

# Eliminating object prior-bias from sparse-projection tomographic reconstructions

Preeti Gopal, Sharat Chandran, Imants Svalbe, and Ajit Rajwade

**Abstract**—Tomographic reconstruction from undersampled measurements is a necessity when the measurement process is potentially harmful, needs to be rapid, or is resource-expensive. In such cases, information from previously existing longitudinal scans of the same object ('object-prior') helps in the reconstruction of the current object ('test') from its significantly fewer measurements. A common problem with these techniques is the strong influence of object-priors in the reconstruction of new regions in the test. In this work, we mitigate this problem by first estimating the location of new regions and then imposing object-prior in *only* old regions which are similar to the prior.

Our work is based on longitudinal data acquisition scenarios where we wish to study new changes that evolve within an object over time, such as in repeated scanning for disease monitoring, or in tomography-guided surgical procedures. While reconstruction is easily feasible when measurements are acquired from a large number of projection angles ('views'), it is challenging when the number of views is limited (sub-Nyquist). We show that in the latter case, a 'spatially-varying' technique is appropriate in order to prevent the prior from adversely affecting the reconstruction of new structures that are absent in any of the earlier scans. The reconstruction of new regions is safeguarded from the bias of the prior by computing regional weights that moderate the local influence of the priors. We are thus able to effectively reconstruct both the old and the new structures in the test. We have tested the efficacy of our method on synthetic as well as real projection data, in both 2D and 3D. Our method significantly improves the overall quality of reconstructions while minimizing the number of measurements needed for imaging in longitudinal studies.

**Index Terms**—Limited-view tomographic reconstruction, regularization priors, object priors, longitudinal studies.

## I. INTRODUCTION

Computed Tomography (CT) deals with the recovery of details of an object's interior from a limited set of projection data acquired by passing X-rays at different orientations ('views'). It is preferable to minimize the radiation exposed in order to prevent any potential damage to it and in order to reduce the acquisition time. Therefore, current research seeks to either significantly reduce the radiation intensity required to reconstruct with adequate fidelity [1], [2], [3], [4] or significantly reduce measurements required to reconstruct with adequate fidelity. For the latter case, there are two lines of pursuit. One is to intelligently choose those sets of projection views that capture most information [5], [6], [7], [8], [9], and the other, which is the focus of this paper, is to design the

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reconstruction algorithm in order to achieve the most accurate recovery of the underlying slice, given the measurements from any limited set of views [1], [10], [11].

In conventional data acquisition techniques, tomographic measurements  $\mathbf{y}$  are acquired by sampling the physical object  $\mathbf{x}$  uniformly with a substantial number of views, ideally above the Nyquist rate. In such a case when there are sufficient measurements, reconstruction using conventional filtered back-projection (FBP) suffices, as seen in Fig. 1 (first column, 800 views). The figure shows the ground truth image ( $260 \times 260$ ) of naturally growing sprouts at the top left (the details of this dataset is postponed to Section IV).

However, reconstruction from reduced views has been made possible by assuming the data to exhibit certain properties such as smoothness in image space, sparsity in gradient space [12] or sparsity under certain mathematical transforms [13], [14] such as the Discrete Cosine Transform (DCT). There are multiple such regularization priors such as the Compressed Sensing based  $l_1$ -ls prior [15]. For the purposes of demonstrating the efficacy of our method, we use the well known total Variation (TV) regularization prior. If  $\mathcal{R}$  represents the acquisition model, then the TV-regularized solution is described by one that minimizes a combination of least squares error and the TV-norm of the object  $\mathbf{x}$ . Mathematically, we choose to minimize  $J_{TV}(\mathbf{x})$  where

$$J_{TV}(\mathbf{x}) = \|\mathcal{R}\mathbf{x} - \mathbf{y}\|_2^2 + \lambda_{TV} TV(\mathbf{x}) \quad (1)$$

and

$$TV(\mathbf{x}) = \sum_{i,j} \nabla_{ij}(\mathbf{x}), \quad (2)$$

$\nabla$  representing the gradient operator.

Fig. 1 (second column) demonstrates the benefit of using a regularization prior when the number of projection views is limited (100 views). We solve this cost function using an optimal first-order method for large-scale TV regularization presented in [16] and available in [17].

### A. Difficulties

When the number of views is significantly reduced (this paper), a regularization prior alone is not sufficient. In such cases additional information ('object-prior') specific to the current object being scanned (called the 'test' henceforth) is useful in further improving the reconstruction. While this has been done in the context of dynamic CT-scans (discussed more in Section II), in this paper we use a set  $S$  of previous scans of

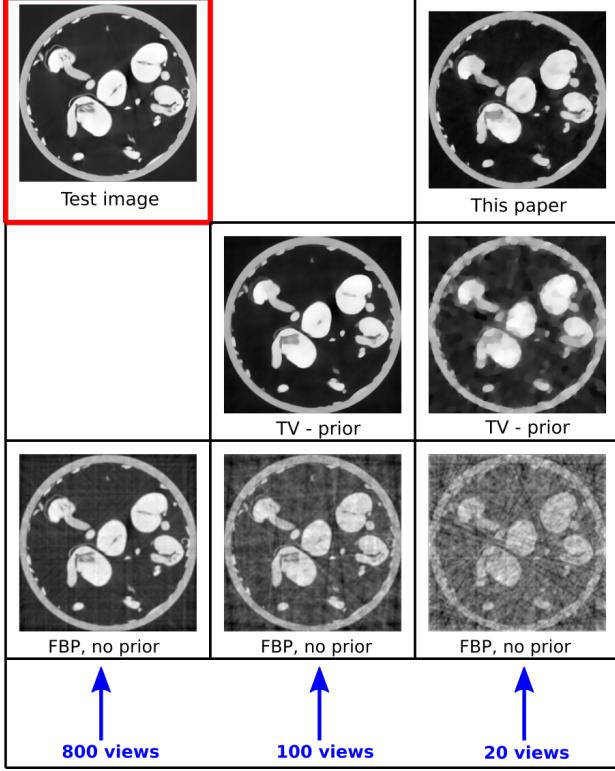


Fig. 1: Illustration of the use of priors in reconstruction as the number of views reduces. For all cases, the ground truth ( $260 \times 260$ ) appears in the top left. For extremely large number of views (800 views), the standard FBP reconstruction (first column) is of very good quality (Structural Similarity Index Metric, SSIM=0.92). As the number of views become limited (100 views), reconstruction using the Total Variation regularization prior (TV, middle column) is better both visually and quantitatively than FBP (SSIM of 0.96 vs 0.82). If the number of views is drastically reduced (20 views), the presence of both TV prior and object- prior together with the spatially-varying technique of this paper improves the reconstruction (SSIM=0.91) when compared to presence of TV-only prior (SSIM=0.87) or no prior (SSIM=0.64).

the same object. This is relevant in a longitudinal context: the acquisition of sequential CT scans of the same subject in order to track time-evolving changes within the subject’s interior.

With the use of an object-prior, a *new* challenge emerges. The prior set  $S$  may potentially overwhelm the necessary details, and when several prior scans are available, finding the right one is an issue. We therefore devise an algorithm that estimates the location and magnitude of new changes in the (unknown) test. As we show in Section V, this eventually prevents the prior from adversely affecting the reconstruction of new regions in the test. We refer to this method as a *spatially-varying* prior-based reconstruction routine, that still uses *all* the previous scans in the set  $S$ , without using any of the prior scans *as is*, or endeavoring to choose one “right” prior. In this way, this work is novel and contrasted with earlier work. As seen in Fig. 1 (last column, 20 views) our method uses the set  $S$  to yield a better reconstruction than the regularization-only, or the backprojection-only methods.

### B. Paper organization

This paper is organized as follows. After discussing related work in Section II, Section III lays down the *key contributions* of this work.

Most datasets from commercial scanners do not provide raw measurements. In order to validate our method quantitatively, we created new cone-beam longitudinal CT datasets, details of which are described in Section IV – these will be offered to the community in the public domain. (In addition, Section VI-C1 demonstrates the utility of our method on a longitudinal medical dataset involving the liver in a radio-ablation study.)

We now move to the details. Section V describes the computation of a spatially-varying weights map in order to preserve new changes in the test. Results are presented in Section VI. Section VII discusses tuning of the hyperparameters involved. Finally, we conclude with key inferences that can be drawn from our work in Section VIII.

## II. RELATED WORK

The idea of using a reduced number of views is most pronounced in specialized applications. For example, in [18], sparsity-constrained optimization is presented for angiography. Here, the regions of interest are the vessels alone and they are highlighted by physically inserting a contrast agent, and therefore there is an inherent sharper contrast between the vessels and the background. In other applications where the spatial-gradient of the underlying volume is known to be sparse, the ‘total variation’ method, as used in [19], [20], can be applied. However, as shown earlier in Fig. 1, these regularization priors are not sufficient when very few views are acquired, thus needing information from stronger priors such as those obtained from previous scans of the same or similar object.

Most object-prior based reconstruction have been in the areas of dynamic CT and 4D-CT. The object-prior usually consists of an object reconstructed to high quality from a large number of projection angles. One of the earliest pieces of work in this area is the well-cited PICCS method [21] for dynamic CT, which enforces a robust-norm based similarity between the test and the object prior, in addition to a sparsity prior on the test. Its limitation is the unchecked over-emphasis of the prior on the reconstruction of the test at hand. A closely related method called PIRPLE was proposed in [22], with additional steps to register the object prior and the test and a somewhat more flexible combination of data fidelity terms, sparsity prior and object prior. However, this method too has similar limitations as PICCS.

In order to prevent the object-prior from overwhelming the reconstruction of test, a few other approaches have either estimated the motion between the object-prior and the test or have applied very specific object-properties for identifying new changes in test. Examples for the former include [23], where changes across the successive scan volumes are assumed to be continuous, thus enabling the use of optical flow to model the motion between corresponding voxels of different scan volumes. In another study [24], changes between successive scans were modeled by affine deformation whose parameters

were computed for motion-correction, which in turn enables acquisition from fewer views. In [25], all the scan volumes are reconstructed together using spatio-temporal regularization. Here again, the inherent assumption is the continuity across volumes in space or time.

The methods that use object-specific knowledge are [26] and [27]. In [26], knowledge of attenuation coefficient of the fluid is used as a prior for reconstructing its flow through gravel. However, this relies on additional expert domain knowledge which may not be available or which may be infeasible to acquire. Other approaches variously use robust principal components analysis assuming that the object consists of a large static part (background) and a sparse moving part (foreground) [28], spline-based models for tracking the regions of change in an SIRT framework [29].

All of these techniques [25], [23], [26], [28], [29] essentially reconstruct all the different stages of the object *simultaneously*, and use the extra redundancy across time simultaneously. In the scenario of a longitudinal study (where projection measurements are acquired at instants that are several weeks apart) which we consider in this paper, such approaches are not feasible. A longitudinal study will necessarily require reconstructions soon after acquisition of each set of measurements.

There also exist methods for estimating new changes (in the test) directly in projection space. In such techniques which can be used in longitudinal studies, the object-prior is projected in the same set of views as the current set of test measurements, and a set of projection differences is computed. These projection differences are essentially tomographic projections of differences between the unknown test and the known object-prior. Hence, a variety of reconstruction algorithms can be used to estimate such a difference image which can then be added to the object-prior to yield the final estimate of the test. Such methods have been proposed in [30], [31]. However, reconstruction of the difference image will inherently contain sub-sampling artefacts, and these artefacts will appear in the final reconstruction as there is no mechanism to mitigate them (unlike the method that we propose). This can be seen in our supplementary material [32], where we have compared our method with [31].

In this paper, we present a technique that makes use of multiple object priors, corresponding to high quality reconstructions of similar objects across time, typically in a longitudinal study. We use these priors to improve the reconstruction of the test from largely undersampled measurements. In particular, we define regions of change between the test and the eigenspace spanned by previous high quality object-priors in a purely data driven manner. Additionally, our technique differentiates between genuine structural changes and “changes” that appear due to undersampling artefacts. Besides this, since we use a statistical model with multiple object-priors, we avoid the problem of selecting an appropriate single object-prior unlike [21], [30], [31], [22], [27] and also make use of the additional information available in multiple priors. Our technique also has the advantage of not requiring any expert-domain knowledge about the object being scanned.

### III. CONTRIBUTIONS

This paper focuses on few-views reconstruction with an emphasis on longitudinal studies. In contrast to the object-prior based studies mentioned above, we reconstruct the current test object without any assumption of continuity of changes or knowledge of the attenuation coefficients of the structures. We also do not make any temporal assumption in terms of time intervals – prior scans could be months apart. We use the current measurements from few-views and previous scans of the same object.

A key idea in our work is this starting point – the new test volume is close to the space spanned by the eigenvectors of multiple representative previously scanned objects. We then present a method to accommodate new changes in the test that may lie outside of this eigenspace. This will prevent an unnecessary bias when reconstructing the data. As a solution, our method moderates the control of the prior by estimating and imposing spatially-varying weights to the prior in order to reconstruct new structures accurately. This spatially-varying prior tunes the effect of the templates in different regions of reconstruction.

Fig. 2 provides an overview.

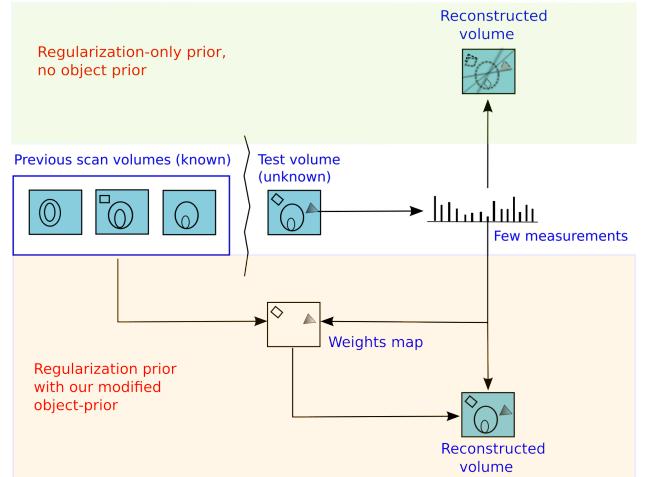


Fig. 2: Overview of our work. (Upper) When the number of measurements is vastly lower than what is conventionally used (last column of Fig. ??) a regularization prior such as Total Variation alone is not sufficient to mitigate the sub-sampling artefacts. (Lower) An object-prior must be used to provide additional information, but in addition, new bias should not be introduced. To this end, here we propose using a spatially-varying weights-map to modulate the effect of object-prior on the final reconstruction.

In summary, the key contributions are

- We create new 3D biological datasets (Section IV) and present results on real cone-beam projections. Our datasets and code will be made available to the community.
- We design a weights-map to depict the location and strength of new changes in the test at every voxel. A novel algorithm is presented to build this map from sub-sampled measurements of the test and a set of high quality templates. Once the weights-map is built, it is

used for accurate reconstruction of those changes in the test that are absent in all of the templates. Results appear in Section VI.

- We show the efficacy of our results in a real-life medical longitudinal study with data obtained in a clinical setting from a live teaching and research hospital.

#### IV. DATASETS

In this section, we present details of the biological and medical datasets created and used for this work. For the first two datasets namely, Potato and Okra, the forward imaging model is available, and hence all reconstructions were performed from raw measurements. For the other two datasets namely, Sprouts and Liver, the forward imaging model was not available to us (due to propriety concerns) and hence only reconstructed volumes of real data were at hand; we therefore generated measurements from these datasets assuming a circular cone-beam imaging model for 3D and a parallel-beam imaging model for 2D.

In each case, we remark on the design of the choice of the test chosen in the dataset.

##### A. Datasets with the forward model available

These consist of longitudinal scans of a Potato and an Okra, both acquired by the first author at the Australian National University (ANU).

**Potato:** This dataset consisted of four scans of the humble potato (Fig. 3). While the first scan was taken of the undistorted potato, subsequent scans were taken of the same specimen, each time after drilling a new hole halfway into the potato. Measurements from each scan consisted of real circular cone-beam projections from 900 views, each of size  $150 \times 150$ . In our experiments, the first 3 volumes were used to build the object-prior and the last volume was used as the test. The ground truth consists of volumes of size  $150 \times 150 \times 100$  reconstructed using the Feldkamp-Davis-Kress (FDK) from the full set of 900 projection views.

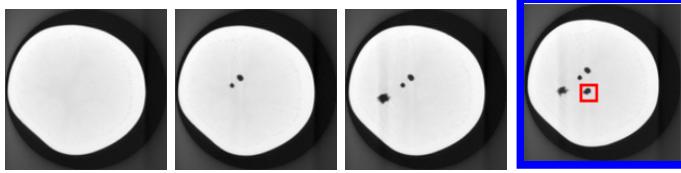


Fig. 3: Potato 3D dataset: One slice each from the scanned objects that are used as templates (the first three from left) and a slice from the test volume (extreme right). Notice the appearance of the fourth hole in the test slice. 3D visualization is available in the supplementary material [32].

*Remark:* The test slice (unlike the Okra below) was designed to show the presence of a new feature (last picture in Fig. 3). Although the potato is visually bland, paradoxically, this becomes the reason that the Potato is challenging if a new method has to be shown to be effective. Simpler methods may suffice. Nevertheless, we show super performance with

our method by vastly reducing the number of measurements corresponding to the ground truth.

**Okra:** This dataset is that of an Okra specimen consisting of its five scans (Fig. 4). Each of the scanned volumes consists of a varying number of cuts (or deformations) on the surface of the specimen. The specimen was kept in the same position throughout the acquisitions. Measurements consisted of real circular cone-beam projections from 450 views, each of size  $336 \times 156$ . In our experiments, the first 4 volumes shown in Fig. 4 were used to build the object-prior and the last volume was used as the test. Note that in the region marked red, all the templates have a deformation, and hence an unchecked reliance of object-prior for the reconstruction of test in this region will result in a deformation. The ground truth consists of volumes of size  $338 \times 338 \times 123$  reconstructed using the FDK algorithm [33] from the full set of 450 view projections.

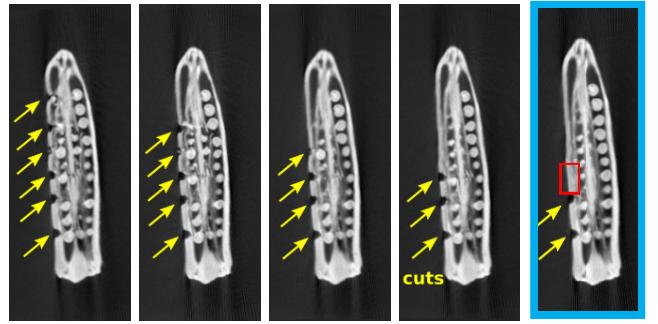


Fig. 4: Okra 3D dataset: One slice each from the scanned objects that are used as templates (the first four from left) and a slice from the test volume (extreme right). Notice the region marked in red box; while all slices have deformities here, the test has none.

*Remark:* Unlike the potato, the test was designed as one with the absence of a feature. Although mathematically one is the conjugate of the other, in practice, experimental methods may have difficulties which explains the choice of our test. Notice also that the structure of the Okra is intricate as compared to Potato, and this brings with it a different challenge.

##### B. Datasets without the forward model

These consist of longitudinal scans of a portion of a liver obtained from Tata Memorial Centre (TMC) [34] in Parel, Mumbai, and longitudinal scans of Sprouts obtained from Australian National University.

**Liver:** TMC is the national comprehensive centre for the prevention, treatment, education and research in cancer, and is recognized as one of the leading cancer centres in India. The dataset (Fig. 5) from this longitudinal medical study consists of 7 scans taken at different stages of radio-frequency ablation study of a liver. In such a procedure [35], the physician inserts a thin needle-like probe into the organ. Repeated CT scans of the patient are acquired in order to track the movement of the needle and to ensure that it is reaching the appropriate target tumor. Once the needle hits the tumor, a high-frequency electric current is passed through the tip of the probe and this burns the malignant tumor (ablation).

In our experiments, we generate parallel beam measurements from 2D slices from each of the 7 volumes. Note that all these 7 slices are located at the *same* index (slice number corresponding to the same depth) within each of the respective volumes. Observe (in Fig. 5) that the needle is seen in all of the 7 slices.

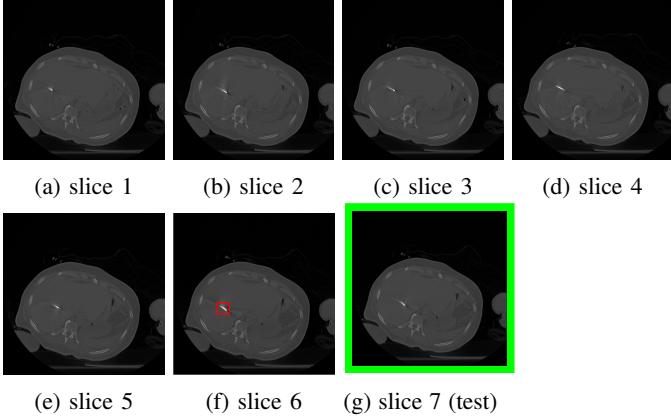


Fig. 5: Radio-frequency ablation dataset: One of the slices ( $512 \times 512$ ) from each of the 7 scan volumes of a longitudinal study dataset of the liver. Note that in volumes (a) through (g), the needle (shown in red in (f)) approaches the target tumor.

**Remark:** Liver is interesting because the data comes from live data from a clinical setting. The oncologist has to be aware of the precise position of the needle contrasted with the site of the tumor, yet we do not want to create new mutant DNA by further bombarding the patient with X-rays.

**Sprouts:** This dataset consists of six volumes corresponding to six scans of an in-vivo sprout specimen imaged at its various stages of growth (Fig. 6). In our experiments, the first 5 volumes were used to build the object-prior and the last volume was used as the test. The ground truth consists of FDK reconstructed volumes of size  $130 \times 130 \times 130$  from a set of 1800 view projections. Test measurements were generated assuming a circular cone-beam imaging model.

**Remark:** In contrast to the scientific experiment performed for the case of the okra and potato where we introduced man-made defects, the changes in the sprouts here are purely the work of nature. This dataset therefore has a different set of challenges.

## V. SPATIALLY-VARYING PRIOR-BASED RECONSTRUCTION

Our method modifies the algorithm presented in [15] and additionally overcomes one of its major limitations by introducing an extra computational step. We first review the eigenspace-cum-CS prior-based reconstruction algorithm which was shown [15] to be better when compared to dictionary-based priors. PCA has been traditionally used to find the significant modes of (Gaussian corrupted) data. In this regard, it has been widely applied in the context of data compression. However, PCA can also be seen as a tool to provide an orthogonal basis to represent the space in which most of the test data could lie (except the new changes). This space is constructed from the available set of previously

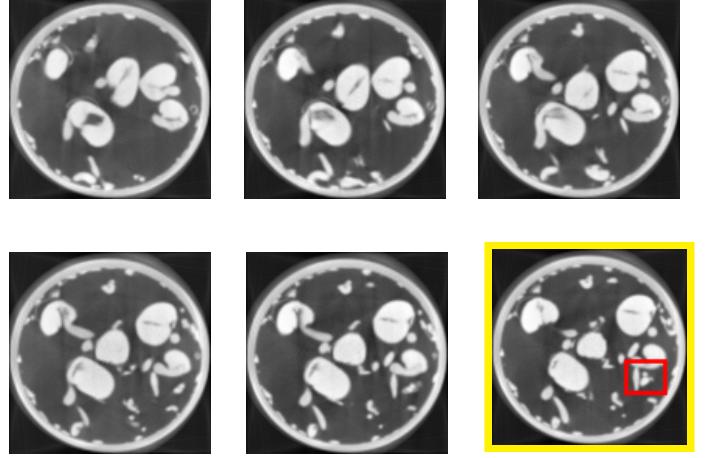


Fig. 6: Sprouts 3D dataset: One slice each from the scanned objects that are used as templates (the first five from left) and a slice from the test (extreme right). One of the many regions in the test that is different from all other templates is the structure within the red box.

scanned objects which must cover a realistically representative range of structures.

To begin with, when an object is scanned multiple times, a set of high quality reconstructions (i.e., reconstructions from a dense set of projection views) may be chosen as object-prior for the reconstruction of future scan volumes, which in turn, may be scanned using far fewer measurements. The eigenspace  $E_{\text{high}}$  of the  $L$  previously scanned objects  $Q_1, Q_2, \dots, Q_L$  is pre-computed. Here, it is assumed that most of the test volume, barring the new changes, can be expressed as a sparse linear combination of the principal components (eigenvectors of the covariance matrix) obtained from a group of structurally similar volumes. Hence, the object-prior is represented by means of PCA. For the eigenspace to encompass a range of possible structures in the test slice, the object-prior must represent a wide structural range. Moreover, if these volumes are not aligned, then they must be first registered before computing the prior.

The prior is built by computing the covariance matrix from the template set  $\{Q_i\}_{i=1}^L$ . The space spanned by the eigenvectors  $\{V_k\}_{k=1}^{L-1}$  (eigenspace) of the covariance matrix is the object prior and is assumed to contain most of the test slice (or volume in the case of 3D) that is similar, but not necessarily identical to the object-prior. We use all of the  $L - 1$  orthogonal eigenvectors as a basis to represent the unknown test volume. Let  $\mu$  denote the mean of the previously scanned objects, and  $\alpha$  the unknown vector of eigen-coefficients of the test scan, of which  $\alpha_k$  is the  $k^{\text{th}}$  element. Then, once the eigenspace is pre-computed, the test was reconstructed in [15] by imposing a penalty if the estimated slice does not lie within the eigen-space of the object prior. Although such a prior can be very useful in some circumstances, it poses a limitation when we want accurate details of a portion of the new changes. While the object prior compensates very well for the possible artefacts due to sparse measurements, it may dominate the regions with new changes masking the signal.

Ideally, we will want to impose the prior only in the regions that are common between the test and object-prior. Our spatially-varying prior based reconstruction overcomes this limitation by minimizing the following cost function:

$$\begin{aligned} J_3(\mathbf{x}, \boldsymbol{\alpha}) = & \| \mathcal{R}\mathbf{x} - \mathbf{y} \|_2^2 + \lambda_1 TV(\mathbf{x}) + \\ & \lambda_2 \| \mathbf{W}(\mathbf{x} - (\boldsymbol{\mu} + \sum_k \mathbf{V}_k \boldsymbol{\alpha}_k)) \|_2^2. \end{aligned} \quad (3)$$

The key to our method is the discovery of a diagonal weights matrix  $\mathbf{W}$ , where  $W_{ii}$  contains the (non-negative) weight assigned to the  $i^{\text{th}}$  voxel of the prior.  $\mathbf{W}$  is first constructed using some preliminary reconstruction methods, following which Eq. 3 is used to obtain the final reconstruction. In regions of change in test data, we want lower weights for the prior when compared to regions that are similar to the prior.

$\lambda_1, \lambda_2$  are tunable weights given to the TV-prior and object-prior terms respectively. The unknowns  $\mathbf{x}$  and  $\boldsymbol{\alpha}$  are solved by alternately minimizing  $J_{\boldsymbol{\alpha}}(\mathbf{x})$  using a fixed  $\boldsymbol{\alpha}$ , and  $J_{\mathbf{x}}(\boldsymbol{\alpha})$  using the resultant  $\mathbf{x}$ .  $J_{\boldsymbol{\alpha}}(\mathbf{x})$  is solved for using an optimal first-order method for large-scale TV regularization presented in [16] and available in [17].

$$\begin{aligned} J_{\boldsymbol{\alpha}}(\mathbf{x}) \triangleq & \| \mathcal{R}\mathbf{x} - \mathbf{y} \|_2^2 + \lambda_1 TV(\mathbf{x}) \\ & + \lambda_2 \| (\mathbf{W}(\mathbf{x} - (\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\alpha})) \|_2^2 \end{aligned} \quad (4)$$

$$J_{\mathbf{x}}(\boldsymbol{\alpha}) \triangleq \| \mathbf{W}(\mathbf{x} - (\boldsymbol{\mu} + \mathbf{V}\boldsymbol{\alpha})) \|_2^2. \quad (5)$$

Solving for  $\boldsymbol{\alpha}$  leads to the closed form update:

$$\boldsymbol{\alpha} = [(\mathbf{W}\mathbf{V})^T \mathbf{W}\mathbf{V}]^{-1} \mathbf{V}^T (\mathbf{W}^T \mathbf{W}(\mathbf{x} - \boldsymbol{\mu})) \quad (6)$$

The cost function described in Eq. 3 is biconvex and the convergence of this optimization is guaranteed by the monotone convergence theorem [36].

**Computation of weights matrix  $\mathbf{W}$ :** Since the test is unknown to begin with, it is not possible to decipher the precise regions in  $\mathbf{x}$  that are different from all the previously scanned objects ('object-prior'). We start with  $X^{\text{fdk}}$ , the initial backprojection reconstruction of the test volume using FDK in an attempt to discover the difference between the object-prior and the test volume. Let  $\mathbf{V}_{\text{high}}$  be the eigenspace constructed from high-quality object-prior. However, the difference between  $X^{\text{fdk}}$  and its projection onto the eigenspace  $\mathbf{V}_{\text{high}}$  will detect the new regions along with many false positives (false new regions). This is because,  $X^{\text{fdk}}$  will contain many geometric-specific artefacts arising from sparse measurements (angle undersampling), which are absent in the high quality object-prior used to construct the eigenspace  $\mathbf{V}_{\text{high}}$ . To discover unwanted artefacts of the imaging geometry, in a counter-intuitive way, we generate *low quality* reconstruction of the object-prior as described below. The algorithm is discussed next (a simplified version appears as a schematic in the supplementary material and may be used to understand the intuition.)

#### Algorithm to compute weights-map $\mathbf{W}$ :

- 1) Perform a pilot reconstruction  $X^{\text{fdk}}$  of the test volume  $\mathbf{x}$  using FDK.

- 2) Compute low quality template volumes  $Y^{\text{fdk}}$ . We assume  $L$  previously scanned objects from which we build an eigenspace.
  - a) Generate simulated measurements  $\mathbf{y}_{Q_i}$  for every template  $Q_i$ , using the exact same projections views and imaging geometry with which the measurements  $\mathbf{y}$  of the test volume  $\mathbf{x}$  were acquired, and
  - b) Perform  $L$  preliminary FDK reconstructions of each of the  $L$  object-prior from  $\mathbf{y}_{Q_i}$ . Let this be denoted by  $\{Y_i^{\text{fdk}}\}_{i=1}^L$ .
- 3) Build eigenspace  $\mathbf{V}_{\text{low}}$  from  $\{Y_i^{\text{fdk}}\}_{i=1}^L$ . Let  $P^{\text{fdk}}$  denote projection of  $X^{\text{fdk}}$  onto  $\mathbf{V}_{\text{low}}$ . The difference between  $P^{\text{fdk}}$  and  $X^{\text{fdk}}$  will not contain false positives due to imaging geometry, but will have false positives due to artefacts that are specific to the reconstruction method used. To resolve this, perform steps 4 and 5.
- 4) Project with multiple methods.
  - a) Perform pilot reconstructions of the test using  $M$  different reconstruction algorithms like the CS [37] with any suitable sparsity domain such as wavelets (Haar) or DCT, Total Variation [12], Algebraic Reconstruction Technique (ART) [38], Simultaneous Algebraic Reconstruction Technique (SART) [39] and Simultaneous Iterative Reconstruction Technique (SIRT) [40]. Let this set of pilot reconstructions be denoted as  $X \triangleq \{X_j\}_{j=1}^M$  where  $j$  is an index for the reconstruction method, and  $X^1 = X^{\text{fdk}}$ .
  - b) From  $\mathbf{y}_{Q_i}$ , perform reconstructions of the template  $Q_i$  using the  $M$  different algorithms, for each of the  $L$  previously scanned objects. Let this set be denoted by  $Y \triangleq \{Y_i^j\}_{j=1}^M\}_{i=1}^L$  where  $Y_i^1 = Y_i^{\text{fdk}}$ ,  $\forall i \in \{1, \dots, L\}$ .
  - c) For each of the  $M$  algorithms (indexed by  $j$ ), build an eigenspace  $\mathbf{V}_{\text{low}}^j$  from  $\{Y_1^j, Y_2^j, \dots, Y_L^j\}$ .
  - d) Next, for each  $j$ , project  $X^j$  onto  $\mathbf{V}_{\text{low}}^j$ . Let this projection be denoted by  $P^j$ . To reiterate, this captures those parts of the test volume that lie in the subspace  $\mathbf{V}_{\text{low}}^j$  (i.e., are similar to the template reconstructions). The rest, i.e., new changes and their reconstruction method-dependent-artefacts, are not captured by this projection and need to be eliminated.
- 5) To remove all reconstruction method dependent false positives, we compute  $\min_j(|X^j(x, y, z) - P^j(x, y, z)|)$ . (The intuition for using the 'min' is provided in the paragraph immediately following step 6 of this procedure.)
- 6) Finally, the weight to prior for each voxel coordinate  $(x, y, z)$  is given by

$$\mathbf{W}_v(x, y, z) = (1 + k(\min_j |X^j(x, y, z) - P^j(x, y, z)|))^{-1}. \quad (7)$$

Note that here  $\mathbf{W}_v(x, y, z)$  represents the weight to the prior in the  $(x, y, z)^{\text{th}}$  voxel.  $\mathbf{W}_v(x, y, z)$  must be low whenever the preliminary test reconstruction  $X^j(x, y, z)$  is different from its projection  $P^j(x, y, z)$  onto the prior eigenspace, for

every method  $j \in \{1, \dots, M\}$ . This is because it is unlikely that *every* algorithm would produce a significant artefact at a voxel, and hence we hypothesize that the large difference has arisen due to genuine structural changes. This hypothesis is empirically and quantitatively proven in the next section. The parameter  $k$  decides the sensitivity of the weights to the difference  $|X^j(x, y, z) - P^j(x, y, z)|$ . Selection of  $k$  is discussed in detail in Sec. VII.

## VI. RESULTS

In this section, we first illustrate the advantage of using multiple types of eigenspaces for the computation of weights map, and then present reconstruction results on all datasets. For quantitative evaluation, we compute the Structural Similarity Index Metric<sup>1</sup> (SSIM) [41].

### A. Motivation for the use of multiple types of eigenspaces for the computation of weights

The changes and new structures present in the test data will generate different artifacts for different reconstruction techniques. These artifacts would not be captured by reconstructions of the object-prior since the underlying new changes and structures may be absent in all of the previously scanned objects. We aim to let the weights be independent of the type of artifact. Hence, we use a combination of different reconstruction techniques to generate different types of eigenspaces and combine information from all of them to compute weights. To illustrate the benefit of this method, we first performed 2D reconstruction of a test slice from the potato dataset. Fig. 7 shows the test and template slices. Fig. 8 shows the weights-maps generated using Eq. 7 by various reconstruction methods. It can be seen that the weights are low in the region of the new change in test data. Because all the iterative methods are computationally expensive, we chose only FBP and TV for computing weights-maps for all 3D reconstructions.

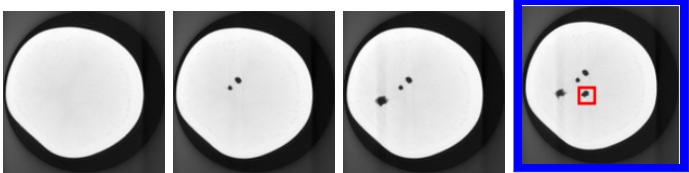


Fig. 7: Dataset for illustrating the use of multiple eigenspaces: One slice each from the scanned objects (the first three from left) and a slice from the test volume (extreme right). Notice the appearance of the fourth hole in the test slice.

TABLE I: SSIM of the reconstructions shown in Fig. 9:(a)-(g). The SSIM of ground-truth (ideal reconstruction) is 1.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
0.64	0.72	0.68	0.83	0.81	0.69	0.83	<b>0.88</b>

<sup>1</sup>Here, we assign higher importance to the preservation of structure in a reconstruction by assigning the following weights to (structure, contrast, intensity): (0.7, 0.2, 0.1).

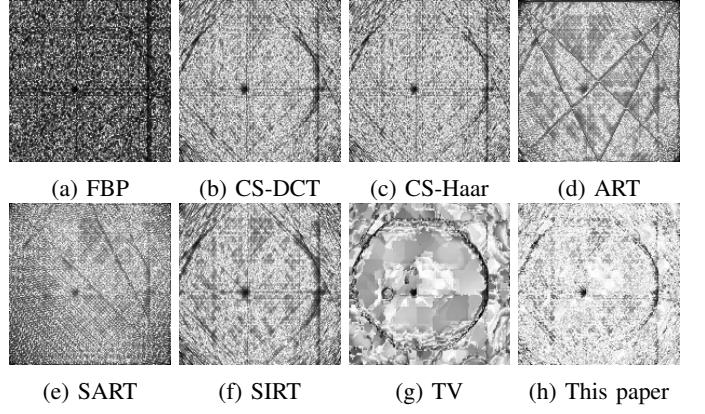


Fig. 8: The motivation for multiple methods is demonstrated with a slice of the data in Fig. 7. The test image in Fig. 7(d) is reconstructed using only 6 paltry views. The weights-maps constructed using different reconstruction methods individually (a-g) are inadequate and represent different false positives. By fusing information from all reconstruction methods, as specified in Eq. 7 we get the final weights map (h). The corresponding reconstructions appear in Fig. 9 .

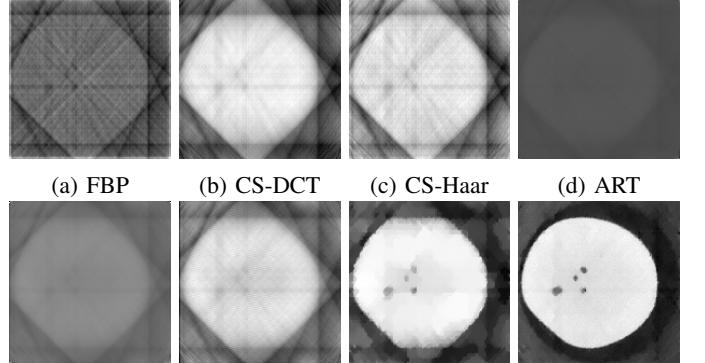


Fig. 9: (a)-(f): Reconstructions of 7(d) from 6 views. The magnitude and sharpness of the artefacts are different for distinct methods. (h) Spatially-Varying-prior method (from this paper) combines weights-map information from all other methods. The SSIM of these reconstructions is shown in Table. I.

### B. Reconstruction of datasets with forward model available

1) **Potato:** The test volume was reconstructed from a partial set of 45 real cone-beam projections, i.e., 5% of the 900 projection views from which ground truth was reconstructed (see remarks in Sec. IV-A). 2D reconstructions of one of the slices is shown in Fig. 10. A complete 3D reconstruction can be found in the supplementary material [32]. The red ROI spans 7 consecutive slices where the test is different from all of the previously scanned objects. The zoomed-in images around the major region of change (red ROI) is shown in Fig. 11.

We observe that our method reconstructs new structures while simultaneously reducing streak artefacts. Table II shows SSIM of the reconstructed new regions using various methods.

2) **Okra:** The test volume was reconstructed from a partial set of 45 real cone-beam projections, i.e., 10% of the 450 projection views from which ground truth was reconstructed (refer Sec. IV). 2D reconstructions of one of the slices is

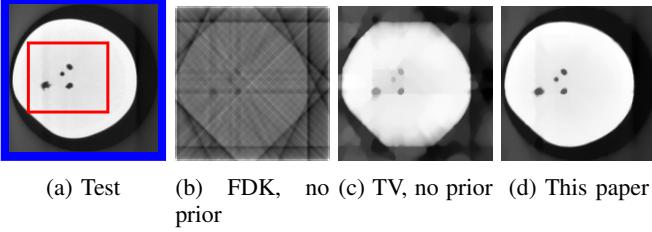


Fig. 10: Reconstruction of potato from 6 projection views—(b) has strong streak artefacts with unclear structure of the potato, (c) largely blurred, and (d) is sharper with significantly reduced streak artefacts.

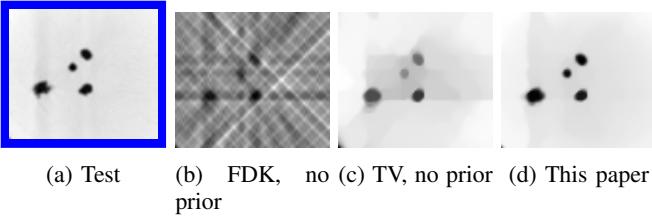


Fig. 11: A zoomed-in version of the regions around the ROI of the reconstructions shown in Fig. 10.

TABLE II: SSIM of reconstructed potato from various methods. SSIM of ground-truth (ideal reconstruction) is 1.

	Backprojection	TV	This paper
2D	0.52	0.92	0.97
3D	0.58	0.75	0.84

shown in Fig. 12. A complete 3D reconstruction can be found in the supplementary material [32]. The red ROI spans 7 consecutive slices where the test is different from all of the previously scanned objects. A zoomed-in region around the major region of change (red ROI) is shown in Fig. 13. As in the test, the reconstruction by spatially-varying method shows the absence of the deformity and better removal of sub-sampling artefacts when compared to FDK and TV. This is also seen in the SSIM values in Table III.

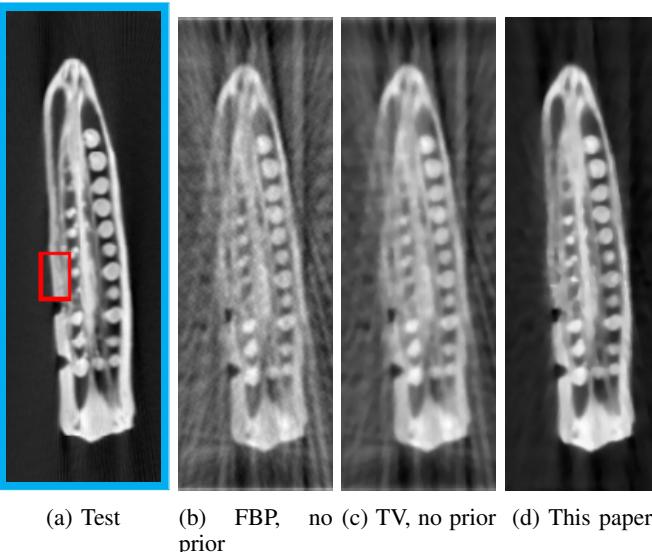


Fig. 12: Reconstruction of okra from 48 projection views (b) has streaky artefacts, (c) has blurred structures and (d) is sharper with significantly less streaky artefacts.

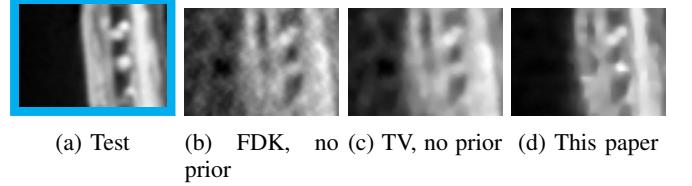


Fig. 13: A zoomed-in version of the regions around the ROI of the reconstructions shown in Fig. 12.

TABLE III: SSIM of reconstructed okra from various methods. SSIM of ground-truth (ideal reconstruction) is 1.

	Backprojection	TV	This paper
2D	0.46	0.50	0.74
3D	0.69	0.74	0.78

### C. Reconstruction of datasets without forward model

1) *Liver*:: Here we show how our technique is useful in a real-life medical longitudinal study. Our data consists of successive scans of the liver taken during a radio-frequency ablation procedure as described in Sec. IV. Our goal is to track the position of the needle in a relatively stationary background, while simultaneously reducing sub-sampling artefacts. Specifically, we choose slices 1-6 as our object-prior, and reconstruct slice 7 from few-views with the specific goal of tracking the needle and simultaneously reducing artefacts. Fig. 14 and Fig. 15 show reconstruction of the test slice from its measurements from only 30 views and a zoomed-in version around the ROI. The reconstructions are quantitatively compared using SSIM.

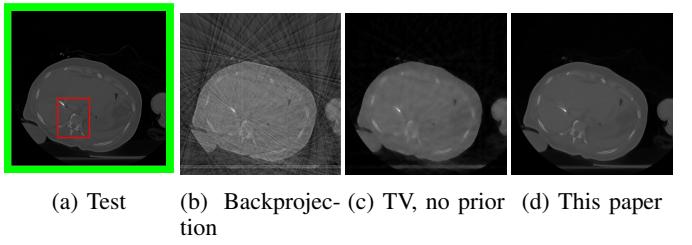


Fig. 14: Reconstruction of ‘test’ (slice 7) from Fig. 5 from only 30 views, using (b) FBP and no prior resulting in streaks (c) TV resulting in blurred bone structures and (d) spatially varying object-prior (slices 1-6 of Fig. 5 are used as object-prior) resulting in clear bone structures with less streaks. The region enclosed in red rectangle is our Region of Interest (ROI) as it contains both the new position of the needle and some background.

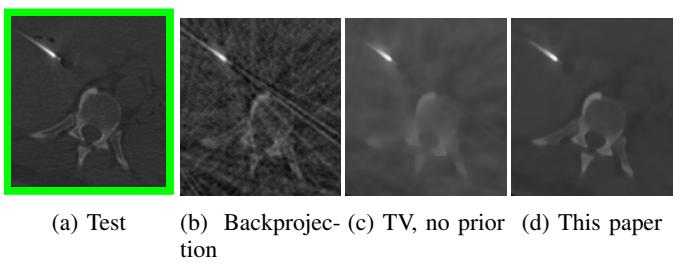
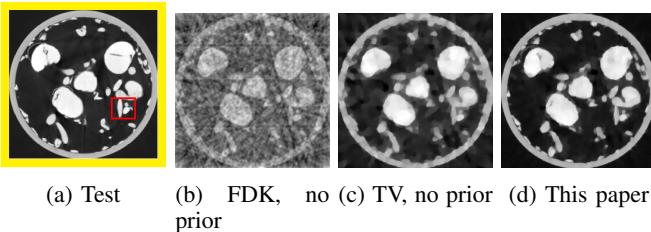


Fig. 15: A zoomed-in version of the regions around the ROI of the reconstructions shown in Fig. 14.

TABLE IV: SSIM of 2D reconstruction of radio-frequency ablation data from various methods. Values have been computed within the ROI. For ground-truth (ideal reconstruction), SSIM equals 1.

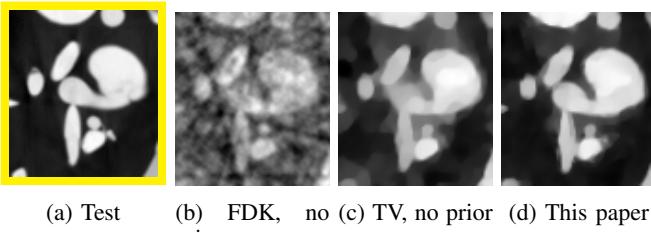
Backprojection	TV	This paper
0.73	0.91	0.95

2) *Sprouts*: The test volume was reconstructed from partial set of 45 real cone-beam projections, i.e., 2.5% of the 1800 projection views from which ground truth was reconstructed (refer Sec. IV). The selected 3D ground truth of template volumes, test volume, as well as the 3D reconstructions are shown in the supplementary material [32]. 2D reconstruction of one of the slices and a zoomed-in region around the ROI are shown in Figs. 16 and 17 respectively. For the sake of exposition, the red region of interest (RoI) has been culled out from 7 consecutive slices in the 3D volume to indicate new structures; other changes can be viewed in the video. Table V shows the improvement in SSIM of the reconstructed new regions as compared to other methods.



(a) Test (b) FDK, no prior (c) TV, no prior (d) This paper

Fig. 16: Reconstruction of sprouts from 20 projection views (b) has streaky artefacts, (c) has blurred structures and (d) is sharper with significantly less streaky artefacts.



(a) Test (b) FDK, no prior (c) TV, no prior (d) This paper

Fig. 17: A zoomed-in version of the regions around the ROI of the reconstructions shown in Fig. 16

TABLE V: SSIM of reconstructed sprouts from various methods. The SSIM of ground-truth (ideal reconstruction) is 1.

	Backprojection	TV	This paper
2D	0.67	0.82	0.86
3D	0.67	0.73	0.82

## VII. DISCUSSION

In this section, we discuss tuning of the parameters used in our method.  $\lambda_1$ ,  $\lambda_2$  and  $k$  are the three hyper-parameters that need to be chosen carefully for optimal reconstruction. In our experiments,  $\lambda_1$  was tuned to maximize SSIM of the whole reconstructed test volume for TV reconstruction of each dataset. This value was retained for spatially-varying prior-based reconstruction as well. The value of  $\lambda_2$  largely depends

on the amount of artefacts we aim to remove by using object-prior at the cost of their dominance in the new regions. This value was chosen to lie between 0–3 for our datasets. Finally, the hyper-parameter  $k$  defines the sensitivity of the weights map to the difference between test image and object-prior (projection of test onto the space of object-prior). When  $k = 0$ , our method behaves similar to the reconstruction using TV-only regularization. As  $k$  increases, the weights map starts capturing new changes in the test, at the cost of detecting a few false positives i.e., false new changes. In other words, as the weights-map becomes more sensitive to the difference between the test and object-prior, it becomes more noisy. In order to visualize the effect of the hyper-parameter  $k$ , we performed 2D reconstructions on okra dataset for different values of  $k$ . Fig. 18 shows the weights map obtained for each of the  $k$  values and Fig. 19 shows the corresponding final reconstructions. We run our experiments for a few values of  $k$  and choose the best reconstruction based on our tolerance for noise in the weights-map. When a large number of templates are available, this parameter can be chosen by assuming one of the templates as test and choosing the  $k$  that gave the best reconstruction. Alternatively, in cases where one wishes to completely avoid the use of this hyper-parameter, one can construct a binary weights-map using a learning based method described in the supplementary material [32].

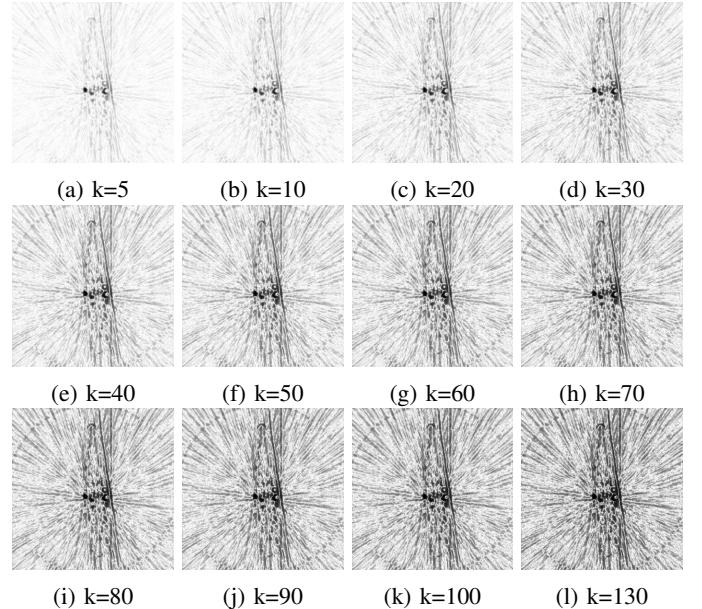


Fig. 18: Different weights-maps for okra reconstruction. Low intensity denotes regions of new changes in test.

## VIII. CONCLUSIONS

This work deals with the effective use of priors for tomographic reconstruction in longitudinal studies. When the number of measurements is substantially low, we discuss the utility of using information from previous scans of the same object, as outlined in Fig. 2. In order to observe details of new changes accurately, we use a novel spatially-varying prior-based method. This method ensures that the reconstruction of localized new information in the data is not affected by

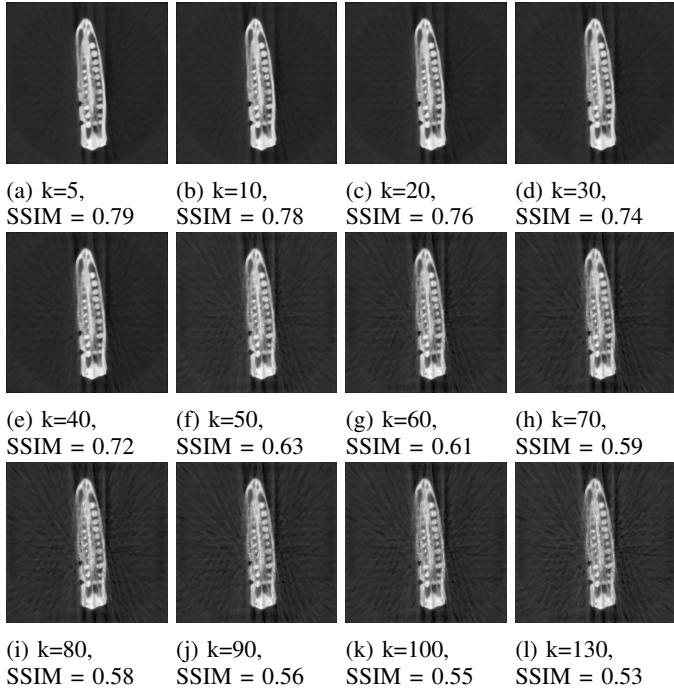


Fig. 19: 2D reconstructions for varying values of  $k$ . The SSIM values for all images are computed within the red ROI (shown in Fig. 12(a)), the region where the test is different from all of the previously scanned objects.

the priors, while the other relatively stationary regions benefit from the extra prior information. We have thus improved state of the art by detecting the strength and location of these regions of change and assigning low prior weights wherever necessary. The probability of presence of a ‘new region’ is enhanced considerably by a novel combination of different reconstruction techniques. We have validated our technique on medical 2D and real, biological 3D datasets for longitudinal studies. The method is also largely robust to the number of previously scanned objects used. We urge the reader to see the videos of reconstructed volumes in the supplementary material [32].

## IX. ACKNOWLEDGMENT

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