INVERSE LAPLACE TRANSFORM

If $f(t)$ is given $(t > 0)$
Then Laplace transform is defined as

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t) \, dt$$

Laplace Inverse

$$L^{-1}\{\bar{f}(s)\} = f(t)$$

Example

$L\{t^1\} = \frac{1}{s^3}$	$L^{-1}\left\{\frac{1}{s^2}\right\} = t$
$L\{1\} = \frac{1}{s}$	$L^{-1}\left\{\frac{1}{s}\right\} = 1$
$L\{t^3\} = \frac{6}{s^4}$	$L^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{6}$

Properties of Inverse Laplace Transform

1) $L\{t^n\} = \frac{\lceil n+1}{s^{n+1}} \Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\lceil n+1}$

2) $L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3) $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

4) $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

5) $L\{\sinh(at)\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh(at)}{a}$

6) $L\{\cosh(at)\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

7) $L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\left\{\frac{1}{s}\right\} = 1$ ($n=0$ in 1st Property)

8) $L\{t\} = \frac{1}{s^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2}\right\} = t$ ($n=1$ in 1st Property)

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Ex $L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

Ex $L^{-1}\left\{\frac{1}{s+3}\right\} = L^{-1}\left\{\frac{1}{s-(-3)}\right\} = e^{-3t}$

Ex $L^{-1}\left\{\frac{1}{s^6}\right\} = L^{-1}\left\{\frac{1}{s^{5+1}}\right\} = \frac{t^5}{\Gamma 5+1}$

$= \frac{t^5}{5!} = \frac{t^5}{120}$

Ex $L^{-1}\left\{\frac{1}{s^2+4}\right\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$

Ex $L^{-1}\left\{\frac{s}{s^2+9}\right\} = L^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos 3t$

Ex $L^{-1}\left\{\frac{2}{s^2-9}\right\} = L^{-1}\left\{\frac{2}{s^2-3^2}\right\} = 2L^{-1}\left\{\frac{1}{s^2-3^2}\right\}$

$= 2 \cdot \frac{1}{3} \sinh(3t)$

$= \frac{2}{3} \sinh(3t)$

Ex $L^{-1}\left\{\frac{1}{s^{5/2}}\right\} = L^{-1}\left\{\frac{1}{s^{3/2+1}}\right\}$

$= \frac{t^{3/2}}{\Gamma 5/2}$

$$= \frac{t^{3/2}}{\frac{3}{2}\Gamma\frac{3}{2}} = \frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4t^{3/2}}{3\sqrt{\pi}}$$

Properties of Laplace ~~Transform~~ Inverse

1) Shifting property: If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{e^{at}\cdot f(t)\} = \bar{f}(s-a)$$

$\Rightarrow$ If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L\{\bar{f}(s-a)\} = e^{at}\cdot f(t)$$

Example Evaluate: $L^{-1}\left\{\frac{1}{(s+2)^3}\right\}$

Solution

$$L^{-1}\left\{\frac{1}{(s+2)^3}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-2t} \frac{t^2}{2!}$$

$$= \frac{e^{-2t}\cdot t^2}{2}$$

Example $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

Solution $= \mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+2s+1)+1}\right\}$

$= \mathcal{L}^{-1}\left[\dfrac{(s+1)-1}{(s+1)^2+1}\right]$

$= e^{-t}\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1} - \dfrac{1}{s^2+1}\right\}$

$= e^{-t}\left\{\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}\right\}$

$= e^{-t}\left\{\cos t - \sin t\right\}$ //

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2) Multiply by t Property,

If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then

$\mathcal{L}\{t\,f(t)\} = -\dfrac{d}{ds}\bar{F}(s)$

$\Rightarrow$ If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ then

$\mathcal{L}^{-1}\left\{\dfrac{d}{ds}\bar{f}(s)\right\} = -t\,\mathcal{L}^{-1}\{\bar{f}(s)\}$

$= -t\,f(t).$

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(3) Divide by t Property.

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s).ds$$

$\Rightarrow$ If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\left\{\int_s^\infty \bar{f}(s).ds\right\} = \frac{f(t)}{t}$$

(4) Derivative Property.

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\left\{\frac{d}{dt}f(t)\right\} = s\bar{f}(s) - f(0)$$

$\Rightarrow$ If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\{s\bar{f}(s)\} = \frac{d}{dt}f(t) \quad \{\text{if } f(0) = 0\}$$

Note: Here we assume $f(0) = 0$ Because Laplace Inverse ~~for~~ does not exist for a constant Value.

$L^{-1}\{\text{constant}\} = \text{dos not exist}$.

(5) Integral Property

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\left\{\int_0^t f(t).dt\right\} = \frac{\bar{f}(s)}{s}$$

$\Rightarrow$ If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\, dt$$

Ex $L^{-1}\left\{\frac{1}{(s+2)(s-3)}\right\}$

Solution Using Partial Fraction,

$$\frac{1}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)}$$

$$A(s-3) + B(s+2) = 1$$
$$As - 3A + Bs + 2B = 1$$

$$s(A+B) - 3A + 2B = 1$$

$A+B = 0$ $\qquad$ $-3A + 2B = 1$

On solving these,

$3B + 2B = 1$ $\qquad$ $B = \frac{1}{5}$ $\qquad$ $A = -\frac{1}{5}$

$$= L^{-1}\left\{ \frac{-\frac{1}{5}}{(s+2)} + \frac{\frac{1}{5}}{(s-3)} \right\}$$

$$= \frac{1}{5} L^{-1}\left\{ \frac{1}{(s-3)} \right\} - \frac{1}{5} L^{-1}\left\{ \frac{1}{(s+2)} \right\}$$

$$= \frac{1}{5} e^{3t} - \frac{1}{5} e^{-2t}$$

$$= \frac{1}{5} \left(e^{3t} - e^{-2t} \right)$$

Example (1) Evaluate: $L^{-1}\left\{ \log \frac{s+1}{s-1} \right\}$

Example (2) Evaluate: $L^{-1}\left\{ \log \frac{s^2+1}{s(s+1)} \right\}$

For such Questions we use Derivative Property.

$$\boxed{L^{-1}\left\{ \frac{d}{ds} F(s) \right\} = -t\, f(t)}$$

P.T.O.

Example (1)

$$L^{-1}\left\{\log \frac{s+1}{s-1}\right\}$$

Solution Let $\bar{f}(s) = \log\left(\frac{(s+1)}{(s-1)}\right)$

$$f(s) = \log(s+1) - \log(s-1)$$

Now, differentiating both sides w.r.t 's'.

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \frac{1}{s+1} - \frac{1}{s-1}$$

Taking Laplace of inverse both sides.

$$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}$$

$$-t\,f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$-t\,f(t) = e^{-t} - e^{t}$$

$$f(t) = \frac{e^{-t} - e^{t}}{-t}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{e^{t} - e^{-t}}{t}$$

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$$L^{-1}\left\{\log \frac{s+1}{s-1}\right\} = \frac{e^{t}-e^{-t}}{t}$$

Example (2) $L^{-1}\left\{\log \frac{s^2+1}{s(s+1)}\right\}$

Solution ~~Let $\bar{f}(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$~~

Let $\bar{f}(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$

$$= \log(s^2+1) - \log(s) - \log(s+1)$$

Now, differentiating both sides w.r.t. 's'

$$\frac{d}{ds}(\bar{f}(s)) = \frac{1}{s^2+1} \cdot 2s - \frac{1}{s} - \frac{1}{s+1}$$

$$\frac{d}{ds}(\bar{f}(s)) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

Taking Laplace Inverse on both the sides.

$$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{2s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

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$$-t f(t) = 2\cos t - 1 - e^{-t}$$

$$f(t) = \frac{e^{-t} + 1 - 2\cos t}{t}$$

$$L^{-1}\{f(s)\} = \frac{e^{-t} + 1 - 2\cos t}{t}$$

$$L^{-1}\left\{\log \frac{s^2+1}{s(s+1)}\right\} = \frac{e^{-t} + 1 - 2\cos t}{t}$$

Convolution Theorem:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$

then $L^{-1}\{\bar{g}(s) \cdot \bar{f}(s)\} = \int_0^t f(u)\, g(t-u) \cdot du$

$$= \int_0^t g(u)\, f(t-u) \cdot du$$

Proof: Let $\phi(t) = \int_0^t g(u) \cdot f(t-u) \cdot du$

$$\Rightarrow L\{\phi(t)\} = \int_0^\infty e^{-st}\left[\int_0^t g(u) \cdot f(t-u) \cdot du\right] \cdot dt$$

$$= \int_{t=0}^\infty \int_{u=0}^t e^{-st} g(u) \cdot f(t-u) \cdot du\, dt$$

Now if we change the order of integration limits will change as

limits $t = u$ to $t \to \infty$
and $u = 0$ to $u \to \infty$

$$= \int_0^\infty \int_u^\infty e^{-st} g(u).f(t-u).dt.du$$

$$= \int_0^\infty g(u) \left[\int_u^\infty e^{-st} f(t-u).dt\right].du$$

put $t-u = x$
$\Rightarrow dt = dx$

limits will become
$x=0$ to $x = \infty$

$$\therefore \mathcal{L}\{\phi(t)\} = \int_0^\infty g(u)\left[\int_0^\infty e^{-s(x+u)} f(x).dx\right].du$$

$$= \int_0^\infty g(u) \left[\int_0^\infty e^{-sx}.e^{-su} f(x).dx\right].du$$

$$\cancel{= \int_0^\infty e^{-su} g(u) \left[\int_0\right.}$$

$$= \int_0^\infty e^{-su} g(u).du \ . \int_0^\infty e^{-sx} f(x).dx$$

$$\mathcal{L}\{\phi(t)\} = \bar{g}(s).\bar{f}(s)$$

$$\phi(t) = \mathcal{L}^{-1}\{\bar{g}(s)\bar{f}(s)\}$$

$$\int_0^t g(u).f(t-u).du = \mathcal{L}^{-1}\{\bar{g}(s)\bar{f}(s)\}$$

Hence Proved.

Example: Evaluate using convolution Theorem.

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$$

Solution: $L^{-1}\left\{\frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+9)}\right\}$

We know,

$$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$$

$$L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t = g(t)$$

Therefore, $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{3}\int_0^t \sin u \cdot \sin 3(t-u) \cdot du$

$$= \frac{1}{3\times 2}\int_0^t 2\sin u \cdot \sin 3(t-u) \cdot du$$

$$= \frac{1}{6}\int_0^t \left(\cos(u-3t+3u) - \cos(u+3t-3u)\right)$$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{1}{6}\int_0^t \left(\cos(4u-3t) - \cos(3t-2u)\right) du$$

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$$= \frac{1}{6\cancel{3}}\left[\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{(-2)}\right]_0^t$$

$$= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4} + \frac{\sin(3t-2u)}{2}\right]_0^t$$

$$= \frac{1}{6}\left(\frac{\sin t}{4} + \frac{\sin t}{2} - \frac{\sin(-3t)}{4} - \frac{\sin(3t)}{2}\right)$$

$$= \frac{1}{6}\left[\frac{\sin t + 2\sin t + \sin 3t - 2\sin 3t}{4}\right]$$

$$= \frac{1}{24}\left[3\sin t - \sin 3t\right]$$

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Q// $L^{-1}\left\{\dfrac{s}{(s^2+4)(s^2+9)}\right\}$

solution $L^{-1}\left\{\dfrac{s}{(s^2+4)} \cdot \dfrac{1}{(s^2+9)}\right\}$

$\bar{f}(s) = \dfrac{s}{s^2+4}$

$\bar{g}(s) = \dfrac{1}{s^2+9}$

$L^{-1}\left\{\bar{f}(s)\right\} = L^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos 2t = f(t)$

$L^{-1}\left\{\bar{g}(s)\right\} = L^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \dfrac{1}{3}\sin 3t = g(t)$

Using Convolution Theorem

$L^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} = \dfrac{1}{3}\int_0^t \sin 3u \cdot \cos 2(t-u) \cdot du$

$= \dfrac{1}{2\times 3}\int_0^t 2\sin 3u \cdot \cos 2(t-u) \cdot du$

$= \dfrac{1}{6}\int_0^t \left(\sin(3u+2t-2u) - \sin(3u-2t+2u)\right)$

$$= \frac{1}{6} \int_0^t \left[\sin(u+2t) - \sin(5u-2t) \right] . du$$

$$= \frac{1}{6} \left[\frac{-\cos(u+2t)}{1} + \frac{\cos(5u-2t)}{5} \right]_0^t$$

$$= \frac{1}{6} \left[\frac{\cos 3t}{5} - \cos(3t) + \cos 2t - \frac{\cos(-2t)}{5} \right]$$

$$= \frac{1}{6} \left[\frac{\cos 3t}{5} - \cos 3t + \cos 2t - \frac{\cos 2t}{5} \right]$$

$$= \frac{1}{6} \left[\frac{4\cos 2t}{5} - \frac{4\cos 3t}{5} \right]$$

$$= \frac{\cancel{4}^{2}}{5} \times \frac{1}{\cancel{6}_{3}} \left[\cos 2t - \cos 3t \right]$$

$$= \frac{2}{15} \left(\cos 2t - \cos 3t \right)$$