STATISTICS AND DISTRIBUTIONS

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Phy 40
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Probabilities

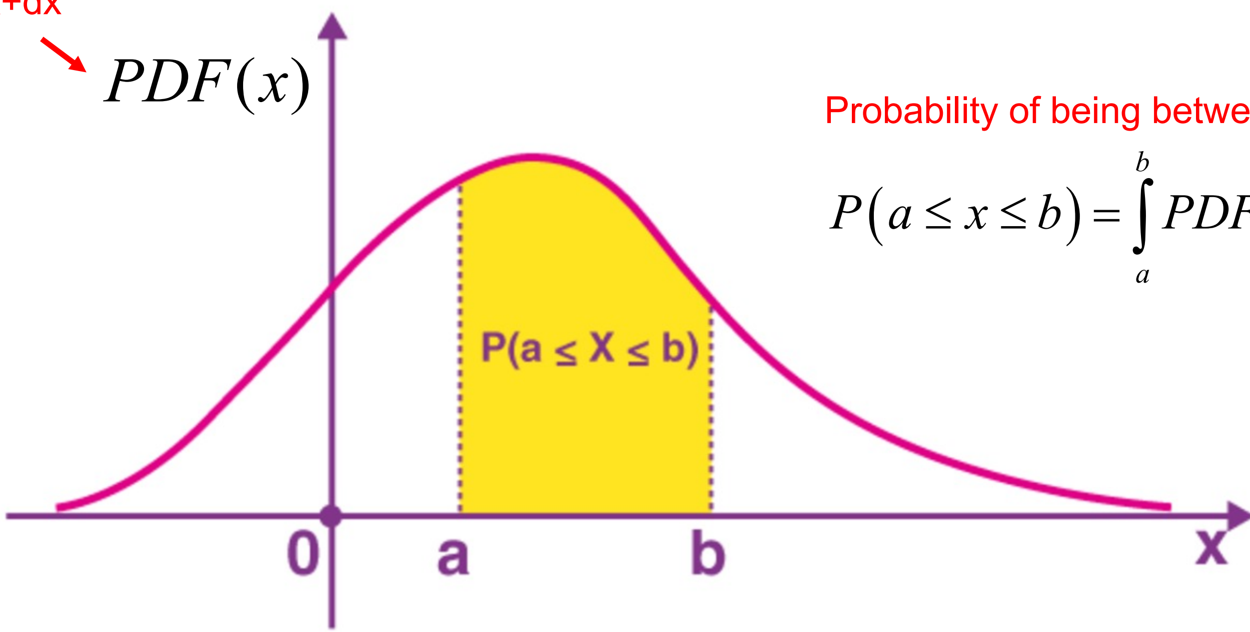
- At the microscopic level, everything in physics is governed by probabilities.
- When statistics become large enough, the uncertainties become small enough that probable outcomes become virtual certainties
  - This is how casinos make money!
- In this course, we will analyze some of the statistical properties most often encountered in physics.

Probability Distributions

- In general, we can characterize a probability by a "Probability Distribution Density Function" (PDF). In one dimension, this is defined as

Probability of being between x and x+dx

$$PDF(x)$$



Probability of being between *a* and *b* is

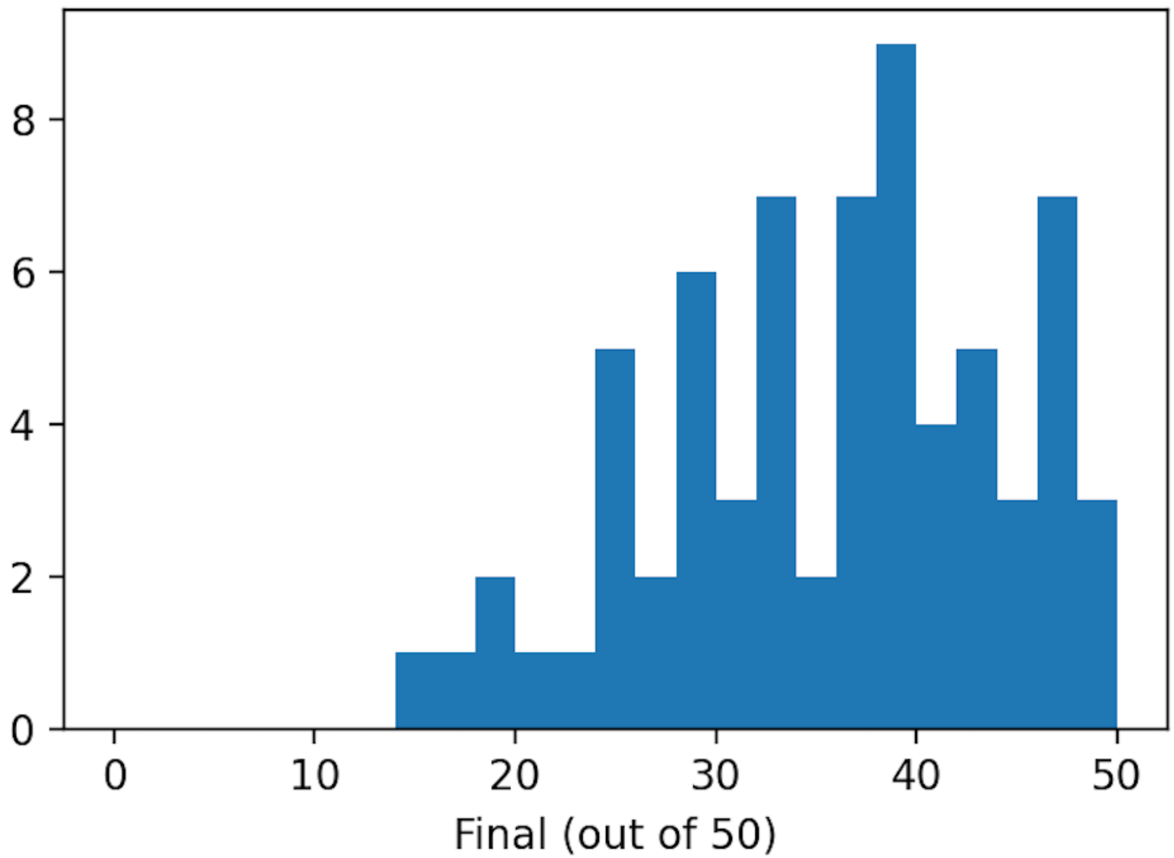
$$P(a \le x \le b) = \int_a^b PDF(x)\,dx$$

Normalization:

$$\int_{-\infty}^{\infty} PDF(x)\,dx = 1$$

Histograms

- We generally plot distributions using “histograms”, which record the number of entries falling within particular ranges.
  - Example: score on Phy 009D final exam



Histogramming Tools in Python

- Python has two basic tools for histogramming

numpy.histogram

```
numpy.histogram(a, bins=10, range=None, density=None, weights=None)
```

Compute the histogram of a dataset. [source]

  - **bins** can either be a number of (equal) bins or a list of bin edges, which can have unequal width.
  - Returns the bin contents and the bin edges
    - Note: an N bin histogram will have N+1 edges
  - Have to use another tool to plot the histogram

```
matplotlib.pyplot.hist(x, bins=None, *, range=None, density=False,
weights=None, cumulative=False, bottom=None, histtype='bar', align='mid',
orientation='vertical', rwidth=None, log=False, color=None, label=None,
stacked=False, data=None, **kwargs) #
```

[source]

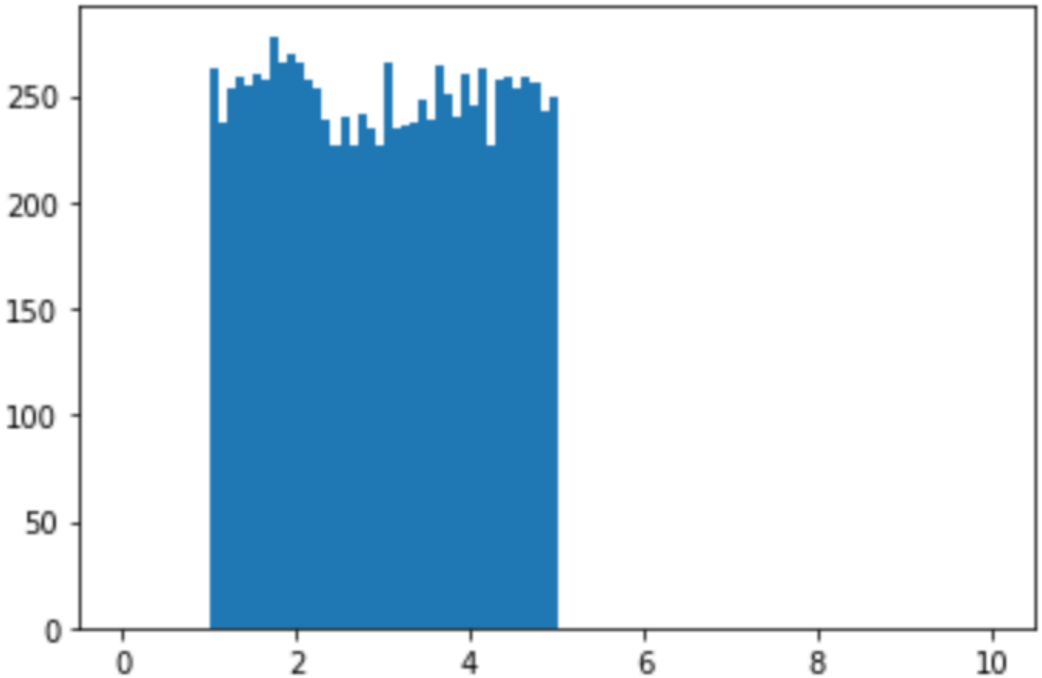
  - Calls **numpy.histogram** and then plots the histogram
- Generally can use the latter unless:
  - You want to process the data in some way before plotting it
  - You want to plot a histogram with error bars!

Random Numbers in Python

- The numpy.random package contains a powerful set of routines to generate random numbers, including
  - Uniform random numbers in a specified range
  - Distributions based on user-specified probabilities
  - Normal (Gaussian) distributions
  - Binomial distributions (we'll discuss this shortly)
  - Many, many, more
- Without a physical inputs, computers cannot generate truly random numbers. They generate "pseudo-random" numbers, based on a starting "seed".
  - If unspecified, they usually use the unique time as the seed.
  - If a seed is specified (np.seed(int)), then *identical* subsequent random number calls will produce identical numbers each time
    - This can be very important for debugging code!
- Do uniform and Gaussian demo…

Uniform Probability

- A uniform probability is a distribution within a specified range in which every value in the range has an equal probability.
- Example: a uniform distribution from 1 to 5



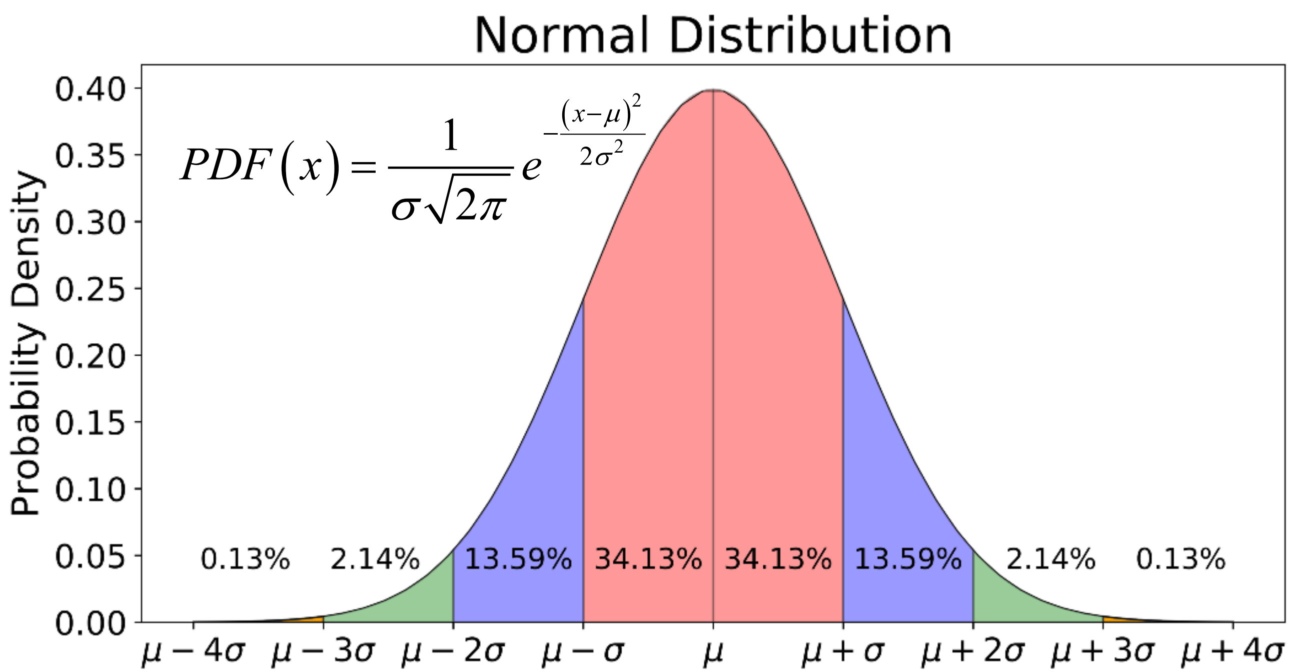
$$x < x_{min} : PDF(x) = 0$$

$$x_{min} \leq x \leq x_{max} : PDF(x) = \frac{1}{(x_{max} - x_{min})}$$

$$x > x_{max} : PDF(x) = 0$$

Normal Distribution

- A normal (or “Gaussian”) distribution is a probability characterized by a mean (μ) and a standard deviation (σ), according to the following PDF



- As we will see, in some limit, all errors can be characterized by this distribution (“central value theorem”)

Binomial Distributions

- Let's imagine I flip a coin 10 times, and it comes up heads 4.
- If I look at the first heads, there are 10 possibilities for which throw it came from, 9 for the second, 8, for the third, and 7 for the fourth, for a total of

$$10\times 9\times 8\times 7=\frac{10!}{6!}=\frac{10!}{(10-4)!}$$

possibilities
- But I don't actually care which throw each came for, so for the first heads there are 4 possibilities that are identical, 3 for the second, the for a total of 4! Indistinguishable "permutations", so the total number of cases with 4 heads is

$$\frac{10!}{4!(10-4)!}$$

- Generalizing this, the number of times, we get n_H out of N throws is given by

$$W_{n_H}^{N}=\frac{N!}{n_H!(N-n_h)!}\equiv\binom{N}{n_H}$$

"Binomial coefficient"

Probabilities

- If I flip a coin *N* times, there are 2^N possible unique sequences, so the probability of finding n_H after *N* throws is

$$P_{n_H}^N = \frac{1}{2^N}\frac{N!}{n_H!\left(N-n_h\right)!} \equiv \frac{1}{2^N}\binom{N}{n_H}$$

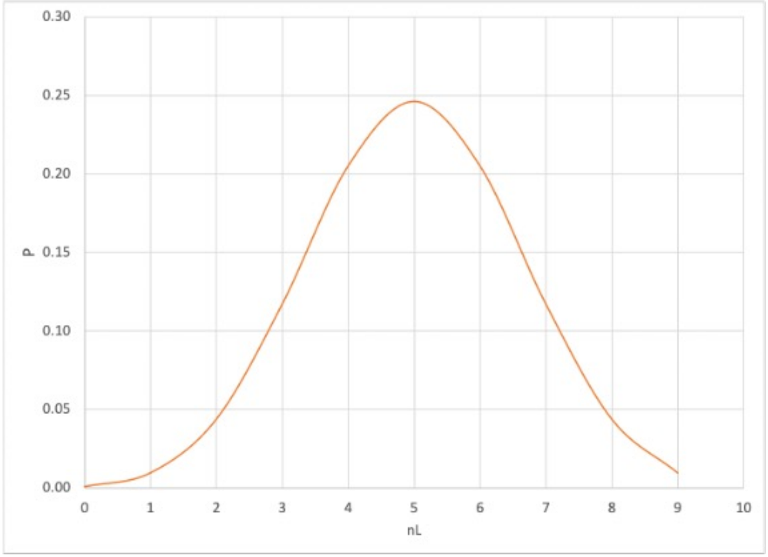
- For 10 throws, there are 2^{10}=1024 possible combinations, so the table of probabilities is

n_H	$W(10,n_H)$	$P(n_H)$
0	1	0.0010
1	10	0.0098
2	45	0.0439
3	120	0.1172
4	210	0.2051
5	252	0.2461
6	210	0.2051
7	120	0.1172
8	45	0.0439
9	10	0.0098
10	1	0.0010

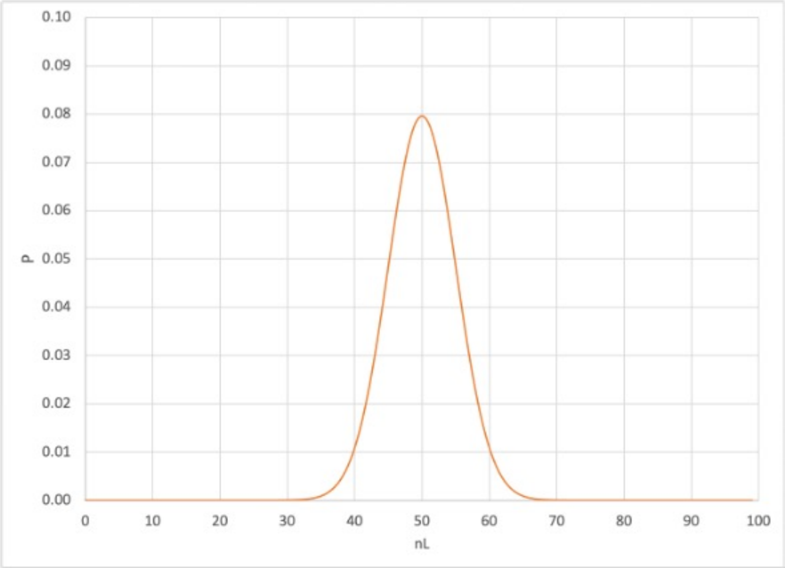
Going to More Throws…

- The more throws, the narrower the distribution becomes

10 Throws



100 Throws



- Of course, the number of particles in a box is much bigger than 100, so the fraction on the left will be *very* close to ½ , but will have a distribution.

Generalized Binomial Distribution

- In our example, a coin had an equal probability (50%) of coming up heads or tails.
- If instead we have two distinct outcomes, **A** and **B**, in which the probability of **A** is *p*, the probability of **B** is therefore *(1-p)*, and the normalized binomial distribution is modified as follows
  - The probability of n_A outcomes in state A out of *N* total trials is

$$P_{n_A}^{N} = \frac{N!}{n_A!\left(N-n_A\right)!}p^{n_a}\left(1-p\right)^{\left(N-n_A\right)} \equiv \begin{pmatrix} N \\ n_H \end{pmatrix} p^{n_a}\left(1-p\right)^{\left(N-n_A\right)}$$

- The average value over many trials of *N* "throws" will be

$$\lambda = pN$$

- And the RMS of the distribution will be

Relative width gets narrower

$$\sigma = \sqrt{Np(1-p)} = \sqrt{\lambda(1-p)} \qquad \frac{\sigma}{\lambda} = \sqrt{\frac{(1-p)}{\lambda}} = \sqrt{\frac{(1-p)}{Np}}$$

Simulating Binomial Distributions in Python

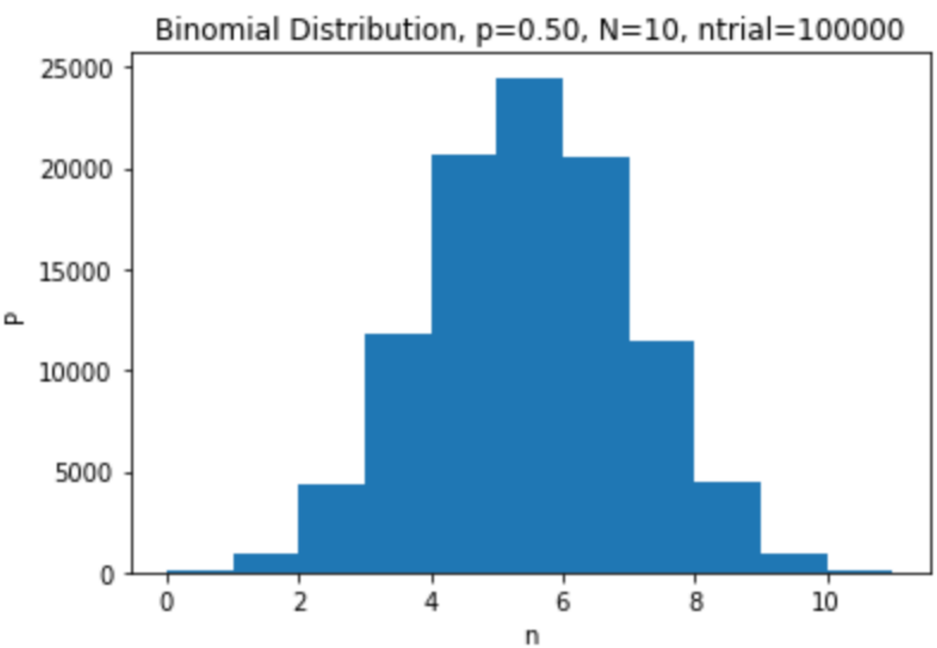
- We will simulate binomial distributions with
  - numpy.random.binomial(N,p,Ntrial)

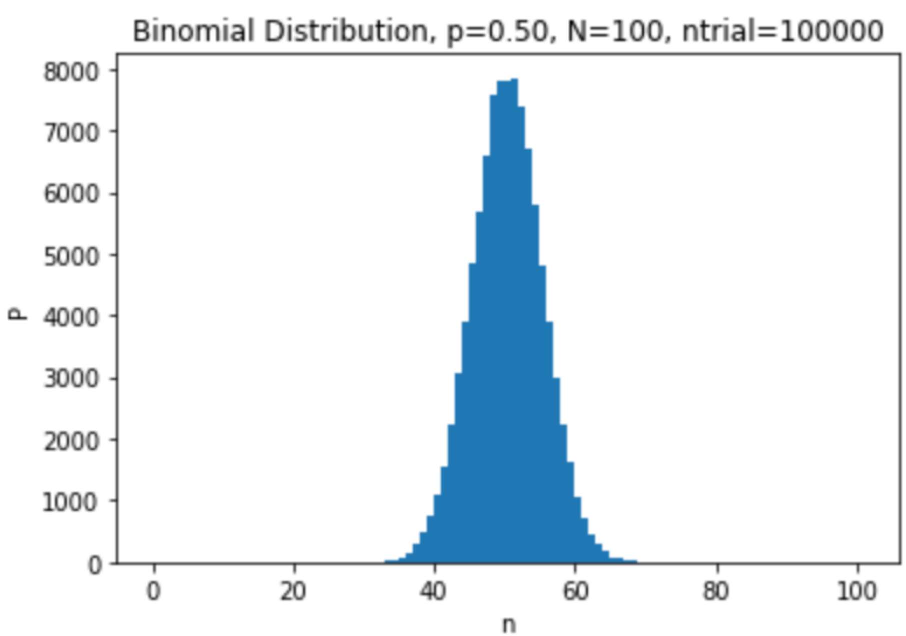
Number of "throws"

Probability each throw

Number of times I repeat the test.
If 1 (or missing), it returns a scalar,
otherwise it returns a vector

- For a coin, p=1/2





Die Example…

- If *p* is the probability of an event occurring, then the expected number of events for *N* “throws” is

$$\lambda = pN$$

- But what if $\lambda<1$?
- Example: throwing a 6-sided die three times
- The chance of it coming of any particular number, say 4, is 1/6, so the average number of will come up 6 in 3 throw is

$$\lambda = \frac{1}{6}3 = .5$$

- Of course, it can’t come up .5 times, it can only come up 0,1,2, or 3 times.

Binomial Distribution for Die Throws

- In this case, our options are coming up 4 (*p=1/6*) or coming up any other number (*q=(1-p)=5/6*)
- The binomial theorem tells is that the probability of coming with 4 a particular number of times is given by

$$P_{n_A}^N = \binom{N}{n_A} p^{n_A} (1-p)^{(N-n_A)} \rightarrow P_{n_4}^3 = \frac{3!}{n_4!\,(3-n_4)!}\left(\frac{1}{6}\right)^{n_4}\left(\frac{5}{6}\right)^{(3-n_4)}$$

$$\mathrm{P}_0^3 = \frac{3!}{0!(3)!}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = 0.5787$$

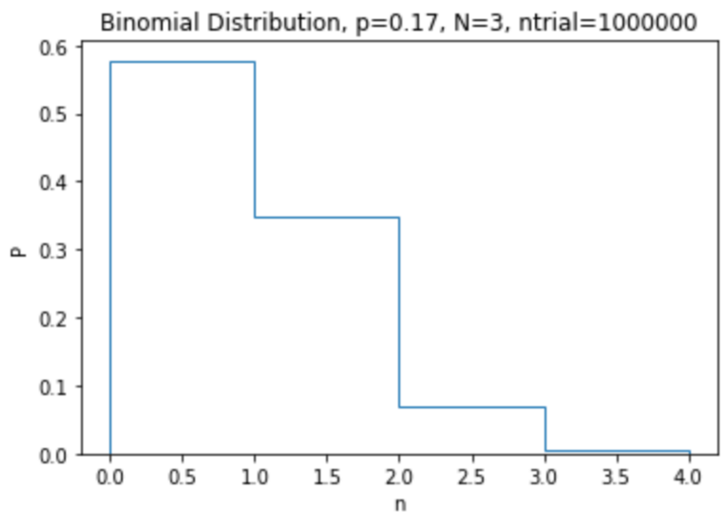
$$\mathrm{P}_1^3 = \frac{3!}{1!(2)!}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = 0.3472$$

$$\mathrm{P}_2^3 = \frac{3!}{2!(1)!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = 0.0694$$

$$\mathrm{P}_3^3 = \frac{3!}{3!(0)!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = 0.0046$$

$$P_0^3 + P_1^3 + P_2^3 + P_3^3 = 1$$

```
Hist values =  [0.577866 0.347917 0.06961  0.004607]
Predicted average = 0.5000
Measured average = 0.5010
Predicted standard deviation = 0.6455
Measured standard deviation = 0.6456
Standard devation/N =  0.21521593348593457
```



Approximations

- The binomial expression is exact, but factorials get problematic pretty fast

N	FACT(N)
168	2.5261E+302
169	4.2691E+304
170	7.2574E+306
171	#NUM!

170! is the biggest number that can fit in a 64-bit floating point, and we deal with numbers in the millions, billions, or more

- Generally, we use two approximations, depending on the expectation values on the counts
  - "Small statistics" ($\lambda<10$): Poisson statistics
    - Example: throwing a die 3 times
  - "Large statistics" ($\lambda\geq 10$): Gaussian statistics
    - Example: flipping a coin at least 10 times

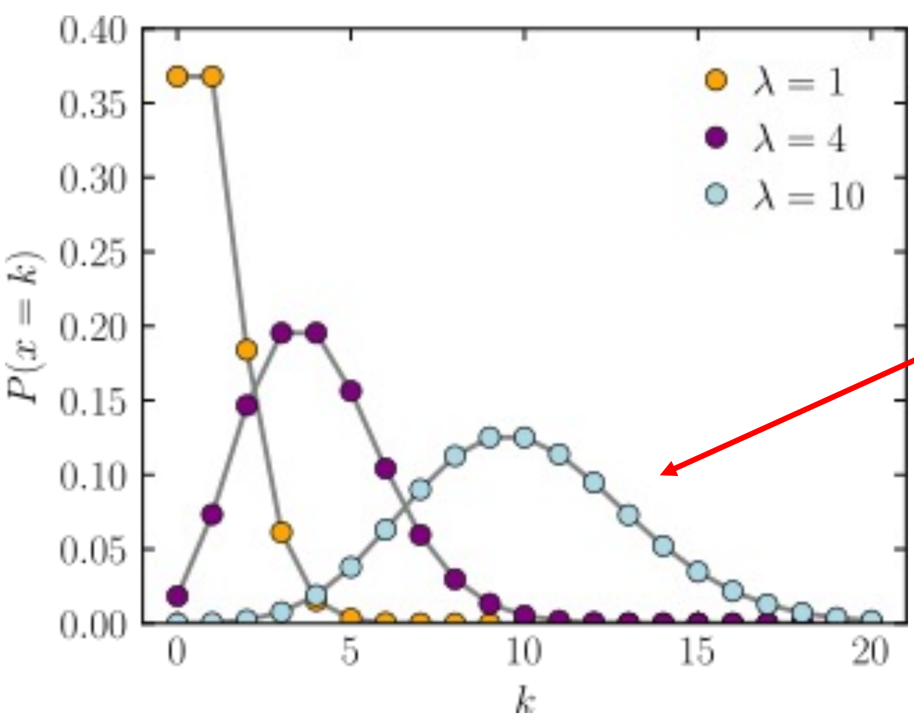
Poisson Distribution

Probability *Mass* Function

- For small statistics (λ<10), the PMF can be approximated by

$$P(k) = \frac{1}{k!}\lambda^k e^{-\lambda}$$

This equation will also continue to work, but again, these terms will quickly become problematic for large λ



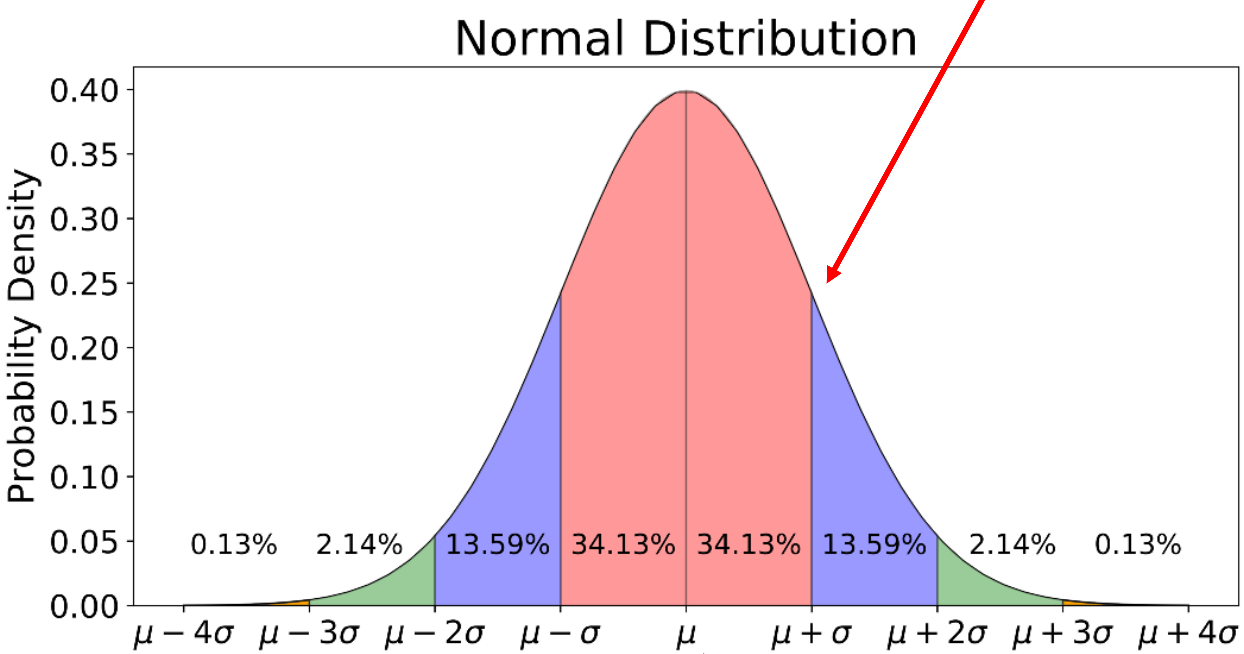
Note the equation becomes symmetric for a λ of 10 or more

Gaussian Distribution

- For larger values of l, we can approximate things with a gaussian distribution.

$$P(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\lambda)^2}{2\sigma^2}}$$

$\sigma = \sqrt{\lambda(1-p)}$



$\mu = \lambda$

Generating These Distributions Functions in Python

- We will use the scipy.stats package
  - Poisson distribution

    "probability mass function"

    - scipy.stats.poisson.pmf(k,lam)

      Point or array of points

      λ

    - PMF designed for discrete integer points
  - Gaussian (normal) distribution

    "probability density function"

    - scipy.stats.norm.pdf(x,loc=mean,scale=sigma)

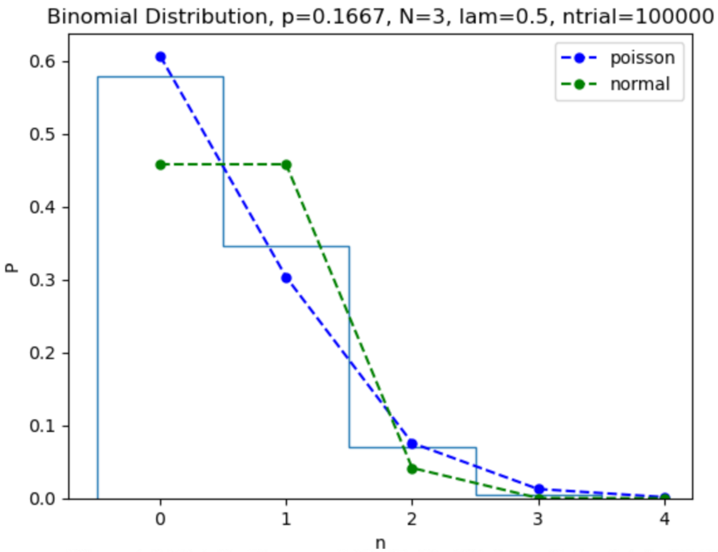
      Point or array of points

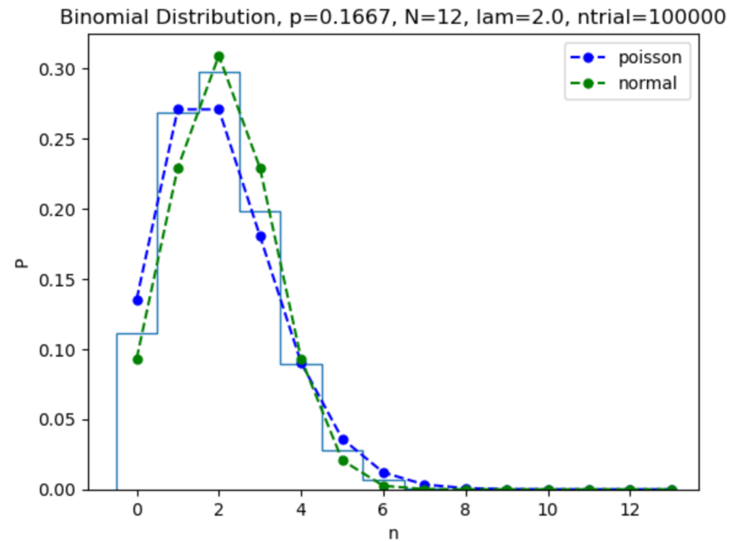
      λ

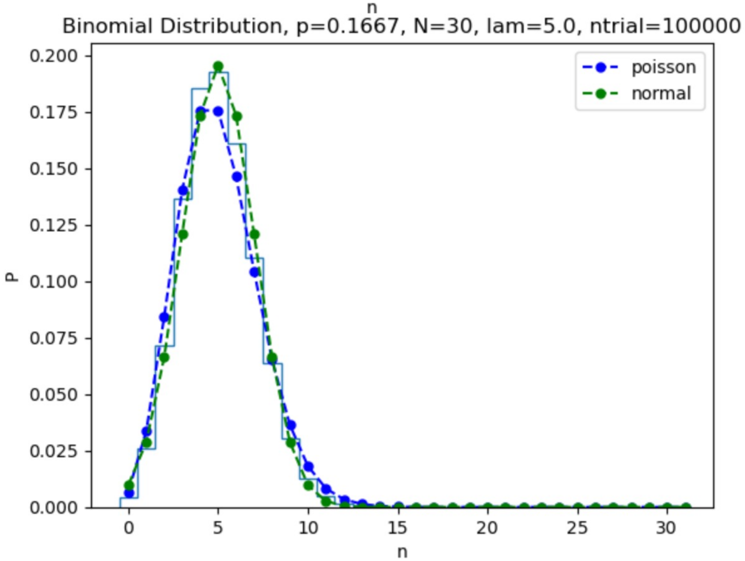
      σ

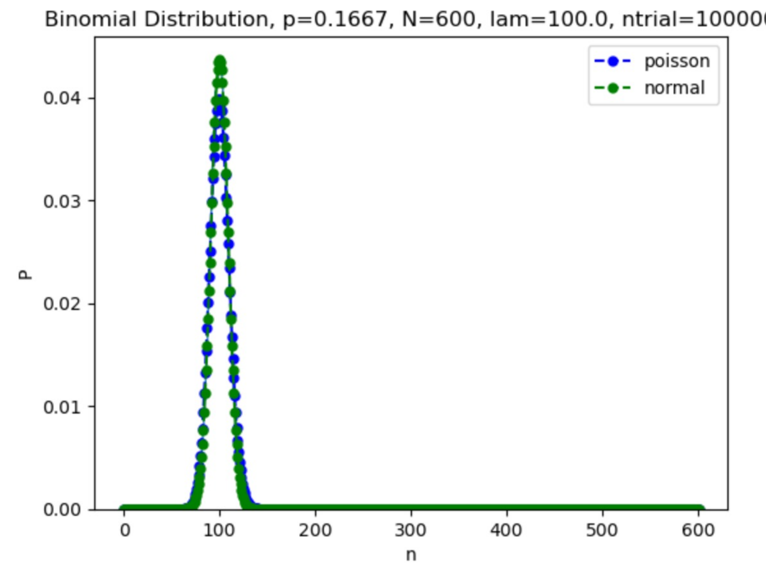
    - PDF is a continuous function
      - *strictly speaking*, we should integrate it over each integer value, but we usually won't do that

Back to Dice









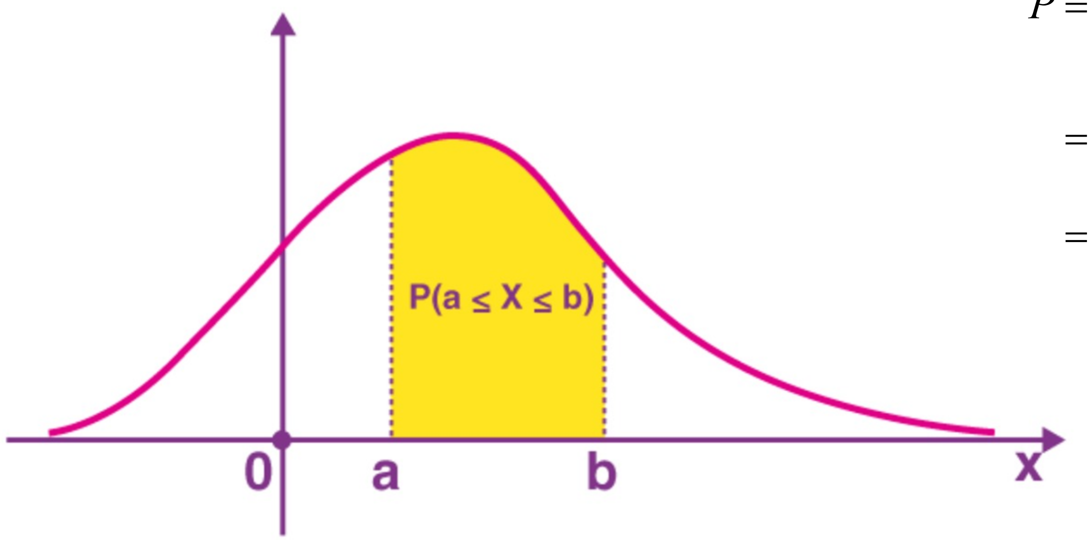
"Central Limit Theorem" = "Everything eventually becomes Gaussian"

Cumulative Density (or Mass) Functions

- For any probability distribution, we can define the ``Cumulative Density function as the integral up to a particular value.

$$CDF(x) = \int_{-\infty}^{x} PDF(z)dz$$

- So we can get the probability of falling between any two points as



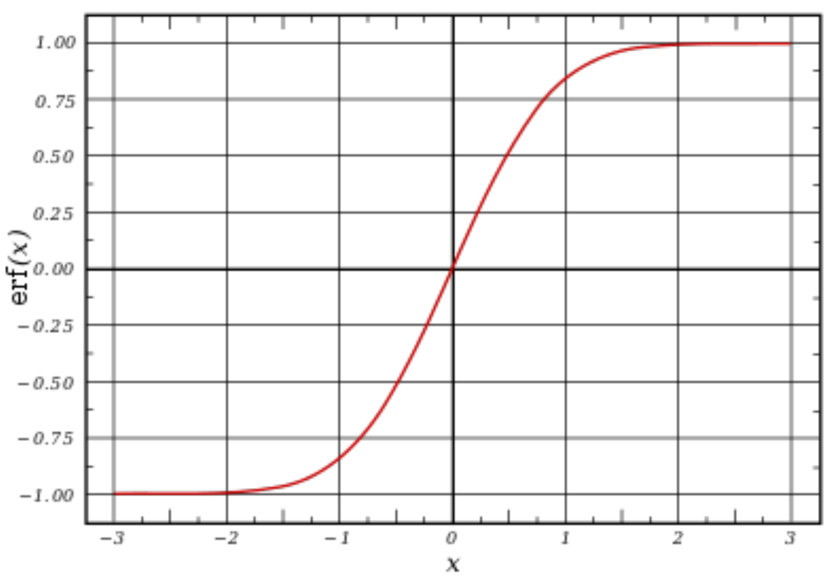
$$\begin{aligned} P &= \int_a^b PDF(x)dx \\ &= \int_{-\infty}^{b} PDF(x)dx - \int_{-\infty}^{a} PDF(x)dx \\ &= CDF(b) - CDF(a) \end{aligned}$$

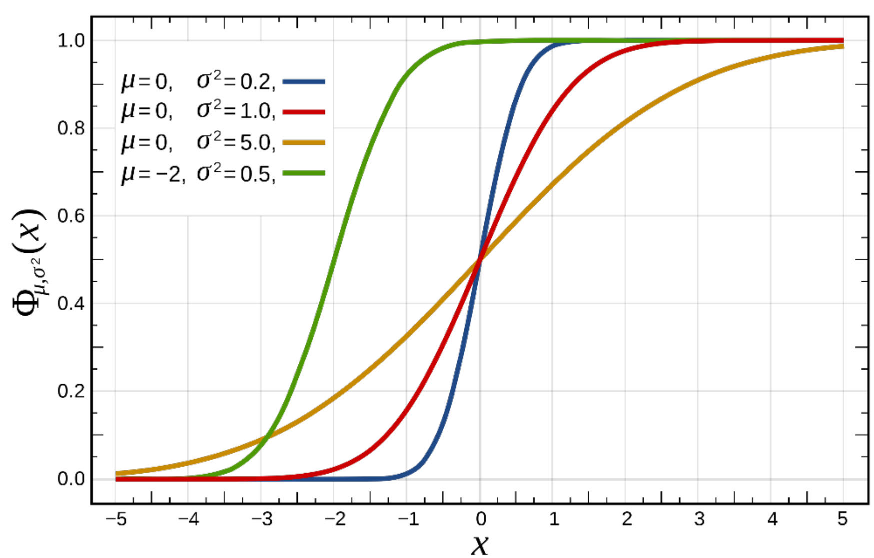
CDF for a Gaussian

- Unfortunately, the indefinite integral of a Gaussian is not defined, so we define it in terms of the "error function" (erf())

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\mathrm{CDF}_{\mathrm{norm}}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz$$
$$= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right)$$





Gaussian CDF in Python

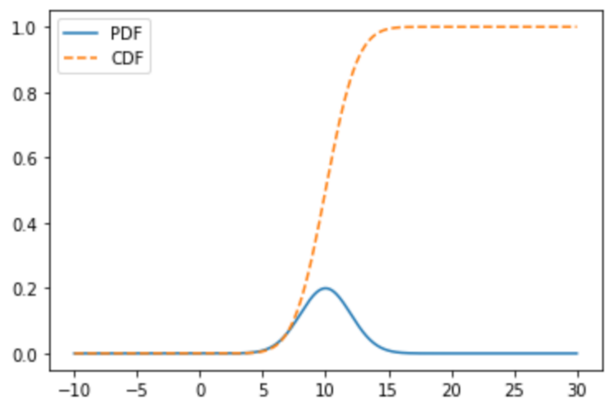
- In Python, the CDF is extracted through the cdf method of stats.norm

[84…
```
# Generate Nsignal events according to a Guassian distribution
mu = 10.
sig = 2.
Nsignal=1000
ssig  = np.random.normal(mu,sig,Nsignal)
```

[85…
```
xlin = np.linspace(-10.,30.,1000)
```

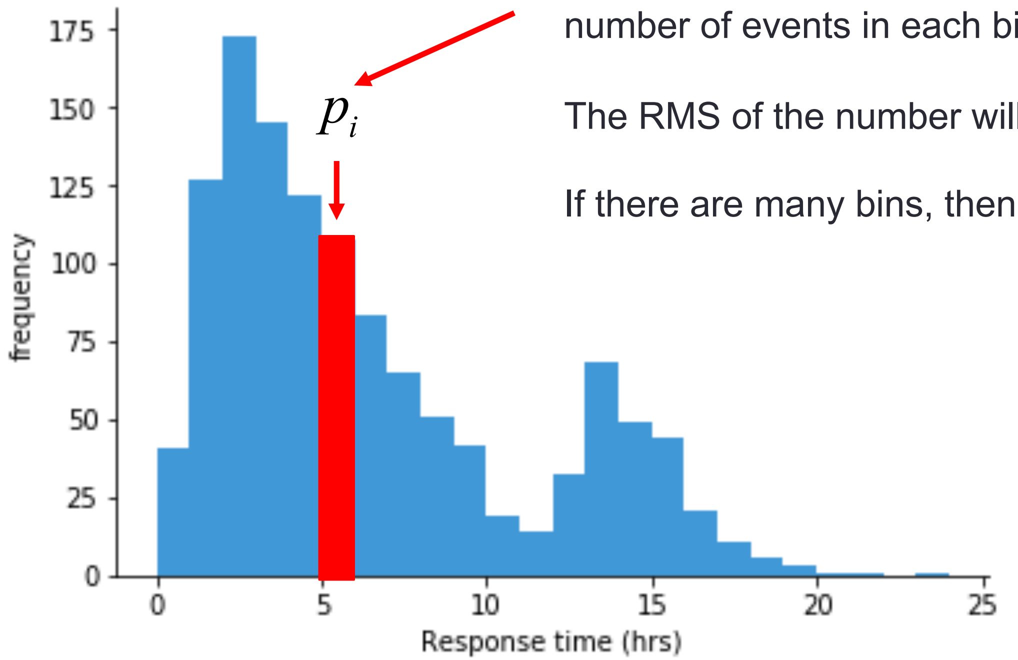
[86…
```
ypdf = stats.norm.pdf(xlin,loc=mu,scale=sig)
```

[88…
```
ycdf = stats.norm.cdf(xlin,loc=mu,scale=sig)
pl = plt.plot(xlin,ypdf,label='PDF')
pl = plt.plot(xlin,ycdf,"--",label='CDF')
leg=plt.legend()
```



Error on Histogram Bin Contents

- If entries are distributed over many histogram bins, we can look at any single bin and consider the binary choice between that bin an all other bins.



Probability that an event will land in bin *i*.
If there are *N* events total, then the expected number of events in each bin will be $\lambda_i = p_i N$.

The RMS of the number will be $\sigma = \sqrt{\lambda(1-p)}$

If there are many bins, then $p << 1$ and $\sigma \approx \sqrt{\lambda_i} \approx \sqrt{n_i}$

Number of events observed in bin *i*.

Limit of Small Statistics

- We generally set the error on the contents of a bin as the square root on the number of the events observed in the bin.
- This isn't really accurate when the number of events in the bin is small (say <5).
- In particular, if there are no events in the bin, this would say

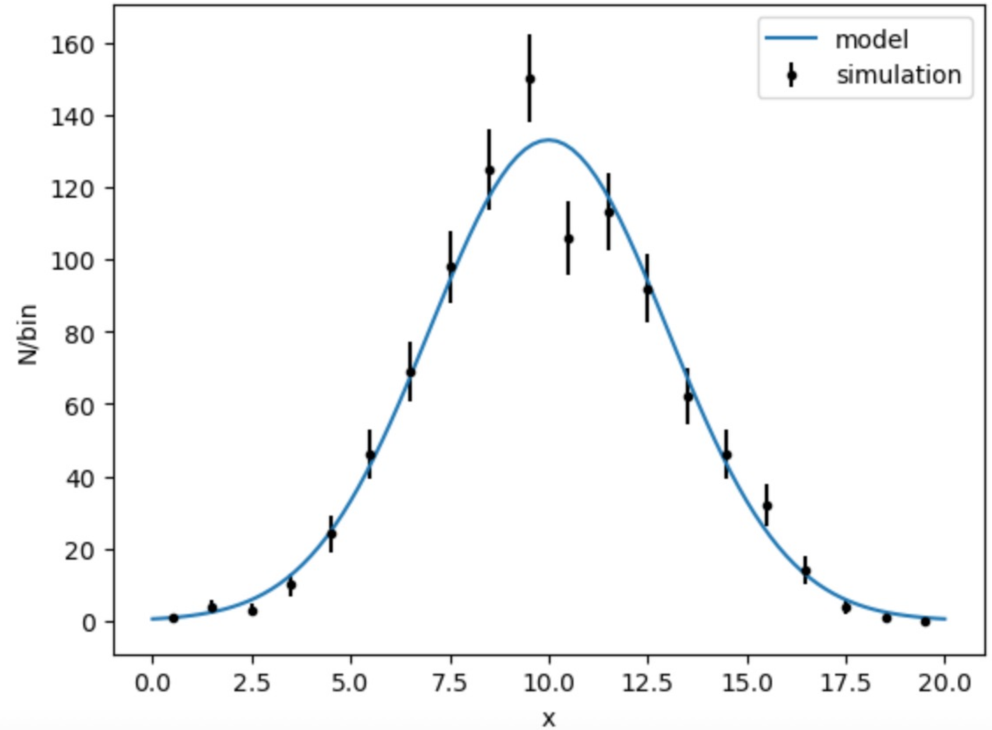
$$\sigma_i \approx \sqrt{n_i} = \sqrt{0} = 0$$

No error on point!

- This will cause any fit to fail if it can't go exactly through the point.
- If we have a small number of bins with zero, we typically set the error to 1.
- If there are a significant number of bins with zero, we need to use a different approach to fitting.

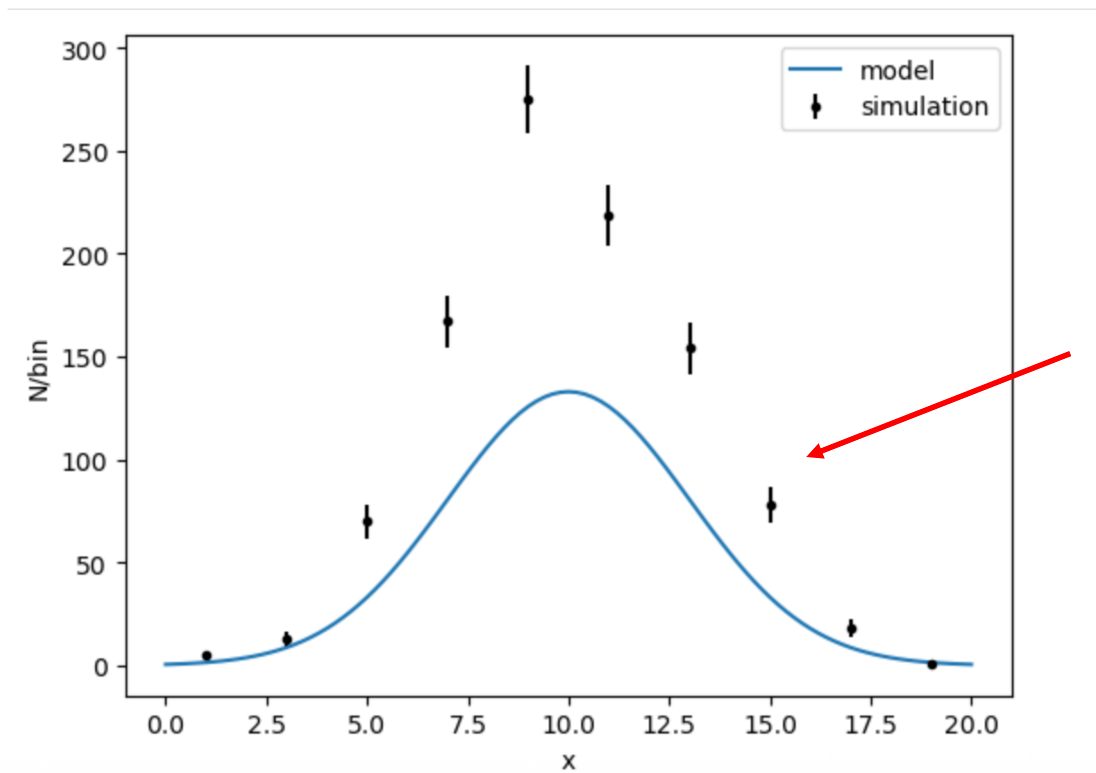
Plotting with Errors

```python
# Histogram them with Unit bin sizes
Nbins = 20
hist,bins = np.histogram(ssig,bins=Nbins,range=(0,Nbins))
hist_error = np.sqrt(hist)
hist_error = np.clip(hist_error,1.,None) # If the sqrt is 0, set it to 1.
xcent = .5*(bins[:-1]+bins[1:])          # Center of the bins
plt.errorbar(xcent, hist, hist_error, fmt="k.",
             label="simulation")
xlin = np.linspace(0.,20.,101)
pl=plt.plot(xlin,Nsignal*stats.norm.pdf(xlin,loc=mu,scale=sig),label='model')
plt.xlabel('x')
plt.ylabel('N/bin')
plt.legend()
plt.show()
```



Normalizing Histograms (Important Concept!)

- In case you haven't noticed, in most of our examples, we've always made our bin size 1. What happens if we change that. Let's redo the last histogram with a bin size of 2



Vhat happened?

Ve added pairs of bins together, which loubled their contents, but my physics houldn't depend on how I bin things.

Iow can I present things in a consistent way?

- Our PDF is defined by the normalization condition

$$\int_{-\infty}^{\infty} PDF(x)dx = 1$$

- The probability of something landing in a particular bin is

$$\int_{x_{lo}}^{x_{hi}} PDF(x)dx = \langle PDF \rangle \left(x_{hi} - x_{lo} \right)$$

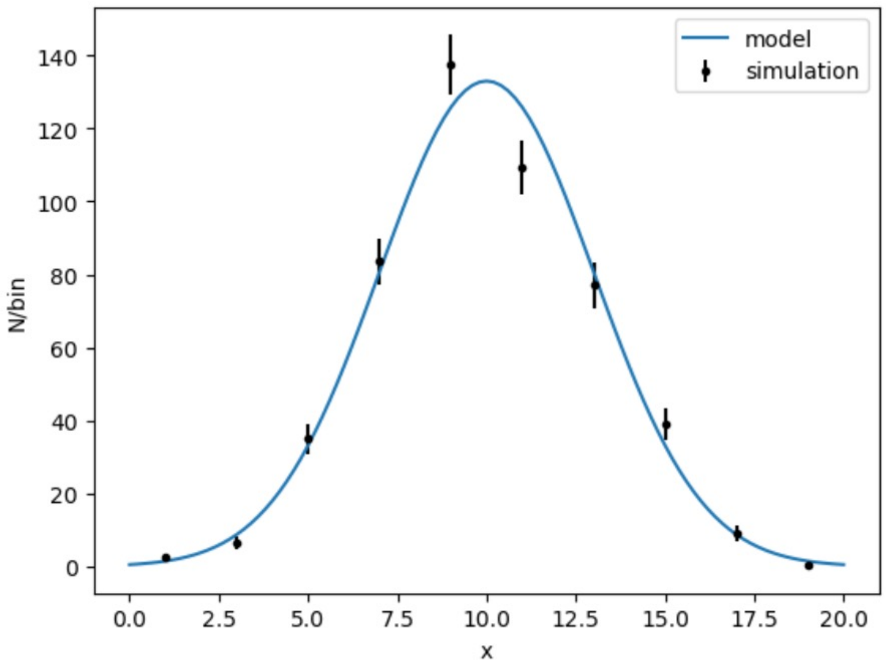
Average value over the bin

- If I divide the bin contents (and the error!) by the size of the bin, I should get a result that's independent of bin size.

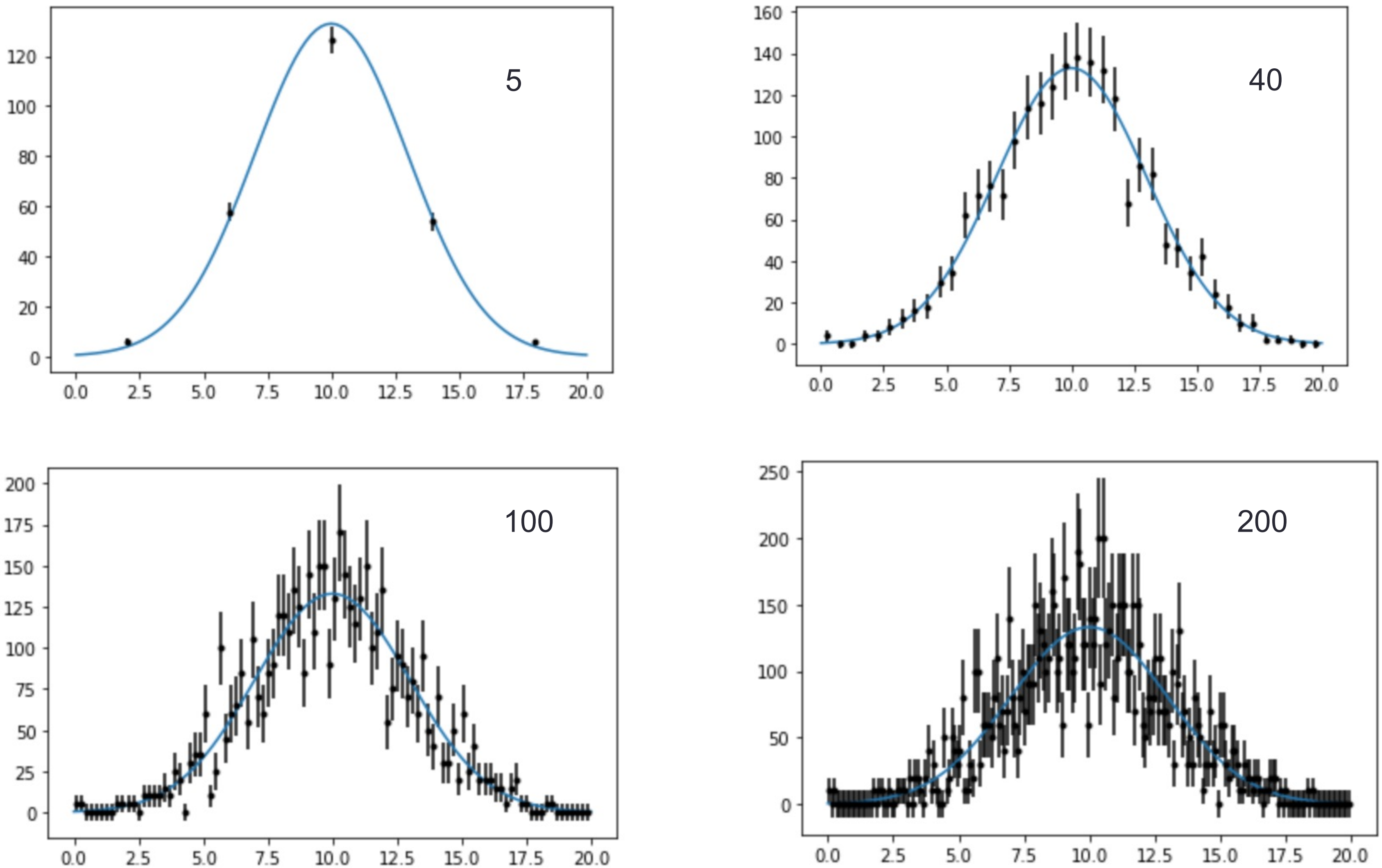
Check

```
# Histogram them with arbitrary bin sizes
plotrange = (0.,20)
Nbins = 10
hist,bins = np.histogram(ssig,Nbins,plotrange)
hist_error = np.sqrt(hist)
hist_error = np.clip(hist_error,1.,None) # If the sqrt is 0, set it to 1.
xcent = .5*(bins[:-1]+bins[1:])
# Normalize
# Let's do it in a way that will work with arbitary bins
binsize = bins[1:]-bins[:-1]
hist = hist/binsize
hist_error /= binsize

plt.errorbar(xcent, hist, hist_error, fmt="k.",
             label="simulation")
xlin = np.linspace(0.,20.,101)
pl=plt.plot(xlin,Nsignal*stats.norm.pdf(xlin,loc=mu,scale=sig),label='model')
plt.xlabel('x')
plt.ylabel('N/bin')
plt.legend()
plt.show()
```



Other Binnings

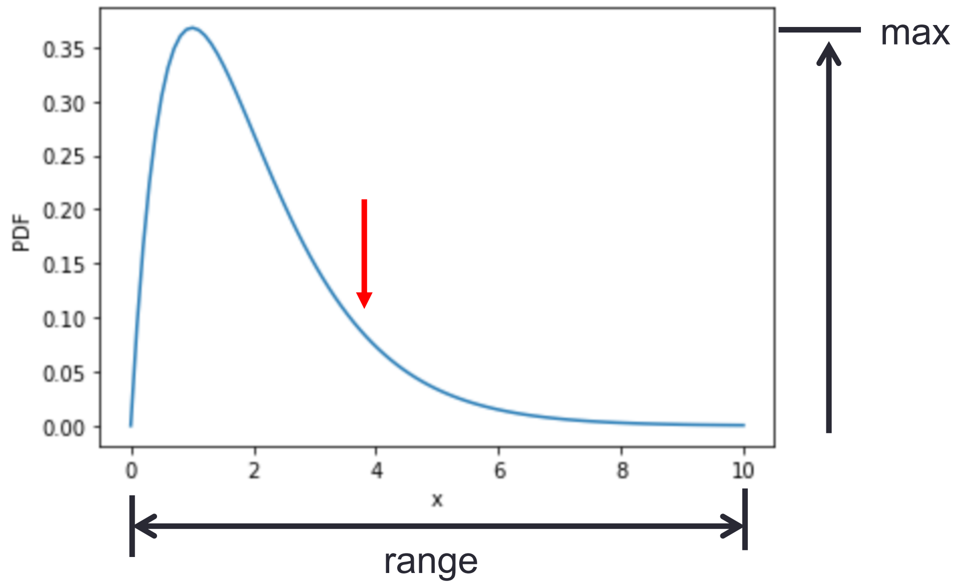


Arbitrary Distributions

- In many cases, we want to generate a distribution based on an arbitrary parametrization.
  - Example, simulating background to a physics signal
- In this case, we can use a “Monte Carlo” technique to simulate any distribution

Procedure

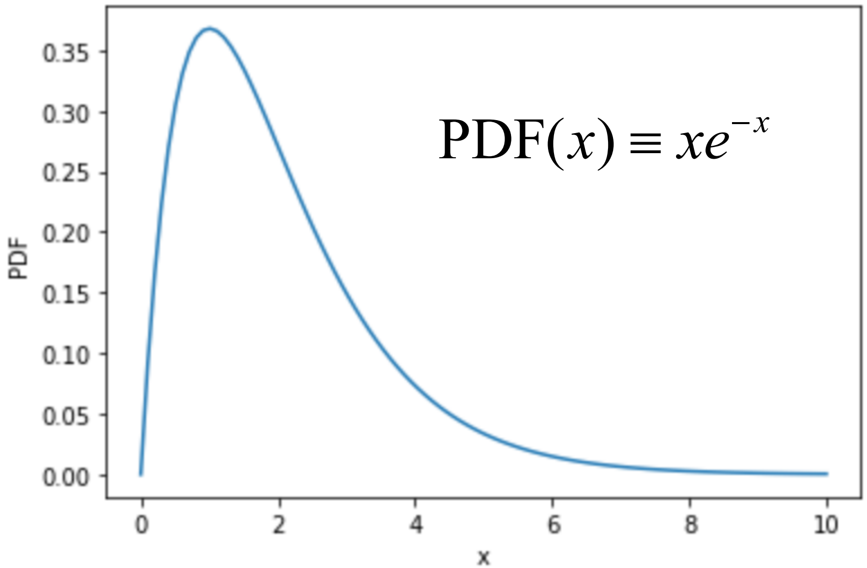
- Example:



- Step 0: Choose a range for your random numbers
- Step 1: Generate a random value of *x* which is *uniformly* distributed over the range.
- Step 2: Evaluate the PDF at this point
- Step 3: Is the PDF value at this point greater than a random number between 0 and that maximum value of the function?
  - Yes? Return the value of *x*
  - No? Go back to step 1

Example

- Generate random numbers with a distribution



Already normalized.

$$\int_0^\infty \mathrm{PDF}dx = \int_0^\infty xe^{-x}dx$$
$$= 1$$

$$\frac{d}{dx}\mathrm{PDF}(x) = (1-x)e^{-x}$$
$$\to x_{max} = 1$$
$$(PDF)_{max} = e^{-1}$$

- It looks like a range of 0 to 10 is reasonable, so let's code this up…

Example Functions

[24…

```
# Returns N 1-D random numbesr based on the function myfunc, in the range xlo to xhi
# funcmax is the maximum value of funcmax
#
def mydist(func,N=1,xlo=0.,xhi=1.,funcmax=1.):
    s = np.empty(N)          # Generate an empty vector N long
    for i in range(0,N):
        while (True):        # Loop until we find a "good" number
            x = np.random.uniform(xlo,xhi)
            if(func(x)>=funcmax*np.random.uniform()):
                break        # Exit the generation loop
        s[i]=x
    return s
```

[25…

```
# Test distribution function
# function x*exp(-x)
#
def distfunc(x):
    val = x*np.exp(-x)  # This is already normalized to 1
    return val
```

[28…

```
# Plot a histogram of my test function
Nsignal = 10000
xlo=0.
xhi=10.
plotrange = (xlo,xhi)
Nbins = 50
ssig = mydist(distfunc,Nsignal,xlo=xlo,xhi=xhi,funcmax=np.exp(-1.))
# Now do everything the way we did it before
hist,bins = np.histogram(ssig,Nbins,plotrange)
hist_error = np.sqrt(hist)
hist_error = np.clip(hist_error,1.,None) # If the sqrt is 0, set it to 1.
binsize = bins[1]-bins[0]
binloc = bins[0:-1]+.5*binsize    # Move location to the middle of the bin
hist = hist/binsize               # Normalize
hist_error = hist_error/binsize   # Normalize error
plt.errorbar(binloc, hist, hist_error, fmt="k.",
             label="simulation")
xlin = np.linspace(xlo,xhi,100)
pl=plt.plot(xlin,Nsignal*distfunc(xlin))
lx = plt.xlabel("x")
ly = plt.ylabel("dN/dx")
```