Senior Thesis in Mathematics

The Art Gallery Problem and Variations

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Submitted to Pomona College in Partial Fulfillment
of the Degree of Bachelor of Arts

Department of Mathematics

Contents

Chapter 1

Introduction

Suppose we are presented with the floor plan of an art gallery as in Figure 1.1. We wish to investigate how to position guards in our gallery so that the gallery is "protected". First, we define some key concepts.

Definition 1.1 *We say an art gallery is protected if every point inside our gallery is* ***visible*** *to at least one guard.*

Here, we need to establish a definition for visibility.

Definition 1.2 *A point is visible to a guard if the straight line segment connecting them does not extend outside the boundary of the art gallery.*

We can look at some examples that explain visibility. Figure 1.2 shows a point, A, in our gallery which is visible to the guard positioned at the vertex. Figure 1.3 shows a point, B, that's not visible to the guard in the corner. If there is a point such that the line segment connecting the two is along the boundary of the art gallery, that point is also visible to the guard. For example, point C is visible to our guard in Figure 1.4.

We return to my floor plan with this information and try to place guards so that our gallery is protected. We observe that the configuration of guards in Figure 1.5 doesn't work to protect the gallery. This is because not every point is visible to at least one of the guards I have positioned, as indicated by the blue point in Figure 1.6.

We move things around a few times and finally arrive at a configuration that works (see Figure 1.7).

Once I have been successful in protecting my art gallery, I am curious about some broader questions with regards to this problem. The following are some things I think about:

1. Is there any configuration of fewer than four guards which would also protect my art gallery? In other words, is four guards the *best* that I can do?

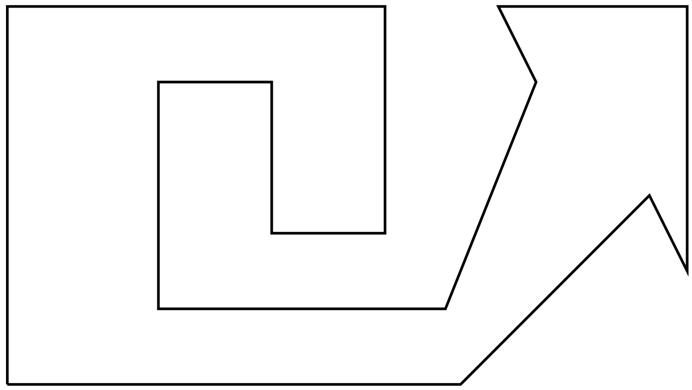


Figure 1.1: The floor plan of my art gallery

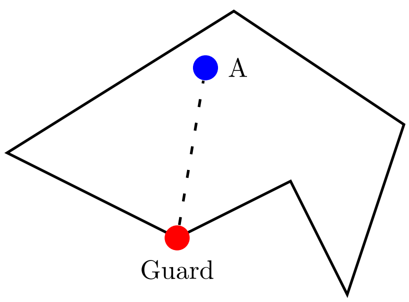


Figure 1.2: a visible point

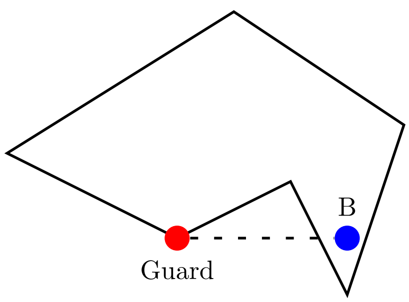


Figure 1.3: a point not visible

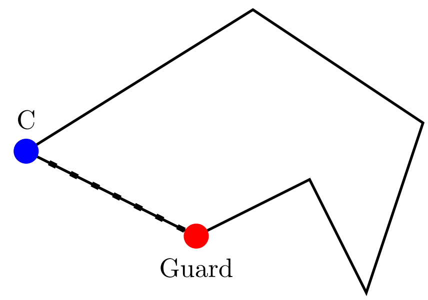


Figure 1.4: A visible point, along the boundary of the gallery

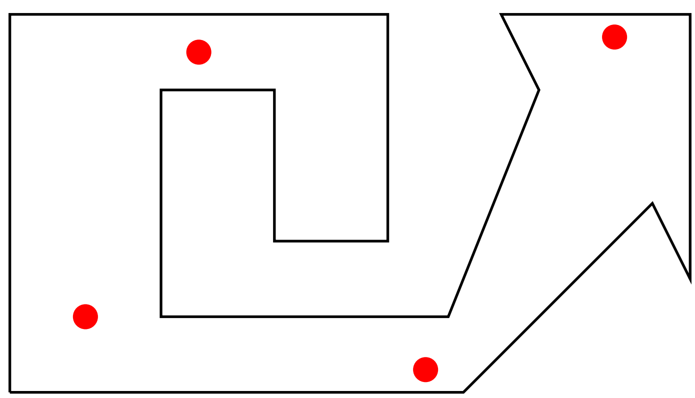


Figure 1.5: a bad configuration of guards

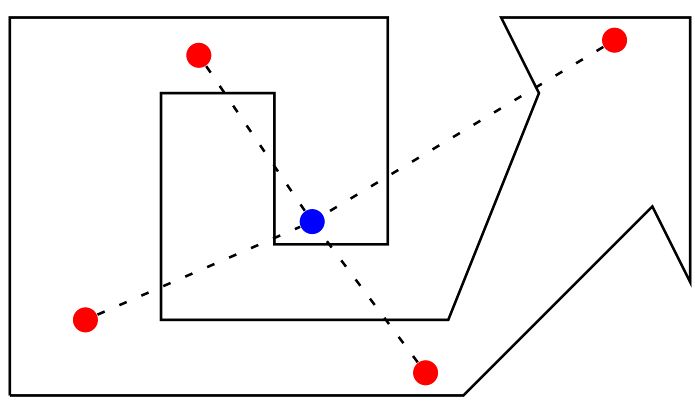


Figure 1.6: A blue point not visible to any guard

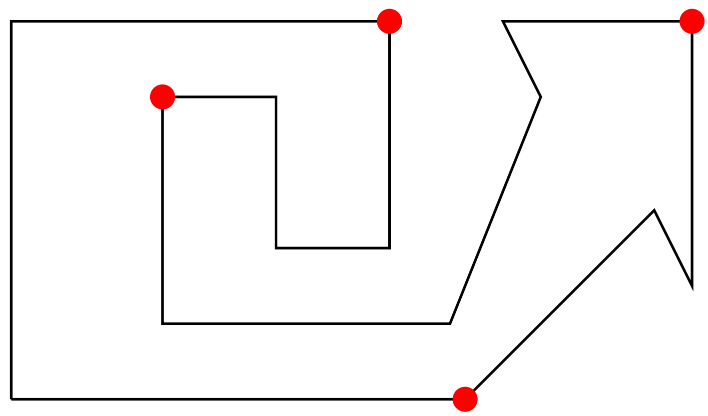


Figure 1.7: A protected art gallery

2. If I don't have the floor plan of my art gallery, but I know something about it's shape, can I say anything about the number of guards that will be sufficient to protect it?

These questions were brought up by Victor Klee in 1973 as a concise mathematical problem. The question was how many points (or guards) are needed to protect a plane, simple, closed polygon. His problem was solved in 1975 by Vasek Chvátal, a Czech mathematician. This resulted in the following theorem:

Theorem 1.3 (Art Gallery Theorem) *For art galleries with n walls, $\left\lfloor \frac{n}{3} \right\rfloor$ guards are always sufficient and* sometimes necessary*.*

This is exciting because it claims that no matter how complicated your n-sided gallery is, you will never need more than $\left\lfloor \frac{n}{3} \right\rfloor$ guards to ensure it is protected.

In 1978, Steve Fisk created an innovative, visual proof for the Art Gallery Theorem using polygon triangulations and proper 3-colorings that was published in *Proofs of THE BOOK*. Fisk's algorithm is significant because it not only proves the art gallery theorem, it also tells us where to position our guards. In the next chapter, we look at the algorithm and it's intricacies.

Chapter 2

Polygons and Properties

For the purpose of the art gallery problem, art galleries are **simple polygons** in the plane. In this chapter, we will explore some properties of these kinds of polygons. When we talk about polygons for the entirety of this thesis, we will be referring to simple, closed polygons.

Definition 2.1 (Simple polygons) *A simple polygon is a connected, closed region whose boundary is defined by a finite number of straight, non-intersecting line segments.*

2.1 Triangulation

Definition 2.2 (Triangulate) *Triangulating a polygon is decomposing it into triangles by drawing non-intersecting diagonals between its vertices.*

Example 2.3 *Figure 2.1 shows a simple, closed polygon that is triangulated in Figure 2.2.*

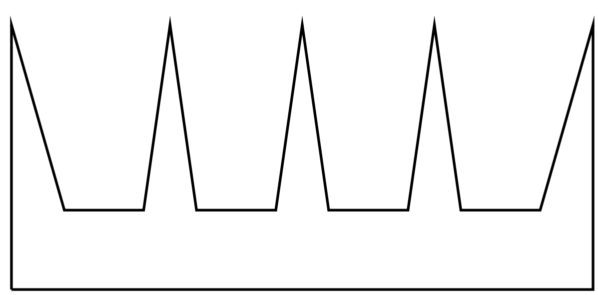


Figure 2.1: A simple, closed polygon

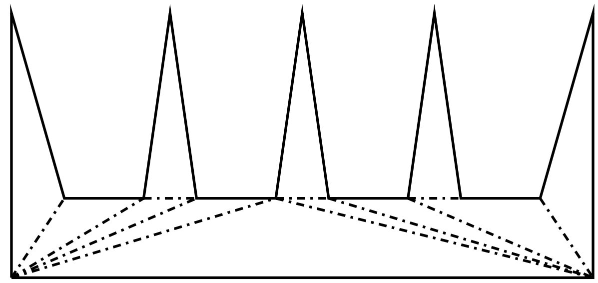


Figure 2.2: Triangulated Polygon

2.1.1 Convex Vertices

In this section, we look at an interesting theorem about the interior angles of polygons that form at the vertices. If these angles are smaller than 180°, we call them **convex**.

Theorem 2.4 (Convex vertices) *Every simple, closed polygon must have at least 3 convex angles.*

Proof Consider a triangulated polygon, P, with n outside edges, as in Figure 2.3, for example. Let x be the number of triangles in the triangulated polygon. Then, the number of inside edges is $x - 1$.

The total number of inside and outside edges can be written as $3x$, which denotes 3 edges for each traingle. It can also be written as $n + 2(x - 1)$, the sum of outside edges and twice the number of inside edges. Then:

$$3x = n + 2(x - 1)$$
$$3x = n + 2x - 2$$
$$x = n - 2$$

The sum of all the angles in the triangulation of P is then $180x = 180(n - 2)$. Assuming only two of the n interior angles of P are convex, the remaining $n - 2 \geq 180°$. The sum of these $n - 2$ angles $\geq 180(n - 2)$. Hence, the total sum of the n angles, **must** be greater than $180(n-2)$. In order for the derived relation above to stand, at least 3 angles have to be less than 180°. This concludes our proof that any polygon, P, has at least 3 convex angles. ∎

2.1.2 Proving the existence of a triangulation

Intuitively, it makes sense that every simple, closed polygon should have a triangulation. However, in order to be rigorous, we need to prove this.

Theorem 2.5 (Triangulation) *A polygon of n vertices may be partitioned into $n - 2$ triangles by the addition of $n - 3$ internal diagonals.*

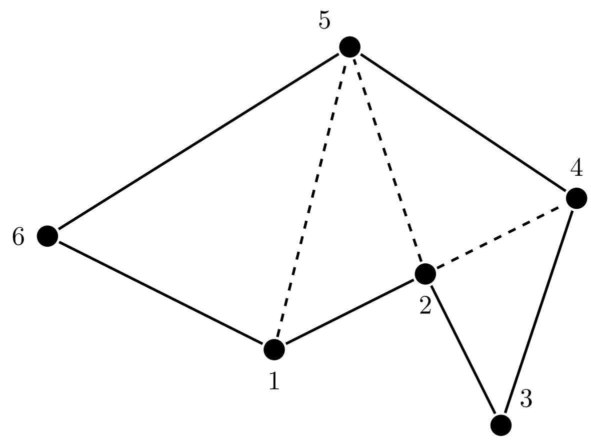


Figure 2.3: A triangulated polygon, P, with 6 edges

Proof This theorem can be proved by induction. The base case, where $n = 3$, is trivial. A triangle is already triangulated.

For the inductive case, let P be a polygon where $n \geq 4$. Let v_2 be a convex vertex of P and v_1, v_2 and v_3 be three consecutive vertices of P. From theorem 2.4, we know such a vertex will always exist. We seek an internal diagonal, d, such that:

- $d = v_1v_3$ if the line segment v_1v_3 is completely interior to P.
- or $d = v_2x$ if the line segment v_1v_3 intersects the exterior of P. In this case, there must be at least one other vertex in the closed triangle $v_1v_2v_3$. We pick the one closest in perpendicular distance from v_1v_3 to v_2. We call this vertex x.

In both cases, d divides P into two, smaller polygons, P_1 and P_2.

Let n_i be the number of vertices that P_i has, for $i = 1, 2$. Then $n_1 + n_2 = n + 2$, since the vertices on the shared edge, d, are counted twice. We know that $n_i \geq 3$ for $i = 1, 2$. Since $n_1 \not< 3$ and $n_2 \not< 3$, $n_i < n$ for $i = 1, 2$. Using the inductive hypothesis, we assume the theorem to hold true for all $n_i < n$. The total number of triangles in P is then :

$$\begin{gathered}(n_1 - 2) + (n_2 - 2) \\ = (n_1 + n_2) - 4 \\ = (n + 2) - 4 \\ = n - 2\end{gathered}$$

Similarly, the number of diagonals drawn in P is:

$$\begin{gathered}(n_1 - 3) + (n_2 - 3) + 1 \\ = (n + 2) - 3 - 3 + 1 \\ = n - 3\end{gathered}$$

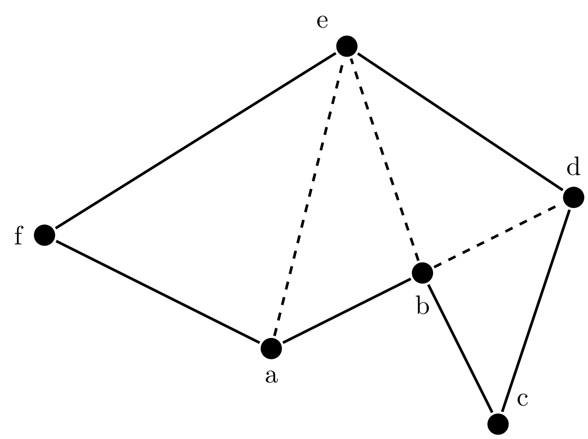


Figure 2.4: A triangulated polygon, P, with two ears

This concludes the proof that every polygon, P, can be triangulated into $n-2$ triangles using $n-3$ diagonals. ∎

2.2 Ears

An interesting feature of polygon vertices are ears. It is helpful to imagine a flap-like property while wrapping our head around the concept of ears.

Definition 2.6 (Ears) *Let u, v, w be three consecutive vertices of polygon P. Then vertex v is an ear vertex if line segment uw is a internal diagonal of P.*

Ears can be understood without triangulating a polygon, but here, we will work with triangulated polygons.

Example 2.7 *In Figure 2.4, vertex c and f are ear vertices.*

Here, we want to question whether the occurrence of two ears in Example 2.7 was a coincidence. Can we always count on the existence of ears in a polygon?

2.2.1 Two Ears Theorem

It seems fascinating (and rather convenient) that for every polygon, we always have at least two ears. The theorem was first proved by Gary Meisters in 1975. Joseph O'Rourke's proof of the two ears theorem was published in his book, "Art Gallery Theorems and Algorithms". O'Rourke's proof used graph theory in an elaborate way to show that simple, closed polygons will always have at least two ears.

First, we define a dual-tree from graph theory.

Definition 2.8 (Dual-tree) *Given a triangulated simple polygon, the dual-tree is the graph generated by plotting a node in each triangle and edges joining these nodes between adjacent triangles. See Figure 2.5 as an example.*

Theorem 2.9 (Two Ears Theorem) *Every polygon of $n \geq 4$ vertices has at least two non-overlapping ears.*

Proof We begin with a triangulated polygon, P, as in Figure 2.4. We know that every polygon can be triangulated from Section 2.1.2. We draw a dual-tree graph, G, for P. Let

- $x =$ the number of triangles in $P =$ the number of nodes in G
- $n =$ the number of outside edges in P
- $y =$ the number of ear triangles

Then, it follows that:

- $(x-1) =$ the number of triangle edges inside $P =$ the number of edges in graph G
- $x - y =$ the number of non-ear triangles

The number of inside edges is $(x-1)$ because each inside edge creates a triangle, except the last one, which creates two triangles.

Recall also from Section 2.1.2 that the number of triangles is equal to $n-2$. In other words, $x = n-2$.

Consider the number of outside edges in P. It can be written as a sum of the outside edges in ear triangles, x, and non-ear triangles, $x-y$. For each ear triangle, we have two outside edges. For each non-ear triangle, we have one or no outside edges since it is possible for a triangle to have all its edges inside P. We can write this sum as an inequality to n, the total number of outside edges, as follows:

$$\begin{gathered} n \leq 2y + (x-y) \\ n \leq x + y \\ n \leq (n-2) + y \\ n \leq n - 2 + y \\ y \geq 2 \end{gathered}$$

This proves P must have at least two ear triangles. ∎

2.3 Proper Coloring

Consider an average 9×9 Sudoku puzzle. Suppose we draw a vertex for every cell in the puzzle and draw an edge between the vertices if they are in the same column, row or block. Now we use nine different colors to color the vertices such that no two vertices that are connected by an edge are the same color. This problem is known as a graph coloring problem.

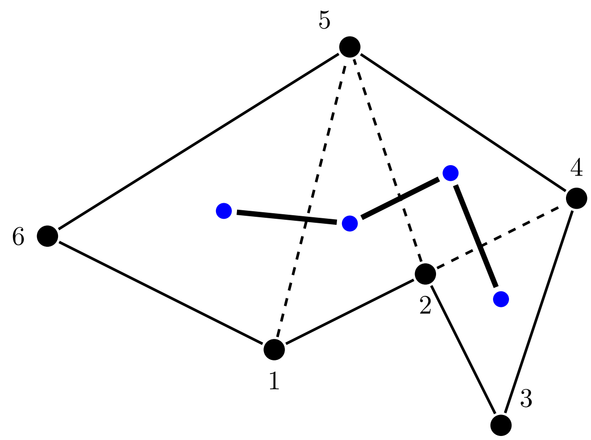


Figure 2.5: Graph G inside a polygon, P

Definition 2.10 (Proper Coloring) *An assignment of colors to the vertices of the graph such that no two adjacent vertices have the same color.*

Definition 2.11 (Proper k-coloring) *A proper k-coloring of a graph is a proper coloring using k colors. If a graph has a k-coloring, it is said to be k-colorable.*

2.3.1 Proper 3-coloring

In this section, we focus on proper 3-colorings of the vertices of our polygons and observe some an interesting theorem about them. We first define proper 3-coloring in the context of triangulated polygons.

Definition 2.12 (Proper 3-coloring) *Assigning one of three colors to the vertices of a triangulated polygon such that no two adjacent vertices have the same color.*

Example 2.13 *See Figure 2.6.*

A useful theorem to prove about proper 3-coloring is that one always exists for a triangulated polygon.

Theorem 2.14 *A triangulated polygon is 3-colorable.*

Proof We can prove this using induction. For the base case, $n = 3$. We assign a different color to each vertex in the triangle and we are done. For the inductive case, we assume a k-polygon has a proper 3-coloring. Let our polygon, P, be a $k+1$-gon where $k+1 \geq 4$. From a triangulation of P, we remove an "ear". The

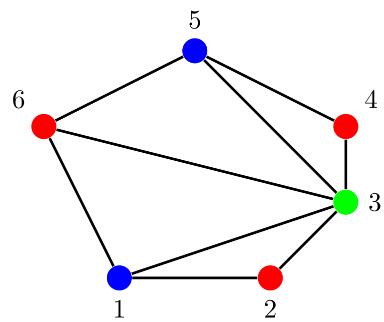


Figure 2.6: A *proper* 3-coloring

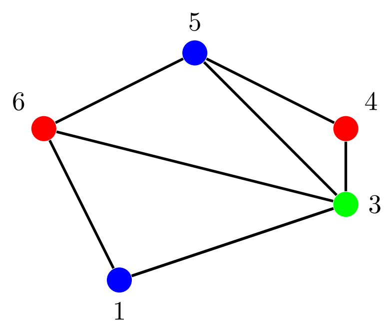


Figure 2.7: A *proper* 3-coloring without vertex 2

two ears theorem from Section 2.2 proves that we will always be able to find at least two such ears.

Once we remove an ear vertex, we are left with a k-gon. By the inductive hypothesis, this has a proper 3-coloring. Now, we add our ear back and assign the color to it that is not used by its adjacent vertices. Hence, our $k + 1$-gon is properly 3-colorable. This concludes our proof that any triangulated polygon is properly 3-colorable. ∎

We can use an example to see the inductive step of the proof above. In Figure 2.6, the polygon has two ears. One is at vertex 2 and the other at vertex 4. We can remove either one of them. For example, we remove vertex 2, as done in Figure 2.7. The pentagon left behind is properly 3-colorable by our hypothesis. Now, we can add vertex 2 back to the polygon and color is red since it cannot be green or blue. Hence, the hexagon is 3-colorable.

Another interesting kind of coloring that we will use later is proper 4-coloring. (See Chapter 5).

Definition 2.15 *Proper 4-coloring Assigning one of four colors to the vertices of a polygon in such a way that no two adjacent vertices have the same color.*

Chapter 3

Fisk's Algorithm

In this chapter, we return to the art gallery problem that was posed in Chapter 1. We use Steve Fisk's algorithm to position guards in the polygonal art gallery in Figure 3.1. The intricacies and layers of Fisk's algorithm have been addressed in Chapter 2. We have all the pieces we need to prove that Fisk's algorithm always works.

Let's look at the steps of the algorithm. We begin with a simple, closed polygon which we consider to be our 'art gallery'. We establish the rule that our guards must be fixed in their position. They cannot move from their place but can rotate in it. The algorithm has three steps to it:

1. **Triangulate** the polygon.
2. Perform a **proper 3-colouring** of its vertices.
3. Assign guards to the vertices with the least frequently occurring colour.

Theorem 3.1 (Art Gallery Theorem) *For an art gallery with n walls, $\left\lfloor \frac{n}{3} \right\rfloor$ guards are sufficient and sometimes necessary.*

Proof We first triangulate our art gallery. From section 2.1.2 we know that a polygon with n edges can always be triangulated into $n - 2$ triangles. Section 2.3 shows that a proper 3-coloring of the vertices is always possible. Now, we assign our guards to the vertices where the least frequently occurring color is.

Since there are n vertices colored with three different colours, there must exist at least one colour that does not appear more than $\frac{n}{3}$ times. This can be proved by contradiction. If all colors appeared more than $\frac{n}{3}$ times, we would have a total sum of vertices greater than n. The number of vertices must be an integer. Consequently, the least frequently occurring color does not appear more than $\left\lfloor \frac{n}{3} \right\rfloor$ times. This ensures that the number of guards is an integer.

By our definition of visibility, a guard positioned at the vertex of the triangle can protect that particular triangle. Since every triangle in our art gallery has a

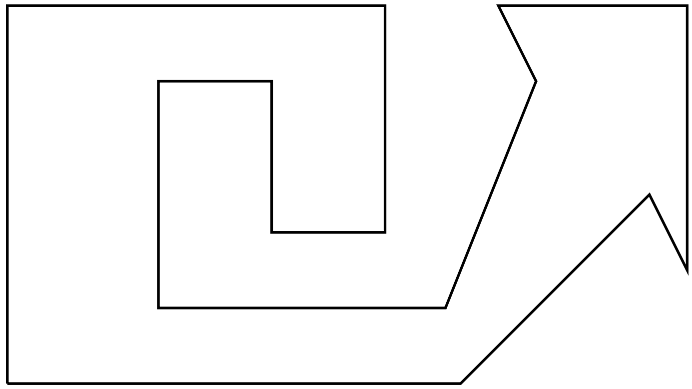


Figure 3.1: The polygon P representing the floor plan of an art gallery

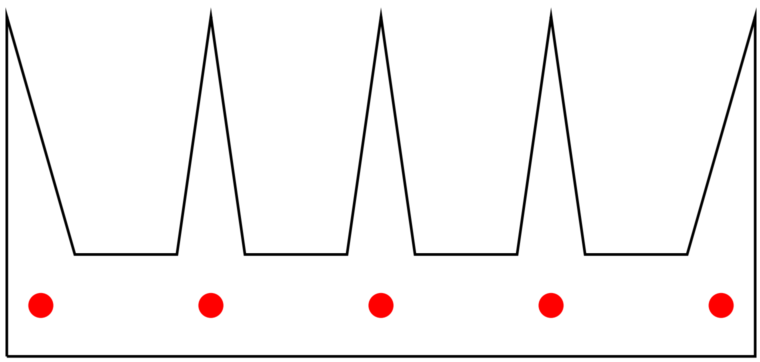


Figure 3.2: A fifteen-walled art gallery that needs five guards to be protected

vertex of each color, all the triangles are protected. Hence, the entire art gallery is protected.

In some cases, $\left\lfloor \frac{n}{3} \right\rfloor$ guards are **necessary** for the gallery to be protected [3]. For example, the art gallery in Figure 3.2. ∎

3.1 Seeing Fisk's Algorithm in Action

The proof above is elegant and pieced together by the building blocks we have used in the previous sections. In the rest of this chapter, I will go through an example of using Fisk's algorithm to protect the art gallery in figure 3.1 which has fifteen walls. Let's call the art gallery P. We number the vertices of P (see Figure 3.3).

The first step to Fisk's algorithm is triangulation. It does not matter *how* we triangulate the art gallery. I begin by connecting vertex 7 to vertices 1 and 2 as shown in Figure 3.4. I continue the process until I have a finished triangulation

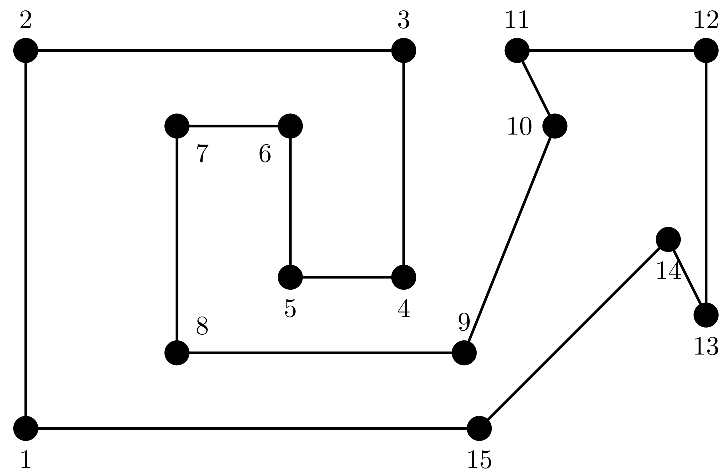


Figure 3.3: P with numbered vertices

like in Figure 3.5.

I move on to the second step of the algorithm which is performing a proper 3-coloring of the vertices of P. Again, it does not matter where we start the coloring from. I color vertex 1 red and vertex 2 blue. Vertex 3 must be green since vertices joined by an edge cannot have the same color. See Figure 3.6.

I continue the process of assigning red, blue and green to the vertices in this manner until I have a finished proper 3-coloring as in Figure 3.7.

I count the number of times each color appears in Figure 3.7. Red appears six times, blue appears five times and green appears just four times. In this case, green is the least frequently occurring color. I position my guards at these green vertices. Note that every triangle has a green vertex and so every triangle is protected by at least one guard. My art gallery as a whole is protected in Figure 3.8.

The algorithm above has paved the way for an algorithmic approach to variant problems of the art gallery problem. In the next two chapters, I explore two variations that I find particularly interesting. They have results that may be rather surprising to learn.

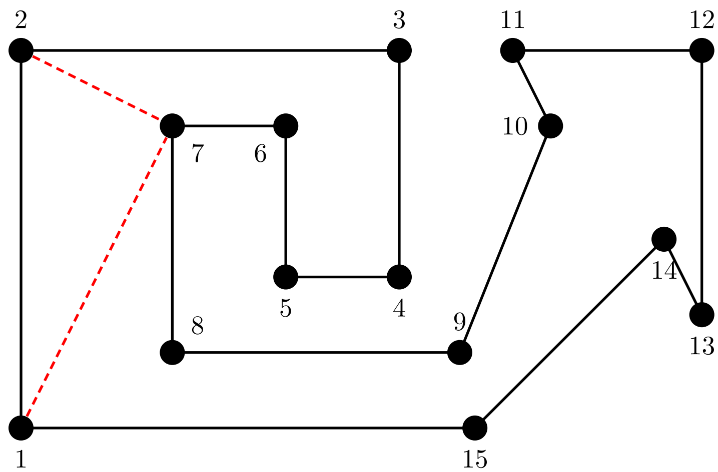


Figure 3.4: Triangulating P

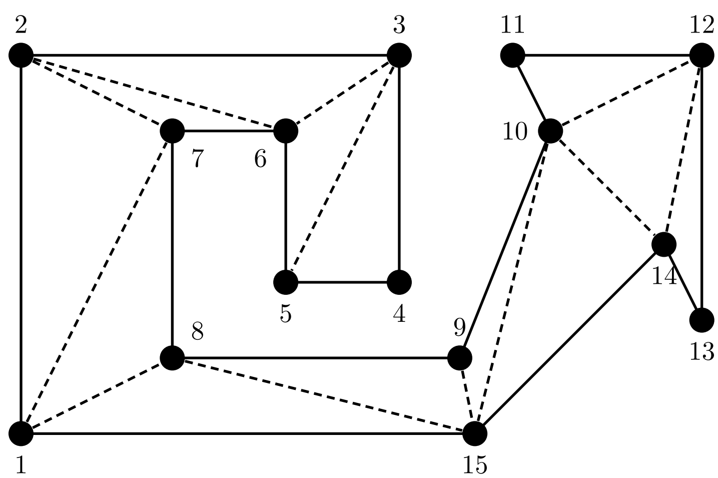


Figure 3.5: Finished triangulation of P

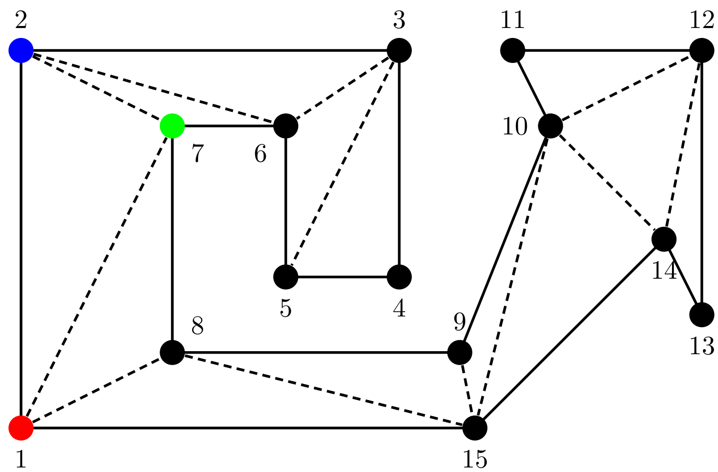


Figure 3.6: Starting the proper 3-coloring

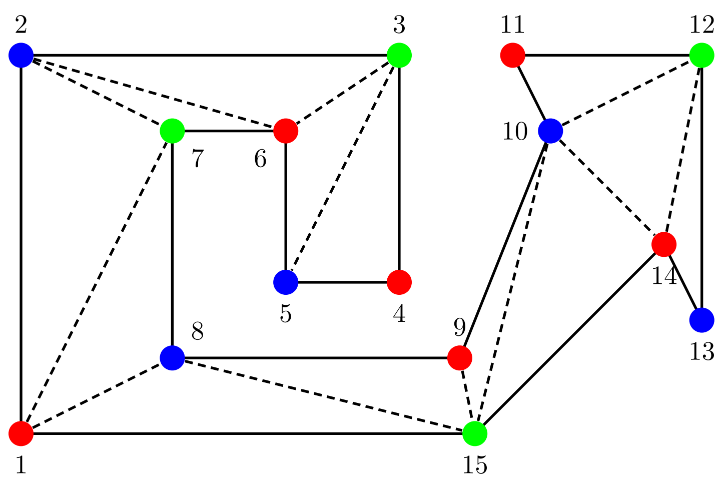


Figure 3.7: Finished proper 3-coloring of P's vertices

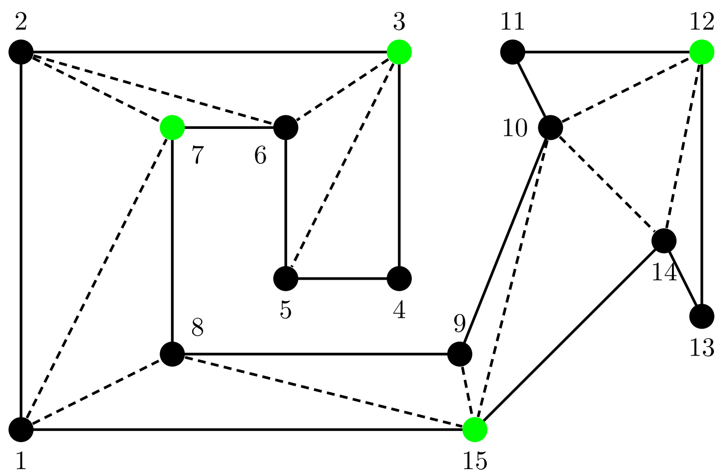


Figure 3.8: P with guards at green vertices

Chapter 4

Guards and Reflex Angles

So far we have looked at Klee's problem as a function of n, the number of walls of the art gallery. Mathematicians have explored whether measures of r, the number of reflex angles in an art gallery give us a more accurate understanding of the "shape" of our polygon. If that is the case, then the number of guards required, written as a function of the number of reflex angles, r, might give us better intuition as to how many guards would be sufficient.

Definition 4.1 *Reflex vertices are the interior angles made at polygon vertices that are greater than 180°.*

4.1 Naive Convex Partitioning

A great way to understand complex shapes is to break them down into simpler parts. Bernard Chazelle, in his P.h.D thesis [1] talks about a polygon decomposition as a means of efficiently solving large problems in computational geometry. A particularly useful decomposition of polygons that he wrote about was the "naive" convex partitioning.

Definition 4.2 (Convex Polygon) *A polygon is convex if all its interior angles are smaller than 180°.*

Definition 4.3 (Convex Decomposition) *A decomposition of a polygon, P, is said to be convex if all the polygons in it are convex.*

Chazelle's paper formalises this kind of decomposition with the following theorem:

Theorem 4.4 (Chazelle 1980) *Any polygon, P, can be partitioned into at most $r + 1$ convex pieces, where r is the number of reflex vertices the polygon has.*

Proof The proof is by induction. For the base case, $r = 0$, the theorem holds true. The polygon is already convex.

For the inductive case, where $r \geq 1$, we begin from any reflex vertex and draw a ray bisecting the interior angle until it intersects the polygon boundary for the first time. This ray divides P into two polygons, P_1 and P_2, with r_1 and r_2 reflex vertices, respectively. Since bisecting one of the reflex angles resolves it, $r_1 + r_2 \leq r - 1$. The sum of reflex vertices in P_1 and P_2 may be less than $r - 1$ since the ray might resolve another reflex vertex at its first intersection with the polygon boundary. Hence, $r_1 < r$ and $r_2 < r$. We can apply the inductive hypothesis to P_1 and P_2 to get the number of convex pieces in P as follows:

$$\begin{aligned} &(r_1 + 1) + (r_2 + 1) \\ &= (r_1 + r_2) + 1 + 1 \\ &\leq (r - 1) + 2 \\ &= r + 1 \end{aligned}$$

This concludes our proof that any polygon can be partitioned into **at most** $r + 1$ convex pieces. ∎

4.2 Reflex Angle Theorem

In this section, we return to floor plans of art galleries. Placing guards at reflex angles seems like a good idea in order to cover a larger region with one guard. Joseph O'Rourke's theorem (1982) [4] confirms this intuition.

Theorem 4.5 (O'Rourke 1982) *r guards are occasionally necessary and always sufficient to see the interior of a simple n-gon of $r \geq 1$ reflex vertices.*

Proof To prove that r guards are always sufficient, we use Chazelle's theorem from section 4.1 to partition our polygon into convex pieces. Each convex piece must have at least one reflex vertex on its boundary. Placing a guard at each reflex vertex ensures that at least one guard can see into each convex piece.

Sometimes, r guards are necessary to protect galleries with $r \geq 1$ reflex vertices. Some examples of such shapes, called "shutter" shapes, can be seen in Figure 4.1. ∎

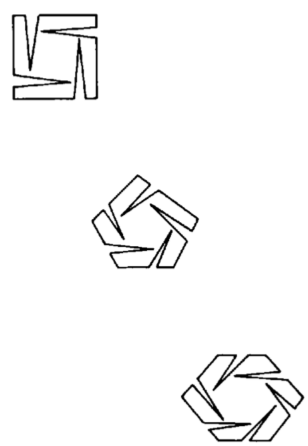


Figure 4.1: Shapes that show the necessity of r guards in some cases

Chapter 5

Orthogonal Art Gallery Theorem

A question mathematicians asked with regards to the art gallery theorem is if we make specific the rules of how we draw our polygon, P, does our theorem change? [2] Some examples of rules are:

- Polygons constructed with just horizontal and vertical lines, called orthogonal polygons
- Polygons with holes in them
- Relaxing the constraint that guards have to be fixed in their position and allowing them to walk up and down

In this chapter, I will look at how the art gallery theorem changes for orthogonal polygons, drawing proofs mainly from Joseph O'Rourke's book, *Art Gallery Theorems and Algorithms* [4]. I will provide an overview of the proof for the orthogonal art gallery theorem.

5.1 Orthogonal and 1-Orthogonal Polygons

Definition 5.1 (Orthogonal Polygons) *A simple, closed polygon where all the interior angles are either $90°$ or $270°$.*

Definition 5.2 (Nose) *Let P be a polygon with a slanted edge e. The nose of e is the right angle triangle towards the interior of P whose hypotenuse is e. The nose is formed using a horizontal and vertical line. It includes the interior of e but excludes the boundary.*

Example 5.3 *See Figure as an example of the nose of a slanted edge.*

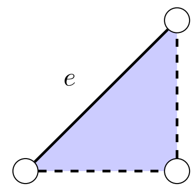


Figure 5.1: The shaded region is the nose of the slanted edge e

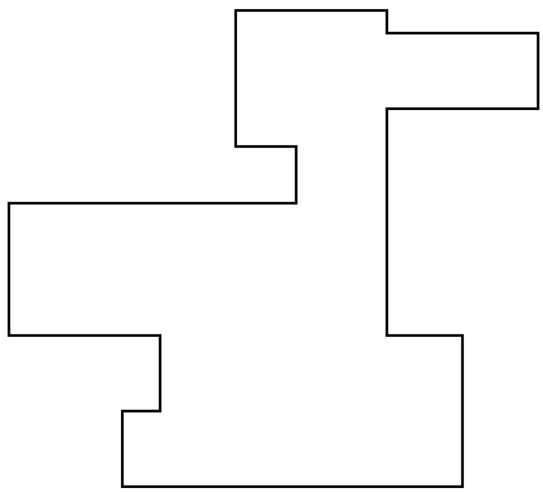


Figure 5.2: An orthogonal art gallery

Definition 5.4 (1-Orthogonal Polygons) *A polygon with a distinguished edge e called the* ***slanted edge****, such that the polygon satisfies four conditions:*

1. *There are an even number of edges*
2. *Except for possibly e, the edges are alternately horizontal and vertical*
3. *All interior angles are less than or equal to 270°*
4. *The nose of the slanted edge contains no vertices*

Example 5.5 *See Figure 5.2 for an example of an orthogonal art gallery.*

5.2 Convex Quadrilateralization

A common theme with art gallery theorems is breaking down the galleries into bite-sized pieces. This lets us protect our gallery piece by piece, whether it is triangles or convex pieces. In this section we talk about decomposing the 1-orthogonal polygons into **convex quadrilaterals**.

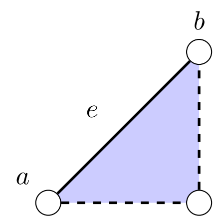


Figure 5.3: Slanted edge $e = ab$

Definition 5.6 (Convex Quadrilateral) *A four-sided polygon where each angle measures less that 180°.*

Definition 5.7 (Convex Quadrilateralization) *The decomposition of a polygon into convex quadrilaterals by connecting vertices using non-intersecting diagonals.*

Theorem 5.8 (Lubiw 1985) *Any 1-orthogonal polygon is convexly quadrilateralizable.*

The proof of Lubiw's theorem is by induction on the number of edges n of polygon P. Lubiw argues that 1-orthogonal polygons are a special class of polygons such that if you remove a convex quadrilateral in a specific manner, you will be left with smaller polygons that are also 1-orthogonal. This is why the proof works by induction.

For the base case, where $n = 4$, regardless of the orientation of e, the slanted edge, P will be convex and we are done.

For the inductive step, we find a convex quadrilateral *abcd*. Once I remove *abcd* from P, I am left with three or less 1-orthogonal polygons. For my inductive hypothesis, I assume that these smaller polygons are convexly quadrilateralizable. Hence, adding *abcd* back maintains that P is also convexly quadrilateralizable.

Instead of a complex and rigourous proof for the theorem, I will go over examples of different 1-orthogonal polygons as we find *abcd*. The kinds of cases can be categorized and I will try to convince myself that they are exhaustive. This means whatever kind of polygon we have, one of the examples I illustrate will happen as I find convex quadrilateral *abcd*.

5.2.1 Examples

The first step to convexly quadrilateralizing a 1-orthogonal polygon is finding the slanted edge e. It is important to note that e *may* be slanted as in Figure 5.1 or it may not. The latter would be the case when we begin with orthogonal polygons. We label $e = ab$ as in Figure 5.3.

Here, we can have two cases for what the edges incident to a and b may look like. From properties (1) and (2) of 1-orthogonal polygons, we know that these

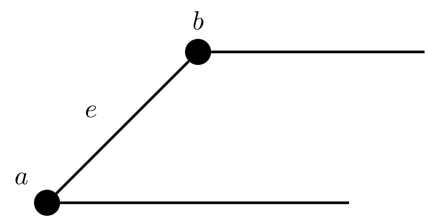


Figure 5.4: Case 1 of edges incident to a and b

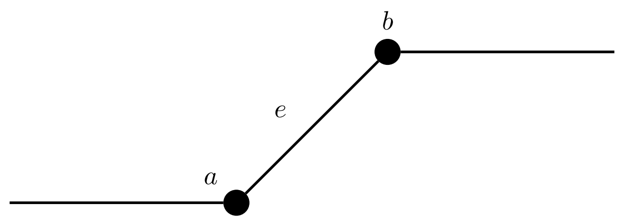
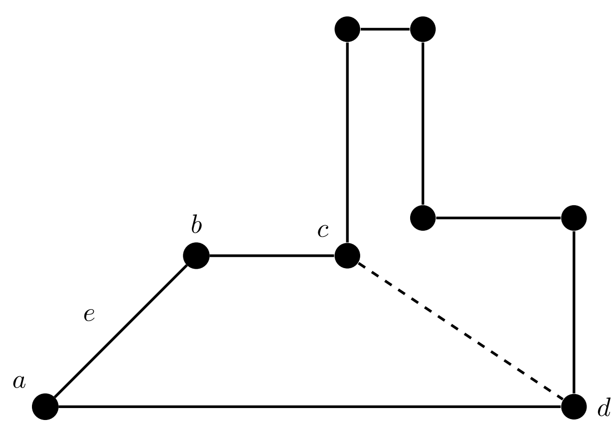


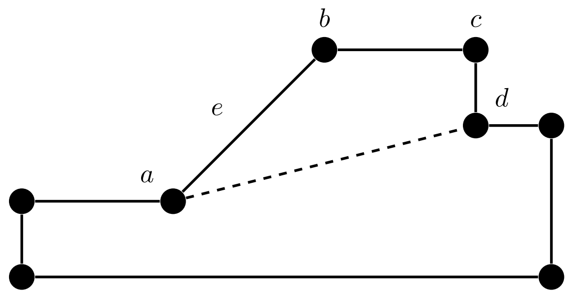
Figure 5.5: Case 2 of edges incident to a and b

edges are either both horizontal or both vertical. The vertical case can be 90°to get the horizontal case. Hence, we have a situation like one in Figure 5.4 or one in Figure 5.5. These are the only two cases possible of edges incident to a and b. They can form c-like shape or an inverted z-like shape. The shaded blue region is the nose of $e = ab$.

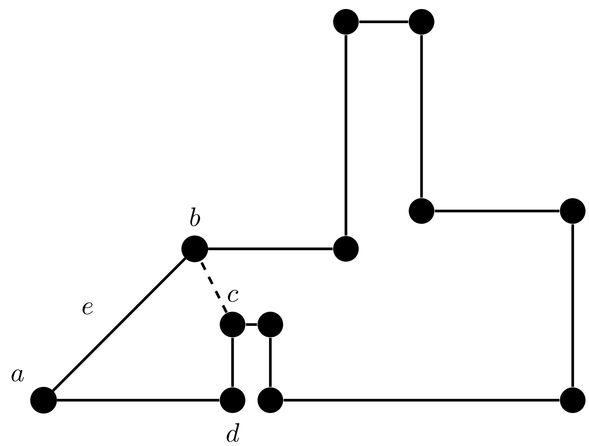
The next step of the quadrilateralization is finding the third and fourth vertex c and d of the convex polygon $abcd$ that we are looking to remove from P. Let's look at some examples of where c and d can be located.

Example 5.9

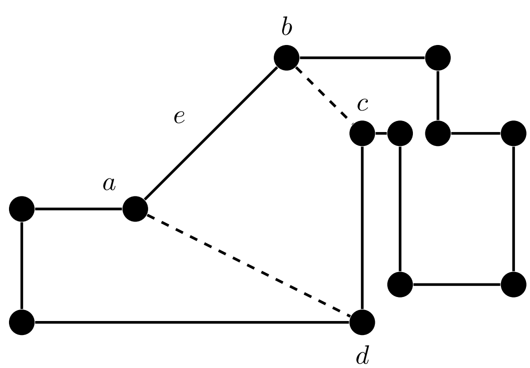




Example 5.10



Example 5.11



Example 5.12

5.3 Extended Example

In this section we perform the orthogonal art gallery algorithm on an orthogonal polygon to see what the algorithm looks like.

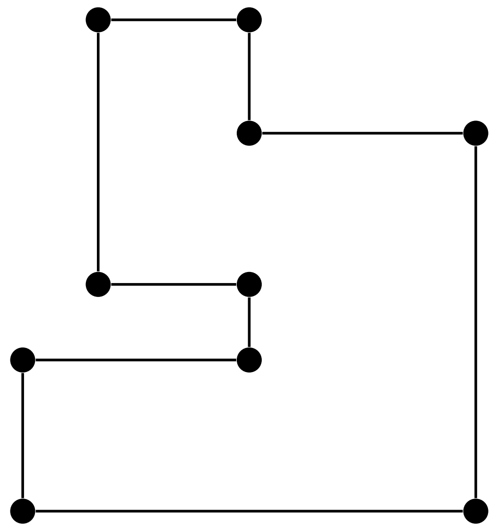


Figure 5.6: An orthogonal polygon

5.3.1 Convexly Quadrilateralizing

We begin with the polygon in Figure 5.6. We remove the convex quadrilateral *abcd* shown in Figure 5.7.

We now find a new convex piece to remove from the slanted edge we have just cut at. This new piece is shown in Figure 5.8.

After flipping and rotating the leftover 1-orthogonal polygon, we get our new removable convex polygon *abcd* as in Figure 5.9.

The piece that is now left over is already a convex quadrilateral. The convex quadrilateralization of our 1-orthogonal polygon is complete. See Figure 5.10 for the finished pieces.

5.3.2 Placing the Guards

Once we have convexly quadrilateralized our polygon, we perform a proper 4-coloring of its vertices (see Figure 5.11).

We position our guards at either the blue vertices or the red vertices since those colors appear least frequently. Suppose we choose to place the guards at the red vertices. Since every quadrilateral has a red vertex, each quadrilateral is protected and, therefore, the entire art gallery is protected.

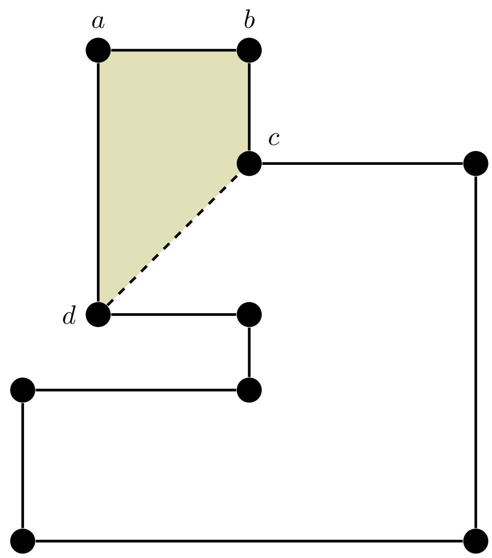


Figure 5.7: The first convex piece *abcd* that we remove

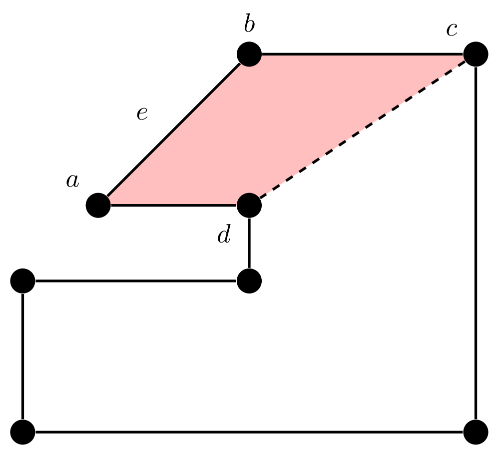


Figure 5.8: The second removable convex polygon

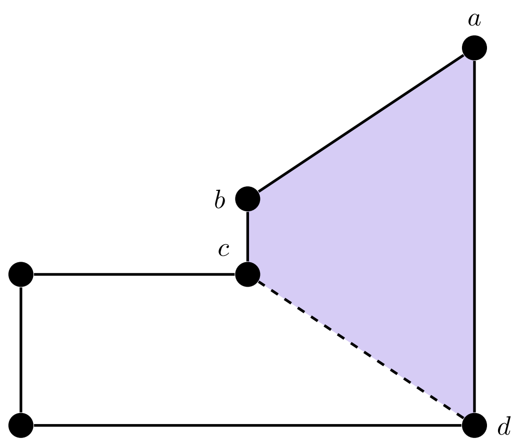


Figure 5.9: The third removable convex piece

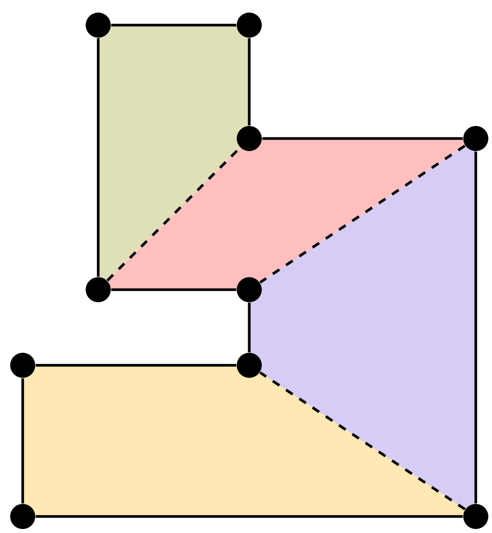


Figure 5.10: Finished convex quadrilateralisation

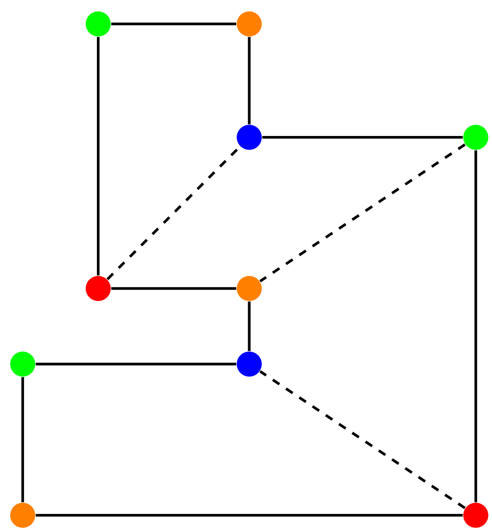


Figure 5.11: A proper 4-coloring

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