



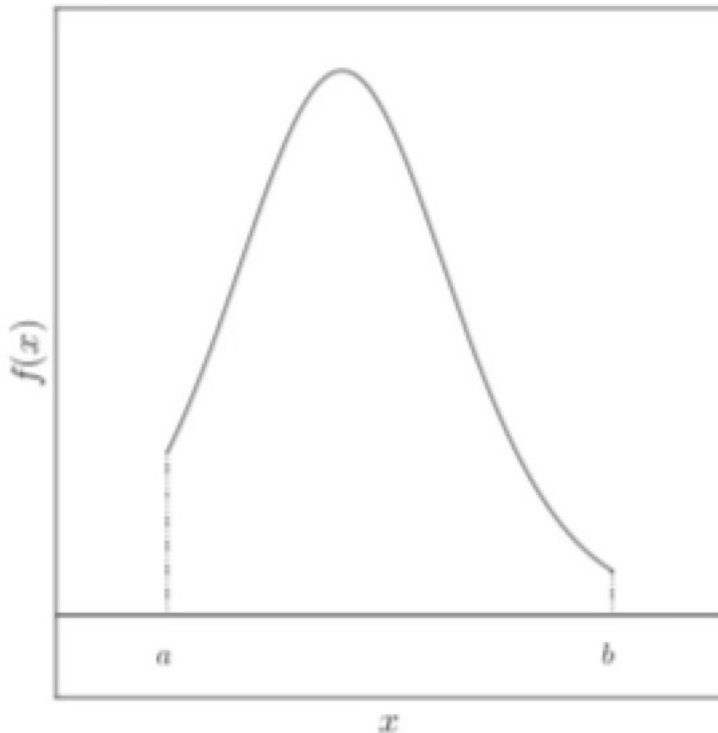
INTEGRALS

Eric Prebys



General Problem

- Reduce an integral to a discrete sums



Start with n discrete points

$$x_i = a + ih \quad h = \frac{b - a}{n - 1}$$

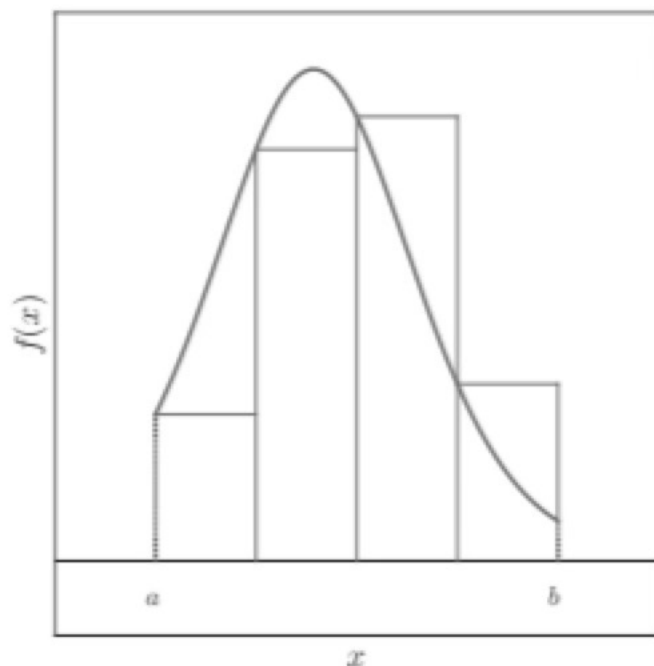
Approximate as a sum

$$\int_a^b f(x) dx \approx \sum_{i=0}^{n-1} c_i f(x_i)$$



Left-side Rule Integrals

- The simplest



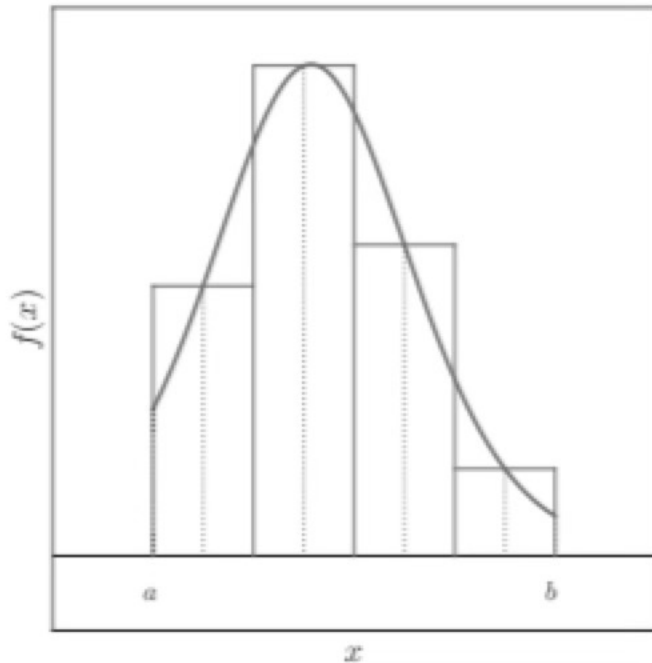
$$\int_{x_i}^{x_{i+1}} f(x) dx \approx h f(x_i)$$
$$\rightarrow \int_a^b f(x) dx \approx h \sum_{i=0}^{n-2} f(x_i)$$

Skip last point



Midpoint Rule

- We can obviously do better if we use the midpoint



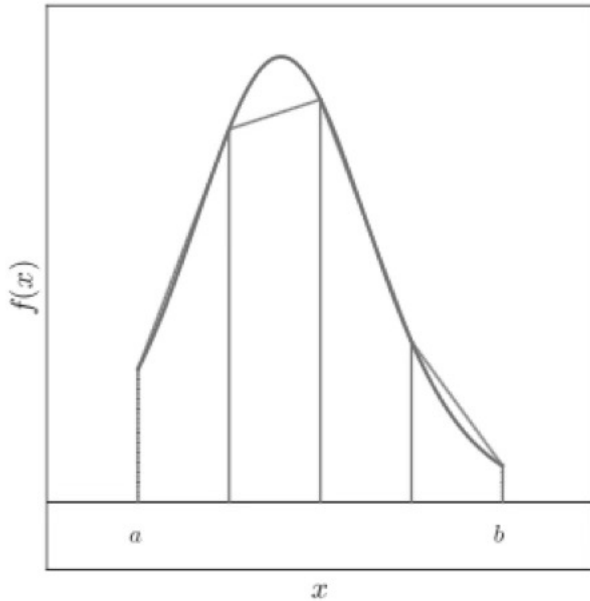
$$\int_{x_i}^{x_{i+1}} f(x) dx \approx h f\left(x_i + \frac{h}{2}\right)$$

$$\rightarrow \int_a^b f(x) dx \approx h \sum_{i=0}^{n-2} f\left(x_i + \frac{h}{2}\right)$$

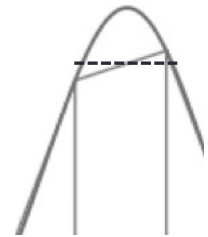


Trapezoidal Rule

- This is based on approximating an integral by a trapezoid



Note: the integral of the trapezoid is just the integral of the mean.



$$\int_{x_i}^{x_{i+1}} f(x) dx \approx \frac{1}{2} h [f(x_i) + f(x_{i+1})]$$

$$\rightarrow \int_a^b f(x) dx \approx h \sum_{i=0}^{n-2} \frac{1}{2} [f(x_i) + f(x_{i+1})]$$

$$= \frac{h}{2} [f(x_0) + f(x_{n-1})] + h \sum_{i=1}^{n-3} f(x_i)$$

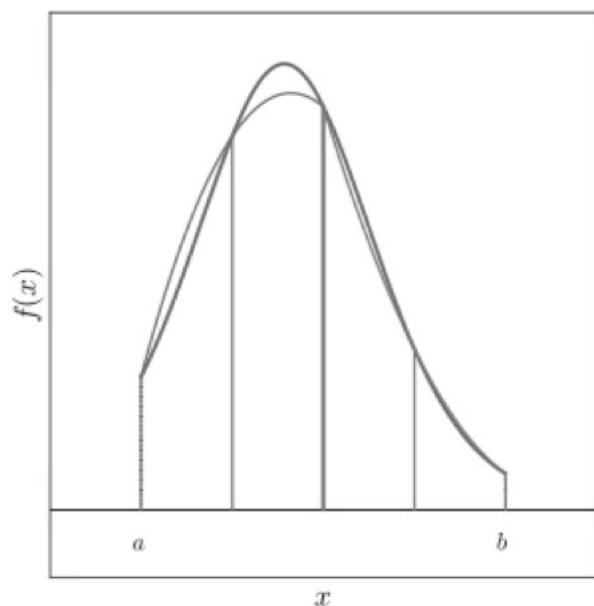


Simpson's Rule

- Simpson's Rule involves fitting a quadratic between three points

Assume three points lie on a parabola about $x_i = 0$

$$\begin{aligned}
 A &= \int_{-h}^h ax^2 + bx + c dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right]_{x=-h}^{x=h} \\
 &= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch \right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch \right) \\
 &= \frac{2ah^3}{3} + 2ch \\
 &= \frac{h}{3} (2ah^2 + 6c)
 \end{aligned}$$



$$y_{i-1} = ah^2 - bh + c$$

$$y_i = c$$

$$y_{i+1} = ah^2 + bh + c$$



Simpson's Rule (con'd)

- The equations. From the second

$$c = y_1$$

- Adding the first and third

$$2ah^2 = y_0 - 2y_1 + y_2$$

- Going back to the original equation

$$\begin{aligned} A &= \frac{h}{3} (2ah^2 + 6c) \\ &= \frac{h}{3} (y_0 - 2y_1 + y_2 + 6y_1) \\ &= \frac{h}{3} (y_0 + 4y_1 + y_2) \end{aligned}$$

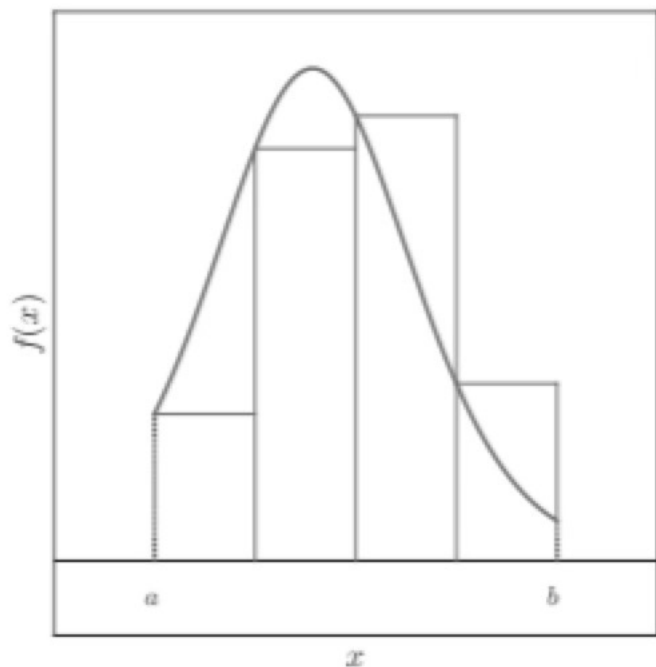
$$\int_{x_i}^{x_i+2} f(x) dx \approx \frac{h}{3} [f(x_i) + 4f(x_{i+1}) + f(x_{i+2})]$$

$$\rightarrow \int_a^b f(x) dx = \frac{h}{3} \sum_{i=0,2,4,\dots}^{n-3} [f(x_i) + 4f(x_{i+1}) + f(x_{i+2})]$$



Error on Left-side Rule

- If we Taylor expand about the starting point



$$\begin{aligned}
 f(x) &= f(x_i) + (x - x_i)f'(\xi_i) \\
 \rightarrow \int_{x_i}^{x_{i+1}} f(x)dx &= \int_{x_i}^{x_{i+1}} dx [f(x_i) + (x - x_i)f'(\xi_i)] \\
 &= h \int_0^1 du [f(x_i) + uhf'(\xi_i)] \\
 &= hf(x_i) + \frac{1}{2}h^2 f'(\xi_i) \\
 \rightarrow \epsilon_i &= \frac{1}{2}h^2 f'(\xi_i)
 \end{aligned}$$

$u \equiv \frac{x - x_i}{h}; hdu = dx$

The accumulated error then becomes

$$\begin{aligned}
 \epsilon &= \frac{n-1}{2}h^2 f'(\xi) \\
 &= \frac{b-a}{2}hf'(\xi)
 \end{aligned}$$

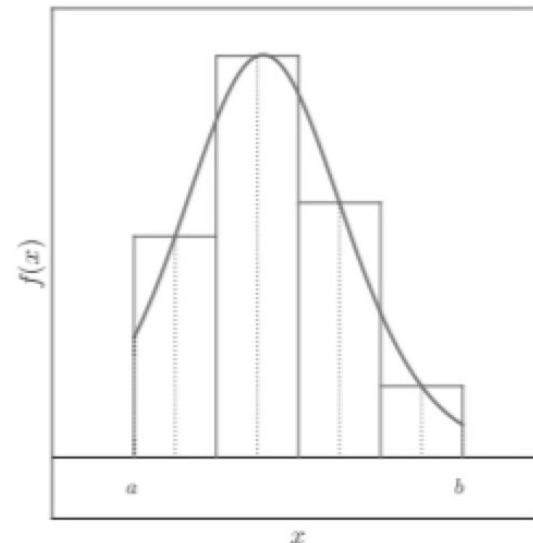


Error on Midpoint Rule

Set the midpoint at $x=0$ and expand

$$f(x) = f(c_i) + f'(c_i)(x - c_i) + \frac{1}{2}f''(c_i)(x - c_i)^2 + \frac{1}{6}f'''(c_i)(x - c_i)^3 + \dots$$

integrate from $c_i - h/2$ to $+h/2$ (odd terms vanish)



$$\begin{aligned} \int_{c_i - \frac{h}{2}}^{c_i + \frac{h}{2}} f(x) dx &= f(c_i)x + \frac{1}{2}f'(c_i)(x - c_i)^2 + \frac{1}{6}f''(c_i)(x - c_i)^3 + \dots \Big|_{c_i - \frac{h}{2}}^{c_i + \frac{h}{2}} \\ &= hf(c_i) + \frac{1}{24}h^3 f''(c_i) + \dots \end{aligned}$$

$$\rightarrow \int_{c_i - \frac{h}{2}}^{c_i + \frac{h}{2}} f(x) dx - hf(c_i) = \frac{1}{24}h^3 f''(c_i) + \dots$$

$$= \frac{1}{24}h^3 f''(\xi_i)$$

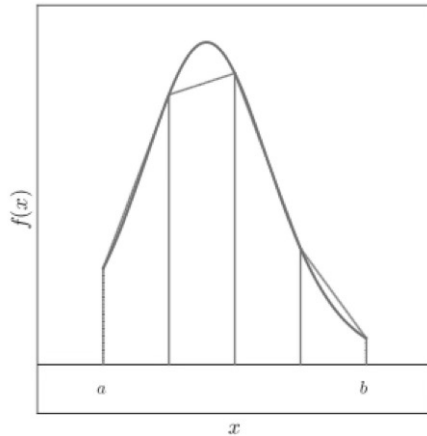
Some point on the curve

$$\rightarrow \varepsilon = \frac{b-a}{24} h^2 f''(\xi)$$



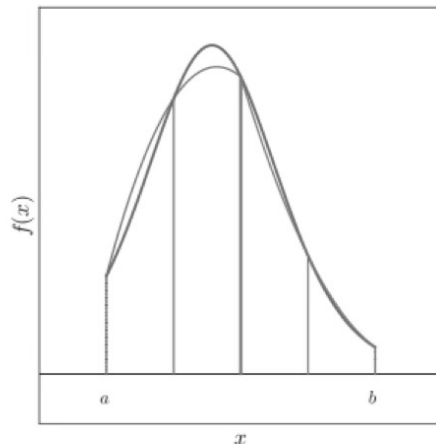
Trapezoidal and Simpson's rule (derived in book)

- Trapezoidal Rule



$$\varepsilon = -\frac{b-a}{12}h^2 f''(\xi)$$

- Simpson's Rule



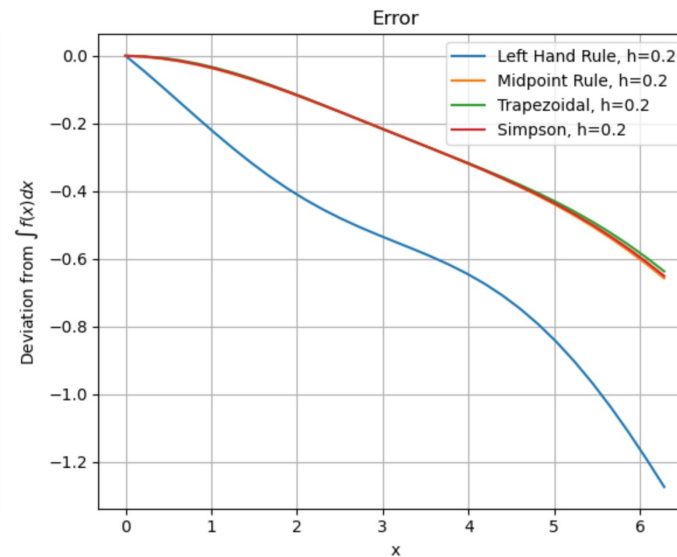
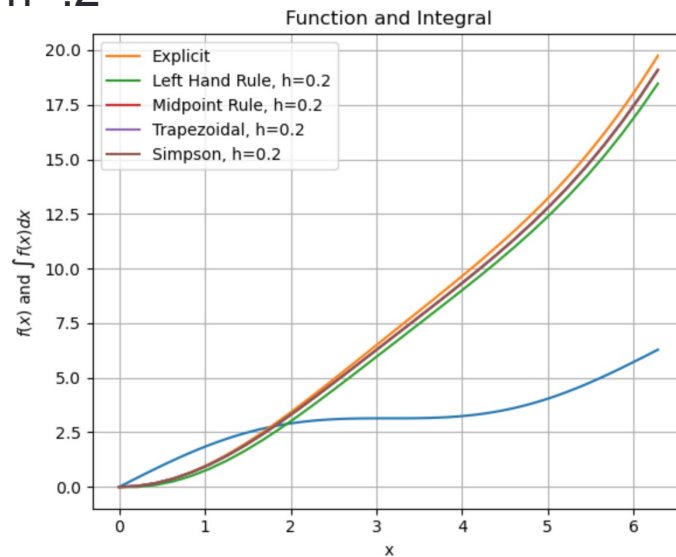
$$\varepsilon = -\frac{b-a}{180}h^4 f''''(\xi)$$

(go to notebook...)

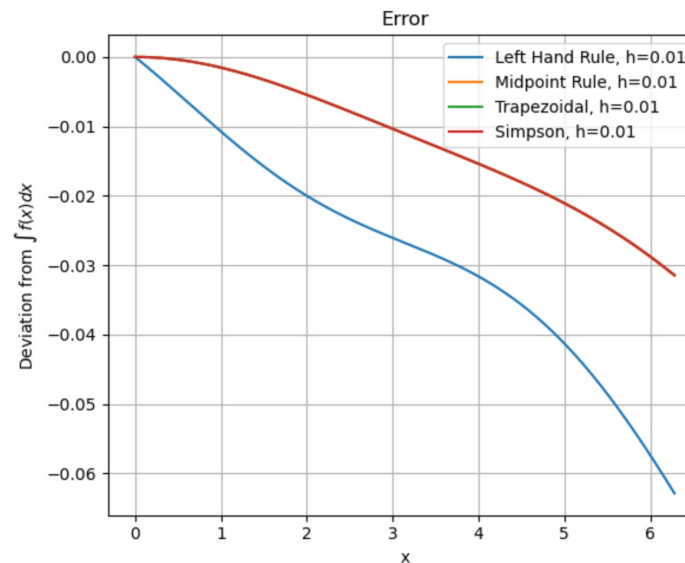
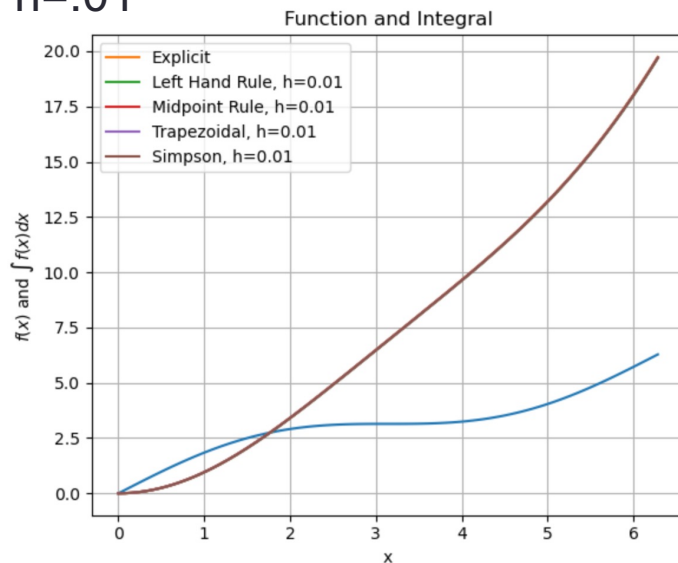


Simulation Results

$h=0.2$



$h=0.01$






Iterative Trapezoid (Next Week's Lab)

- In most cases, rather than calculate the error, it's easier to simply reduce the step size until the integral stops changing.
- In the case of the Trapezoid Rule, you can begin dividing the interval by two each iteration.

$$h_1 = b - a; \quad h_2 = \frac{b - a}{2}; \quad h_m = \frac{b - a}{2^{m-1}}$$

- For each value h_m , calculate the integral I_m . Continue until

$$\frac{|I_m - I_{m-1}|}{I_{m-1}} < \epsilon$$

Predetermined accuracy target 

- Note that for each iteration, half the points were points you used before, so you could save a teeny bit of time by basing each integral on the previous integral

$$I_m = \frac{1}{2} I_{m-1} + h_m \sum_{i=1}^{2^m} f(a + (2i - 1)h_m).$$

but to be honest, it would probably take you more time to get the algorithm right than you would save!



Built-in Integration tools

`scipy.integrate.`

`quad`

`quad(func, a, b, args=(), full_output=0, epsabs=1.49e-08, epsrel=1.49e-08, limit=50, points=None, weight=None, wvar=None, wopts=None, maxpl=50, limlst=50, complex_func=False)` [\[source\]](#)

Compute a definite integral.

Integrate `func` from `a` to `b` (possibly infinite interval) using a technique from the Fortran library **QUADPACK**.

