DIFFERENTIAL EQUATIONS

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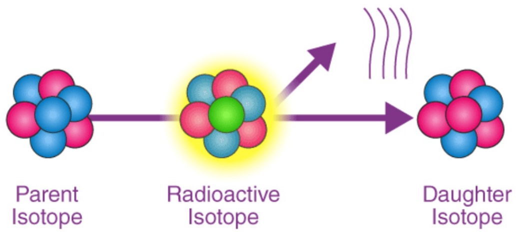
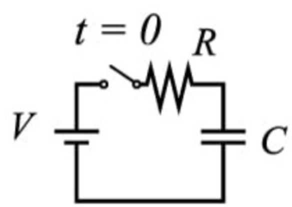
Differential Equations in Physics

- Virtually everything in physics is described by differential equations. Some one dimensional examples
  - Radioactive decay



$$N = N(t)$$
$$\frac{dN}{dt} = -N$$

  - Charging a capacitor



$$Q = Q(t)$$
$$\frac{dQ}{dt} = \frac{V - \frac{Q}{C}}{R}$$

  - Gravitational motion with turbulent drag

$$x = x(t)$$
$$\frac{d^2x}{dt^2} = -g \pm \frac{k}{m}\left(\frac{dx}{dt}\right)^2$$

Sign depends on direction!

Formalism

- You've dealt with a number of specific differential equations in physics, which typically have specific forms.
- For example a damped-driven harmonic oscillator (or LC circuit, quantum state, etc) has the form

$$m\frac{d^2x}{dt} + \gamma\frac{dx}{dt} + kx = F(t)$$

which is a linear differential equation with constant coefficient. A non-zero F(t) term makes in "inhomogeneous".

- But this is not the most general form of differential equation.
- We will start with ordinary differential equations (ODE), in which one variable *y* is uniquely dependent on another variable *x*, ie

$$y = y(x)$$

- Note that in practice, the dependent and independent variables may have different names. In the oscillator example $x \rightarrow t$ and $y \rightarrow x$

Formalism (cont')

- The "order" of an ODE is the highest order derivative of *y*, and the standard format is to express that as a function of *x,y*, and all lower order derivatives

$$\frac{d^n y}{dx^n} \equiv y^{(n \cdot ')}(x) = f(x, y, y' ... y^{((n-1) \cdot '))}))$$

- Fpr example

$$y' = f(x, y)$$

First order

$$y'' = f(x, y, y')$$

Second Order

- In general, I'll need a number initial conditions equal to the order of the equation. For my previous examples.
  - For the first two first order equations, I'll need to know the initial amount or radioactive material *N(0)* or charge *Q(0)*.
  - For the third, second order example, I'll need to know
    - The initial position *x(0)*
    - The initial velocity *x'(0)=v(0)*

Explicit Solutions

- You've learned many tricks for solving differential equations.
- For example, if a first order ODE can be expressed as

$$y' = f(x, y) = \theta(x)\phi(y)$$

Then we can express the solution as integrals.

$$\frac{1}{\phi(y)}dy = \theta(x)dx$$

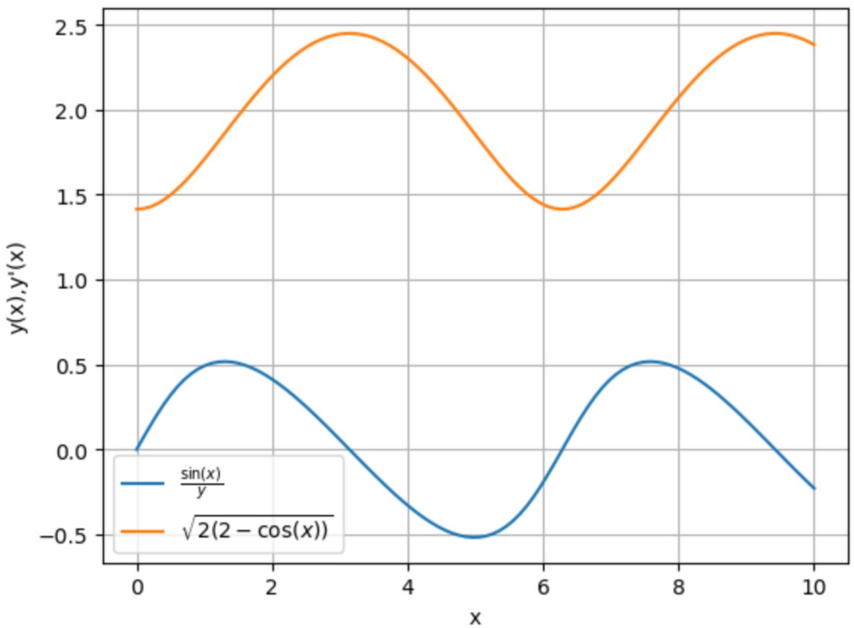
$$\rightarrow \int \frac{1}{\phi(y)}dy = \int \theta(x)dx$$

- IF both sides are explicitly integrable, we will have a least a transient solution, where the constant of integration is determined by the initial condition.

Example

- Example

$$\boxed{\frac{dy}{dx} = \frac{\sin(x)}{y}; \quad y\left(\frac{\pi}{2}\right) = 2}$$



$$\int y\,dy = \int \sin(x)dx$$ Integrate

$$\frac{1}{2}y^2 = -\cos(x) + K$$ General solution

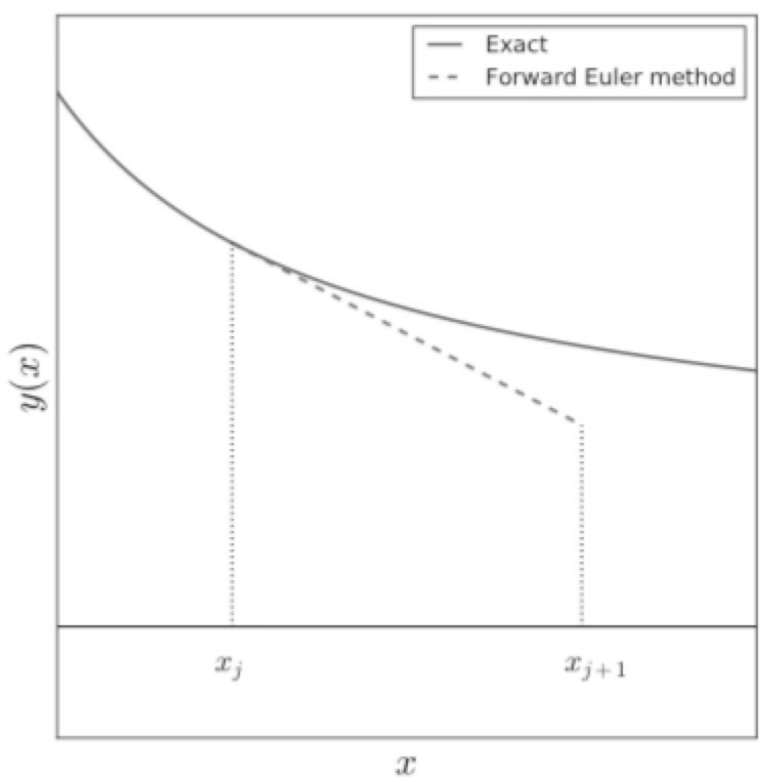
$$\frac{1}{2}(2)^2 = K$$ Solve for initial condition

$$\rightarrow K = 2$$

$$\rightarrow \boxed{y = \sqrt{2(2 - \cos(x))}}$$ Explicit solution

Numerical Solutions: Euler Method

- General approach:
  - Start at the initial point
  - Use discrete steps to estimate the derivative and evolve the function to the next step.



- Start at point a
- Step to point b in discrete steps

$$x_j = a + jh$$

$$j = 0, 2, 3, ..., n-1$$

$$\text{(step)} \qquad h = \frac{b-a}{n-1}$$

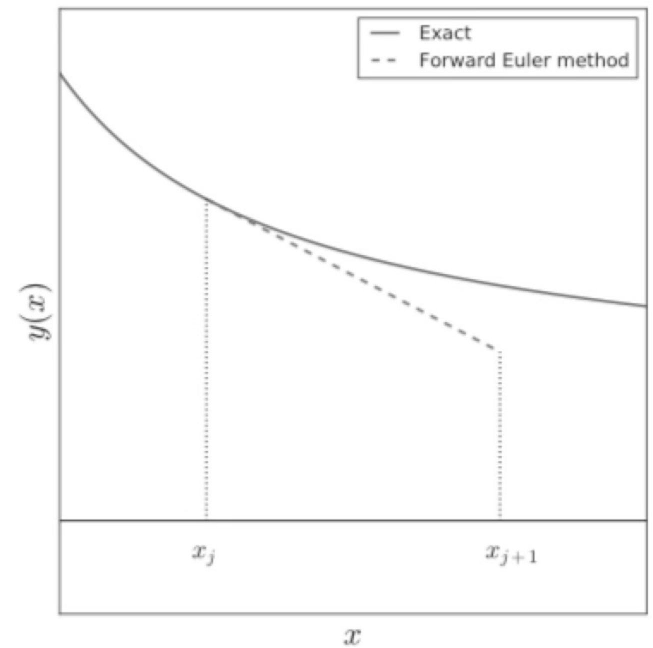
Extrapolating

- Generalizing forward derivative techniques

$$y'(x_j) = f\left(x, y(x_j)\right) = \frac{\boxed{y(x_{j+1})} - y(x_j)}{h} + \mathcal{O}(h)$$

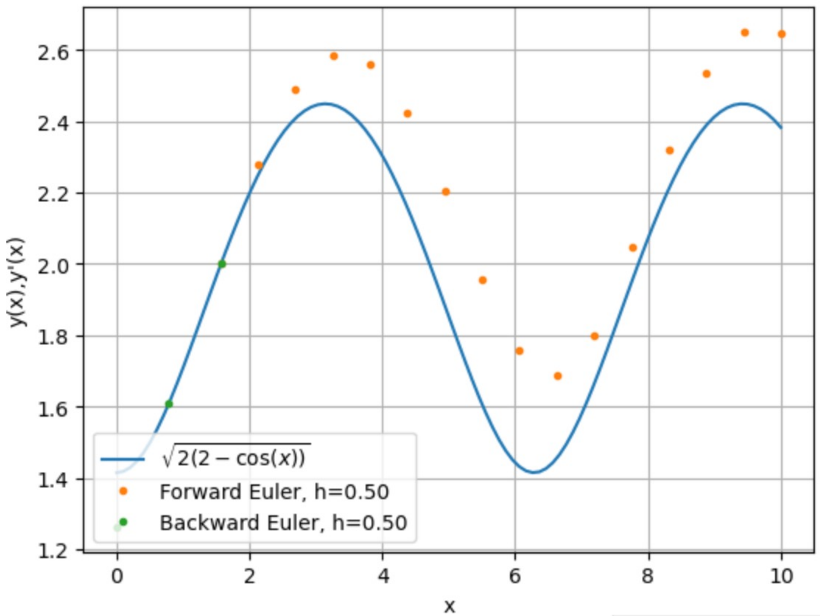
- We can turn around and extrapolate to the next value of *y* with

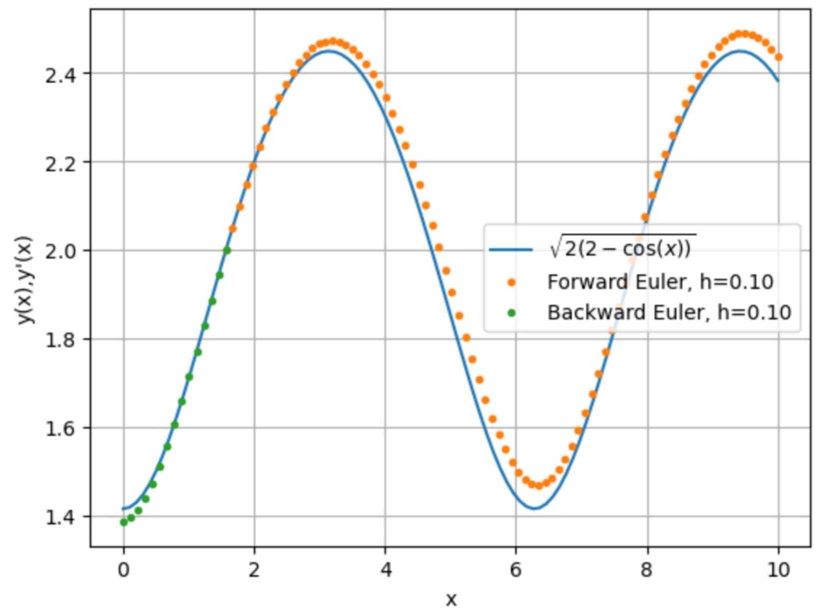
$$y(x_{j+1}) = y(x_j) + hf\left(x, y(x_j)\right) + \mathcal{O}(h^2)$$

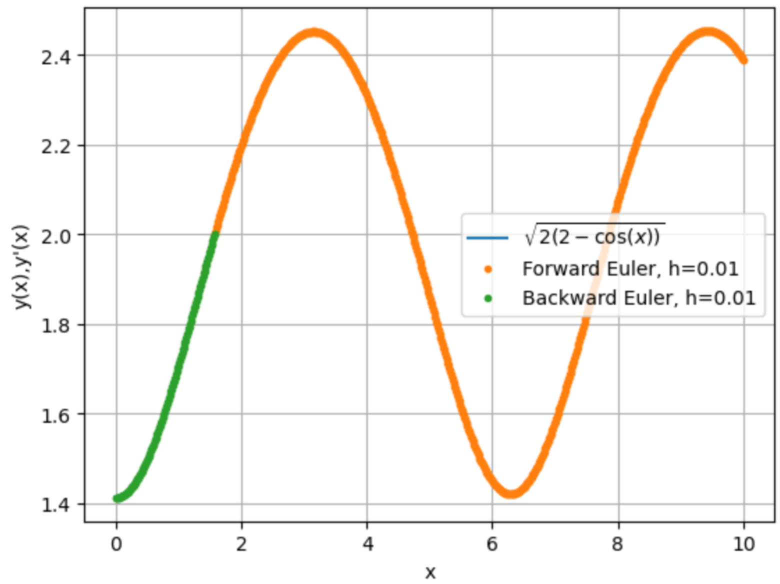


- Note, we can also work backwards by simply flipping the sign of *h*

Test it on our explicit example (Notebook)







Numerical Solutions to 2nd Order ODE

- We can apply our Euler technique to 2nd order ODEs as well, by extrapolating *both* the position and the first derivative. Starting with *y(0)* and *y'(0)*, extrapolate with

$$y''(x) = \frac{dy'}{dx} = f(x, y, y')$$

$$\rightarrow y'_{j+1} = y'_i + hf(x_i, y_i, y'_i)$$

$$y_{j+1} = y_i + hy'_i$$

Extrapolate *y'* the same way I extrapolated *y* before

Extrapolated y using the extrapolated values of y'

Example: Harmonic Oscillator

- Our usual way to write this is

$$m\frac{d^2x}{dt^2} + kx = 0 \to \frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

- And we know the solution is

$$a\cos(\omega t) + b\sin(\omega t); \qquad \omega = \sqrt{\frac{k}{m}}$$

- Initial conditions:

$$x(0) = a = x_0 \to a = x_0$$

$$x'(0) = \omega b = v_0 \to b = \frac{v_0}{\omega}$$

$$\to x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$

$$v(t) = -\omega x_0\sin(\omega t) + v_0\cos(\omega t)$$

Expressing it in our formalism…

- We rearrange our expression:

$$m\frac{d^2x}{dt^2} + kx = 0 \rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

- Note that in our usual formalism

$$\begin{aligned} t &\rightarrow x \\ x &\rightarrow y \\ \rightarrow y''(x) &= f(x, y, y') \\ \text{where} \quad f(x, y, y') &= -y \end{aligned}$$

- You won't believe how much time I wasted getting this wrong!

Dimentionless Time

- For an explicit solution, I can always just plug in different values for *k* and *m*, but if I'm doing a numerical solution, I'd rather not re-run it for different values.

$$\tau \equiv \omega_0 t$$

$$\rightarrow dt = \frac{1}{\omega_0} dt$$

- Then if I find *x* as a function of τ, I can express it as a function of time as

$$x(\tau) = x(\omega t)$$

- In these units, our equation and solutions become

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$$\rightarrow \omega_0^2 \frac{d^2x}{d\tau^2} + \omega_0^2 x = 0$$

$$\rightarrow \frac{d^2x}{d\tau^2} + x = 0$$

$$x(\tau) = x_0 \cos(\tau) + \frac{v_0}{\omega} \sin(\tau)$$

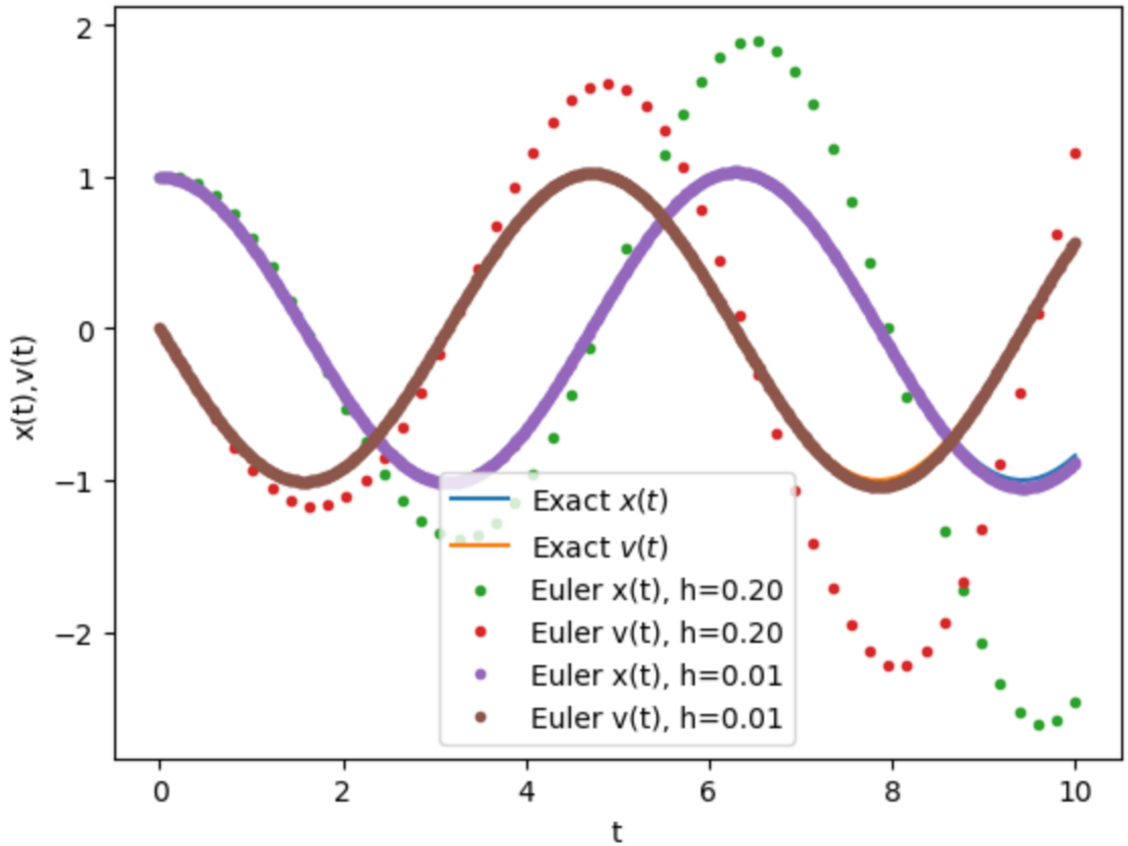
$$v(\tau) = -\omega x_0 \sin(\tau) + v_0 \cos(\tau)$$

Example

- Use

$$m = k = 1.$$
$$\rightarrow \omega = 1.$$
$$x_0 = 1.$$
$$v_0 = 0.$$
$$\rightarrow x(t) = \cos(t)$$
$$\rightarrow v(t) = -\sin(t)$$

- Go to notebook…



Adding Damping

- We add a damping force given by

$$F = -\gamma v$$

- And our equation becomes

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

$$\rightarrow \frac{d^x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\eta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0$$

$$\frac{d^2x}{d\tau^2} + 2\eta\frac{dx}{d\tau} + x = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\eta = \frac{\gamma}{2\sqrt{km}}$$

Damping factor

Explicit Solutions

- We can solve this explicitly for particular values of η…
  - Underdamping (η<<1):

$$x(\tau) = \left(x_0 \cos(\tau) + \frac{v_0}{\omega}\sin(\tau)\right) e^{-\eta\tau}$$

  - Critical Damping (η=1):

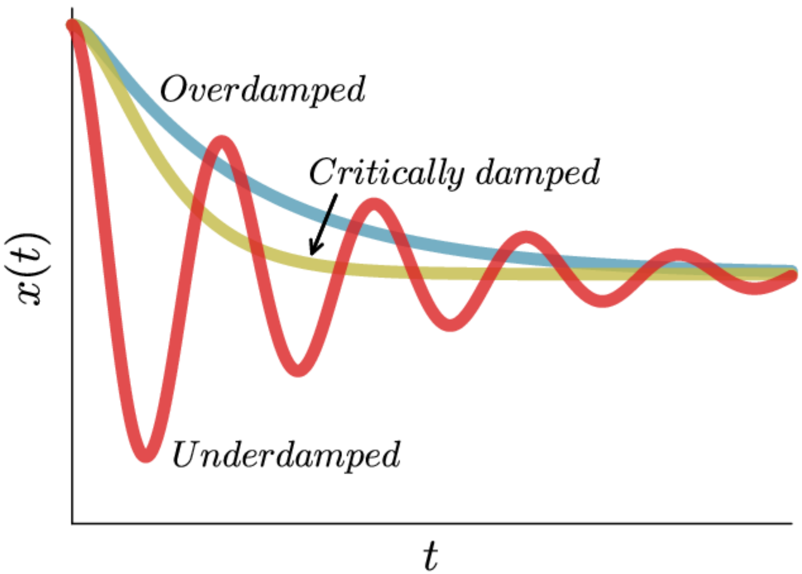
$$x(\tau) = (A + B\tau)\, e^{-\tau}$$

  - Overdamping (η=1):

$$x(\tau) = Ae^{-\tau} + Be^{-\frac{\tau}{\eta}}$$

Fast

Slow

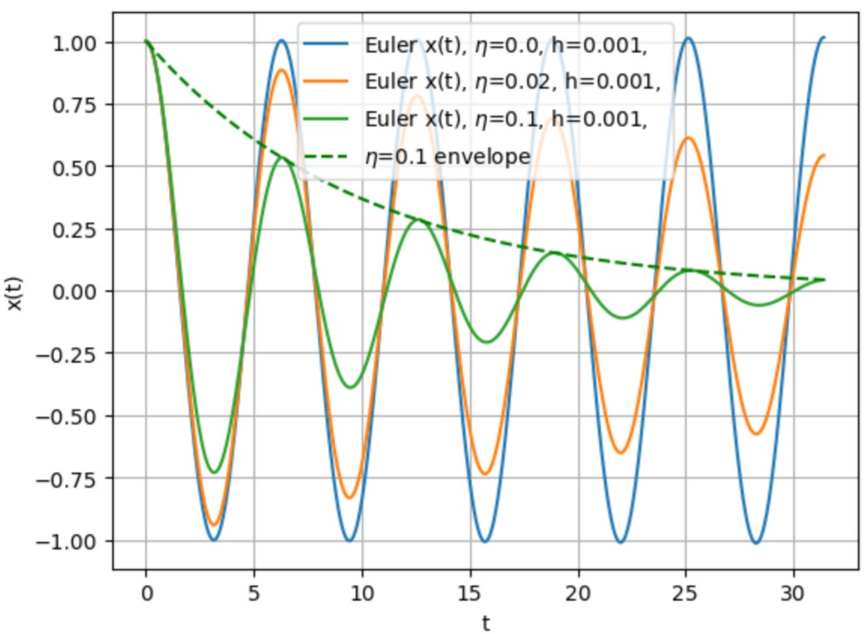


Go to Notebook

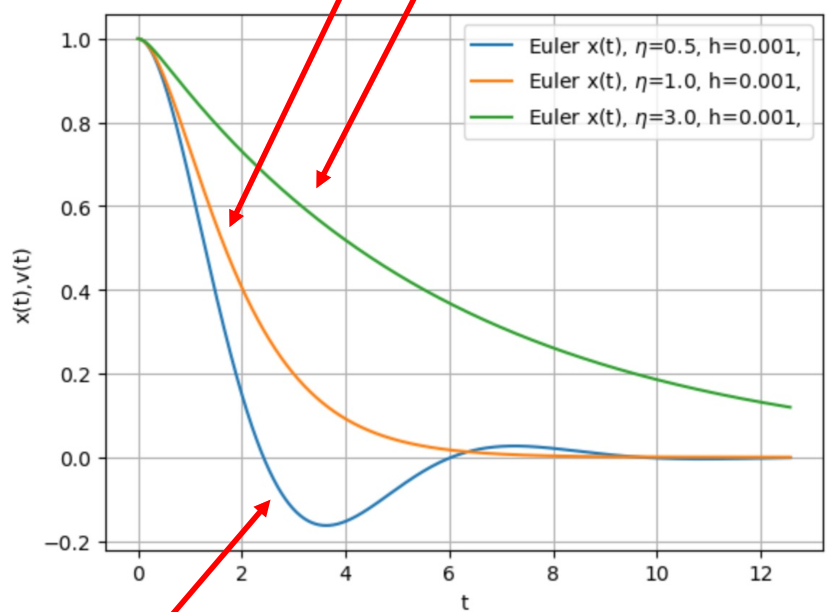
Numerical Examples

Critically damped

Overdamped



Underdamping



Transition to overdamping

Slightly underdamped

Transition to Turbulent Drag

- Turbulent drag has the form

$$F = -\gamma v^2 \cdot \mathrm{sign}(v)$$

- Our equation now becomes

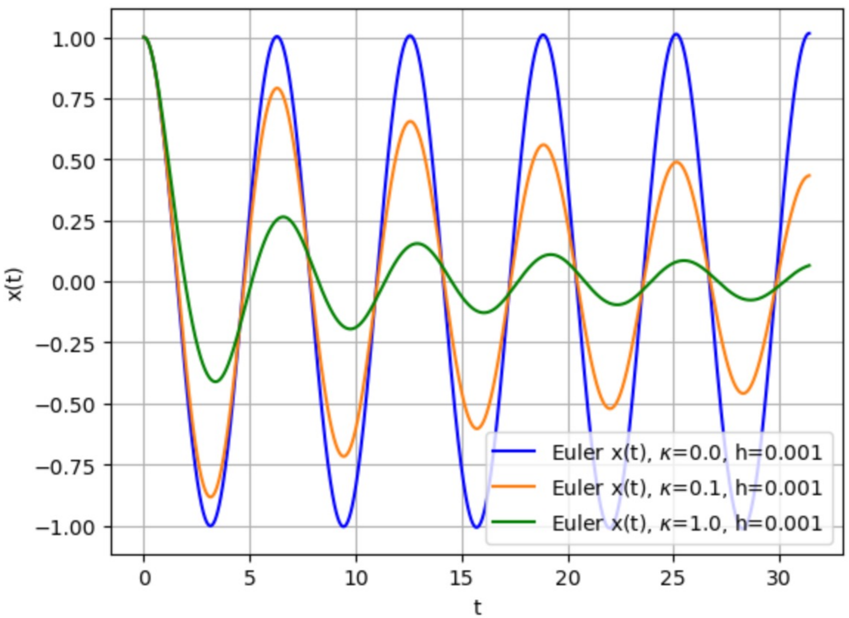
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{\gamma}{m}\left(\frac{dx}{dt}\right)^2 \cdot \mathrm{sign}(v)$$

$$\rightarrow \omega_0^2 \frac{d^2x}{d\tau^2} = -\omega_0^2 x - \frac{\gamma}{m}\omega_0^2 \left(\frac{dx}{dt}\right)^2 \cdot \mathrm{sign}(v) \qquad \kappa \equiv \frac{\gamma}{m}$$

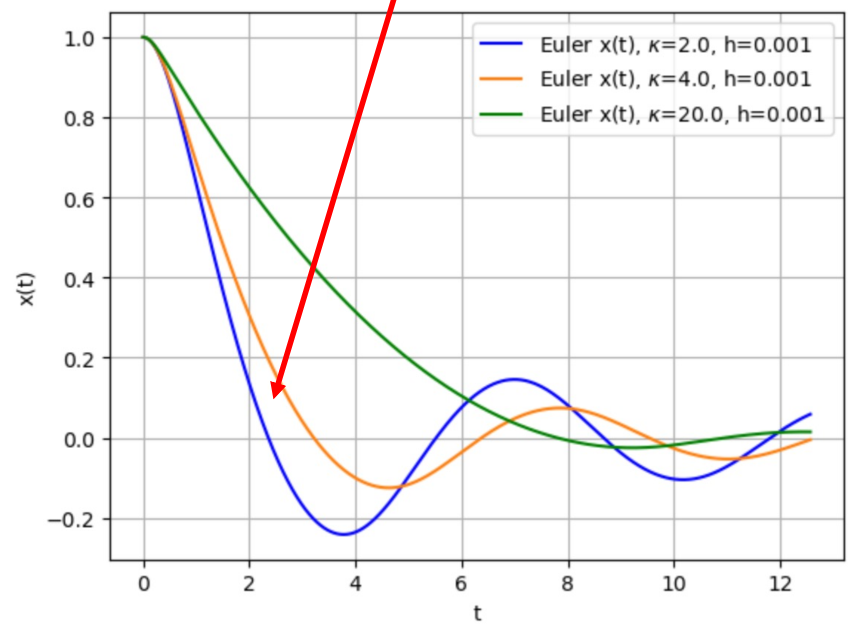
$$\rightarrow \frac{d^2x}{d\tau^2} = -x - \kappa \left(\frac{dx}{dt}\right)^2 \cdot \mathrm{sign}(v)$$

Numerical Examples

No critical damping



Light Damping



Heavy Damping

Generic Representation of ODEs

- To avoid yet more confusion, I'm going to switch to the Python formalism for ODEs
  - Independent variable *t*
  - Dependent variable *y*
- For our Euler extrapolation of an nth order ODE, we extrapolate each point to the next based on its derivative
- More generally, we can calculate the infinitesimal change of the vector of dependent variables and derivatives as a function of their current values and the independent variable

$$y_{j+1} = \frac{dy}{dt}_j \Delta t$$

$$\frac{dy}{dt}_{j+1} = \frac{d^2y}{dt^2}_j \Delta t$$

$$\frac{d^2y}{dt^2}_{j+1} = \frac{dy^3}{dt^3}_j \Delta t$$

$$\vdots$$

$$\frac{d^{n-1}y}{dt^{n-1}}_{j+1} = \frac{dy^n}{dt^n}_j \Delta t$$

$$\mathbf{y} = \begin{pmatrix} y \\ y' \\ \vdots \end{pmatrix}$$

$$\rightarrow \frac{d}{dt}\mathbf{y} = f(t, \mathbf{y})$$

Applying this to our Oscillator

- We define our vector as

$$\mathbf{y} = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} y \\ v \end{pmatrix}$$

- And our rate of change will be

$$\frac{d}{dt}\mathbf{y} = \begin{pmatrix} v \\ -\omega_0^2 x - 2\eta\omega v \end{pmatrix}$$

- Which in this *linear* case can be represented by a matrix operation.

$$\frac{d}{dt}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\eta\omega \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$$

- But that's not a requirement to solve the problem

Using scipy.integrate.solve_ivp()

- We need to define an ode function of the form

```
def ode(t,y):
# t = independent variable
# y = *array* of dependent variable and its derivatives
#     up to (n-1)
#
#
    return dy  # dy/dt (can be tuple, list, or array)
```

- We then solve it with the initial value problem solver

```
from scipy.integrate import solve_ivp
solution = solve_ivp(ode, (tmin,tmax), y0, t_eval=t)
```

ode function

Time range

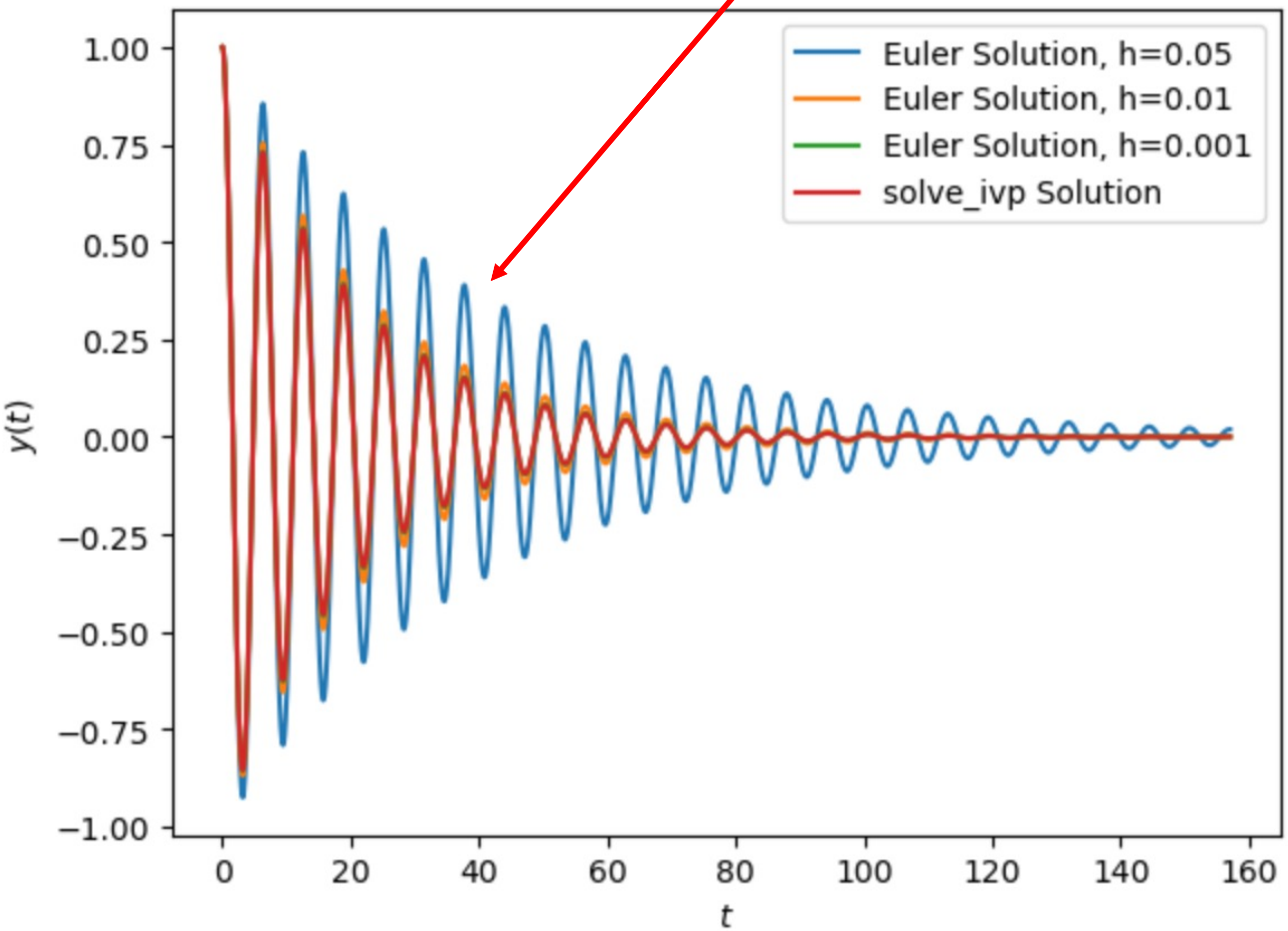
Initial values

Time points to evaluate

Go to notebook…

Euler vs. ivp_solver

Euler not accurate for larger *h*



Adding a driving term

- If we add a driving term, our equation becomes

$$m\ddot{y} + \gamma\dot{y} + ky = F(t)$$

$$\ddot{y} + 2\eta\omega_0\dot{y} + \omega_0^2 y = \tilde{F}(t)$$

Explicit time dependence

$$\rightarrow \frac{d\dot{y}}{dt} = -\omega_0^2 y - 2\eta\omega_0\dot{y} + \tilde{F}(t)$$

- Use a driving term of the form

$$\tilde{F} \propto \sin(\omega_d t)$$

- Go to notebook…

Resonant Behavior