Investigating the differences between traffic models and their applicability to reality

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**Introduction**

Traffic congestion is a major issue that affects countless urban areas across the globe, not only increasing travel time for commuters, but also increasing pollution. This problem has begun to affect me as well, as I have recently started driving myself. Traffic flow may seem random but can be modelled through analysis of mathematical patterns. As I researched traffic flow and the mathematics of traffic, I found multiple models that could be used to explain traffic. I wondered what the differences between them were and whether any of them were actually accurate if there were so many of them. The two main models I researched were Greenshield’s model and Greenberg’s model. I wanted to explore the differences in their predictions and how applicable they are to real, urban environments. This investigation will explore the mathematics behind modelling traffic flow, mainly focusing on how traffic flow can be maximized and how models can be applied to the real world. The aim of this investigation is to investigate the differences in traffic models and then to test how they hold up when used to make predictions in reality. In this investigation, I will use derivatives—both first and second—to find maximums for equations. Derivatives are essential for this exploration, as they assist in finding the slope of an equation. At maximum and minimum points, the slope is 0, so finding the derivative equation and then finding when it is equivalent to 0 will allow me to find when there is a maximum or minimum, which is exactly what I want to find. Second derivatives are also important, as they will allow me to know if an equation is concave up or concave down. This is necessary, as whether it is concave up or down will allow me to know if a stationary point (when the slope is 0) is a maximum or a minimum, since I want to find maximums. I will also use data found from my state’s Department of Transportation website for plugging into the traffic models to test them in a real situation to find out which model is more accurate.

**Exploration**

Traffic flow can be analyzed by using mathematical models to relate three key variables: traffic density (), velocity (), and traffic flow (). Traffic density is a measure of the number of vehicles per unit length of road (for example: ), velocity is the average speed of the vehicles within the modelled system represented as unit length per unit time (), and traffic flow is the number of vehicles that go through a point per unit time ( These three variables are related to each other with the fundamental equation:

which states that the traffic flow rate is directly dependent on both vehicle density and vehicle velocity.

**Greenshield’s Model:**

Using Greenshield’s model which assumes a linear relationship between velocity and traffic density, we can model velocity as:

where velocity in terms of traffic density is equivalent to the maximum velocity multiplied by the quantity where is the maximum density. In this model, when density , velocity , so when there are no cars in the system, the velocity is the maximum. This is called the free-flow speed, representing the maximum speed vehicles can ideally maintain without external influences. To find how a change in density affects the traffic flow, we can substitute Greenshield’s model for velocity into , resulting in:

Then, we can find how changes with respect to by differentiating in terms of to find the derivative function :

We can take out of the brackets since it is a constant being multiplied to simplify the equation:

Using the product rule:

To find the stationary points of we can then set and solve for the that results in a derivative of

Now that we know there is a stationary point when , we can check to see if it is a maximum or minimum point by finding the second derivative :

Since , , and are always positive, as they are constants, multiplication and division with them will always produce a positive number. Therefore, when multiplied by , the product will become negative, making always negative. Since the second derivative of the function is always negative, that means that the shape of the graph of in terms of is always concave down. Because it is always concave down, the stationary point when the first derivative is a maximum, meaning that traffic flow is maximized when traffic density is half of its maximum given by: . By plugging the optimal density into Greenshield’s model, we find that the velocity at maximum traffic flow is:

Thus, according to Greenshield’s model, at maximum traffic flow, the velocity of the vehicles is half of the maximum velocity. For example, assuming a road with a speed limit of and a maximum density of , .

A graph of a function with Gateway Arch in the background

Description automatically generatedThen, the velocity . So, the maximum traffic flow . Traffic flow can also be represented as a graph:

in which represents the density and represents the traffic flow. So, we know that in this example, the maximum point of in terms of is .

**Greenberg’s Model:**

We can use another model of traffic called Greenberg’s logarithmic model. In Greenberg’s model, the velocity in terms of density is instead defined as:

We can use the same density model as before with Greenshield’s model to find the optimal density at which traffic flow is maximized. Substituting in into :

Differentiating with respect to to get :

Using product rule:

Setting to find stationary point(s):

To check if the stationary point is a maximum or minimum, we find the second derivative :

Using the product rule:

Since will always be positive since and are always positive, will always be negative since it is multiplied by . Therefore, the graph of with respect to is always concave down, meaning the stationary point is a maximum. Thus, is maximized at which is when . To find the velocity during maximum traffic flow, we plug into Greenberg’s model:

Therefore, the velocity during maximum flow is equivalent to the maximum velocity. Using the previous example, assuming a road with a speed limit of and a maximum density of , .

Then, the velocity . We can see this is the same as , proving that . Thus, the maximum traffic flow is

Representing the traffic flow with respect to density as a graph with the Greenberg model results in:  
A graph of a function

Description automatically generatedwhere represents density and represents flow . So, given Greenberg’s model, we know the traffic flow is maximized at the point . However, a limitation with Greenberg’s model exists, as when density nears , speed tends towards infinity:

We can see that as gets closer to , the equation essentially becomes:

Since as tends towards , will tend towards infinity since does not change.

Therefore, as density approaches 0 the predicted velocity based on Greenberg’s model is essentially equivalent to infinity, which does not occur in real life. Thus, Greenberg’s model is limited when applied to extremely small densities.

However, while being limited when analyzing small densities, Greenberg’s model is much better than Greenshield’s model at predicting traffic flow at high densities which are much more common in urban areas.

Plugging in as the density value for the velocity equation in Greenshield’s model gives:

Thus, we can see that according to Greenshield’s model, when traffic reaches its maximum density, velocity becomes 0. In reality this is not always correct, demonstrating Greenshield’s inaccurate portrayal of speed at high traffic density.

**Application to reality:**

In order to actually test these models, I used my state’s traffic count data from the Washington State Department of Transportation. I was able to find the Annual Average Daily Traffic (AADT) for many roads I frequent. AADT is the total volume of traffic for a year divided by 365 days, giving the average number of vehicles passing through a point per day. However, AADT is not enough for a precise measurement of vehicles per hour, as it is an average of vehicles per day. Thus, I will use the K-factor. The K-factor is used to estimate the traffic flow during the peak hour (the maximum traffic flow). The K-factor is a percentage of how much of the traffic in a day occurs during the peak hour. For the road I will be modelling, I found that the AADT was (“WSDOT - Traffic Counts (AADT) Current”). Finding the K-factor was more complicated, however. I was unable to find an exact K-factor value for the road I was modelling, so I instead researched what kind of road it was classified as by my state. I found it was classified as “Other Principal Arterial, Urban” (*WSDOT GeoPortal*). Then, I found that the K-factor for these kinds of roads is generally around 0.09-0.12. Thus, I will use the average of the two as my K-factor: . To test how close Greenshield’s and Greenberg’s models are, I will estimate maximum density and maximum velocity of that road. The road is a 2-way road with 2 lanes for each direction, meaning 4 lanes total. Density for maximum congestion usually ranges around 40-60 vehicles per mile for one lane, so I will estimate maximum density for one lane of traffic as 50 vehicles per mile. Since there are 4 lanes in the road, the maximum density is: vehicles per mile. On this road, the speed limit is 55 miles per hour. However, drivers usually drive above the speed limit, so my estimation will be that the maximum velocity is 5 miles per hour greater than the speed limit: miles per hour as the maximum velocity.

Using Greenshield’s model, we know that the optimal density and the velocity during this is as found previously. Thus, using the equation :

Thus, according to Greenshield’s model of the situation, the peak traffic flow for this road should be 3,000 vehicles per hour.

Using Greenberg’s model, we know that the optimal density and the velocity during this is as previously found. Thus, using the traffic flow equation again:

Thus, according to Greenberg’s traffic model, the peak traffic flow is predicted to be about 4,414 vehicles per hour.

From these results, we can see that Greenshield’s and Greenberg’s predictions for peak traffic flow are very different, as the difference between them is . To find which one was more accurate, we will find the peak traffic flow of the road in real life. This can be found using the K-factor and the AADT for the road in the following equation:

This is true, because AADT is a measure of vehicles in a day, while the K-factor is the percent of vehicles during the peak hour, meaning that multiplying them gives the number of vehicles during the peak hour, which is what we want.

Because the AADT is 29,000 vehicles per day, and the K-factor for this road is assumed to be 0.105:

Therefore, with the measured AADT of 29,000 vehicles per day, and the assumed K-factor of 0.105, the maximum flow was about 3,045 vehicles per hour. Comparing these results with Greenshield’s model’s prediction of 3,000 vehicles per hour and Greenberg’s Model’s prediction of 4,415 vehicles per hour, we can see Greenshield’s model’s prediction is very close to the actual value. Percent error is a measure of the discrepancy between an observed/measured value and an actual value. Percent error is given by the following equation:

Percent error for Greenshield’s model’s prediction of 3,000 vehicles per hour:

Thus, the percent error for Greenshield’s model’s prediction is about 1.5%.

Percent error for Greenshield’s model’s prediction of 4,415 vehicles per hour:

Thus, the percent error for Greenberg’s model’s prediction is about 45%.

Therefore, there is much less discrepancy between Greenshield’s model’s estimated value and the actual value than between Greenberg’s model’s estimated value and the actual value. Thus, we can say that in this case, Greenshield’s model was much more accurate than Greenberg’s model was for estimating peak traffic flow.

**Conclusion**

The aim of this exploration was to investigate the differences between traffic flow models and how applicable they are to reality. I was able to analyze two different models of traffic flow: Greenshield’s model and Greenberg’s model. I used derivatives and second derivatives to find when traffic flow is maximized in both models and then applied my findings to a real road and data, I was able to find through my research. The results of my investigation demonstrated that Greenshield’s model for traffic flow is much more accurate than Greenberg’s model for traffic flow, as the value predicted by Greenshield’s model was only off from the actual value by 1.5%, while the vale predicted by Greenberg’s model was off by 45%.Therefore, based on my findings, the relationship between velocity and traffic density is closer to a linear relationship rather than a logarithmic one. However, this was only found using data of one type of road, classified by my state’s Department of Transportation as “Other Principal Arterial, Urban”. Thus, this exploration was limited, as there were multiple other types of roads that exist and were not modelled. The road I used is classified as urban, but it is not as congested as many inter-city roads, which Greenberg’s model is better at modelling. This means that although Greenberg’s model was not accurate for predicting the peak flow rate of the modelled road, it could be more accurate than Greenshield’s model for other types of roads that are usually more congested, since Greenshields’s model would predict no speed at a maximum density, while Greenberg’s model would be more realistic. If I were to do this investigation again, I would analyze multiple different types of roads, both urban and rural. I was also limited, as there were multiple values I had to assume, such as maximum velocity, since driver behavior is very unpredictable. I also had to assume the K-factor for the road I modelled, as it was not available in any data I found. However, I was able to overcome this challenge by finding out the type of road I was modelling and then finding what a K-factor for that kind of road would be. This investigation allowed me to explore the factors that determine traffic flow, as well as the different ways it can be modelled. I now can use the traffic models I learned about to analyze what peak traffic flow could look like for a road. By also using real data, I also can apply this to my own experiences, furthering my understanding of the topic.

**References**

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