

# Counterexamples in (Introductory) Algebra

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## Isomorphism of quotients does not imply isomorphism of quotient groups

ie:  $H \cong K \not\Rightarrow G/H \cong G/K$

Let  $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ , with  $H = \langle (2, \bar{0}) \rangle$  and  $K = (\bar{0}, \bar{1})$ .

Then  $H \cong K \cong \mathbb{Z}_2$  but  $G/K \cong \mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong G/H$

## Isomorphism of quotient groups does not imply isomorphism of quotients

ie:  $G/H \cong G/K \not\Rightarrow H \cong K$

(D&F 3.3.8): For prime  $p$ , let  $G$  be the group of  $p$ -power roots of unity. And  $\phi : G \rightarrow G$  be the surjective homomorphism  $z \mapsto z^p$ . Then  $G/\ker\phi \cong G$ .

So let  $K = \ker\phi$  and  $H$  be trivial. Then  $G/K \cong G \cong G/H$ , but  $H \not\cong K$  (because  $\ker\phi$  is non-trivial).

## A group can be isomorphic to a proper quotient of itself

Same example as above.

## An infinite group in which every element has finite order but for each positive integer $n$ there is an element of order $n$

$$\prod_{n \in \mathbb{N}} \mathbb{Z}_n$$

## A group such that every finite group is isomorphic to some subgroup

1) The direct product of all finite groups, or 2) The group of all bijections  $\mathbb{N} \rightarrow \mathbb{N}$  (then applying Cayley's Theorem)

## A nontrivial group $G$ s.t. $G \cong G \times G$

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ , with isomorphism  $(g_1, g_2, g_3, \dots) \mapsto ((g_1, g_3, g_5, \dots), (g_2, g_4, g_6, \dots))$

## A group of order $n$ may not have a subgroup of order $k$ for all $k|n$

The alternating group  $A_4$  has order 12, but no element of order 6 (all elements are order 1, 2, or 4).

## Direct product of Hamiltonian Groups<sup>1</sup> may not be Hamiltonian

In  $Q_8 \times Q_8$ , the subgroup  $\langle (i, j) \rangle$  is not normal because  $\langle (i, j) \rangle = \{(1, 1), (i, j), (-1, -1), (-i, -j)\}$  but  $(j, 1)(i, j)(j, 1)^{-1} = (-i, j) \notin \langle (i, j) \rangle$

## Subgroups of finitely-generated groups may not be finitely generated

The commutator subgroup of the free group on two elements  $F(\{x, y\})$  cannot be finitely generated (proof omitted).

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<sup>1</sup>non-abelian group where every subgroup is normal