MATHEMATICAL SERIES AND SUMS

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Digression: Euler Relations

- If we put an imaginary argument into the exponential expansion

$$
\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \ldots \\
&= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \ldots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots\right) \\
&= \cos\theta + i\sin\theta \\
e^{-\theta} &= \cos\theta - i\sin\theta \\
\rightarrow \cos\theta &= \frac{e^{i\theta} + e^{-\theta}}{2} \\
\sin\theta &= \frac{e^{i\theta} - e^{-\theta}}{2i}
\end{aligned}
$$

- You'll use these relationships a lot!

Mathematical Sequence

- In mathematics, a sequence is an enumerated collection of objects in which repetitions are allowed and order matters
  - May be finite or infinite
- Example: Prime numbers
  - 1,3,5,7,11,13,17,19
  - No known generation or recursion algorithm for ALL prime numbers.
- Generation algorithms
  - Index-based: Each element is determined by some function of the index, independently from the other terms
  - Recursive: Each element is determined from some combination of the previous terms

Recursive Example: Fibonacci Series

- From Wikipedia:
  - “The Fibonacci numbers were first described in Indian mathematics as early as 200 BC in work by Pingala on enumerating possible patterns of Sanskrit poetry formed from syllables of two lengths. They are named after the Italian mathematician Leonardo of Pisa, also known as Fibonacci, who introduced the sequence to Western European mathematics in his 1202 book Liber Abaci.”
- The Fibonacci Series begins with 0 and 1, and after that, each term is the sum of the previous two terms:

$$F_1 = 0$$
$$F_2 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$
$$\rightarrow 1, 1, 2, 3, 5, 8, 13, 21, 34, 55...$$

Programming hints to generate Fibonacci Sequence

- The first two elements are special

$$F_a = F_1 = 0$$
$$F_b = F_2 = 1$$

- If you want values beyond the 2nd, you will need to repeat a recursive algorithm *n-2* times

$$F_c = F_a + F_b$$
$$F_a = F_b$$
$$F_b = F_c$$

Why is it important that I do it in exactly this order?

- Then return the final value of F_c

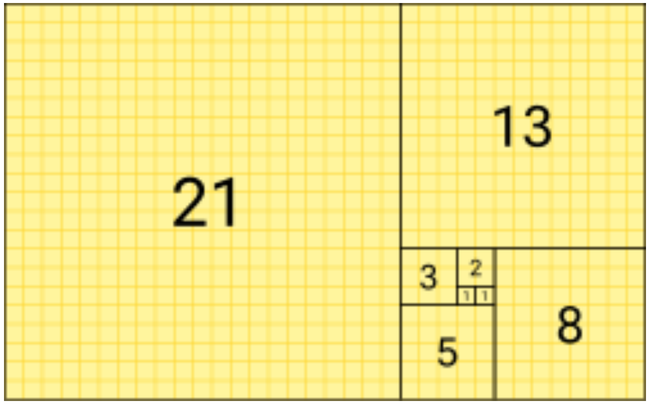
Alternative Approaches

- You're not being asked to do this yet, but you could generate the entire Fibonacci Sequence up to n by filling an array

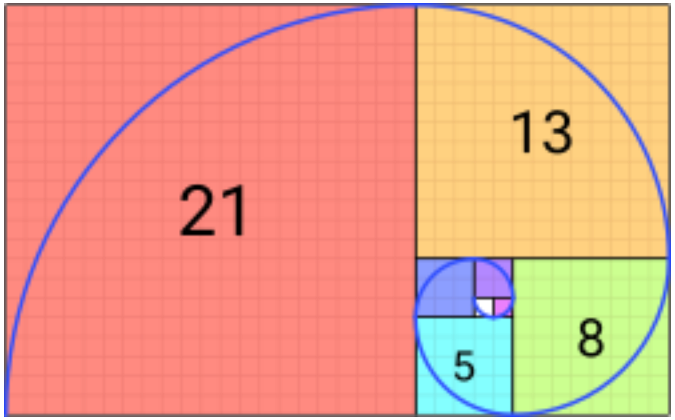
(Go to notebook)

Visualizing the Fibonacci Series

- Fibonacci Tiling



- Fibonacci Spiral (Approximation of Golden Spiral)



Digression: The Golden Ratio

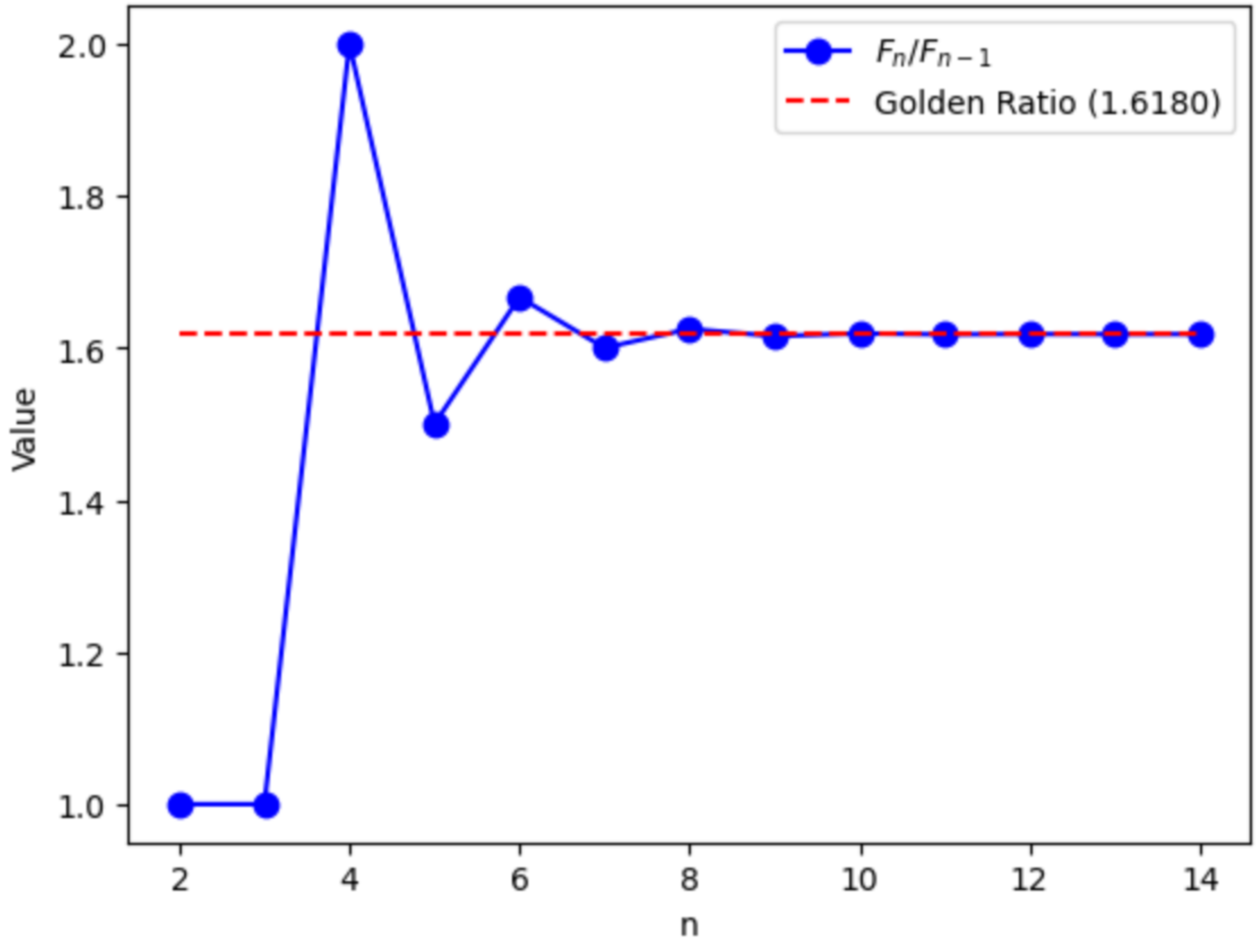
- In mathematics, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities.

$$
\begin{aligned}
\frac{a+b}{a} = \frac{a}{b} &\equiv \varphi \\
\rightarrow 1 + \frac{1}{\varphi} &= \varphi \\
\rightarrow \varphi^2 - \varphi - 1 &= 0 \\
\rightarrow \varphi &= \frac{1 \pm \sqrt{1+4}}{2} \quad \text{(take larger value)} \\
&= \frac{1 \pm \sqrt{5}}{2} = 1.618033988749
\end{aligned}
$$

The Fibonacci Series and the Golden Ratio

- The ratio of subsequent terms in the Fibonacci Sequence quickly converges on the Golden Ratio (go to notebook)

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi$$



Fibonacci Series and Perfect Triangles

- Starting with the value 5, every 2nd number in the Fibonacci sequence is a perfect triangle, ie

$$c^2 = a^2 + b^2 \qquad (a, b \text{ and } c \text{ are integers})$$

- First two

$$5^2 = 3^2 + 4^2$$

$$13^2 = 5^2 + 12^2$$

- You'll find the next two in lab, and I recommend you just write a loop that looks at

$$\sqrt{F_n^2 - i^2}$$

for i<n and see which value you gives you and integer

Arithmetic Series

- An arithmetic series is one in which the difference between successive terms are constant, then the terms are all summed.

$$S_n = \sum_{k=0}^{n} (a + kd) = a + (a + d) + (a + 2d) + \ldots + (a + nd)$$

- Which sums to the average of the first and last terms times the number of terms.

$$S_n = (n + 1)\frac{a + (a + nd)}{2}$$

Geometric Series

- A geometric series is one in which the terms are powers

$$f(r) = a + ar + ar^2 + ar^3 \ldots = \sum_{n=0}^{\infty} ar^n$$

- Note

$$rf(r) = ar + ar^2 + ar^3 + \ldots = f(r) - a$$

$$\rightarrow f(r) = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Taylor Expansion (you'll use this a lot)

- A power series provides a basis to represent any function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots = \sum_{n=0}^{\infty} c_n x^n$$

- We can also represent a series about an arbitrary point *a* that is not 0

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

- Note! Obviously, I could expand the terms in the second form and put it in the form of the first, but it's more useful in this form for…

Taylor Expansion (cont'd)

- Two functions are equal if their values and ALL of their derivatives are equal. Considering our expansion

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \ldots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$f(a) = c_0$$

$$\left.\frac{df}{dx}\right|_{x=a} = 1 \cdot c_1$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=a} = 2 \cdot 1 \cdot c_2$$

$$\left.\frac{d^3 f}{dx^3}\right|_{x=a} = 3 \cdot 2 \cdot 1 \cdot a_3 = (3!)\, c_3$$

$$\left.\frac{d^n f}{dx^n}\right|_{x=a} = (n!)\, c_n$$

Calculating the Coefficients

- If I'm trying to represent a known functional form, I can use this to calculate the coefficients

$$c_0 = f(a)$$

$$c_1 = \left.\frac{df}{dx}\right|_{x=a}$$

$$c_2 = \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a}$$

$$c_3 = \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=a}$$

$$c_n = \frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=a}$$

- I can then redefine my series as

$$f(x) = f(a) + \left.\frac{df}{dx}\right|_{x=a}(x-a) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=1}(x-a)^2 + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=a}(x-a)^n$$

Maclaurin Series

- A Maclaurin Series is a special case of a Taylor Series, expanded about zero*.

$$f\left(x\right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots = \sum_{n=0}^{\infty} c_n x^n$$

$$f\left(x\right) = f(0) + \left.\frac{df}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=0} x^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=0} x^n$$

*Most people still just call it a Taylor Series

e^{ax}

$$e^0 = 1$$

$$\left.\frac{d}{dx} e^{ax}\right|_{x=0} = a$$

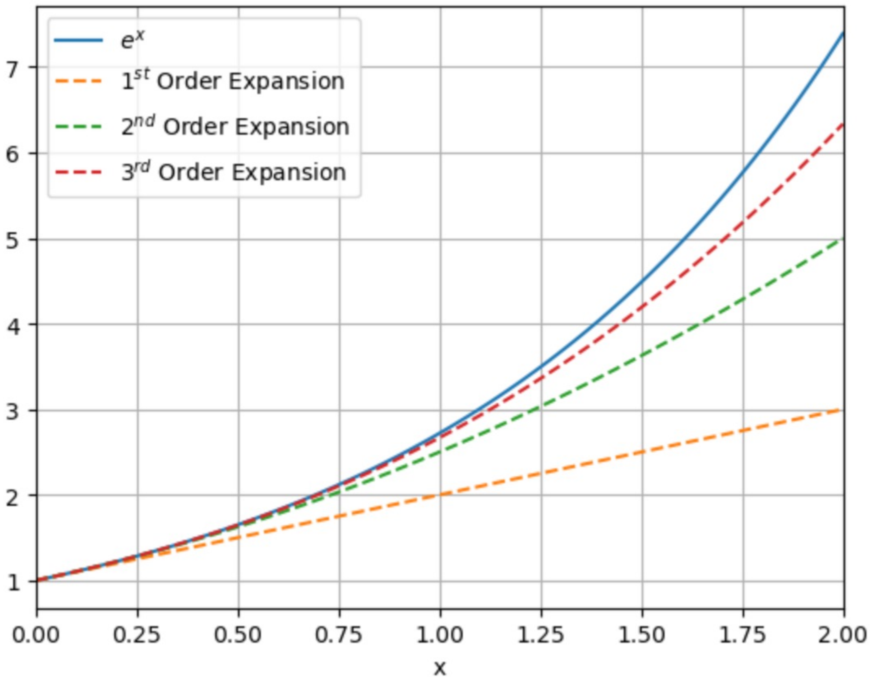
$$\left.\frac{d^2}{dx^2} e^{ax}\right|_{x=0} = a^2$$

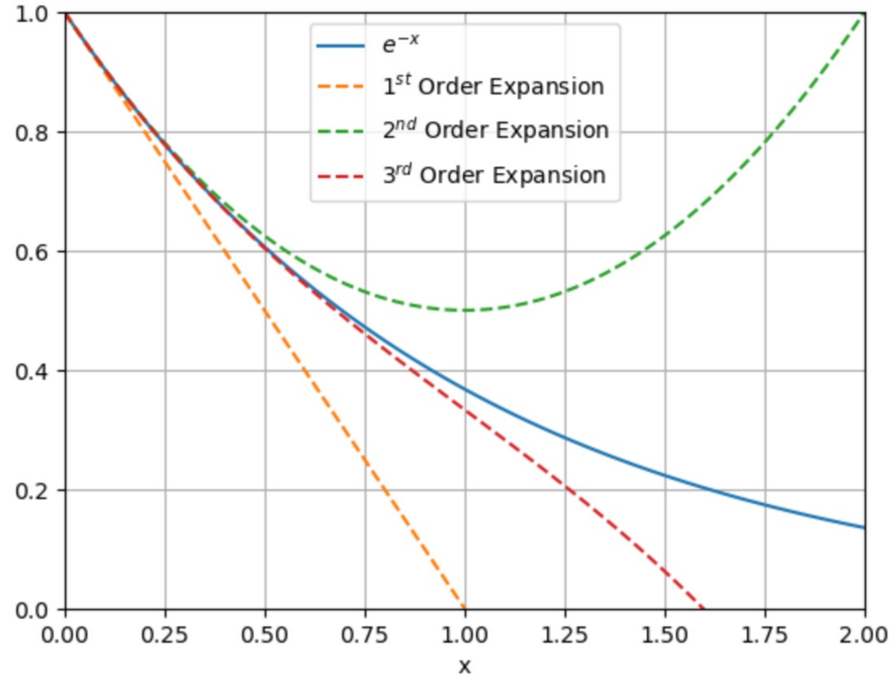
$$\left.\frac{d^3}{dx^3} e^{ax}\right|_{x=0} = a^3$$

$$\rightarrow e^{ax} = 1 + ax + \frac{1}{2!}(ax)^2 + \frac{1}{3!}(ax)^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}(ax)^n$$

Note! Sign will alternate if *a* is negative

Accuracy (See Notebook)





sin(x)

$$\sin(0) = 0$$

$$\left.\frac{d}{dx}\sin\right|_{x=0} = \cos(0) = 1$$

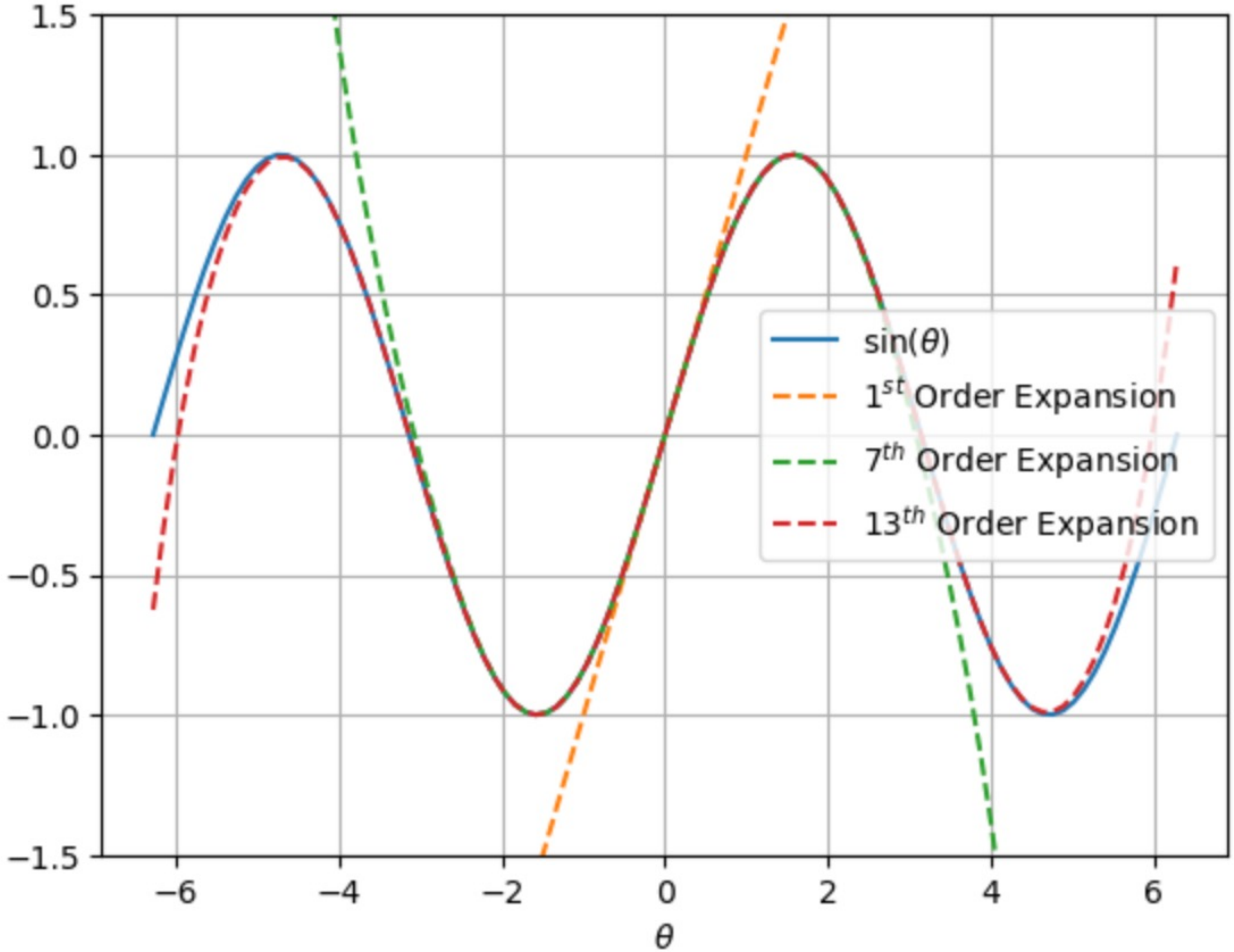
$$\left.\frac{d^2}{dx^2}\sin\right|_{x=0} = -\sin(0) = 0$$

$$\left.\frac{d^3}{dx^3}\sin\right|_{x=0} = -\cos(0) = -1$$

$$\to \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots$$

Only odd terms

Sine Accuracy



Need a lot of terms

cos(x)

$$\cos(0) = 1$$

$$\left.\frac{d}{dx}\cos\right|_{x=0} = -\sin(0) = 0$$

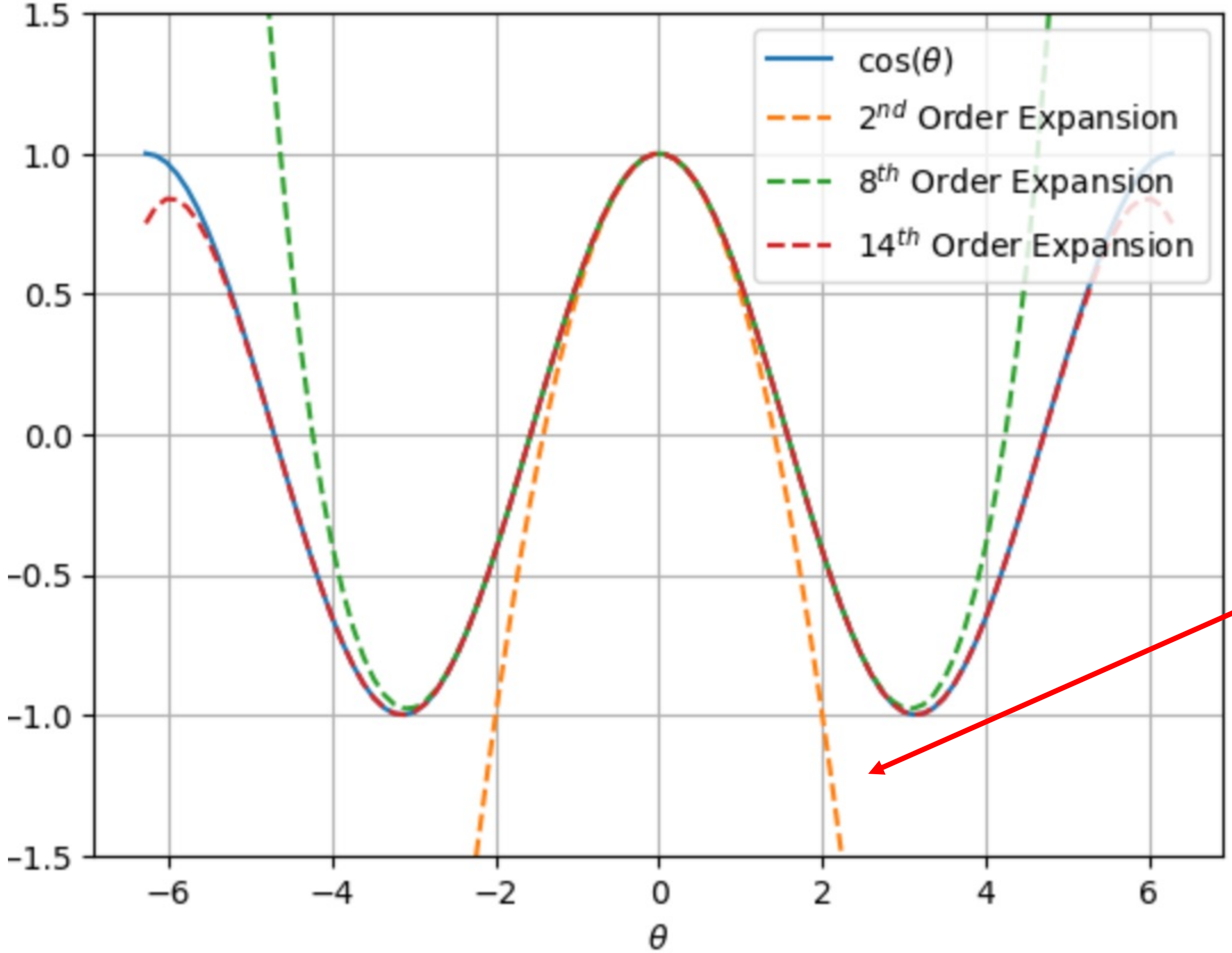
$$\left.\frac{d^2}{dx^2}\cos\right|_{x=0} = -\cos(0) = -1$$

$$\left.\frac{d^3}{dx^3}\cos\right|_{x=0} = \sin(0) = 0$$

$$\rightarrow \cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$$

Only even terms

Cosine Accuracy



2nd order is a parabola