Modelling and Optimization

INF170

#11:Knapsack and BinPacking Problem

AHMAD HEMMATI

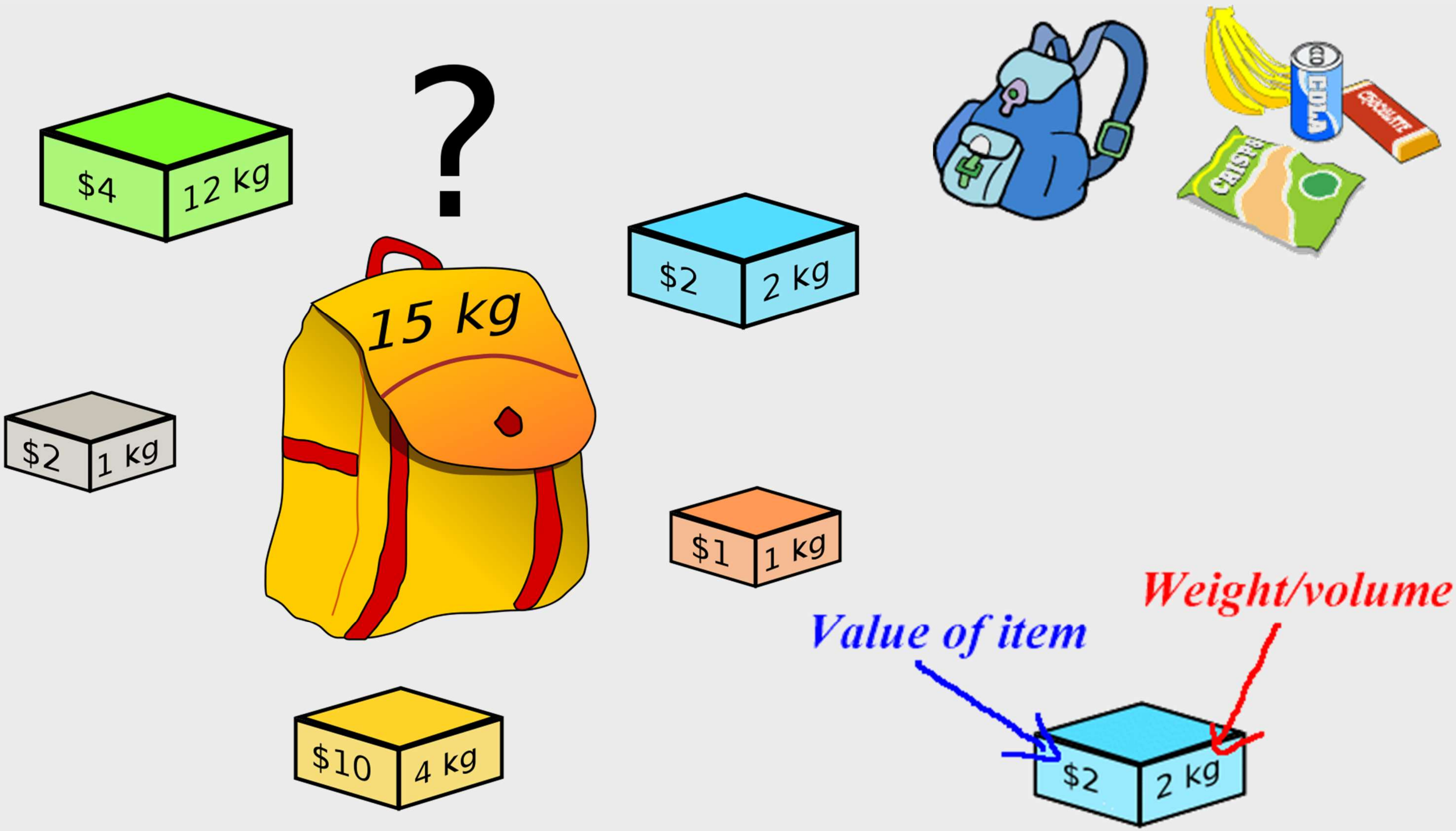
Optimization Group
Dept. of Informatics
University of Bergen

Fall Semester
2018

AGENDA

- Knapsack Problem
    - Knapsack problem with multiple resources
- Packing Problem
- Bin Packing Problem
- Loading problem

Knapsack Problem



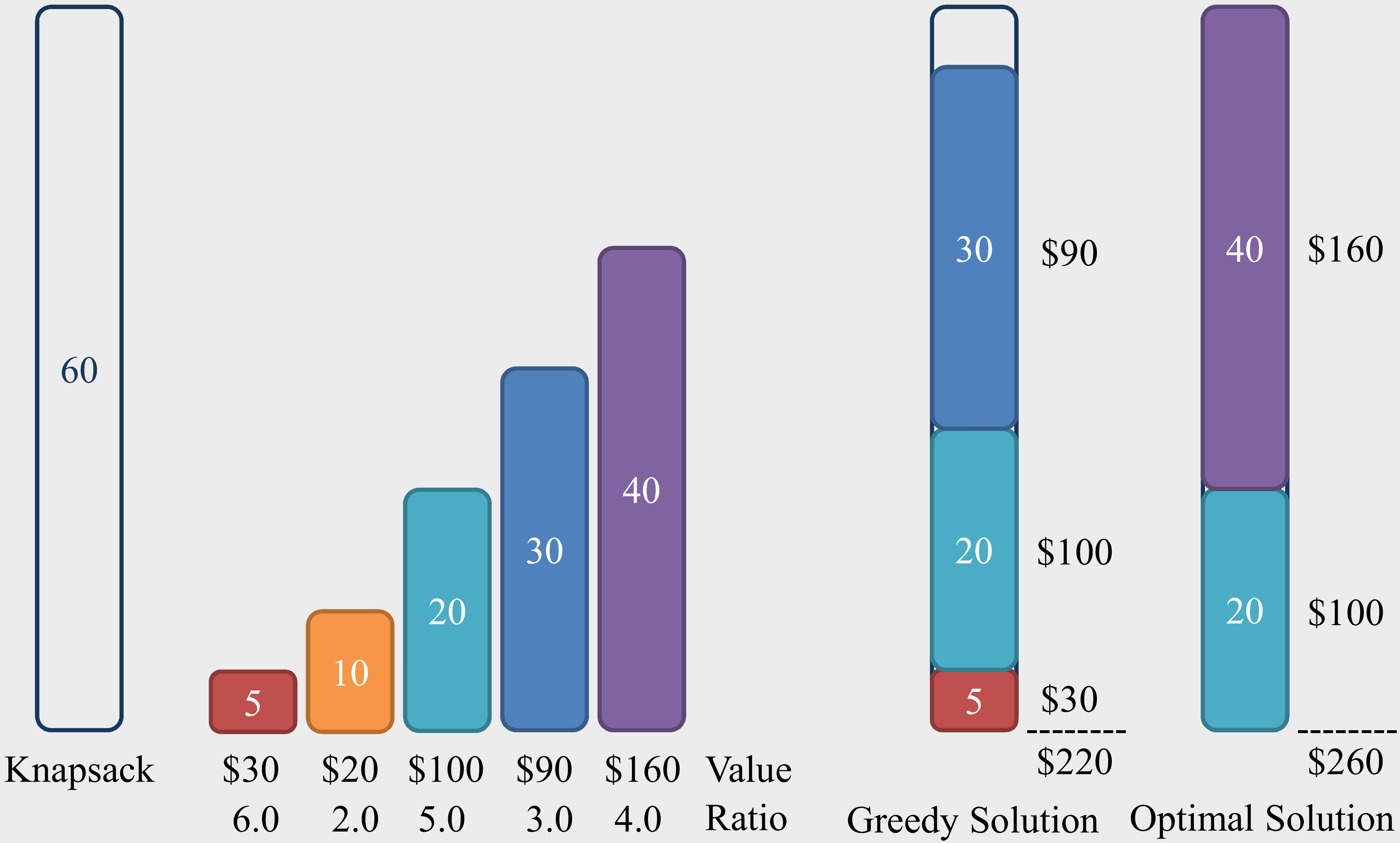
KNAPSACK PROBLEM

- Greedy approximation algorithm

- Proposed by George Dantzig

1. Sort the items in decreasing order of value per unit of weight, V_i/W_i.
2. Insert them into the sack, starting with as many copies as possible of the first kind of item until there is no longer space in the sack for more.

Knapsack Problem



Knapsack Problem

Select the "best" objects (projects, investments), given a resource constraint, such as

- weight
- budget
- time

Requirement: no shares or proportions of full objects or projects can be chosen.

Knapsack Problem

J : set of objects

p_j : benefit added by object j if it is chosen

a_j : weight of object j (amount of resource it uses)

b : maximum weight (or available resource)

Decision variables

For each object j we define a binary variable:

$$x_j = \begin{cases} 1 & \text{if object } j \text{ is } \textit{chosen} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \dots, n \text{ (for } j \in J)$$

Objective function:

Maximize total benefit

$$max \sum_{j \in J} p_j \, x_j$$

Constraints:

Maximum weight the knapsack can hold

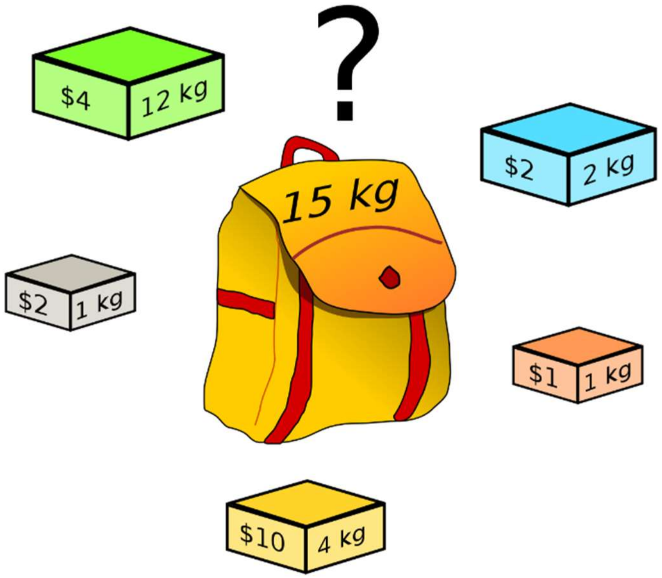
$$\sum_{j \in J} a_j x_j \leq b$$

Binary nature of the variables

$$x_j \in \{0,1\} \qquad \text{for } j \in J$$

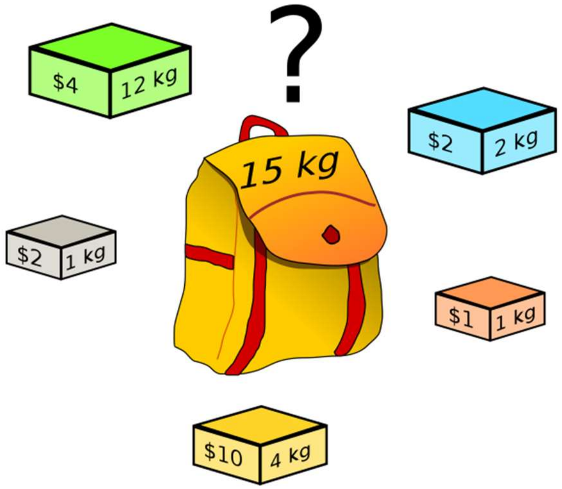
Knapsack Problem

```
set Objects;
param Weight {Objects} > 0;
param Value {Objects} > 0;
param Capacity > 0;
var x {Objects} binary;
maximize Total_Value:
sum {i in Objects} Value[i] * x[i];
subject to Heaviness:
sum {i in Objects} Weight[i] * x[i] <= Capacity;
```



Knapsack Problem

```
set Objects:= Green Blue Orange Yellow Gray;
param Capacity:= 15 ;
param :        Weight Value :=
      Green    12     4
      Blue     2      2
      Orange   1      1
      Yellow   4      10
      Gray     1      2;
```




```
solve; display x;
```

```
Green  = 0
Blue   = 1
Orange = 1
Yellow = 1
Gray   = 1
```

```
objective = 15
```

Knapsack Problem

- A company is considering four alternatives of investments. Each alternative requires a certain cash outflow at the present time and yields a net present value (NPV), given in the table below.

	Alternative 1	Alternative 2	Alternative 3	Alternative 4
NPV	16,000	22,000	12,000	8,000
I_0	5,000	7,000	4,000	3,000

- Currently, \$ 14 000 are available for investment
- Which alternatives should be chosen such that total NPV is maximized?

Knapsack Problem

Decision variables

$$x_j = \begin{cases} 1 & \text{if investment on alternative } j \text{ is } \textit{chosen} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \dots, 4$$

	Alternative 1	Alternative 2	Alternative 3	Alternative 4
NPV	16,000	22,000	12,000	8,000
I_0	5,000	7,000	4,000	3,000

$$max\ z = 16\,x_1 + 22\,x_2 + 12\,x_3 + 8\,x_4$$

s.t.

$$5\,x_1 + 7\,x_2 + 4\,x_3 + 3\,x_4 \leq 14$$

$$x_1, \dots, x_4 \in \{0,1\}$$

Knapsack Problem

- Optimal IP solution: (0,1,1,1), that is, the investment on all alternatives is made except for alternative 1.
- Total NPV = 42,000

- Optimal LP solution (relax binary condition to $0 \leq x \leq 1$): (1,1,0.5,0).
- Total NPV = 44,000

Note the rounded solution is infeasible!

Knapsack problem with multiple resources

- In many application applications there are more than one resource constraint.
- We can still use the same variables but we need to refine the coefficients to handle several resources

Knapsack Problem

J : set of objects

I : set of resources

p_j : benefit added by object j if it is chosen

a_{ij} : amount of resource i used by object j

b_i : availability of resource i

Decision variables

For each object j we define a binary variable:

$$x_j = \begin{cases} 1 & \text{if object } j \text{ is } \textit{chosen} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \dots, n \text{ (for } j \in J)$$

Objective function:

Maximize total benefit

$$max \sum_{j \in J} p_j \, x_j$$

Constraints:

Maximum availability of each resource

$$\sum_{j \in J} a_{ij} x_j \leq b_i \qquad \text{for } \; i \in I$$

Binary nature of the variables

$$x_j \in \{0,1\} \qquad\qquad \text{for } j \in J$$

Knapsack Problem

Investment with logical constraints:

- The company optimalInvest is to decide which building project to cary out during the next year.
- Each project requires a number of staff members for project management and a given sum of funding.
- Some projects can not be carried out together and some require that other projects are carried out.
- There is a limited budget of 225 million NOK and there are 28 staff members available.
- No more than 9 projects can be funded

Knapsack Problem

No.	Object	Value (kNOK)	Budget (MNOK)	Staff	Not with	Require also
1	Ice/hockey arena	600	35	5	10	
2	Sports arena	400	34	3		
3	Hotel	100	26	4		15
4	Restaurant	150	12	2		15
5	Office A	80	10	2	6	
6	Office B	120	18	2	5	
7	School	200	32	4		
8	Daycare center	220	11	1		7
9	Storage	90	10	1		
10	Swimming pool	380	22	5	1	
11	Apartment house	290	27	3	15	
12	Car garage	130	18	2		
13	Tennis arena	80	16	2		2
14	Track & field	270	29	4		2
15	Boat harbour	280	22	3	11	

Knapsack Problem

Decision variables

$$x_j = \begin{cases} 1 & \text{if project } j \text{ is } \mathit{chosen} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \dots, 15$$

We have three resource limitations:

- Budget
- Staff
- Number of funded projects

We also logical restrictions:

- Some projects can not be carried out at the same time
- Some projects must be funded in order to fund another project

Knapsack Problem

- Some projects can not be carried out at the same time

$$x_i + x_j \leq 1$$

- Some projects must be funded in order to fund another project

$$x_j \leq x_i$$

KNAPSACK PROBLEM

$$\max z = 600\,x_1 + 400\,x_2 + \ldots + 270\,x_{14} + 280\,x_{15}$$

s.t.

$$35\,x_1 + 34\,x_2 + \ldots + 29\,x_{14} + 22\,x_{15} \leq 225 \qquad \text{(budget)}$$

$$5\,x_1 + 3\,x_2 + \ldots + 4\,x_{14} + 3\,x_{15} \leq 28 \qquad \text{(staff)}$$

$$x_1 + x_2 + \ldots + x_{14} + x_{15} \leq 9 \qquad \text{(number of projects)}$$

$$x_1 + x_{10} \leq 1 \qquad \text{(not both 1 and 10)}$$

$$x_5 + x_6 \leq 1 \qquad \text{(not both 5 and 6)}$$

$$x_{11} + x_{15} \leq 1 \qquad \text{(not both 11 and 15)}$$

$$x_3 - x_{15} \leq 0 \qquad \text{(3 requires 15)}$$

$$x_4 - x_{15} \leq 0 \qquad \text{(4 requires 15)}$$

$$x_8 - x_7 \leq 0 \qquad \text{(8 requires 7)}$$

$$x_{13} - x_2 \leq 0 \qquad \text{(13 requires 2)}$$

$$x_{14} - x_2 \leq 0 \qquad \text{(14 requires 2)}$$

$$x_1, \ldots, x_{15} \in \{0,1\}$$

The optimal solution:

$$x_1 , x_2 , x_4 , x_6 , x_7 , x_8 , x_{12} , x_{14} , x_{15} = 1$$

$$x_3 , x_5 , x_9 , x_{10} , x_{11} , x_{13} = 0$$

The optimal profit is

$$z^* = 2370 \text{ kNOK}$$

Packing Problem

- Three railway wagons with a carrying capacity of 100 quintals (1 quintal = 100 kg) have been reserved to transport sixteen boxes.
- The weight of the boxes in quintals is given.

Box	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Weight	34	6	8	17	16	5	13	21
Box	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Weight	25	31	14	13	33	9	25	25

Packing Problem

Box	1	2	3	4	5	6	7	8
Weight	34	6	8	17	16	5	13	21
Box	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Weight	25	31	14	13	33	9	25	25

- How shall the boxes be assigned to the wagons in order to
  - keep to the limits on the maximum carrying capacity and to
  - minimize the heaviest wagon load?

Packing Problem

- Decision variables?

$$x_{b,w} = \begin{cases} 1 & \text{if box } b \text{ is assigned to wagon } w \\ 0 & \text{otherwise} \end{cases}$$

- Objective: minimize the the maximum weight over all the wagon loads

y : represent the maximum weight over all the wagon loads

$$minimize\ y$$

Packing Problem

$$minimize\ y$$

s.t.

$$\sum_{b=1}^{16} W_b x_{bw} \leq y \quad \forall w \in \{1,2,3\}$$

$$\sum_{w=1}^{3} x_{bw} = 1 \quad \forall b \in \{1,\ldots,16\}$$

$$x_{bw} \in \{0,1\} \quad \forall\ b \in \{1,\ldots,16\}, w \in \{1,2,3\}$$

$$y \geq 0$$

Bin Packing Problem

- The one dimensional bin packing problem is defined as follows. Given a set $L=\{1,\ldots, n\}$ of items and theirs weights $w_i \in (0,1)$ $i \in L$.

- We wish to partition the set L into minimal number m of subsets B_1, B_2, …, B_m in such a way that

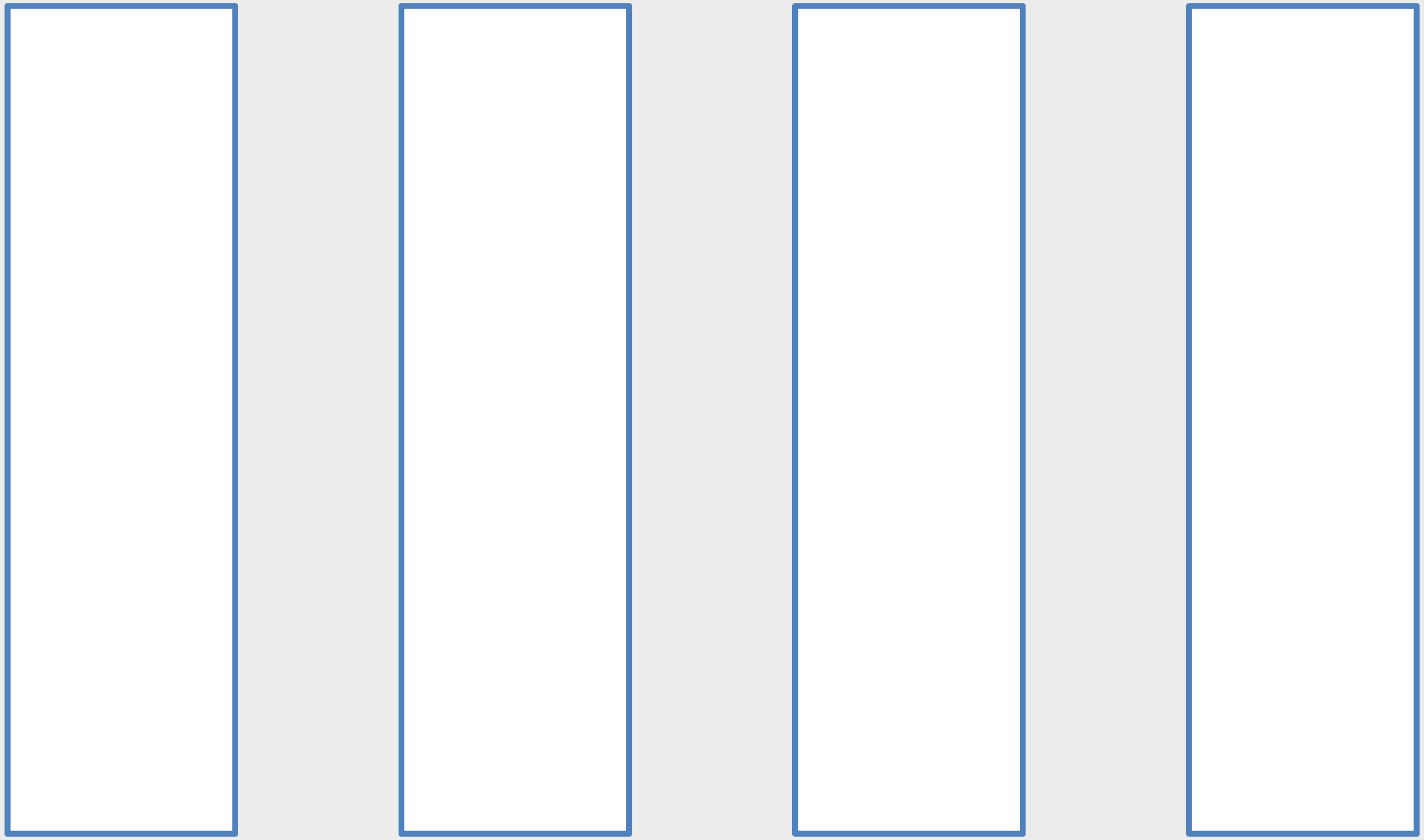
$$\sum_{i \in B_j} w_i \leq 1 \qquad 1 \leq j \leq m$$

- The sets B_j , we call **bins**.

- In other words, we wish to pack all items in a minimal number of bins.

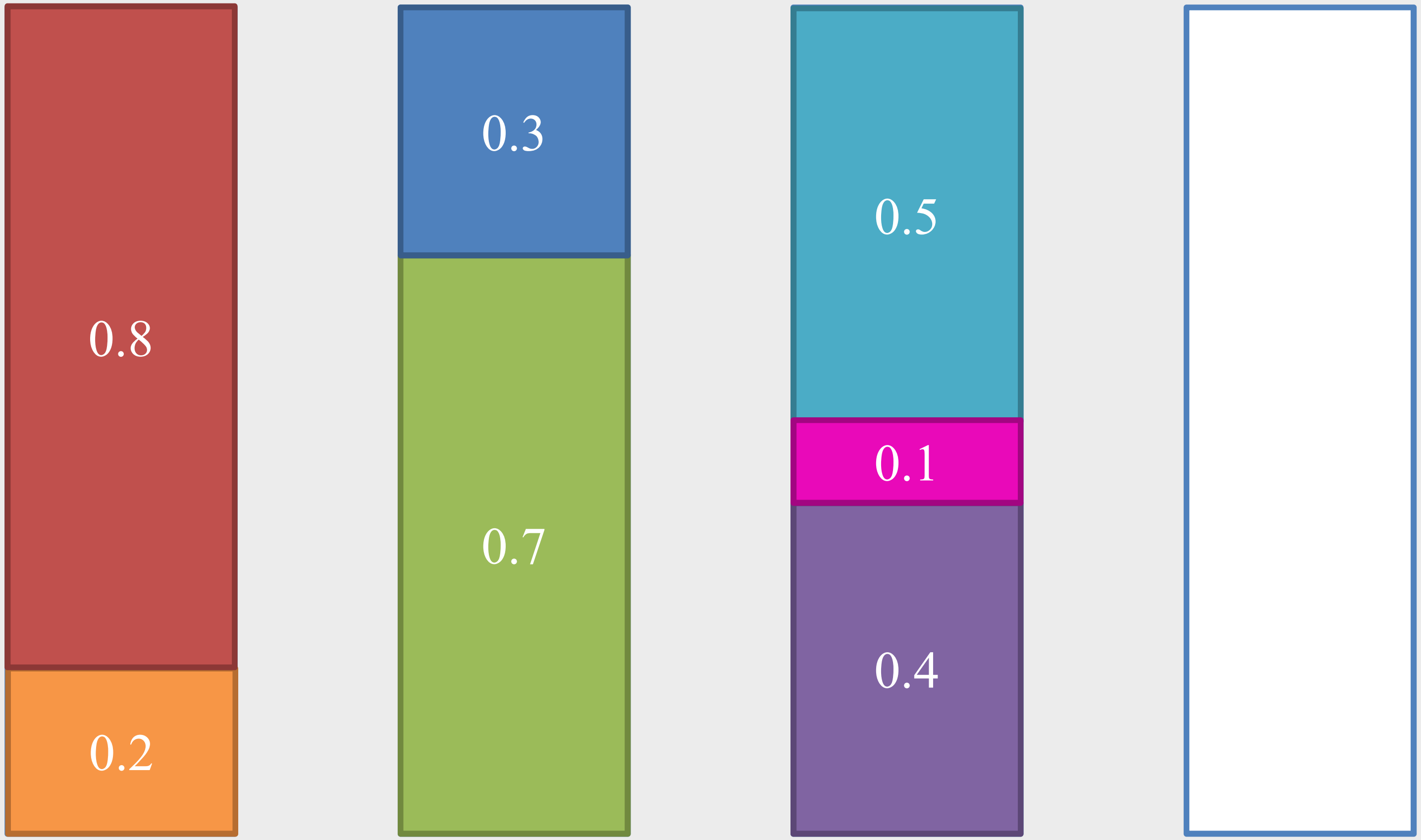
Bin Packing Problem

- 7 items
- Weights: 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8



Bin Packing Problem

- 7 items
- Weights: 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8



Bin Packing Problem

- Decision variables?

$$x_{i,j} = \begin{cases} 1 & \text{if item } j \text{ is in bin } i \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if bin } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

- Objective:

$$minimize \sum_{i=1}^{n} y_i$$

Bin Packing Problem

$$minimize \sum_{i=1}^{n} y_i$$

s.t.

$$\sum_{j=1}^{n} a_j x_{ij} \leq V y_i \quad \forall i \in \{1, \dots, n\}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \in \{1, \dots, n\}$$

$$x_{ij} \in \{0,1\} \quad \forall\, i, j \in \{1, \dots, n\}$$

$$y_i \in \{0,1\} \quad \forall\, i \in \{1, \dots, n\}$$

Backing up files!

- Before leaving on holiday, you wish to backup your most important files onto floppy disks.
- You have got empty disks of 1.44Mb capacity.
- The sixteen files you would like to save have the following sizes: 46kb, 55kb, 62kb, 87kb, 108kb, 114kb, 137kb, 164kb, 253kb, 364kb, 372kb, 388kb, 406kb, 432kb, 461kb, and 851kb.
- Assuming that you do not have any program at hand to compress the files and that you have got a sufficient number of floppy disks to save everything, how should the files be distributed in order to minimize the number of floppy disks used?

Bin Packing Problem

- Decision variables?

$$x_{i,j} = \begin{cases} 1 & \text{if file } j \text{ is in floppy disk } i \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if floppy disk } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

- Objective:

$$minimize \sum_{i=1}^{16} y_i$$

Bin Packing Problem

$$minimize \sum_{i=1}^{16} y_i$$

s.t.

$$\sum_{j=1}^{16} a_j x_{ij} \leq V y_i \quad \forall i \in \{1, \dots, 16\}$$

a_j is the size of the file j

$$\sum_{i=1}^{16} x_{ij} = 1 \quad \forall j \in \{1, \dots, 16\}$$

$$x_{ij} \in \{0,1\} \quad \forall\, i, j \in \{1, \dots, 16\}$$

$$y_i \in \{0,1\} \quad \forall\, i \in \{1, \dots, 16\}$$

Bin Packing Problem

$$minimize\ y$$

s.t.

$$\sum_{i=1}^{n} i\, x_{ij} \leq y \quad \forall j \in \{1, \dots, n\}$$

$$\sum_{j=1}^{n} a_j x_{ij} \leq V \quad \forall i \in \{1, \dots, n\}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \in \{1, \dots, n\}$$

$$x_{ij} \in \{0,1\} \quad \forall\, i, j \in \{1, \dots, n\}$$

$$y \geq 0$$

Bin Packing Problem

- In the optimal solution, three disks are used. The files may be distributed to the disks as shown in the following table (there are several possible solutions).

Disk	File sizes (in kb)	Used space (in Mb)
1	46 87 137 164 253 364 388	1.439
2	55 62 108 372 406 432	1.435
3	114 461 851	1.426

Loading Problem

- Five tanker ships have arrived at a chemical factory.
- They are carrying loads of liquid products that must not be mixed:
  - 1200 tonnes of Benzol,
  - 700 tonnes of Butanol,
  - 1000 tonnes of Propanol,
  - 450 tonnes of Styrene
  - 1200 tonnes of THF

Loading Problem

- Nine tanks of different capacities are available on site.
- Some of them are already partially filled with a liquid.
- The characteristics of the tanks (in tonnes) is given.

Tank	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Capacity	500	400	400	600	600	900	800	800	800
Current product	–	Benzol	–	–	–	–	THF	–	–
Quantity	0	100	0	0	0	0	300	0	0

- Into which tanks should the ships be unloaded

(Q1) to maximize the capacity of the tanks that remain unused

(Q2) to maximize the number of tanks that remain free?

Loading Problem

- Decision variables?

$$x_{l,t} = \begin{cases} 1 & \text{if liquid } l \text{ is unloaded into the tank } t \\ 0 & \text{otherwise} \end{cases}$$

- R_l : the quantity of liquid product l that remain once the capacities of the partially filled tanks are exhausted

Tank	1	2	3	4	5	6	7	8	9
Capacity	500	400	400	600	600	900	800	800	800
Current product	–	Benzol	–	–	–	–	THF	–	–
Quantity	0	100	0	0	0	0	300	0	0

Loading Problem

$$minimize \sum_{l \in L} \sum_{\substack{t \in T \\ Q=0}} C_t x_{lt}$$

s.t.

$$\sum_{\substack{t \in T \\ Q=0}} C_t x_{lt} \geq R_l \quad \forall l \in L$$

$$\sum_{l \in T} x_{lt} \leq 1 \quad \forall t \in T$$

$$x_{lt} \in \{0,1\} \quad \forall\, l \in L\,, t \in T, Q_t = 0$$

Loading Problem

$$minimize \sum_{l \in L} \sum_{\substack{t \in T \\ Q=0}} x_{lt}$$

s.t.

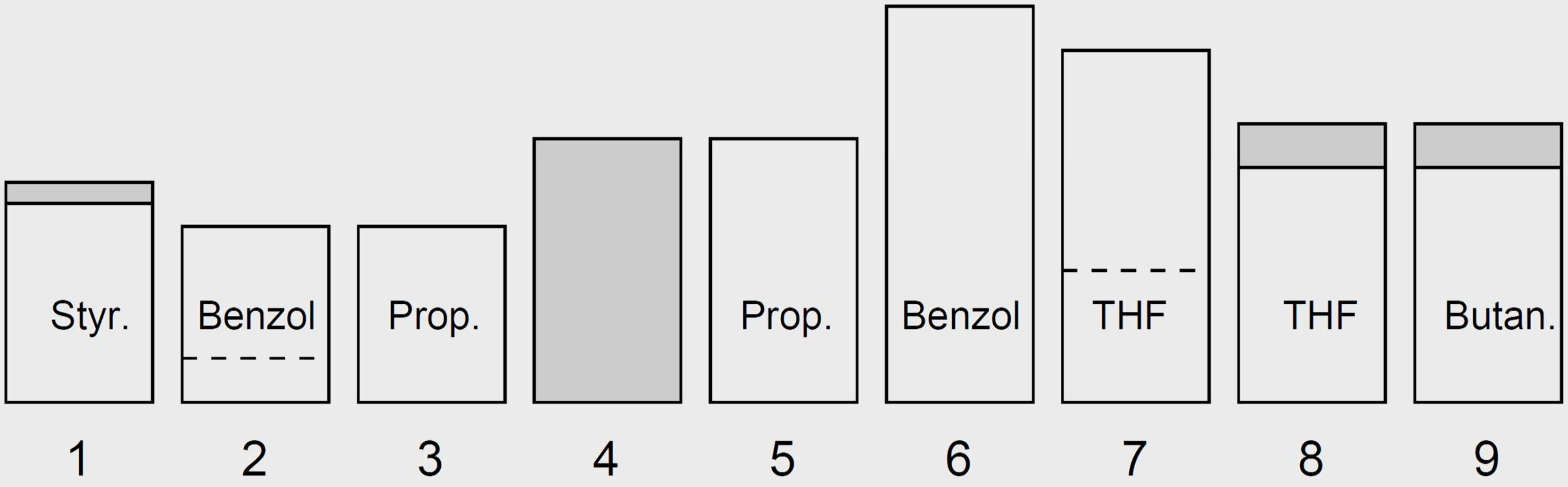
$$\sum_{\substack{t \in T \\ Q=0}} C_t x_{lt} \geq R_l \quad \forall l \in L$$

$$\sum_{l \in T} x_{lt} \leq 1 \quad \forall t \in T$$

$$x_{lt} \in \{0,1\} \quad \forall \, l \in L \,, t \in T, Q_t = 0$$

Loading Problem

Product	Tanks	Remaining capacity
Benzol	2, 6	0
Butanol	9	100
Propanol	3, 5	0
Styrene	1	50
THF	7, 8	100



Assignment #6:

AMPL Book
Chapter 20. Exercises(1-5)

LECTURE #12:

TRAVELLING SALESMAN PROBLEM