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Research

A Framework for Constructing Equity-Risk-Mitigation Portfolios

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The key trade-off among equity-risk-mitigation strategies is their expected return versus their ability to diversify equity risk. In particular, the more reliable a strategy's equity-hedging properties, the lower its expected return, and vice versa. This article proposes a framework for optimal equity-risk-mitigation portfolio construction. In our model, the investor maximizes the portfolio's unconditional expected return, subject to a constraint on its conditional equity beta. We show that the return to a risk-mitigation portfolio can be decomposed into hedging and return-generating components. We then demonstrate that optimal risk-mitigation portfolios exhibit better return-defensiveness properties relative to the underlying strategies.

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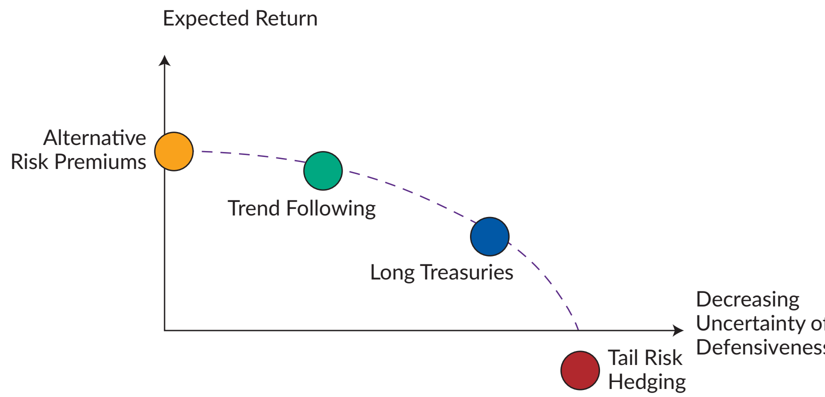
Public equities are typically among the most volatile asset classes in investors' portfolios. Accordingly, for those investors with high or even moderate allocations to equities, how to cost-effectively hedge against large equity market drawdowns is often a topic of great interest. There are, of course, many ways to deal with this issue. Paring back equity exposure is an obvious solution, but it comes with the opportunity cost of failing to earn the equity risk premium, a potentially expensive proposition. Therefore, investors have looked to other styles and asset classes that can help protect against equity sell-offs but typically with a lower opportunity cost than simply selling equities.

Investors have come to accept four general approaches to diversifying and mitigating equity risk: (1) US Treasuries, (2) trend following, (3) tail risk hedging (put buying), and (4) alternative risk premium diversifiers, such as carry and value strategies.[1] Unlike the risk–return trade-off involved in selling equities, which is quite clear, the key trade-off among these equity-risk-mitigation strategies is the "cost" of each approach versus the level of confidence that they will "work" in times of crisis. US Treasuries, for example, have historically been a good hedge against equity risk, particularly in recessions, but of course, there is no guarantee this will be the case in the future.

In this article, we posit a formalized framework for constructing equity-risk-mitigation portfolios. The key insight of our research is that a risk–return trade-off exists among equity-risk-mitigation assets in which higher-returning asset classes and styles have a higher degree of uncertainty with respect to their equity-hedging properties. We can frame this notion in the context of a stylized risk-mitigation frontier, as shown in **Figure 1**, where the *y*-axis is the expected return and the *x*-axis represents the uncertainty that a strategy will work in an equity risk-off event. The *x*-axis represents qualitatively the confidence with which the strategy will hedge risk. In the section "A Framework for Optimal Portfolio Construction," we more formally define a measure

Figure 1. Stylized Risk–Return Trade-Off for Risk-Mitigation Strategies



Notes: This figure illustrates the trade-off between return and defensiveness benefit among various commonly used risk-mitigation assets and strategies. The horizontal axis represents the equity-risk-mitigation benefit, and the vertical axis represents returns.

for this concept. Figure 1 shows that the only way for the investor to have a high degree of confidence that the portfolio will provide protection in large down equity markets is to be willing to pay an *insurance premium* in terms of lower expected returns.[2] The investor's preferences will ultimately determine the specific portfolio on this risk–return frontier that the investor chooses to hold.

Background on Popular Equity-Risk-Mitigation Strategies

Tail risk hedging (put buying) is contractually linked to equity market performance and thus can provide a direct hedge with a high degree of confidence in severe downturns. However, this high degree of confidence comes at the cost of several headwinds, including negative exposure to the equity risk and volatility risk premiums, which often results in negative expected returns. This is why tail risk hedging resides at the rightmost part of the frontier in Figure 1; it exhibits the greatest hedging efficacy but with the highest cost.

Long Treasuries are perhaps the most prominent example of a simple and robust positive-expected-return equity-risk-mitigation strategy. Baz, Sapra, and Ramirez (2019) showed that since the mid-1990s, Treasuries have consistently exhibited a negative correlation with equities, and over the past 50 years, US government bonds have produced positive returns in most recessions. However, the stock–bond correlation has moved considerably over time (Campbell, Sunderam, and Viceira 2017), which increases the uncertainty about using Treasuries for equity diversification. Furthermore, Wright (2011) and Adrian, Crump, and Moench (2013) showed that the term premium has compressed toward zero over the past several decades, casting doubt on the asset's forward-looking return-generating potential.

Trend-following strategies benefit from persistent trends in prices across major markets. Fung and Hsieh (2001) showed that these strategies exhibit payoff profiles that resemble those of long volatility strategies, which tend to be negatively correlated with the equity market but lack the explicit negative exposure to the equity risk and volatility risk premiums that accompanies tail risk hedging. This increased efficiency, however, comes at the cost of uncertainty around equity risk mitigation, because trend following may deliver negative returns when market reversals are abrupt or frequent (Daniel and Moskowitz 2016).

Finally, general alternative risk premium strategies are broadly defined by having (1) limited exposure to traditional equity and bond risk premiums, (2) a clear economic rationale for their existence, (3) empirical validation based on historical data, and (4) implementation that typically requires the use of leverage, via shorting and derivatives. Many alternative risk premium strategies have been studied in the literature. For example, Menkhoff, Sarno, Schmeling, and Schrimpf (2017) assessed currency value strategies. Asness, Moskowitz, and Pedersen (2013) and Koijen, Moskowitz, Pedersen, and Vrugt (2018) studied the value and carry strategies for various asset classes.[3] Given that these strategies are designed to exhibit limited exposure to equity markets, they are natural candidates for risk-mitigation portfolios. However, although the correlation of alternative risk premiums with equities tends to be near zero, as with trend-following approaches, there is wide variation in the conditional correlations of these strategies with equities.

A Framework for Optimal Portfolio Construction

In this section, we develop a framework for equity-risk-mitigation portfolio construction. This framework can be considered an extension of the standard Markowitz (1952) mean–variance optimization (MVO) problem, except that the choice of assets and their respective allocations are affected by the investor's need for downside equity market protection. Specifically, investors optimize the standard risk–return trade-off between the portfolio's unconditional expected return and unconditional volatility while at the same time constraining the portfolio to have certain downside risk-mitigation properties, as measured by its conditional beta. The conditional beta refers to an asset's beta in down equity markets, which may be quite different from its beta in "normal" markets, depending on the asset class or investment style under consideration (Page and Panariello 2018). As described in the previous section, the constraint on conditional beta is particularly relevant because asset classes and styles with the highest unconditional expected returns tend to have the weakest downside equity protection, and vice versa.

Formally, we consider an investor who chooses a set of weights, w, to maximize the portfolio's unconditional expected return, $E[R_p]$, subject to constraints on its unconditional volatility and conditional equity beta:

$$\max_w E\left[R_P\right] = w'\mu, \tag{1}$$

$$0.5w'\Sigma w = 0.5\sigma_p^2, \tag{2.1}$$

and

$$w'\beta_c \le \bar{\beta}_c, \tag{2.2}$$

where

μ is a vector of unconditional expected excess returns

$\mathbf{\Sigma}$ is the unconditional covariance matrix

σ_p^2 is the targeted unconditional portfolio variance

β_c is a vector of conditional betas

$\bar{\beta}_c$ is the conditional beta target of the portfolio

Equations 2.1 and 2.2 thus represent constraints on the portfolio problem imposed by the investor, reflecting both the investor's willingness to assume volatility risk and the need for downside equity market protection.

A detailed solution to the problem is provided in Appendix B. Section 1 of Appendix B shows properties for three special portfolios resulting from the solution to Equation 1: (1) the unconstrained MVO portfolio, w^{MVO}, which has a conditional beta β^{MVO} and a Sharpe ratio SR_{MVO}; (2) the minimum-variance portfolio with a unit conditional beta (unit-beta portfolio), w^B, which has an unconditional variance $\sigma^2_{w^B}$ and a Sharpe ratio denoted by SR_{w^B}; and (3) the zero-beta MVO portfolio, $w^{MVO} - \beta^{MVO}w^B$, which adjusts the weights of the unconstrained MVO portfolio by a set of weights that is proportional to those of the unit-beta portfolio in order to achieve a zero-net equity beta.

We are, of course, most interested in the case in which the conditional beta constraint (Equation 2.2) is binding.[4] In this case, as shown in Appendix B, the optimal portfolio is given by

$$w = \bar{\beta}_c w^B + c\left(w^{MVO} - \beta^{MVO} w^B\right), \tag{3}$$

where c is a constant. Equation 3 shows that the optimal portfolio is a weighted sum of the unit-beta portfolio, w^B, and the zero-beta MVO portfolio, $w^{MVO} - \beta^{MVO}w^B$. To provide intuition, the solution can be thought of as a two-step procedure: First, create the beta-hedging portfolio using $\bar{\beta}_c$ of the unit-beta portfolio in order to achieve the conditional beta target, and then hold some amount in the zero-beta MVO portfolio to maximize the unconditional expected return. The degree of exposure to the zero-beta MVO portfolio (as measured by c) is a function of the overall unconditional volatility target, σ_p, and the fraction of the risk budget remaining after the conditional beta constraint has been satisfied.

As shown in Section 2 of Appendix B, the expected return of the optimal portfolio is given by

$$\begin{aligned} E\left[R_p\right] &= \mu'\left(\bar{\beta}_c w^B\right) \\ &\quad + \left(\sigma_p^2 - \bar{\beta}_c^2 \sigma_{w^B}^2\right)^{1/2}\left(SR_{MVO}^2 - SR_{w^B}^2\right)^{1/2}. \end{aligned} \tag{4}$$

A close inspection of Equation 4 yields some powerful insight. The first term is the unconditional expected excess return on the beta-hedging portfolio. This term arises because of the conditional beta constraint and directly corresponds to the first term in Equation 3.[5] It is useful to think of this first term as a cost, or "insurance premium," associated with achieving a negative conditional beta.[6]

The variance of the beta-hedging portfolio is $\bar{\beta}_c^2 \sigma_{w^B}^2$. Therefore, the second term in the parentheses, $\left(\sigma_p^2 - \bar{\beta}_c^2 \sigma_{w^B}^2\right)^{1/2}$, measures the amount of the volatility budget remaining after the conditional beta constraint has been satisfied. We refer to this term as the "risk budget" because it reflects the quantity of risk that can be allocated to the return-enhancing zero-beta MVO portfolio after the conditional beta constraint has been achieved. If, for example, the investor requires a highly negative conditional beta, this requirement will have the effect of decreasing the amount of the risk budget that can be used for increasing overall returns.

The last term in parentheses in Equation 4 is the difference between the squared unconditional Sharpe ratio of the unconstrained MVO portfolio and that of the beta-hedging portfolio. We label this term "efficiency" because it reflects the extent to which the risk-mitigation assets collectively embody diversifying risk–return properties relative to the pure beta-hedging portfolio. Efficiency is scaled by the risk budget in Equation 4 because only variance that is not used to satisfy the targeted conditional beta can be used to exploit the greater efficiency of the MVO portfolio of risk-mitigation assets.

This intuitive decomposition leads us to a general rule of thumb to describe the expected return on a portfolio of risk-mitigation assets:

$$E[R_p] = -\text{Insurance premium} + (\text{Risk budget} \times \text{Efficiency}), \tag{5}$$

where the insurance premium is $-\mu'\left(\bar{\beta}_c w^B\right)$, the risk budget is $\left(\sigma_p^2 - \bar{\beta}_c^2 \sigma_{w^B}^2\right)^{1/2}$, and efficiency is $\left(SR_{MVO}^2 - SR_{w^B}^2\right)^{1/2}$.

Equation 5 tells us that the risk-mitigation portfolio problem is ultimately about balancing the cost of hedging equity risk with the desire to generate positive unconditional returns. It shows that positive expected returns on a risk-mitigation portfolio can be achieved if one can allocate a sufficient amount of the risk budget to assets with high Sharpe ratios (efficiency) and one does not pay "too high" an "insurance premium" for the equity-hedging component.

Empirical Results

To better understand the practical implications of the model in the previous section, we applied our framework to five asset classes and styles: US Treasuries, trend following, tail risk hedging, carry, and value.[7] The construction methodology for each strategy is detailed in Section 1 of Appendix A. We used quarterly data from the first quarter of 1994 to the fourth quarter of 2018. **Table 1** shows the excess return moment assumptions for each strategy and for the S&P 500 Index.

The conditional moments in Table 1 are computed using quarters in which the S&P 500 excess returns were less than –3.75%, or approximately –15% annualized. For the unconditional returns of the five risk-mitigation strategies, we adjusted the historical realized returns to account for estimates of transaction costs. Specifically, we subtracted estimated transaction costs from the historical returns to calculate a net-of-cost return and Sharpe ratio for each strategy. The net-of-cost Sharpe ratios are 0.40, 0.33, –0.50, 0.86, and 0.58 for long Treasuries, trend following, tail risk hedging, carry, and value, respectively. Further, in order to account for such realities as lower bond yields today, data mining, and the potential for the returns of alternative strategies to be arbitraged away, we applied a "haircut" to the post-transaction-cost Sharpe ratios for long Treasuries, trend following, carry, and value. For simplicity, we reduced all net-of-cost Sharpe ratios by 50%, which yielded our final Sharpe ratio assumptions shown in Table 1: 0.20, 0.15, 0.40, and 0.30 for long Treasuries, trend following, carry, and value, respectively.[8] We assumed an unconditional annualized volatility of 5% for these strategies, resulting in expected excess returns of 1.0%, 0.8%, 2.0%, and 1.5% for the four strategies, respectively. In Section 2 of Appendix A, we provide the details of our transaction cost assumptions.[9]

We used quarterly data, as opposed to monthly or daily data, in order to obtain conditional moments at a frequency that is relevant to institutional investors, who tend to focus more on performance at quarterly, semiannual, and annual horizons and pay less attention to day-to-day variation in portfolio value. Our choice of a 15% annualized equity decline as the conditioning return represents the bottom quintile

Table 1. Assumptions on Conditional and Unconditional Excess Return Moments

	Unconditional Moment Assumptions			Conditional Moment Assumptions			
	Excess Return	Volatility	Sharpe Ratio	Volatility	Equity Covariance	Equity Beta	Equity Correlation
S&P 500	5.2%	15.6%	0.33	10.3%	1.1%	1.00	1.00
Long Treasuries	1.0%	5.0%	0.20	6.7%	–0.4%	–0.42	–0.64
Tail risk hedging	–2.6%	5.4%	–0.50	7.1%	–0.7%	–0.64	–0.92
Trend following	0.8%	5.0%	0.15	6.4%	–0.5%	–0.43	–0.70
Carry	2.0%	5.0%	0.40	5.7%	0.0%	–0.04	–0.06
Value	1.5%	5.0%	0.30	5.1%	0.3%	0.26	0.52

Notes: All moments are annualized and calculated using quarterly returns in excess of cash from 1 March 1994 to 31 December 2018. The "Unconditional Moment Assumptions" columns report the assumptions for each asset's/strategy's unconditional mean, standard deviation, and Sharpe ratio (defined as its mean excess return divided by volatility). Long Treasuries, trend following, carry, and value are adjusted for transaction costs and look-ahead bias as described in the text. The "Conditional Moment Assumptions" columns report each asset's/strategy's conditional moments based on quarters in which the S&P 500's excess returns were less than –3.75%. We report the assets'/strategies' conditional volatility as well as their covariance, beta, and correlation with equities, represented by the S&P 500. Conditional moments are calculated directly from the historical data.
Sources: Bloomberg and PIMCO.

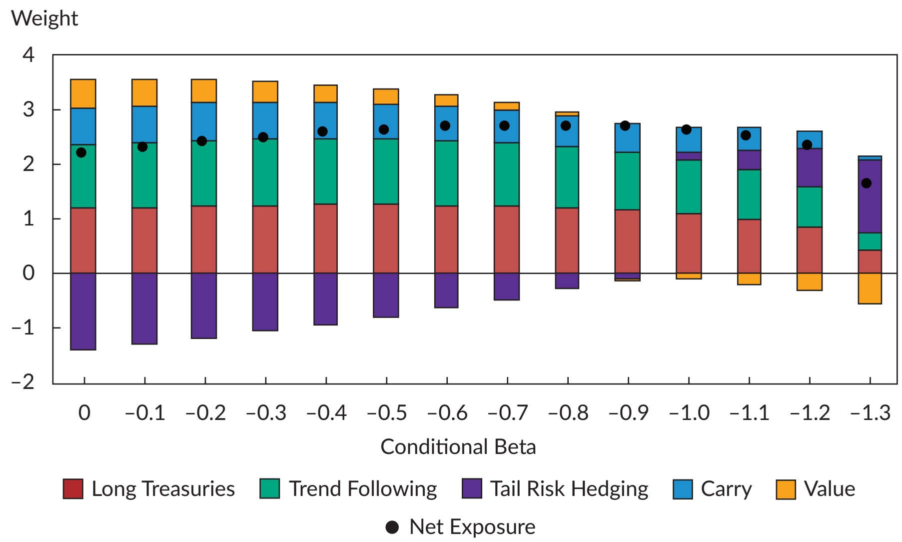
of historical quarterly excess market returns and reflects our subjective view of the threshold beyond which investors would generally expect some offsetting performance from their risk-mitigation portfolio.

The asset allocation framework presented in the previous section can be applied to any arbitrary number of assets, and investors can freely choose the opportunity set that fits their investment objectives. In this empirical analysis, we selected from a set of commonly used equity-risk-mitigation strategies, but our selection is not meant to be an exhaustive representation of all potential risk-mitigation strategies. Low-volatility equity-market-neutral strategies, for example, have also been used as an effective means of risk mitigation in portfolios.

We focused on the case in which an investor has an already-constructed investment portfolio with some degree of equity exposure that the investor wishes to hedge by optimizing an overlay portfolio of risk-mitigation strategies. Although it would be perfectly reasonable to consider the risk-mitigation problem in a single step, in which the underlying portfolio and the overlay are optimized simultaneously, we treat the risk-mitigation portfolio independently, primarily for purposes of intuition. Furthermore, for many investors, the underlying portfolio is constructed with a multitude of objectives that are generally outside the scope of this framework. Indeed, in practice, most institutional investors treat the risk-mitigation portfolio separately from the underlying portfolio decision. Accordingly, we specifically excluded the broad-based equity market portfolio from the opportunity set and formulated equity-risk-mitigation portfolios based solely on the risk-mitigation strategies described in this article. For completeness, in the next section, we show the results when the equity market portfolio is included in the investment opportunity set.

Figure 2 shows the allocations to each asset class and style in Table 1 as a function of the targeted conditional beta for a 10% volatility risk-mitigation portfolio. Allocations on the left side of Figure 2 reflect portfolios for which equity hedging is less of a consideration and that thus have the least negative conditional betas. As expected, such portfolios are made up of higher-returning assets since the need for equity hedging is minimal. In fact, these portfolios actually hold a short position in the tail-risk-hedging strategy (effectively selling puts) as a means of increasing the unconditional expected return. Moving rightward on the graph, the importance of equity hedging increases and the impact of the conditional beta constraint becomes more pronounced. As a result, the investor allocates to more "reliable" hedging sources, such as long Treasuries and equity put options, as the conditional beta constraint becomes increasingly negative.

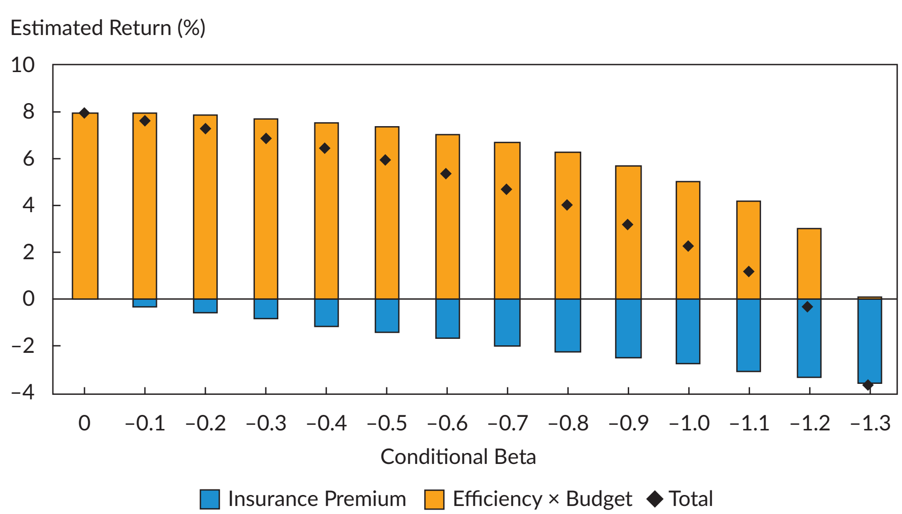
Figure 2. Optimal Portfolio Weights for Various Conditional Beta Targets



Notes: This figure plots the portfolio allocation for various conditional beta targets, given by Equation 3. Each bar represents the optimal allocation that maximizes unconditional expected excess return while being constrained to a 10% unconditional volatility and a specific conditional beta (indicated by the *x*-axis value). The conditional beta constraints bind for all these cases. Net exposure is the sum of all weights.
Source: PIMCO.

Figure 3 shows the decomposition of the portfolio return based on Equations 4 and 5. With a conditional beta of zero, the investor can avoid using any of the risk budget for equity hedging and, as a result, will hold a portfolio that is proportional to the zero-beta MVO portfolio. In this case, all of the return comes from the combination of efficiency and risk budget described in Equation 5. As the conditional beta constraint becomes negative, however, the investor must use an increasingly large fraction of the risk budget purely for hedging purposes. This situation results in a drag on expected return from the increasing "insurance premium." At the rightmost part of Figure 3, the unconditional return actually turns negative because the majority of the portfolio is made up of high "insurance premium" assets, such

Figure 3. Decomposition of Unconditional Expected Excess Return



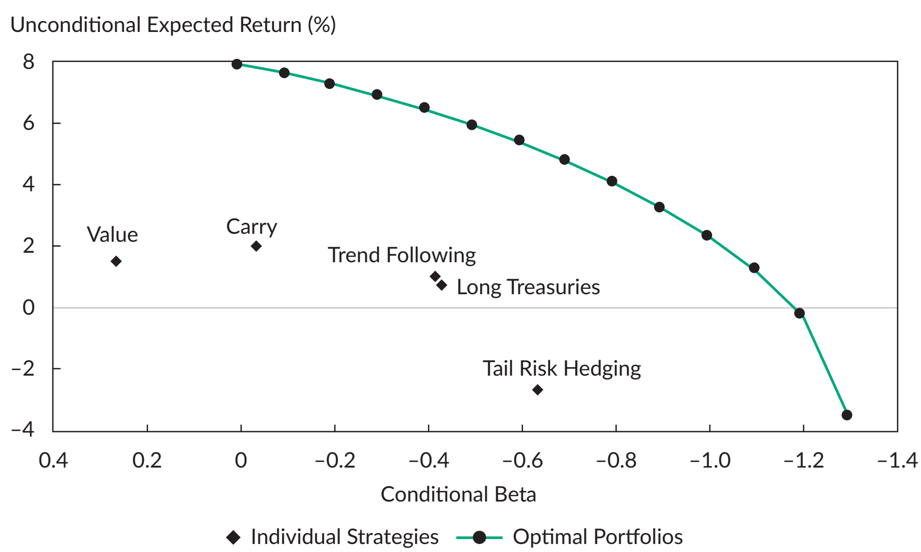
Notes: This figure plots the unconditional expected excess return for each optimal allocation shown in Figure 2 (the black diamonds) and its decomposition into "insurance premium" (blue bars) and efficiency times budget (orange bars), as given by Equations 4 and 5. The *x*-axis values represent the binding conditional beta constraints for the optimizations.
Source: PIMCO.

as put options. At this point, 100% of the risk budget is used to pay the "insurance premium" since the investor holds a portfolio that is negatively proportional to the unit-beta portfolio. Because this portfolio is dominated by equity put options, the expected return on the portfolio is negative.

Figure 2 shows that the optimal risk-mitigation portfolios are indeed *portfolios* in that they comprise different risk-mitigation strategies. A natural question, of course, is how much benefit the investor receives from forming portfolios relative to using individual risk-mitigation strategies. To evaluate this question, **Figure 4** plots the conditional beta versus the expected return for the optimal portfolios and for the five individual strategies. Indeed, similar to the results of Baz, Davis, and Rennison (2017), we found the benefits of forming portfolios to be substantial. Consider long Treasuries as an example. This strategy on its own has a conditional equity beta of about −0.4, an expected excess return of 1.0%, and an unconditional Sharpe ratio of 0.2. In contrast, the optimal risk-mitigation portfolio with a conditional beta target of −0.4 achieves an expected return of 6.4%, or an unconditional Sharpe ratio of 0.64. Hence, compared with long Treasuries on a standalone basis, the optimized portfolio achieves roughly triple the Sharpe ratio with a similar overall level of defensiveness. Furthermore, Figure 4 shows that forming optimal portfolios of equity-risk-mitigation strategies shifts the achievable frontier further outward to the right.

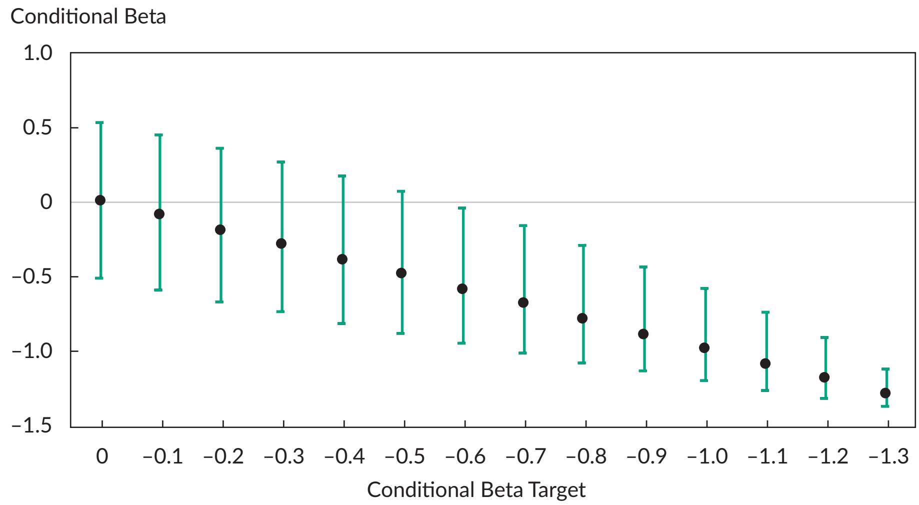
Thus far, we have considered the salient properties of the optimal risk-mitigation portfolio given a conditional beta target. Estimates of conditional betas are generally based on covariance/correlation estimates and are thus subject to estimation error.[10] This means that although the targeted portfolio beta can be achieved on paper, it may not work out in practice because realized betas can vary from their estimated values. The extent to which such a deviation may occur is, of course, a function of the equity-hedging ability of the underlying strategies. Therefore, to assess the uncertainty around these estimates, we calculated the 95% confidence intervals around the conditional betas of the portfolios shown in Figure 2.[11] As shown in **Figure 5**, the hedging properties of these portfolios are far from guaranteed, particularly for portfolios derived from less negative beta constraints. For example, Figure 5 shows that in order to obtain a negative conditional beta with 95% confidence, the conditional beta target would need to be below −0.6. Likewise, targeting a conditional beta of zero could produce realized correlations between approximately −0.5 and +0.5, perhaps a wider range than some investors are comfortable with.

Figure 4. Risk-Mitigation Strategies' Risk–Return Frontier



Notes: This figure plots the conditional beta (*x*-axis) and the unconditional expected return (*y*-axis) for each of the five risk-mitigation strategies/assets (diamonds) and for each optimal portfolio constructed using these five strategies/assets (circles). The properties of the underlying assets/strategies are given in Table 1. The allocation for each optimal portfolio is given in Figure 2, and the unconditional expected excess return is given in Figure 3.
Sources: Bloomberg and PIMCO.

Figure 5. 95% Confidence Interval for Conditional Beta Estimates



Notes: This figure plots the 95% confidence interval of the conditional beta for each optimal portfolio. For each optimal portfolio with a specific conditional beta target (values on the *x*-axis), we first converted the portfolio's conditional beta into a conditional correlation and then calculated the confidence interval of that correlation using Fisher transformation. The $(1 - \alpha)\%$ confidence interval is $\left[\tanh\left(\text{artanh}(r) - \frac{z_{\alpha/2}}{\sqrt{n-3}}\right), \tanh\left(\text{artanh}(r) + \frac{z_{\alpha/2}}{\sqrt{n-3}}\right)\right]$, where r is the sample correlation, n is the sample size, and tanh and artanh are the hyperbolic tangent and the inverse hyperbolic tangent functions, respectively. Subsequently, we converted the correlation back into a conditional beta value for expositional purposes.
Source: PIMCO.

Changing Opportunity Set and Constraints

In this section, we discuss the optimal portfolios when we include the equity market portfolio in the opportunity set and impose a long-only constraint on the optimization problem.

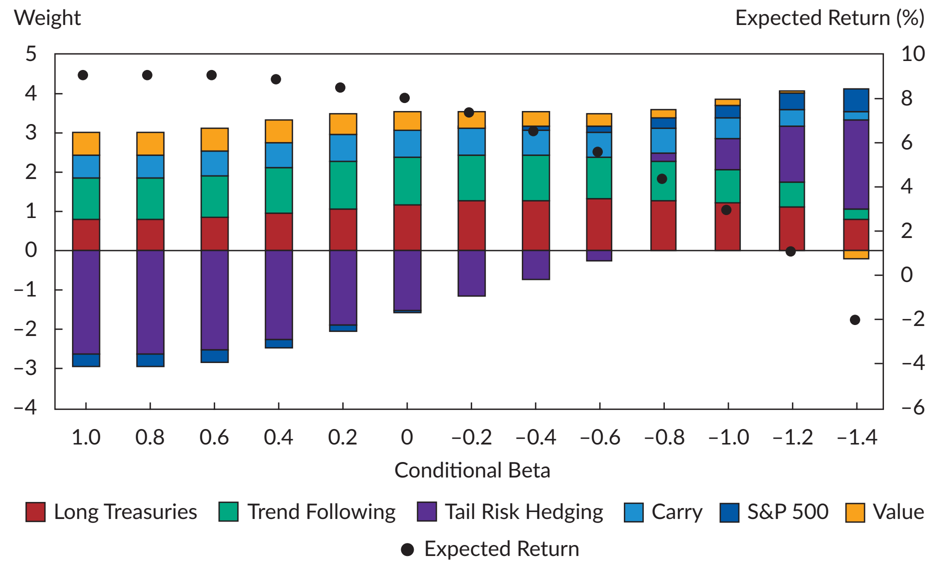
Including the Equity Market Portfolio.

In the benchmark case presented in the previous section, we excluded the broad-based equity portfolio from the opportunity set. Nonetheless, as previously discussed, the framework is flexible and will work with any arbitrary set of asset classes or styles. So, in order to understand the role that the cap-weighted equity market portfolio may play in risk-mitigation portfolios, in this section, we construct optimal portfolios in a fashion identical to that in the previous section but include the equity market portfolio in the opportunity set.[12]

Figure 6 shows the optimal allocations when using the extended opportunity set and after increasing the range of conditional betas to include positive values. These changes allow for two different investor objectives: a purely standalone risk-mitigation overlay portfolio with equities as a potential hedging asset or a jointly optimized portfolio inclusive of both the underlying equity and overlay portfolio decisions simultaneously. We can consider results to the left of zero (positive conditional betas) in Figure 6 to represent a simultaneous "one-step" optimization, incorporating both beta and risk-mitigation asset decisions at once. Results to the right of zero (negative conditional betas) are portfolios more likely to be considered as pure risk-mitigation overlay allocations, given their defensive posture.

Interestingly—and perhaps counterintuitively—the equity weight actually *increases* as the conditional beta becomes more negative. This phenomenon occurs because of the strong negative correlation between passive put buying and the broad-based equity index. Hence, the equity market portfolio and the tail-risk-hedging portfolio are strong diversifiers for one another, which is why both strategies are generally held on the same "side" in each portfolio. When the portfolio conditional beta is positive, shorting the equity index effectively allows for selling more tail risk hedging. Similarly, when the conditional beta is strongly negative, holding some amount of long equity exposure allows for even greater exposure to tail risk hedging, a finding consistent with Bhansali and Davis (2010).[13]

Figure 6. Optimal Portfolio Weights and Expected Returns for Various Conditional Beta Targets with the Equity Market Portfolio in the Opportunity Set



Notes: This figure plots the portfolio allocation (left axis) and expected return (right axis) for various conditional beta targets (x-axis). Each optimal portfolio maximizes unconditional expected excess return while being constrained to a maximum of 10% unconditional volatility and a conditional beta upper bound.
Source: PIMCO.

Short-Sale Constraint. In the benchmark optimization, we allowed the portfolios to have short positions in the risk-mitigation strategies. Figure 2 shows that the optimal allocation takes a short position in tail risk hedging to increase return when the conditional beta targets are high. Although selling these strategies short is certainly possible, it might not be practical in the context of risk-mitigation portfolio construction. For example, investors are unlikely to sell put options when the overall goal is to mitigate equity risk.

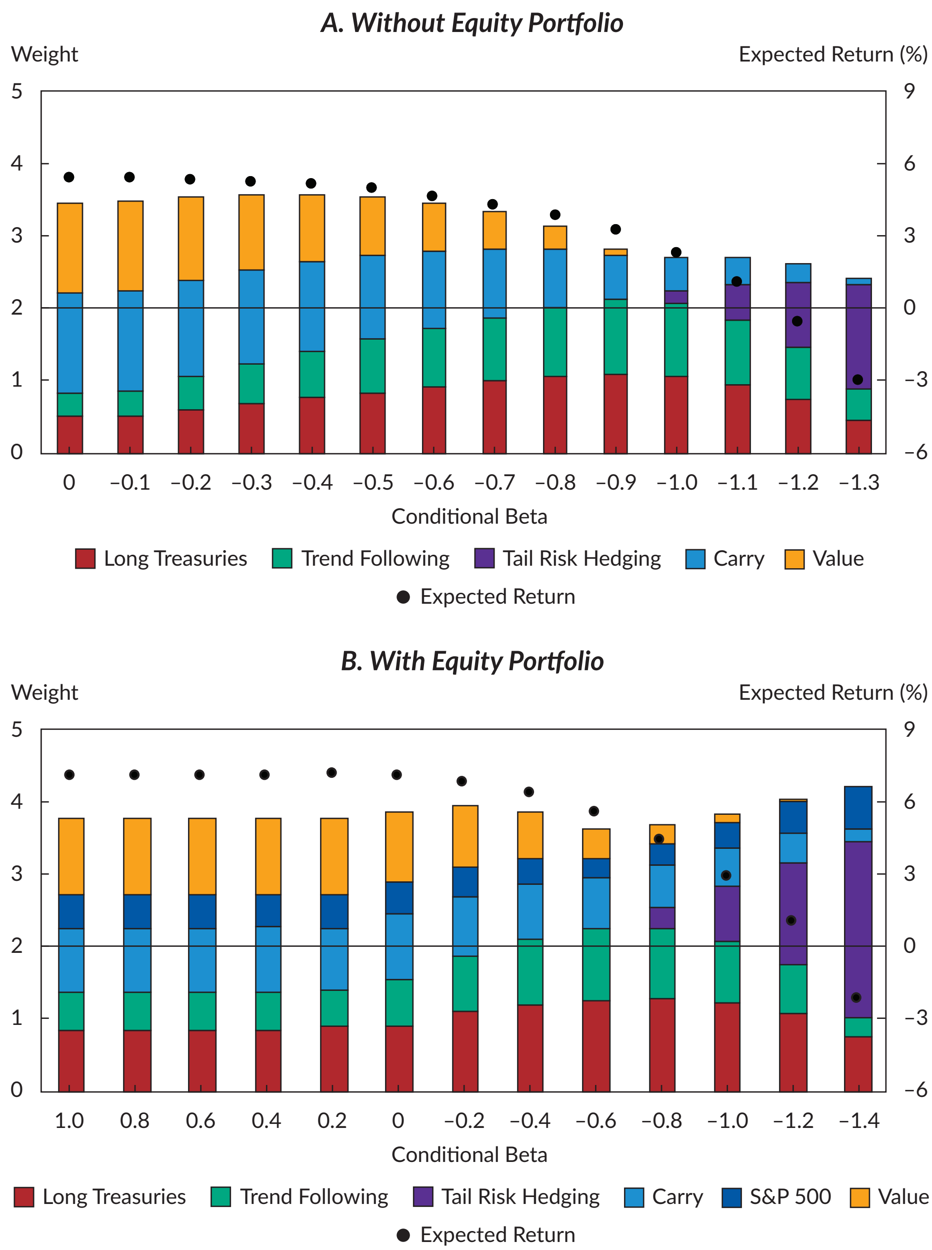
To take this potential real-world consideration into account, we imposed a long-only constraint on the optimization problem. Formally, we considered again the optimization problem from Equations 1, 2.1, and 2.2 with an additional constraint: $w_i \geq 0$ for all strategies i in the opportunity set. **Figure 7** plots the optimal allocation with this additional constraint and shows the results both with and without the broad-based equity index included. A comparison of Panels A and B of Figure 7 with Figures 2 and 6 shows that the main difference is that with a no-short-sale constraint, the investor can no longer increase return by selling puts when the conditional beta targets are high. Because a short position in put options induces a positive equity beta, all else equal, eliminating put selling effectively decreases the beta of the portfolio. As a result, the portfolios allocate less to the lower-returning/more defensive long Treasuries and trend-following strategies and more to the higher-returning/less defensive carry and value strategies. Figure 7 also shows the expected return of each optimal portfolio on the right axis. With the addition of the no-short-sale constraint, expected returns are lower than in the benchmark case. For example, the expected returns in Panel A of Figure 7 and in Figure 3 using the portfolio with a conditional beta of −0.5 are 4.9% and 5.9%, respectively.[14]

Conclusion

In this article, we examined the properties of equity-risk-mitigation strategies and noted that there is a critical trade-off between defensiveness and return potential. As a result, it will be possible to obtain a high and reliable negative conditional beta *only* if the investor is willing to give up some overall expected return.

Although lower-return implications of increasing defensiveness are inevitable, we showed that investors can minimize the “cost” of equity defensiveness with thoughtful portfolio construction. To this end, we have proposed a framework for constructing optimal equity-risk-mitigation portfolios. Optimal portfolios in this context are those that achieve the highest possible expected return given a constraint on the portfolio’s conditional equity beta. The setup is flexible, and investors simply need to decide on the

Figure 7. Optimal Portfolio Weights vs. Conditional Beta Target with Long-Only Constraint



Notes: This figure plots the portfolio allocation (left axis) and expected return (right axis) for various conditional beta targets (*x*-axis). Each optimal portfolio maximizes unconditional expected excess return while being constrained to a maximum of 10% unconditional volatility and a conditional beta upper bound. All the weights are restricted to be positive (no shorting). Panel A shows results with the five risk-mitigation strategies, and Panel B shows results with the five risk-mitigation strategies and the equity market portfolio.
Source: PIMCO.

set of assets/strategies they wish to use and specify the following:

1. Expected returns, the variance–covariance matrix, and the conditional equity betas for the assets/strategies used in the portfolio
2. The desired conditional equity beta and the volatility constraint for the portfolio

We showed that the expected return and optimal portfolio weights can be decomposed into two components: (1) an "insurance premium" component, the size of which is driven by the need for downside equity protection, and (2) a return-seeking component, which the investor seeks to maximize *after* paying the "insurance premium" component. The investor's ability to seek higher returns in this context is driven primarily by the amount of equity protection desired and the amount of overall risk the investor is willing to take in the risk-mitigation portfolio.

In the "Empirical Results" section, we applied this framework and constructed optimal portfolios for a range of conditional beta targets using five common equity-risk-mitigation strategies: long Treasuries, tail risk hedging, trend following, carry, and value. For not-too-negative beta targets, the optimal portfolio consists mainly of strategies with moderate defensiveness and return properties (Treasuries and trend following), with some allocations to higher-return-potential but less reliably defensive alternative strategies (carry and value). As a result of their negative expected returns, put-buying strategies are used only when very negative conditional beta is required.

Our results show that by combining various strategies with different risk and return properties, investors do not need to sacrifice a significant amount of expected return to achieve a high level of defensiveness, relative to, say, the more expensive propositions of put buying or shorting an equity index. In fact, it is possible to achieve positive expected excess returns for conditional beta as low as –1.1. Furthermore, by using a portfolio approach rather than simply relying on individual strategies, the investor may be able to materially improve the expected effectiveness of the equity hedge while at the same time seek to improve overall risk-adjusted returns.

Appendix A. Backtests

1. Strategy Descriptions

Long Treasuries: Returns are calculated daily using the return on the Bloomberg Barclays US Long Treasury Total Return Index Unhedged.

Tail risk hedging: Returns are calculated as follows: The underlying portfolio, V_t, earns the cash rate (daily compounding). Each month on option expiration day, t, buy $V_t/(12 \times S_t)$ contracts of 10% out-of-the-money put options with a one-year expiration (S_t denotes the S&P 500 value on date t). At each point in time, the option portfolio consists of 12 put options with expiration times ranging from 1 to 12 months. Returns are calculated daily using the total portfolio, which includes the underlying portfolio and the option portfolio.

Carry: Target a 1% scale based on the 260-day volatility for each of the three asset classes. For currencies, at the beginning of each month, rank G–10 currencies against USD (AUD, CAD, CHF, EUR, GBP, JPY, NOK, NZD, and SEK) on the basis of their annualized real carry. Take a long position in the top third highest-yielding currencies, and short the bottom third lowest-yielding currencies. For rates, rank G–6 10-year swaps (AUD 10 year, CAD 10 year, EUR 10 year, GBP 10 year, JPY 10 year, and USD 10 year) by carry plus roll-down per year of duration. Take a long position in the top third highest carry, and short the bottom third lowest carry plus roll-down per year of duration. For commodities, calculate carry using commodity forwards for 21 commodities (aluminum, Brent crude, cocoa, coffee, copper, corn, cotton, gasoline, gold, heating oil, lead, low-sulphur gas oil, natural gas, nickel, platinum, silver, soybeans, sugar, wheat, West Texas intermediate crude, and zinc). At the beginning of each month, rank the commodities on the basis of their annualized nominal carry. Take a long position in those with the top third highest carry, and short those with the bottom third lowest carry.

Value: Target a 1% scale based on the 260-day volatility for each of the three asset classes. For currencies, at the beginning of each month, rank the G–10 currencies against USD (AUD, CAD, CHF, EUR, GBP, JPY, NOK, NZD, and SEK) on the basis of their purchasing power parity. Take a long position in the lowest third currencies, and short the highest third currencies. For rates, rank G–6 10-year swaps (AUD 10 year, CAD 10 year, EUR 10 year, GBP 10 year, JPY 10 year, and USD 10 year) by their real yields, and then take a long position in the highest third and short the lowest third. For commodities, divide the assets into four sectors: livestock (live cattle and lean hogs), grain (corn, soybeans, and wheat), softs (cocoa, coffee, cotton, and sugar), and petroleum (Brent crude, gasoline, heating oil, low-sulphur gas oil, and West Texas intermediate crude). At each point in time for each commodity, compute the five-year rolling risk-adjusted return of the front rolling contract. Within each sector, take a long position in the third with the lowest risk-adjusted return and short the third with the highest risk-adjusted return.

Trend following: Target a 1% scale based on the 260-day volatility for each of the four asset classes: equity (EURO STOXX 50, S&P 500, Nikkei 225, FTSE 100, ASX 200, and Swiss Market Index), commodities (aluminum, Brent crude, cocoa, coffee, copper, corn, cotton, gasoline, gold, heating oil, lead, low-sulphur gas oil, natural gas, nickel, platinum, silver, soybeans, sugar, wheat, West Texas intermediate crude, and zinc), currencies (AUD, CAD, CHF, EUR, GBP, JPY, NOK, NZD, and SEK against USD), and rates (10-year swaps for AUD, CAD, EUR, GBP, JPY, and USD). For each asset at each point in time, calculate the signal as the difference between the last value of the excess return index and its 250-day moving average. Assuming the signal has a normal distribution, apply the normal cumulative distribution function and convert the signal to a strength measure between 0 and 1. Then, use a response function to convert signal strength to position. Rescale the negative sine function so that the function is zero when the strength measure is 0.5 and achieves its maximum and minimum at strength measures of 0.8 and 0.2, respectively. The output is scaled by $1\%/\sigma_t$ so that the maximum position has a 1% *ex ante* volatility. Note that the response function used gradually enters a trend; at the extremes, the size of the position declines to "take profits."

Table A1 and **Table A2** show the historical raw conditional and unconditional moments of the five strategies modeled in this section and the time series of each strategy conditional on equity market returns exceeding the –3.75% quarterly drawdown threshold.

2. Transaction Costs

Tail risk hedging: Put option prices are calculated using implied volatility, and we assume the transaction cost is an additional 2.5% of the implied volatility. Specifically, the total cost for an option with implied volatility $v\%$ is equal to the price of an option with implied volatility of $1.025v\%$. The average historical excess return is about 0.2% lower, at –2.6%, and the Sharpe ratio is –0.50.

Trend following, carry, and value: We assume the following percentage transaction costs for each of the four asset classes used in these strategies: 15 basis points (bps) for commodity futures, 2 bps for currency forwards, 3 bps for 10-year interest rate swaps (0.04 bps per year of duration), and 5 bps for equity futures. For each of the three strategies, we first calculate the daily turnover for each instrument. We then calculate the average annual turnover and aggregate these values across instruments within each asset class to obtain the average annual turnover for each asset class for each strategy. Next, we multiply these turnover numbers by the transaction cost and sum the results to arrive at the transaction cost assumption for each strategy. **Table A3** presents

Table A1. Historical Unconditional Excess Return Moments of Various Strategies

	Unconditional Moments			Conditional Moments				
	Excess Return	Volatility	Sharpe Ratio	Excess Return	Volatility	Equity Covariance	Equity Beta	Equity Correlation
S&P 500	5.2%	15.6%	0.33	–43.0%	10.3%	1.1%	1.00	1.00
Long Treasuries	4.4%	11.0%	0.40	18.7%	14.8%	–1.0%	–0.92	–0.64
Tail risk hedging	–2.4%	5.4%	–0.45	11.4%	7.1%	–0.7%	–0.64	–0.92
Trend following	0.4%	0.7%	0.66	1.2%	0.9%	–0.1%	–0.06	–0.70
Carry	1.0%	1.0%	1.01	0.9%	1.2%	0.0%	–0.01	–0.06
Value	0.7%	1.0%	0.63	0.7%	1.1%	0.1%	0.05	0.52

Notes: All moments are annualized and calculated using quarterly returns in excess of cash from 1 March 1994 to 31 December 2018. The "Unconditional Moments" columns report each asset's/strategy's unconditional mean, standard deviation, and Sharpe ratio (defined as its mean excess return divided by volatility). The "Conditional Moments" columns report each asset's/strategy's conditional moments based on quarters in which the S&P 500 excess returns were less than –3.75%. We report the assets'/strategies' conditional mean and volatility, as well as their covariance, beta, and correlation with equities (S&P 500).
Sources: Bloomberg and PIMCO.

Table A2. Historical Conditional Performance of Strategies

Date	Tail Risk Hedging	Trend Following	Long Treasuries	Carry	Value	S&P 500
Mar 1994	0.8%	–5.2%	–2.1%	–2.5%	–1.0%	–4.9%
Sep 1998	2.7%	1.5%	2.9%	–0.9%	0.6%	–11.6%
Sep 1999	0.2%	–0.1%	–0.7%	2.1%	0.6%	–7.8%
Jun 2000	–1.0%	0.2%	–0.3%	2.9%	3.4%	–4.5%
Dec 2000	0.9%	3.3%	2.5%	1.3%	1.2%	–9.6%
Mar 2001	3.8%	3.9%	0.0%	3.5%	0.3%	–13.3%
Sep 2001	7.2%	3.4%	2.6%	2.8%	3.3%	–15.8%
Jun 2002	3.4%	3.0%	2.5%	5.1%	1.9%	–14.1%
Sep 2002	9.1%	4.5%	5.2%	2.5%	–2.7%	–18.0%
Mar 2003	–0.4%	2.1%	0.5%	–1.0%	4.0%	–3.9%
Dec 2007	–0.2%	2.1%	2.0%	3.1%	0.7%	–5.1%
Mar 2008	1.1%	5.9%	1.4%	0.9%	0.5%	–10.7%
Jun 2008	–1.0%	0.7%	–1.3%	4.4%	4.7%	–3.9%
Sep 2008	3.0%	0.7%	1.0%	–5.8%	–3.4%	–9.5%
Dec 2008	12.9%	11.1%	8.2%	0.3%	–4.2%	–23.0%
Mar 2009	4.2%	0.0%	–2.4%	1.6%	2.1%	–11.8%
Jun 2010	2.8%	0.1%	5.5%	0.7%	3.7%	–11.9%
Sep 2011	4.3%	4.8%	11.2%	4.3%	–2.6%	–14.4%
Sep 2015	1.0%	1.1%	2.3%	0.0%	2.6%	–7.0%
Dec 2018	2.3%	0.9%	1.6%	–4.0%	2.1%	–14.5%
Average	2.8%	2.2%	2.1%	1.1%	0.9%	
Minimum	–1.0%	–5.2%	–2.4%	–5.8%	–4.2%	
Maximum	12.9%	11.1%	11.2%	5.1%	4.7%	
Conditional correlation with S&P 500	–0.9	–0.7	–0.6	–0.1	0.5	

Notes: This table reports historical returns in excess of cash for each underlying risk-mitigation strategy in quarters when the S&P 500 excess return was less than –3.75%. The dates in the first column denote the last month of the corresponding quarter; for example, the "Dec 2018" row reports the excess returns of each strategy and of the S&P 500 in the fourth quarter of 2018.
Sources: Bloomberg and PIMCO.

the results. For the trend-following, carry, and value strategies, the Sharpe ratios decline from 0.66, 1.01, and 0.63 to 0.33, 0.86, and 0.58, respectively.

Long Treasuries: The long Treasury strategy is assumed to be implemented using 30-year interest rate swaps, which typically have very low transaction costs. We assume a one-way transaction cost of 2 bps. Assuming annual rebalancing, we subtract a round-trip transaction cost of 4 bps per year from the long Treasury return.

Appendix B. Model Solution

1. MVO, Unit-Beta, and Zero-Beta MVO Portfolios

Unconstrained mean–variance optimization (MVO) portfolio: First, we consider a standard Markowitz MVO portfolio. The optimization is given by

$$\max_w E\left[R_P\right] = w'\mu$$

Table A3. Sample Transaction Costs for Trend-Following, Carry, and Value Strategies

Strategy	Asset Class	Turnover	Asset Class Trans. Cost	Strategy Trans. Cost	Net-of-Cost Return	Net-of-Cost Sharpe Ratio
Trend following	Commodities	0.63	0.10%	0.17%	0.23%	0.33
	Currencies	0.73	0.01%			
	Interest rates	9.34	0.04%			
	Equities	0.47	0.02%			
Carry	Commodities	0.58	0.09%	0.14%	0.86%	0.86
	Currencies	0.98	0.02%			
	Interest rates	8.89	0.04%			
Value	Commodities	0.55	0.08%	0.13%	0.58%	0.58
	Currencies	1.35	0.03%			
	Interest rates	5.34	0.02%			

Notes: This table reports the transaction costs for the trend-following, carry, and value strategies. The "Turnover" column shows the total average annual turnover for all instruments in each asset class (for 1% *ex ante* volatility strategies); the "Asset Class Trans. Cost" column shows the annual transaction costs for each asset class; the "Strategy Trans. Cost" column shows the transaction costs for each strategy. The last two columns show the net-of-cost return and Sharpe ratio, based on volatility estimates given in Table A1.
Sources: Bloomberg and PIMCO.

and

$$0.5w'\Sigma w = 0.5\sigma_p^2.$$

The Lagrangian is given by

$$\mathcal{L} = w'\mu - \frac{\gamma_{MVO}}{2} w'\Sigma w.$$

The first-order condition with respect to w is $w^{MVO} = \gamma_{MVO}^{-1}\Sigma^{-1}\mu$. Substituting this into the volatility constraint yields $\gamma_{MVO} = \sigma_p^{-1}\left(\mu'\Sigma^{-1}\mu\right)^{1/2}$. Hence,

$$w^{MVO} = \gamma_{MVO}^{-1}\Sigma^{-1}\mu = \sigma_p \frac{\Sigma^{-1}\mu}{\left(\mu'\Sigma^{-1}\mu\right)^{1/2}}. \tag{B1}$$

Note that the expected return of the MVO portfolio is $\mu' w^{MVO} = \sigma_p\left(\mu'\Sigma^{-1}\mu\right)^{1/2}$ and the Sharpe ratio of the MVO portfolio—or, in fact, any portfolio that is proportional to the MVO portfolio (cw^{MVO})—is

$$SR_{MVO} = \left(\mu'\Sigma^{-1}\mu\right)^{1/2}. \tag{B2}$$

Unit-beta portfolio: Next, we consider an optimization problem that minimizes variance subject to a unit-conditional-beta target:

$$\min_w \frac{1}{2} w'\Sigma w$$

$$w'\beta_c = 1.$$

The solution to this problem is

$$w^B = \frac{\Sigma^{-1}\beta_c}{\beta_c'\Sigma^{-1}\beta_c}.$$

Note that this portfolio has unconditional expected excess return $\mu^B = \mu' w^B = \frac{\mu'\Sigma^{-1}\beta_c}{\beta_c'\Sigma^{-1}\beta_c}$, unconditional variance $\sigma_{w^B}^2 = \left(\beta_c'\Sigma^{-1}\beta_c\right)^{-1}$, and unconditional Sharpe ratio

$$SR_{w^B} = \frac{\mu'\Sigma^{-1}\beta_c}{\left(\beta_c'\Sigma^{-1}\beta_c\right)^{1/2}}. \tag{B3}$$

It also follows that the minimum-variance portfolio for any target $\hat{\beta}$ is $\hat{\beta} w^B$, which has an unconditional expected excess return $\hat{\beta}\mu^B = \frac{\mu'\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}\hat{\beta}$, unconditional variance $\hat{\beta}^2\sigma^2_{w^B} = \hat{\beta}^2\left(\beta_c{}'\Sigma^{-1}\beta_c\right)^{-1}$, and unconditional Sharpe ratio $\frac{\mu'\Sigma^{-1}\beta_c}{\left(\beta_c{}'\Sigma^{-1}\beta_c\right)^{1/2}} = SR_{w^B}$.

Zero-beta MVO portfolio: Finally, we consider an optimization problem subject to a zero-conditional-beta target:

$$\max_w w'\mu - \frac{\gamma_{MVO}}{2} w'\Sigma w$$

$$w'\beta_c = 0,$$

with $\gamma_{MVO} = \sigma_p^{-1}\left(\mu'\Sigma^{-1}\mu\right)^{1/2}$, the multiplier in the MVO problem. The Lagrangian is

$$\mathcal{L} = w'\mu - \frac{\gamma_{MVO}}{2} w'\Sigma w - \lambda_{ZB} w'\beta_c.$$

The first-order condition with respect to w is $w = \gamma_{MVO}^{-1}\Sigma^{-1}\mu - \lambda_{ZB}\gamma_{MVO}^{-1}\Sigma^{-1}\beta_c$. Substituting this into the beta constraint, $w'\beta_c = 0$, yields $\lambda_{ZB} = \frac{\mu'\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}$. Therefore, this zero-beta portfolio is

$$w^{ZB} = \gamma_{MVO}^{-1}\Sigma^{-1}\mu - \frac{\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}\gamma_{MVO}^{-1}\mu'\Sigma^{-1}\beta_c.$$

Note that $\gamma_{MVO}^{-1}\mu'\Sigma^{-1}\beta_c = w^{MVO}\beta_c \equiv \beta^{MVO}$, the beta of the unconstrained MVO portfolio. We can thus rewrite the zero-beta portfolio as

$$w^{ZB} = w^{MVO} - \beta^{MVO} w^B.$$

2. Solution to the Beta-Constrained Optimization Problem

The Lagrangian for the optimization problem from Equations 1, 2.1, and 2.2 is given by

$$\mathcal{L} = w'\mu - \gamma\left(0.5w'\Sigma w - 0.5\sigma_p^2\right) - \lambda\left(w'\beta_c - \bar{\beta}_c\right). \quad \text{(B4)}$$

The optimal conditions are

$$\mu - \gamma\Sigma w - \lambda\beta_c = 0, \quad \text{(B5)}$$

$$w'\Sigma w - \sigma_p^2 = 0, \quad \text{(B6)}$$

and

$$w'\beta_c - \bar{\beta}_c \le 0, \lambda \ge 0, \lambda\left(w'\beta_c - \bar{\beta}_c\right) = 0. \quad \text{(B7)}$$

If the beta constraint does not bind, then $\lambda = 0$ and Equations 1, 2.1, and 2.2 reduce to a standard unconstrained MVO problem, as shown in Section 1 of Appendix B. If the beta constraint binds, however, then $\lambda > 0$ and Equation B5 can be rewritten as

$$w = \gamma^{-1}\Sigma^{-1}\mu - \gamma^{-1}\lambda\Sigma^{-1}\beta_c. \quad \text{(B8)}$$

Substituting this into the binding beta constraint $w'\beta_c - \bar{\beta}_c = 0$ in Equation B7, we can solve for λ as a function of γ:

$$\lambda = \frac{\gamma^{-1}\mu'\Sigma^{-1}\beta_c - \bar{\beta}_c}{\gamma^{-1}\beta_c{}'\Sigma^{-1}\beta_c}. \quad \text{(B9)}$$

We can then rewrite the optimal weight in terms of γ only:

$$w = \gamma^{-1}\Sigma^{-1}\mu - \left(\gamma^{-1}\mu'\Sigma^{-1}\beta_c - \bar{\beta}_c\right)\frac{\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}. \quad \text{(B10)}$$

Because γ is a constant (the Lagrange multiplier), the first component of Equation B10, $\gamma^{-1}\Sigma^{-1}\mu$, is simply a set of weights that is proportional to that of the unconstrained MVO portfolio (Equation B1) and can be expressed as cw^{MVO}. The first term in parentheses can thus be expressed as $\gamma^{-1}\mu'\Sigma^{-1}\beta = \beta'\left(cw^{MVO}\right) = c\beta^{MVO}$, where $\beta^{MVO} = \beta' w^{MVO}$ is the beta of the unconstrained MVO portfolio and constant $c = \gamma_{MVO}/\gamma$. Finally, $\frac{\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}$ is the unit-beta portfolio w^B. Putting everything together, Equation B10 becomes

$$w = \bar{\beta}_c w^B + c\left(w^{MVO} - \beta^{MVO} w^B\right). \quad \text{(B11)}$$

Substituting Equation B10 into the variance constraint (Equation B6) to solve for γ,

$$\gamma = \left[\frac{\mu'\Sigma^{-1}\mu - \frac{\left(\mu'\Sigma^{-1}\beta_c\right)^2}{\beta_c{}'\Sigma^{-1}\beta_c}}{\sigma_p^2 - \bar{\beta}_c{}^2\left(\beta_c{}'\Sigma^{-1}\beta_c\right)^{-1}} \right]^{1/2}. \qquad \text{(B12)}$$

Section 1 of Appendix B shows that $\mu'\Sigma^{-1}\mu = SR_{MVO}^2$ and $\frac{\left(\mu'\Sigma^{-1}\beta_c\right)^2}{\beta_c{}'\Sigma^{-1}\beta_c} = SR_{w^B}^2$. Therefore, the numerator is the difference between the squared Sharpe ratio of the MVO portfolio and that of a portfolio that levers the unit-beta portfolio to achieve a beta of $\bar{\beta}_c$. Because the unconstrained MVO portfolio achieves the highest possible Sharpe ratio, the numerator is nonnegative and equals zero when μ = Constant × β_c (e.g., if the CAPM holds and conditional beta equals unconditional beta). Section 1 of Appendix B also shows that $\bar{\beta}_c{}^2\left(\beta_c{}'\Sigma^{-1}\beta_c\right)^{-1}$ is the minimum variance for a portfolio with beta $\bar{\beta}_c$. Therefore, for a solution γ > 0 to exist, we need the expected excess returns not to be proportional to the conditional betas and the beta target has to be achievable so that $\sigma_p^2 > \bar{\beta}_c{}^2\left(\beta_c{}'\Sigma^{-1}\beta_c\right)^{-1}$. Using Equation B9 together with $\gamma^{-1}\mu'\Sigma^{-1}\beta_c = c\beta^{MVO}$, we see that the condition for λ > 0 to exist, when γ > 0 exists, is that $c\beta^{MVO} - \bar{\beta}_c > 0$.

Substituting Equation B12 into Equation B10, we obtain the optimal weights, and substituting Equation B12 into

$$\mu'w = \gamma^{-1}\mu'\Sigma^{-1}\mu - \left(\gamma^{-1}\mu'\Sigma^{-1}\beta_c - \bar{\beta}_c\right)\frac{\mu'\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}, \qquad \text{(B13)}$$

we obtain the expected return of the portfolio:

$$\mu'w = \mu'\left(\frac{\Sigma^{-1}\beta_c}{\beta_c{}'\Sigma^{-1}\beta_c}\right)\bar{\beta}_c + \left[\mu'\Sigma^{-1}\mu - \frac{\left(\mu'\Sigma^{-1}\beta_c\right)^2}{\beta_c{}'\Sigma^{-1}\beta_c}\right]^{1/2} \left(\sigma_p^2 - \frac{\bar{\beta}_c^2}{\beta_c{}'\Sigma^{-1}\beta_c}\right)^{1/2}.$$

Using notations introduced earlier,

$$\mu'w = \mu'\left(\bar{\beta}_c w^B\right) + \left(SR_{MVO}^2 - SR_{w^B}^2\right)^{1/2}\left(\sigma_p^2 - \bar{\beta}_c^2\sigma_{w^B}^2\right)^{1/2}. \qquad \text{(B14)}$$

Editor's Note

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Notes

1. Some examples include currency, fixed-income, and commodity carry strategies, as well as other equity risk premium factors, such as quality and value.

2. The term "insurance premium" used herein refers to the lower expected returns due to higher allocations to assets/strategies with better risk-mitigation properties but lower return potential and is not an insurance product backed by the claims-paying ability of an insurance carrier.

3. See also Baz, Granger, Harvey, Le Roux, and Rattray (2015) for a review of trend-following, carry, and value strategies in various asset classes as well as a discussion of their economic rationale.

4. When the constraint in Equation 2.2 is not binding, the solution reduces to the standard MVO result.

5. Under the capital asset pricing model (CAPM), the expected excess return on the unit-beta portfolio will equal the expected equity excess return $\left(\mu' w^{B} = \mu_{equity}\right)$; thus, the expected return on the beta-hedging portfolio, $\bar{\beta}_c \mu_{equity}$, would be negative for $\bar{\beta}_c < 0$. However, this term could be less negative or even positive when the CAPM does not hold (as we assume in our framework) since the investor can hold assets with both negative betas and positive unconditional expected returns.

6. This term can be positive if the unit-beta portfolio loads heavily on negative-beta and positive-expected-return strategies. In general, we should expect the "insurance premium" component to be greater than $\bar{\beta}_c \mu_{equity}$, but it can be negative if we use such assets as put options. We can think of this term as a cost even if the beta-hedging portfolio has a positive expected return, because the beta-hedging portfolio still has a worse risk–return trade-off than the unconstrained MVO portfolio.

7. The value strategy we consider here should not be confused with value investing in the stock market. Our value strategy uses the rationale of "buy cheap, sell rich" for commodities, currencies, and interest rates. The section "Background on Popular Equity-Risk-Mitigation Strategies" provides some examples discussed in the literature.

8. As Harvey and Liu (2015) noted, the rule-of-thumb 50% haircut is generally considered industry standard. We applied this broad assumption largely to focus on the intuition of the final result. In practice, investors and portfolio managers would ideally use strategy-specific assumptions. See also Harvey, Liu, and Zhu (2016).

9. We made the adjustment to the Sharpe ratio because these strategies can generally be implemented using derivatives; therefore, different volatility targets are relatively easy to achieve. We chose the same volatility target for each strategy to make the final allocations more intuitive.

10. In this study, we chose conditional beta as our measure of defensiveness. However, any other measure of defensiveness based on statistical estimates would also be subject to estimation error.

11. We first converted the conditional beta to a conditional correlation, and then we constructed the confidence interval of the conditional correlation using Fisher transformation. The (1 – α)% confidence interval is $\left[\tanh\left(\operatorname{artanh}(r) - \frac{z_{\alpha/2}}{\sqrt{n-3}}\right), \tanh\left(\operatorname{artanh}(r) + \frac{z_{\alpha/2}}{\sqrt{n-3}}\right)\right]$, where *r* is the sample correlation, *n* is the sample size, and tanh and artanh are the hyperbolic tangent and the inverse hyperbolic tangent functions, respectively. Subsequently, we converted the correlation back into a conditional beta value for expositional purposes.

12. We used excess returns on the S&P 500 as a proxy for equity market returns. The return moments are based on historical values and are reported in Table 1.

13. Although these allocations are theoretically correct, using put options to control the equity exposure of the overall portfolio may not be practical for most investors. One way to avoid these short equity positions is to impose a no-short-sale constraint, which we consider in the next section.

14. For the case with equities in the opportunity set, the maximum return portfolio has a conditional beta of 0.1. That is, the conditional beta constraint does not bind for beta targets greater than 0.1 and all these portfolios have the same optimal allocations.

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