Ring Taxonomy

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Fields $\subset$ Euclidean domains $\subset$ PIDs $\subset$ UFDs $\subset$ Integral domains

Examples of property X do not apply to listed property $Y \subset X$. Eg, examples of UFDs are not also PIDs

Ideals, Rings

- maximal $\implies$ prime
- if char(R) = n, then R contains subring $\cong \mathbb{Z}/n\mathbb{Z}$

Integral domains

- prime $\implies$ irreducible
- characteristic 0 or prime
- Ex: $\mathbb{Z}[2i]$ not a UFD: $4 = (2)(2) = (2i)(-2i)$
- Also, $(2i)$ is irreducible but not prime in $\mathbb{Z}[2i]$, since $\mathbb{Z}[2i]/(2i) \cong \mathbb{Z}/4\mathbb{Z}$

UFD

- prime $\iff$ irreducible
- Ex: $\mathbb{Z}[x], \mathbb{Q}[x, y]$

PID

- irreducible $\implies$ prime (also from inclusion in UFD)
- Every nonzero prime ideal is maximal.
- R commutative, $R[x]$ PID $\iff$ R is field. (since (x) is prime $\implies$ maximal)
- Ex: (nontrivial… most are Euclidean)

Euclidean domain

- Ex: $F[x], \mathbb{Z}[i]$

Fields

- Ex: $\mathbb{R}, \mathbb{Q}, \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$