Modelling and Optimization

INF170

#9:IP/MILP models

Ahmad Hemmati

Optimization Group
Dept. of Informatics
University of Bergen

Fall Semester
2018

AGENDA

- Parking problem!
- Matching Problem
  - Personnel assignment
- Graph Coloring Problem
  - Exam scheduling
- Cutting-Stock Problem

Power of Binary Variables – Recap

$$x = \begin{cases} 1 & \text{if I get coffee} \\ 0 & \text{otherwise} \end{cases} \qquad y = \begin{cases} 1 & \text{if I get waffle} \\ 0 & \text{otherwise} \end{cases}$$

I want…

coffee and waffle	$x = 1\ , y = 1 \ \rightarrow x + y = 2$
coffee and/or waffle	$x + y \geq 1$
either coffee or waffle	$x + y = 1$
either coffee or waffle or nothing	$x + y \leq 1$
either both or nothing	$x = y$
if coffee, then waffle	$x \leq y$
if coffee then do not want waffle	$y \leq 1 - x$

Parking Problem

You are organizing a party for around 30 people, who will arrive in 15 cars. Cars can be parked on both sides of the street. The length of the i-th car is λ_i, expressed in meters as follows:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
λ_i	4	4.5	5	4.1	2.4	5.2	3.7	3.5	3.2	4.5	2.3	3.3	3.8	4.6	3

In order to avoid bothering the neighbours, you want to arrange the parking on both sides of the street so that the length of the street occupied by your friends' cars should be minimum.

Parking Problem

Indices:

- i: index on the set $N = \{1, ..., n\}$ of cars
- j: index on the set $M = \{1, 2\}$ of car lines (one per street side)

Parameters:

- λ_i : length of car i

Parking Problem

Variables:

$x_{ij} = 1$ if i is parked on line j and 0 otherwise;

$t_j \geq 0$: length of car line j;

Objective function:

$$Min\ Max_{j \in M}\ t_j$$

Parking Problem

Constraints:

- Car line length definition $\forall j \in M$ $t_j = \sum_{i \in N} \lambda_i x_{ij}$

- Assignment of cars to lines $\forall i \in N$ $\sum_{j \in M} x_{ij} = 1$

Model:

$$Min\ t$$

$$\forall j \in M \qquad t \geq t_j$$

$$\forall j \in M \qquad t_j = \sum_{i \in N} \lambda_i x_{ij}$$

$$\forall i \in N \qquad \sum_{j \in M} x_{ij} = 1$$

Parking Problem

How does the model change if on exactly one of the street sides the cars should not occupy more than 15m?

Parking Problem

Indices:

– i: index on the set $N = \{1, \ldots, n\}$ of cars

– j: index on the set $M = \{1, 2\}$ of car lines (one per street side)

Parameters:

– λ_i : length of car i

– L: upper bound on the car line length

– μ: upper bound for the sum of car lengths

Parking Problem

Variables:

$x_{ij} = 1$ if i is parked on line j and 0 otherwise;

$t_j \geq 0$: length of car line j;

$y_j = 1$ if $t_j \leq L$ and 0 otherwise.

Objective function:

$$Min\ Max_{j \in M}\ t_j$$

Parking Problem

Constraints:

- Car line length definition $\forall j \in M$ $t_j = \sum_{i \in N} \lambda_i x_{ij}$
- Assignment of cars to lines $\forall i \in N$ $\sum_{j \in M} x_{ij} = 1$
- Constraint disjunction $\forall j \in M$ $t_j - L \leq \mu(1 - y_j)$
- Constraint on one line only $\sum_{j \in M} y_j = 1$

Parking Problem

Model:

$$Min\ t$$

$$\forall j \in M \qquad t \geq t_j$$

$$\forall j \in M \qquad t_j = \sum_{i \in N} \lambda_i x_{ij}$$

$$\forall i \in N \qquad \sum_{j \in M} x_{ij} = 1$$

$$\forall j \in M \qquad t_j - L \leq \mu(1 - y_j)$$

$$\sum_{j \in M} y_j = 1$$

Personnel Assignment

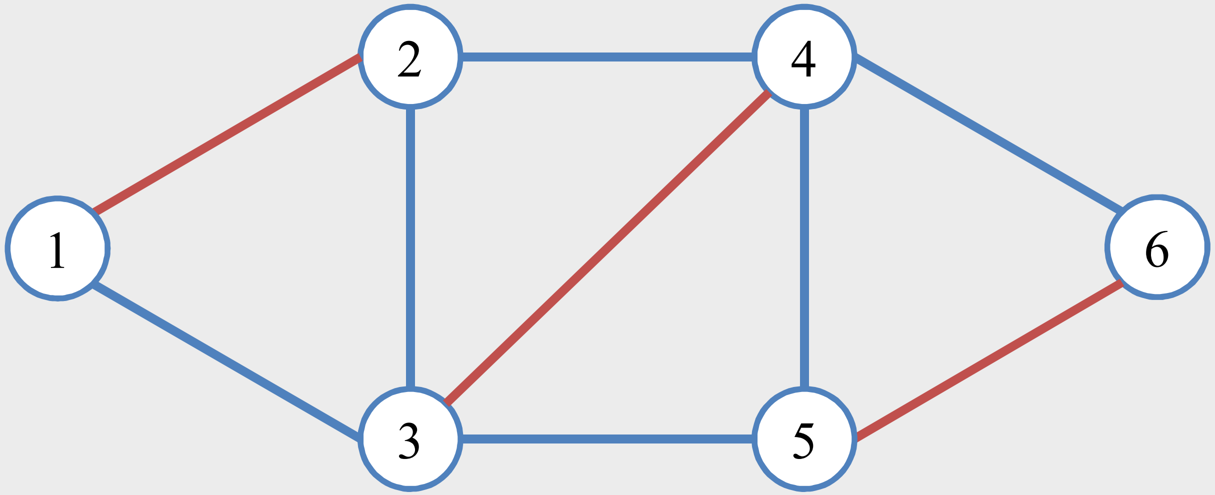
- During World War II, the Royal Air Force (RAF) of Britain contained many pilots from foreign countries who spoke different languages and had different levels of training

- The RAF needed to assign two pilots to each plane, always assigning airing pilots with compatible languages and training

- The RAF wanted to fly as many planes as possible

- How to model this problem mathematically?

Matching on Graphs



- Undirected graph (N, E)
- A **matching** in (N, E) is a set of edges such that <u>no two edges share a common node</u>
- Think of
  - edges of (N, E) as possible pairings
  - matching of (N, E) as a set of feasible pairings

Personnel Assignment

- Relationship between RAF's problem and matchings?

- Think of

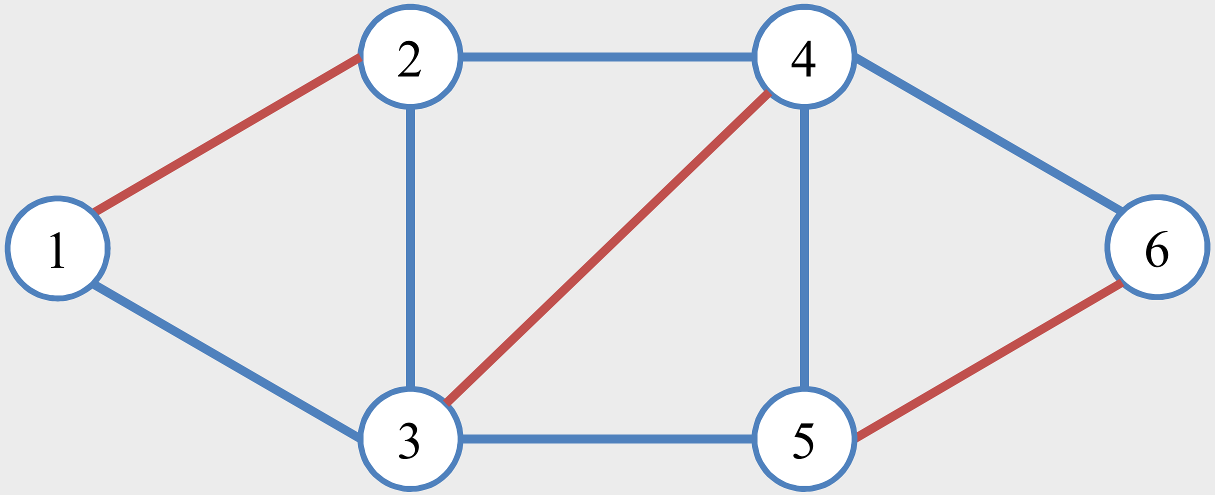
  ✓ pilots = nodes in a graph

  ✓ compatibility between pilots = edge between nodes

- Want to pair up as many compatible pilots as possible

  ↔ Want to find a matching with as many edges as possible

Matching on Graphs



- A matching is a set of <u>edges</u> such that no two edges share a common node
- Decision variables:

$$x_{i,j} = \begin{cases} 1 & \text{if edge } \{i, j\} \text{ is in the matching} \\ 0 & \text{otherwise} \end{cases}$$

- Constraints: for each node, make sure there is at most one adjacent edge selected

$$\sum_{\{i,k\} \in E} x_{i,k} \leq 1 \quad \text{for each node } i \in N$$

Maximum Matching Problem

- The model:

$$max \sum_{\{i,j\}\in E} x_{i,j}$$

$$s.t. \quad \sum_{\{i,k\}\in E} x_{i,k} \leq 1 \quad \text{for each node } i \in N$$

$$x_{i,k} \in \{0,1\} \quad \text{for each } \{i,j\} \in E$$

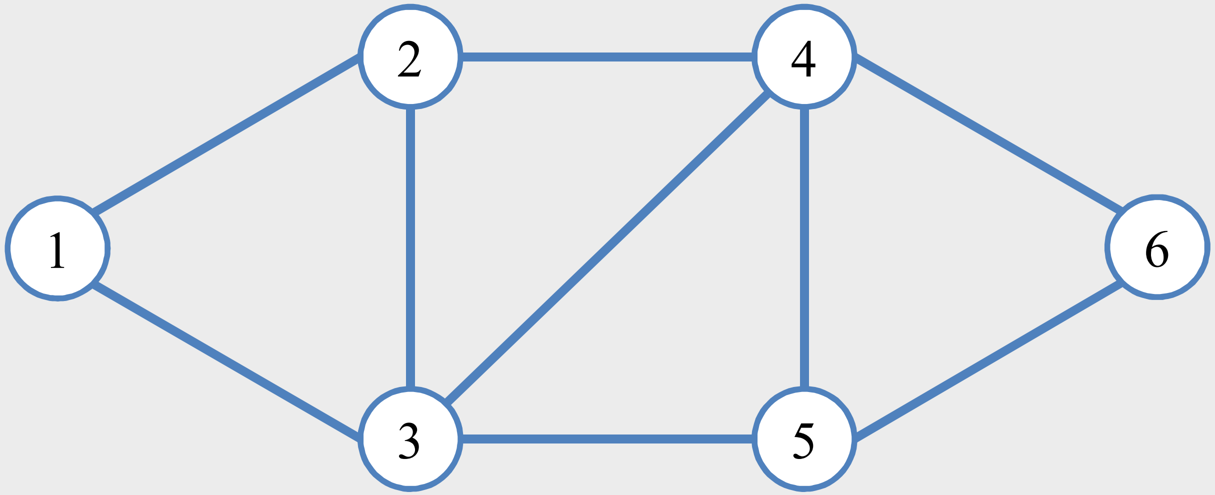
- Variant: suppose each edge $\{i\ ,j\}\in E$ has an associated weight $w_{i,j}$
  - e.g. numerical compatibility ratings between pilots
  - Replace objective function with

$$\sum_{\{i,j\}\in E} w_{i,j} x_{i,j}$$

Exam Scheduling

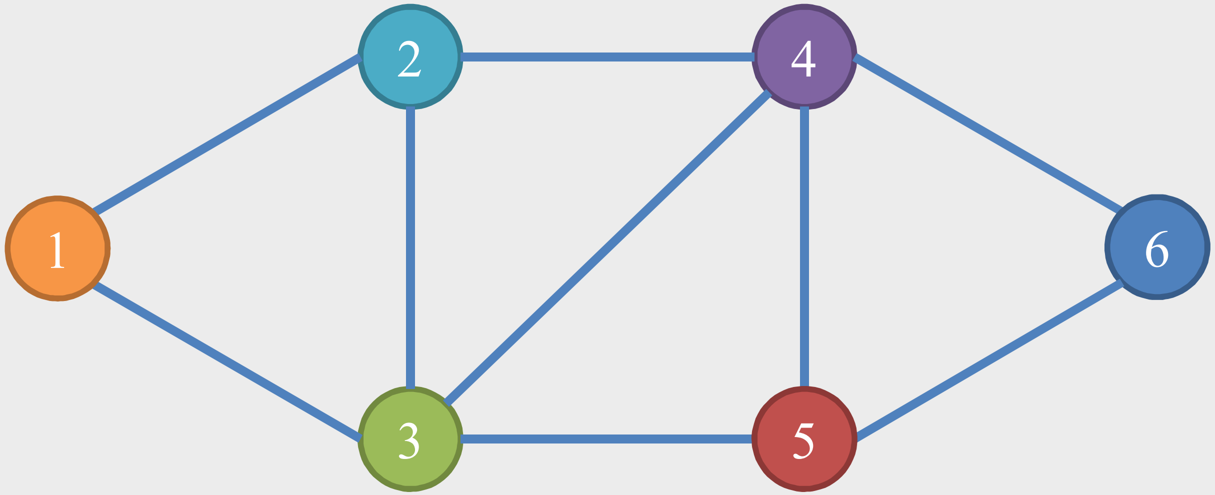
- A university is in the midst of creating its final exam schedule for the semester
- The registrar wants to schedule the exams in the fewest number of exam periods so that no person has to take two exams at the same time
- Set of exams $N = \{1, \dots, n\}$
- Set of conflicts E
  - E consists of pairs of exams
- How can we solve this problem?

Graph Coloring



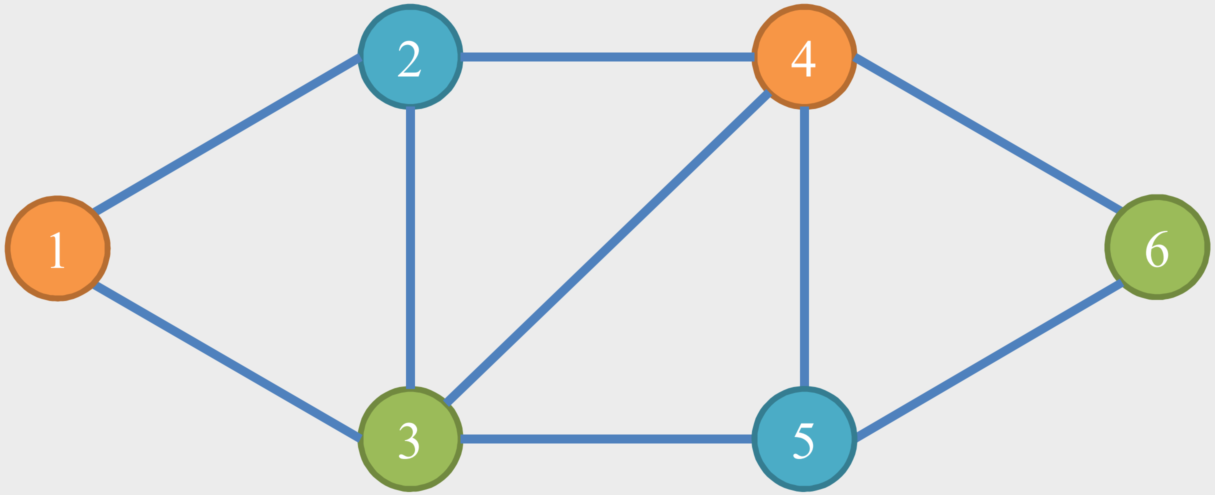
- Undirected graph (N , E)

Graph Coloring



- Undirected graph (N, E)
- A **coloring** of (N, E) is an assignment of "colors" to nodes so that <u>no two adjacent nodes have the same color</u>

Graph Coloring



- Undirected graph (N, E)
- A **coloring** of (N, E) is an assignment of "colors" to nodes so that <u>no two adjacent nodes have the same color</u>
- A graph with n nodes needs at most n colors

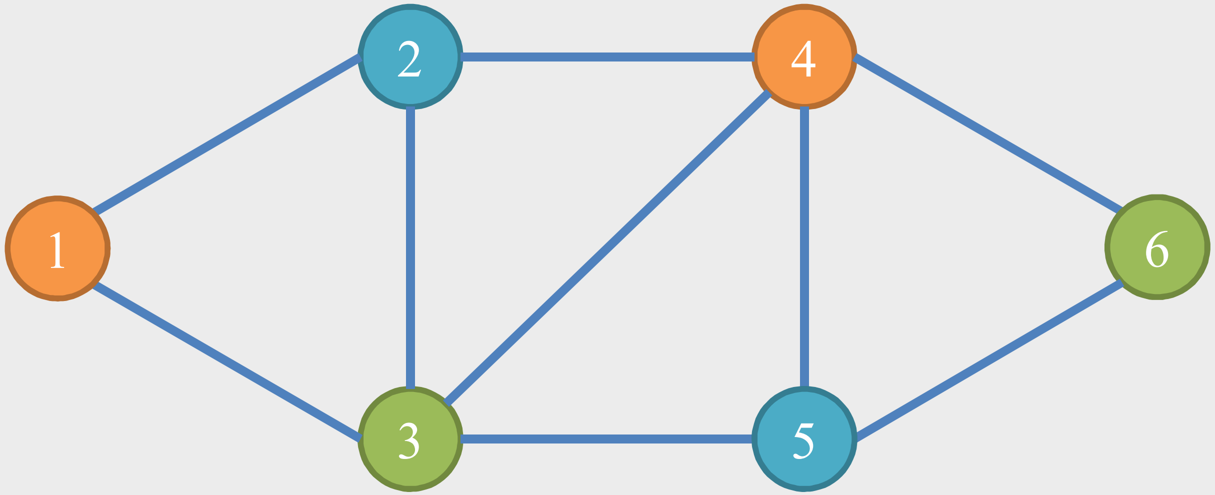
Exam Scheduling

- Relationship between exam scheduling and graph coloring?
- Think of
  - Nodes = exams
  - Edges = conflicts
  - Colors = exam periods
- Valid coloring

  ↔ no two adjacent nodes have same color

  ↔ no two exams that are in conflict are assigned the same exam period
- Exam scheduling problem: want to find a coloring of (N, E) that <u>uses the fewest colors</u>

Graph Coloring



- Undirected graph (N , E)
- Set of colors K: $|K| = |N|$
  - A graph with n nodes needs at most n colors
- Find a valid coloring of (N, E) that minimizes the number of colors used

- Decision variables?
  - Assign colors to nodes:

$$x_{i,c} = \begin{cases} 1 & \text{if node } i \text{ is colored with color } c \\ 0 & \text{otherwise} \end{cases}$$

  - Keep track of colors used:

$$y_c = \begin{cases} 1 & \text{if color } c \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

- Objective:

$$minimize \sum_{c \in K} y_c$$

Graph Coloring

Constraints?

- Every node is assigned exactly one color:

$$\sum_{c \in K} x_{i,c} = 1 \quad \text{for each node } i \in N$$

- The nodes of each edge must be assigned different colors (or, each color can be assigned to an edge (either node) at most once):

$$x_{i,c} + x_{j,c} \leq 1 \qquad \text{for each edge } \{i,j\} \in E \text{ and color } c \in K$$

- Bookkeeping: if color c is used, then $y_c = 1$:

$$x_{i,c} \leq y_c \qquad \text{for each node } i \in N \text{ and color } c \in K$$

GRAPH COLORING

The full model:

$$\min \sum_{c \in K} y_c$$

s.t.

$$\sum_{c \in K} x_{i,c} = 1 \quad \text{for each node } i \in N$$

$$x_{i,c} + x_{j,c} \leq 1 \qquad \text{for each edge } \{i, j\} \in E \text{ and color } c \in K$$

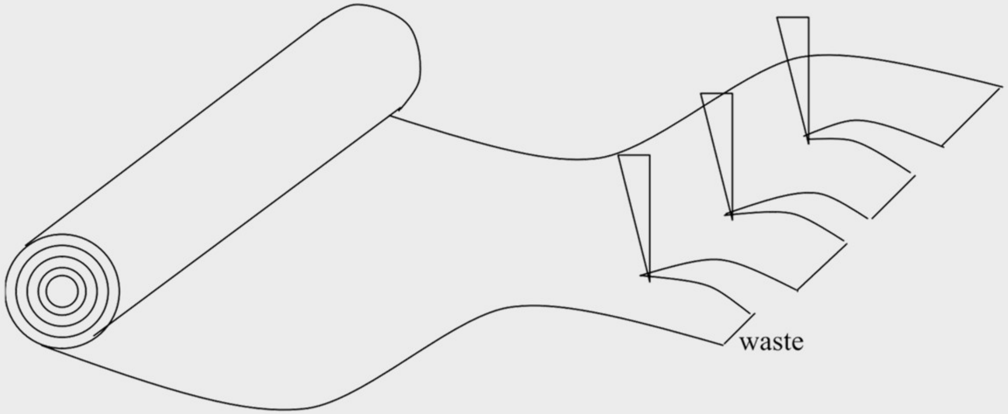
$$x_{i,c} \leq y_c \qquad \text{for each node } i \in N \text{ and color } c \in K$$

$$x_{i,c} \in \{0,1\} \qquad \text{for each edge } \{i, j\} \in E$$

$$y_c \in \{0,1\} \qquad \text{for each color } c \in K$$

Cutting-Stock Problem

- A company makes 110-inch wide rolls of thin sheet metal and slices them in smaller rolls of 12, 15, and 30 inches.
- A cutting pattern is a configuration of the number of smaller rolls of each type that are cut from the raw stock. Six different cutting patterns are used.



	Size of End Item			
Pattern	12 in.	15 in.	30 in.	Scrap
1	0	7	0	5 in.
2	0	1	3	5 in.
3	1	0	3	8 in
4	9	0	0	2 in
5	2	1	2	11 in
6	7	1	0	11 in

- Demands for the coming week are 500 12-inch rolls, 715 15-inch rolls, and 630 30-inch rolls.
- The problem is to develop a model that will determine how many 110-inch rolls to cut into each of the six patterns to meet demand and minimize scrap.

Cutting-Stock Problem

Variables:

$$x_i = \text{number of 110-inch rolls to cut using pattern } i$$

- x_i needs to be a whole number (integer variable). Each roll that is cut generates a different number of end items.
- The only constraints are end-item demand, nonnegativity, and integer restrictions.

Cutting-Stock Problem

Size of End Item				
Pattern	12 in.	15 in.	30 in.	Scrap
1	0	7	0	5 in.
2	0	1	3	5 in.
3	1	0	3	8 in
4	9	0	0	2 in
5	2	1	2	11 in
6	7	1	0	11 in

$$\min 5X_1 + 5X_2 + 8X_3 + 2X_4 + 11X_5 + 11X_6$$

$$0X_1 + 0X_2 + 1X_3 + 9X_4 + 2X_5 + 7X_6 \geq 500 \quad \text{(12-inch rolls)}$$

$$7X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 \geq 715 \quad \text{(15-inch rolls)}$$

$$0X_1 + 3X_2 + 3X_3 + 0X_4 + 2X_5 + 0X_6 \geq 630 \quad \text{(30-inch rolls)}$$

$$X_i \geq 0 \text{ and integer}$$

Cutting-Stock Problem

$$\textit{Minimize} \qquad \sum_{i=1}^{n} c_i x_i$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i \geq q_j \qquad \forall\, j = 1, \ldots, m$$

$$x_i \geq 0\ ,\ \textit{integer} \qquad \forall\, i = 1, \ldots, n$$

LECTURE #10:

SET COVERING/PACKING/PARTITIONING PROBLEMS