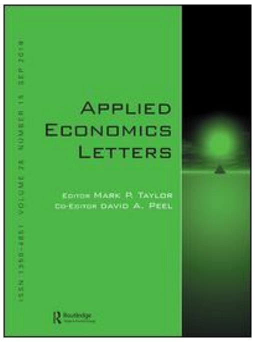


Applied Economics Letters

ISSN: 1350-4851 (Print) 1466-4291 (Online) Journal homepage: https://www.tandfonline.com/loi/rael20

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APPLIED ECONOMICS LETTERS
https://doi.org/10.1080/13504851.2020.1738322

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# A Monte Carlo synthetic sample based performance evaluation method for covariance matrix estimators

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**ABSTRACT**
The evaluation of covariance matrix estimators is very important for portfolio analysis and risk management. The Monte Carlo synthetic sample based performance evaluation method proposed by this article can avoid the main shortcomings of statistical and economic methods which are widely used in the existing literature. The proposed method does not need the true covariance and does not need to introduce the performance of the out-of-sample portfolios. It is an intuitive, effective and robust measure for both simulation and empirical analysis.

**KEYWORDS**
Synthetic sample; performance evaluation; covariance matrix; estimators

**JEL CLASSIFICATION**
C13; C15; C52; C83

## I. Introduction

For empirical researchers, how to evaluate and select the covariance matrix estimators reasonably in practical application is an important problem, while few studies focus on this topic. Elton and Gruber (1973) indicate that statistical and economic measure criteria are the widely used evaluation methods to compare alternative covariance estimators (see, for instance, Sharpe 1963, Eun and Resnick 1992 and Ledoit and Wolf 2003). The statistical criteria measure the average of the differences of the estimated and actual covariance. The popular statistical measures include root-mean-square error (RMSE) and other forms from the same family such as mean square error and mean absolute error. The problem with the statistical measures is that to use such criteria correctly needs to know the true covariance, while it is rarely possible to know the truth in empirical analysis. The general practice is to view a noisy estimator of the covariance matrix using finite out-of-sample data as the pseudo truth. Therefore, this evaluation method is mainly suitable for simulation research but not reliable for empirical analysis. For example, Fan, Fan, and Lv (2008) use the statistical criterion RMSE in simulation to verify their theoretical results.

The economic criteria measure the ability of covariance estimators by producing efficient out-of-sample portfolios. Cohen and Pogue (1967) compare different covariance structures by comparing the locations of the mean-variance efficient frontiers. Chan, Karceski, and Lakonishok (1999), Caldeira et al. (2017) and Trucíos et al. (2019) use the minimum variance portfolio (MVP) method to compare alternative covariance matrices. However, there are two obvious shortcomings in this kind of evaluation method. First, it tries to indirectly reflect the accuracy of estimators through the performance of the out-of-sample portfolios. Second, because of the complexity of the market, the diversity of portfolio allocation methods and the limitation of information, a more accurate covariance matrix estimator does not always guarantee a better performance of out-of-sample portfolios. For example, the discussion of these problems could also be found in Amendola and Candila (2017) and Lan (2007).

The evaluation method proposed by this article can avoid the main shortcomings of the above two kinds of evaluation methods. First, our method does not need the true covariance matrix, which is an effective and robust measure for both simulation and empirical analysis. Second, our intuitive method does not need to introduce the performance of the portfolios to evaluate the performance of estimators indirectly.

## II. Methodology

For evaluation of covariance matrix estimators, one of the biggest challenges the research confronts is that the true covariance matrix is unknown in

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empirical analysis, which makes it impossible to measure the difference between covariance matrix estimators and the truth directly. Considering that the real asset returns are known, we propose the Monte Carlo synthetic (MCS) sample based performance evaluation method to measure the ability of covariance matrix estimators. This method does not focus on measuring the difference between the truth and the estimators directly in the covariance estimation space, but on a converting sample observation space. The difference between the real asset return sample and the synthetic asset return sample generated by the Monte Carlo simulation using the covariance matrix estimator is measured, and the measurement results are used as an indicator of the ability of the covariance matrix estimators (see Figure 1 for an illustration).

Figure 1 shows that the true covariance matrix $\Sigma$ connects with the real sample $Y$ through the sampling model $f(\Sigma, \Theta)$, and only the form of $f(\Sigma, \Theta)$ are assumed known. Thus, a direct comparison of estimator $\hat{\Sigma}$ and $\Sigma$ in the estimation space is impossible. However, $Y$ and $\hat{\Sigma}$ are known, and the unknown parameter set $\Theta$ in $f(\Sigma, \Theta)$ usually could be well determined using a large amount of historical sample data. Based on the estimated sampling model $f(\hat{\Sigma}, \hat{\Theta})$, the synthetic sample $\hat{Y}$ can be generated and compared with $Y$ in the observation space. If $\hat{\Sigma}$ is close to $\Sigma$, $\hat{Y}$ should be close to $Y$ and vice versa. Then, the difference between the covariance matrix estimator $\hat{\Sigma}$ and the true covariance matrix $\Sigma$ can be measured by a distance which can be defined as follows.

**Definition 1**. We defined the distance, $D(\hat{\Sigma}, \Sigma)$, as the measure of a covariance matrix estimator $\hat{\Sigma}$ and the true covariance matrix $\Sigma$:

$$D(\hat{\Sigma}, \Sigma) = d(\hat{Y}, Y) \tag{1}$$

where $Y$ is the real asset return sample, $\hat{Y}$ is the synthetic asset return sample generated from the estimated known distribution $f(\hat{\Sigma}, \hat{\Theta})$ using the Monte Carlo method, where $\hat{\Theta}$ is a known estimator of parameter set $\Theta$. $d(\hat{Y}, Y)$ can be any reasonable distance, such as the well-known Hausdorff distance. The Hausdorff distance between the two sample sets $Y$ and $\hat{Y}$ in a metric space $(\Omega, d)$ is

$$d(\hat{Y}, Y) = \max\{\rho(\hat{Y}, Y), \rho(Y, \hat{Y})\} \tag{2}$$

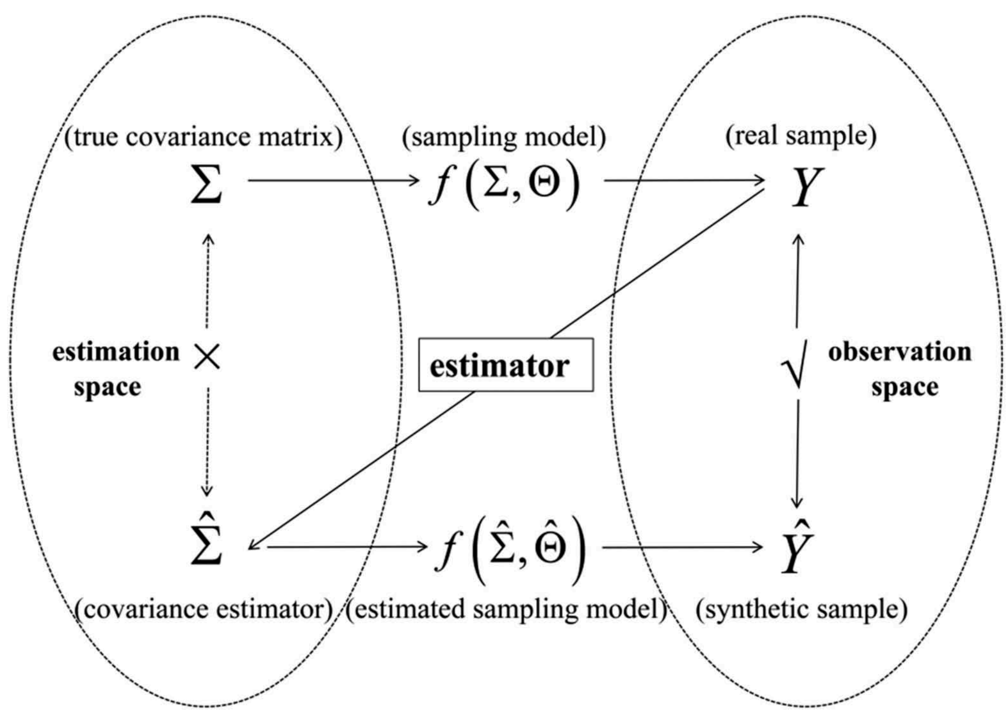


Figure 1. Illustration of the MCS method.

$$\rho(\hat{Y}, Y) = \sup_{\hat{y}\in\hat{Y}} \inf_{y\in Y} d(\hat{y}, y) \tag{3}$$

According to Definition 1, the MCS performance evaluation method of covariance matrix estimators can be described as follows.

**Definition 2**. Given two covariance matrix estimators $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$, the covariance matrix estimator $\hat{\Sigma}_1$ performs better than $\hat{\Sigma}_2$, if and only if

$$D(\hat{\Sigma}_1, \Sigma) < D(\hat{\Sigma}_2, \Sigma) \tag{4}$$

## III. Simulations

We compare our MCS with the RMSE[1] using the true covariance matrix (RMSE-TC), the RMSE using a pseudo covariance matrix estimated from the out-of-sample data (RMSE-PC), and the MVP. Note that the RMSE-TC is viewed as a reference in our simulations since it is one of the most accurate measures in theory. We use three covariance estimators, namely the historical sample covariance matrix estimator (HS), the single-index covariance estimator (SI) of Sharpe (1963) and the covariance estimator of Ledoit and Wolf (2003) (LW) to provide a simple but powerful illustration.

To run our simulation, a known distribution form should be chosen to generate the real and synthetic sample. It is well known that the financial data has the feature of heavy tail (Praetz 1972). Thus, our simulation will be performed using a multivariate *t*-distribution with

$$f(\lambda, \mu, \Sigma) = c_d(\Sigma, \lambda)\left[(\lambda - 2) + (Y - \mu)'\Sigma^{-1} (Y - \mu)\right]^{-\frac{\lambda+d}{2}} \tag{5}$$

where $c_d = \left[\pi^{\frac{d}{2}}\Gamma\left(\frac{\lambda}{2}\right)|\Sigma|^{\frac{1}{2}}\right]^{-1}(\lambda - 2)^{\frac{\lambda}{2}}\Gamma\left(\frac{\lambda+d}{2}\right)$, $\lambda$ is the degree of freedom, $\mu \in R^d$ is the mean vector, covariance matrix $\Sigma$ is a $d \times d$symmetric positive definite matrix and $\Gamma(\cdot)$ is the gamma function (Cornish 1954).

In our simulation, the degree of freedom is set to be a fix value $\lambda$= 20, the true mean vector $\mu$ is set to be zero and we construct the true covariance matrix $\Sigma$ by taking a historical sample covariance matrix using the monthly returns of nine-year periods of January 2010–December 2018 for NYSE 100 Composite Index component stocks. Following Table 1 reports the average measure results of the estimated covariance matrices obtained from three covariance estimators when the estimation windows T are set to 100, 500 and 1000. The reported average results and associated SEs are based on 100 simulations. The pair-wise differences of the evaluation results of the three estimators are also reported, along with the t-statistics. '*', '**' and '***' indicate that the corresponding results are statistically significant at 90%, 95% and 99% confidence levels, respectively.

Table 1 shows that the pair-wise statistical differences of the MCS and the RMSE-TC of the three estimators are almost the same, while the RMSE-TC cannot be implemented in empirical analysis. Table 1 also shows that our MCS is more effective, robust and reliable than the RMSE-PC and the MVP in every estimation window. The inconsistencies of the evaluation results are as follows.

The t-statistic of the MVP difference of SI-LW at T = 100 is 0.1217 which means that the LW estimator performs better than the SI estimator, while the corresponding t-statistic of the MCS difference and the RMSE-TC difference indicates that the SI estimator performs better. The RMSE-PC difference of HS-SI is not found to be statistically different at T = 1000, and the MVP difference of SI-LW is not found to be statistically different at T = 500, while the corresponding MCS and the RMSE-TC differences are both statistically significant at 99% confidence level. In addition, the RMSE-PC difference of SI-LW at T = 100 and the MVP difference of SI-LW at T = 1000 are is statistically different at 95% confidence level; however, the corresponding RMSE-TC difference is statistically significant at 99% confidence level.

## IV. Conclusion

In this article, we proposed an MCS sample based performance evaluation method for various covariance matrix estimators. Our innovative method is more reasonable than the RMSE criterion with out-of-sample estimated covariance matrix and the MVP method. The simulation results also show

[1]$\mathrm{RMSE}(\hat{\Sigma}, \Sigma) = \|\hat{\Sigma} - \Sigma\|$, where$\|\bullet\|$ stands for the Frobenius norm of two matrices.

Table 1. RMSE-TC, MCS, RMSE-PC, MVP and their differences.

| | T = 100 | | T = 500 | | T = 1000 | |
|---|---|---|---|---|---|---|
| | Mean | Std. | Mean | Std. | Mean | Std. |
| RMSE-TC (Reference) | | | | | | |
| HS | 4.0728 | 0.1001 | 2.8827 | 0.0801 | 1.9738 | 0.0679 |
| SI | 3.6011 | 0.0925 | 2.6171 | 0.0793 | 1.8736 | 0.0658 |
| LW | 3.9126 | 0.0901 | 2.4998 | 0.0712 | 1.8008 | 0.0592 |
| HS-SI | 0.4717 (46.72***) | 0.1201 | 0.2656 (23.76***) | 0.0968 | 0.1002 (11.09***) | 0.0884 |
| HS-LW | 0.1602 (15.93***) | 0.1255 | 0.3829 (34.15***) | 0.1012 | 0.1730 (18.04***) | 0.0891 |
| SI-LW | −0.3115 (−31.59***) | 0.1119 | 0.1173 (12.68***) | 0.0897 | 0.0728 (7.23***) | 0.0705 |
| MCS | | | | | | |
| HS | 32.9211 | 0.7899 | 21.0130 | 0.8090 | 15.6521 | 0.5356 |
| SI | 26.9165 | 0.6942 | 17.9315 | 0.5901 | 14.6928 | 0.5063 |
| LW | 30.5419 | 0.7625 | 16.0066 | 0.6677 | 13.8654 | 0.5208 |
| HS-SI | 6.0046 (78.18***) | 0.9971 | 3.0815 (39.11***) | 0.8987 | 0.9593 (12.98***) | 0.7820 |
| HS-LW | 2.3792 (27.95***) | 0.9620 | 5.0064 (62.01***) | 0.9882 | 1.7867 (18.85***) | 0.7546 |
| SI-LW | −3.6254 (−43.12***) | 0.9545 | 1.9249 (21.59***) | 0.9012 | 0.8274 (10.02***) | 0.6897 |
| RMSE-PC | | | | | | |
| HS | 3.7644 | 0.1034 | 2.6865 | 0.0861 | 1.7924 | 0.0768 |
| SI | 3.3132 | 0.0958 | 2.4312 | 0.0810 | 1.7915 | 0.0721 |
| LW | 3.3376 | 0.0892 | 2.1604 | 0.0737 | 1.6995 | 0.0698 |
| HS-SI | 0.4512 (44.17***) | 0.1143 | 0.2553 (21.61***) | 0.0998 | **0.0009 (0.11)** | 0.0873 |
| HS-LW | 0.4268 (40.39***) | 0.1245 | 0.5261 (55.78***) | 0.1209 | 0.0929 (10.79***) | 0.0891 |
| SI-LW | **−0.0244 (−2.51**)** | 0.1196 | 0.2708 (25.03***) | 0.1028 | 0.0920 (9.88***) | 0.0806 |
| MVP | | | | | | |
| HS | 2.3013 | 0.0815 | 1.6219 | 0.0809 | 0.9714 | 0.0645 |
| SI | 2.0918 | 0.0796 | 1.5531 | 0.0755 | 0.9306 | 0.061 |
| LW | 1.9701 | 0.0718 | 1.5406 | 0.0727 | 0.9034 | 0.0710 |
| HS-SI | 0.2095 (17.89***) | 0.1100 | 0.0688 (6.76***) | 0.1021 | 0.0408 (5.93***) | 0.0819 |
| HS-LW | 0.3312 (32.16***) | 0.1021 | 0.0813 (9.16***) | 0.1092 | 0.0680 (7.24***) | 0.0912 |
| SI-LW | **0.1217 (13.63***)** | 0.0935 | **0.0125 (1.32)** | 0.0989 | **0.0272 (2.82***)** | 0.0808 |

that it is an effective and robust evaluation method to distinguish covariance estimators in various estimation windows. The proposed method is intuitive and convenient for the empirical comparison of covariance estimators in practice.

## Disclosure statement

No potential conflict of interest was reported by the authors.

## Funding

This work was partially supported by the Chinese Natural Science Foundations [11631013, 11971372 and 11801433]

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