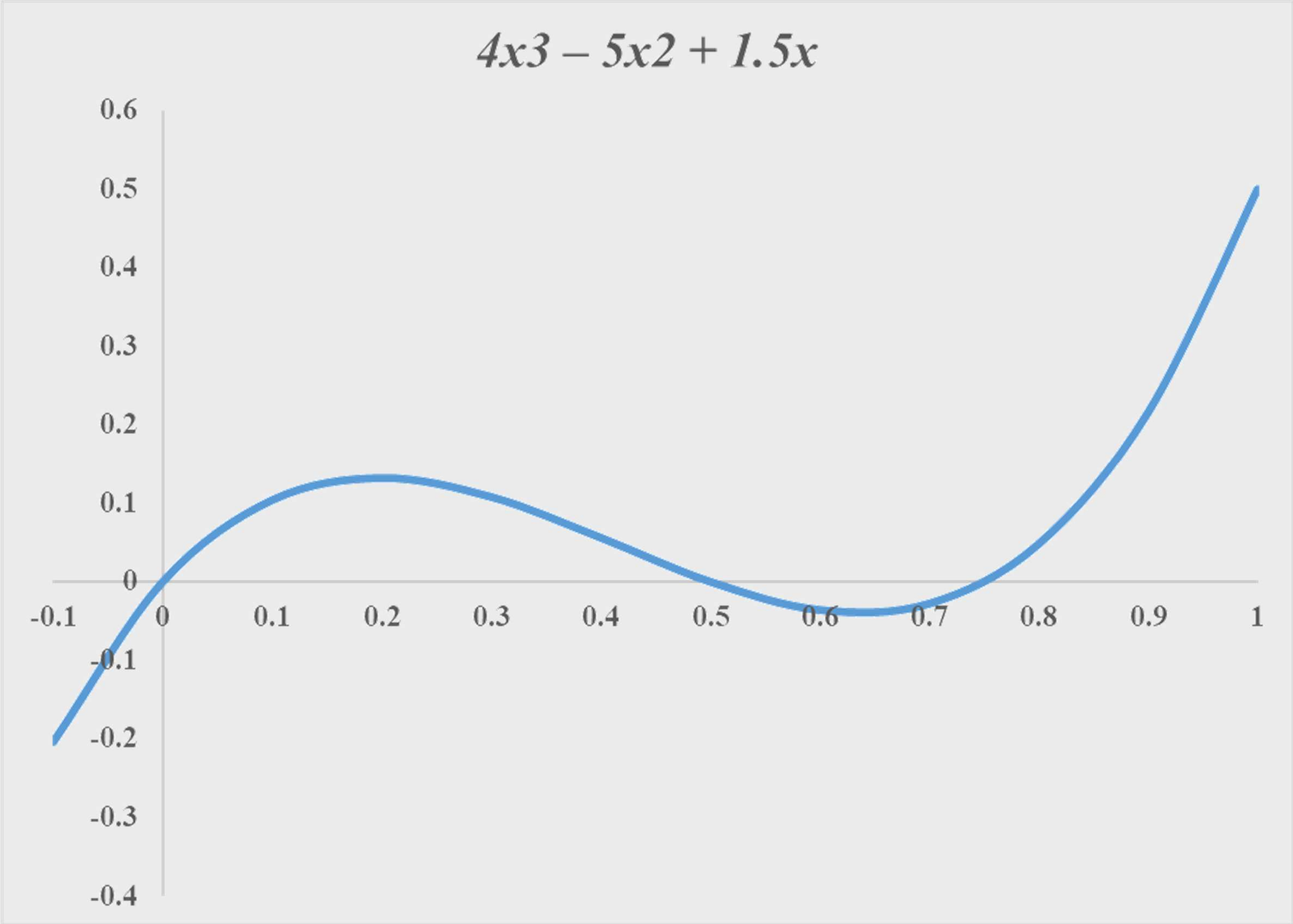
$$r_2 = \frac{10 - \sqrt{28}}{24} = 0.196 \text{ ✓}$$

Boxy.com



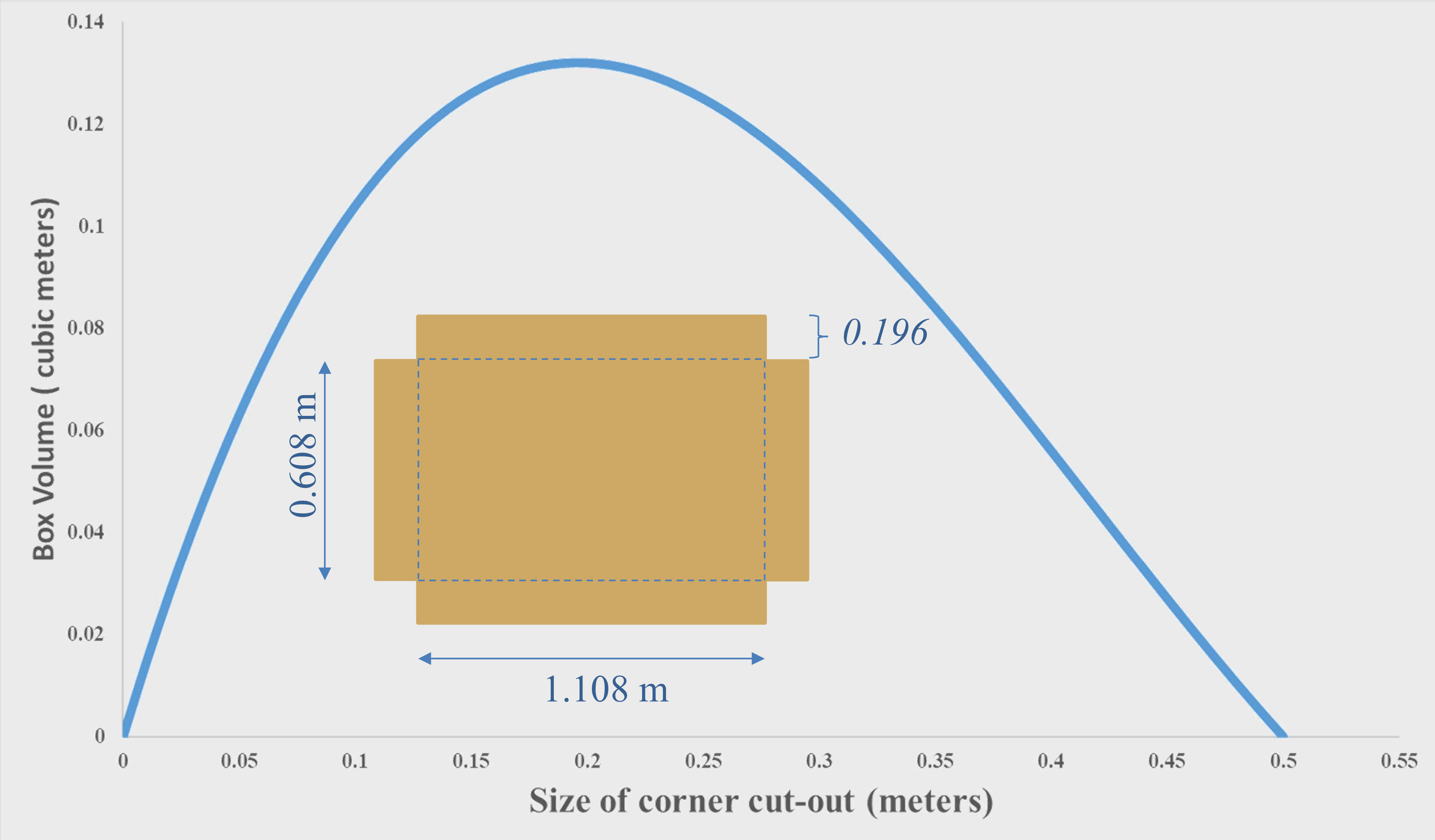
4x3 – 5x2 + 1.5x
400
300
200
100
0
-100
-200
-300
-400
-500
-600
-5
-4
-3
-2
-1
0
1
2
3
4
5

Boxy.com



4x3 – 5x2 + 1.5x
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Boxy.com


0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Box Volume (cubic meters)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
Size of corner cut-out (meters)
0.196
0.608 m
1.108 m

AMPL

- Mathematical Programming Language

- It supports many solvers
  - ✓ MINOS→linear and nonlinear programming
  - ✓ CPLEX→ linear, quadratic, and mixed integer programming
  - ✓ Gurobi → linear, quadratic, and mixed integer programming
  - ✓ Xpress → linear, quadratic, and mixed integer programming
  - ✓ …

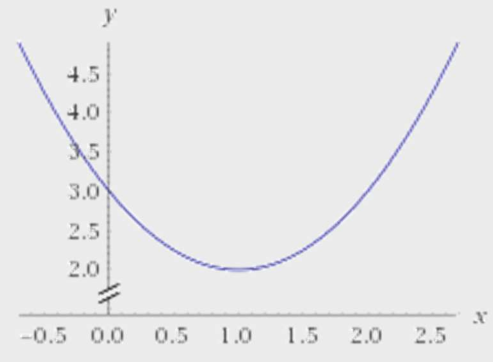
AMPL

- Syntax in AMPL
  - AMPL is very sensitive to syntax
  - Every line must end with a semi colon;
  - Upper case, Lower case
  - Commands must be given in a specified order
  - In case of error, you will have to type "reset;"

AMPL

- Simple optimization syntax
  - Specify variables
  - Specify maximize or minimize and the objective function
  - Solve
  - Display variables of interest

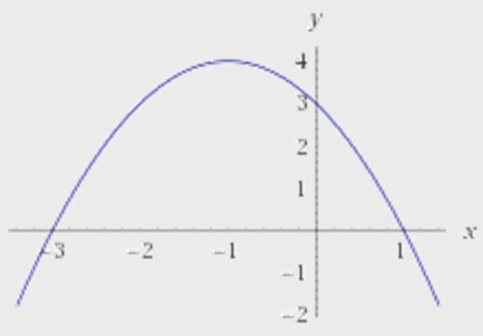
- Minimization




```
var x;
minimize y: x^2-2*x+3;
solve;
display y;
display x;
```

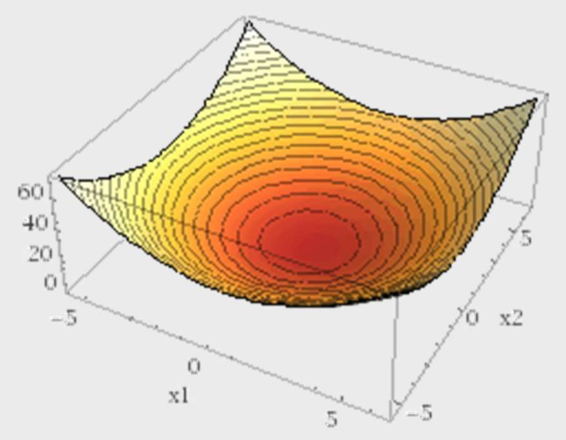
AMPL

- Maximization




```
var x;
maximize y: -x^2-2*x+3;
solve;
display y;
display x;
```

- Multiple Variable Optimization




```
var x1; var x2;
minimize y: x1^2-x1-x2+x2^2;
solve;
display y;
display x1; display x2;
```

Lecture #3:

Maximizing Profits
and
Diet Model