Math H113 - D&F 13.2 Notes

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1. We have an extension K over F, where F is a field.
2. $\alpha \in K$ is **algebraic over F** if it is the root of some nonzero polynomial $f(x) \in F[x]$. Otherwise α is **transcendental over F**.
3. The extension K/F is algebraic if $\forall \alpha \in K$, α is algebraic over F.
4. If α is algebraic, $m_{\alpha,F}$ is the unique monic irreducible (a.k.a **minimal**) polynomial in $F[x]$ with α as a root. Any polynomial in $F[x]$ has α as a root **iff** it's divisible by $m_{\alpha,F}$. Define $\deg \alpha = \deg m_{\alpha,F}$.
5. If L/F is a field extension and α is algebraic over both F and L, $m_{\alpha,L} \mid m_{\alpha,F}$.
6. Given $m_{\alpha,F}$, $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$, where $F(\alpha)$ is the field generated by α over F. We have $[F(\alpha) : F] = \deg m_\alpha(x) = \deg \alpha$.
7. α is algebraic over F **iff** $F(\alpha)/F$ is finite (i.e $\deg \alpha < \infty$). I.e, if $\deg \alpha = n$ a polynomial of deg $\leq n$ has α as a root, and if α satisfies a polynomial of deg $= n$ then $\deg F(\alpha) \leq n$.
8. If the extension K/F is finite, it is algebraic.
9. Let $F \subseteq K \subseteq L$ be fields: $[L : F] = [L : K][K : F]$.
10. If L/F is a finite extension, $K \subseteq L$ is a field and $K \supseteq F$ s.t $F \subseteq K \subseteq L$, then $[K : F] \mid [L : F]$.
11. An extension K/F is **finitely generated** if there are elements $\alpha_1, ..., \alpha_k$ in K s.t $K = F(\alpha_1, ..., \alpha_k)$.
12. The field generated over F by a collection of elements in an extension field K is the smallest subfield of K containing these elements and F.
13. $F(\alpha, \beta) = (F(\alpha))(\beta)$. I.e we may 'adjoin' elements to F and then close the resulting set over addition, subtraction, multiplication, and division to obtain a field.
14. The extension K/F is finite **iff** K is generated by a finite number of algebraic elements over F. If the generators have degree $n_1, ..., n_k$ then $\deg F(\alpha_1, ..., \alpha_k) \leq n_1 ... n_k$.

15. Suppose α and β are algebraic over F. Then $\alpha \pm \beta$, $\alpha\beta$, and α/β $(\beta \neq 0)$ are all algebraic.

16. Let L/F be an arbitrary extension. Then the collection of elements of L that are algebraic over F form a subfield K of L.

17. If K is algebraic over F and L is algebraic over K, then L is algebraic over F.

18. Let K_1 and K_2 be two subfields of a field K. Then the **composite field** of K_1 and K_2 denoted by K_1K_2 is the smallest subfield of K containing both K_1 and K_2.

19. Let K_1 and K_2 be two finite extensions of a field F contained in K. Then $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$, with equality **iff** an F-basis for one of the fields remains linearly independent over the other field.

20. Suppose that $[K_1 : F] = n$ and $[K_2 : F] = m$ where $(n, m) = 1$. Then $[K_1K_2 : F] = [K_1 : F][K_2 : F] = nm$.