Counterexamples in (Introductory) Algebra

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Isomorphism of factors does not imply isomorphism of quotient groups

ie: $H \cong K \not\Longrightarrow G/H \cong G/K$

Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, with $H =< (\bar{2}, \bar{0}) >$ and $K = (\bar{0}, \bar{1})$.

Then $H \cong K \cong Z_2$ but $G/K \cong Z_4 \not\cong Z_2 \times Z_2 \cong G/H$

Isomorphism of quotient groups does not imply isomorphism of factors

ie: $G/H \cong G/K \not\Longrightarrow H \cong K$

(D&F 3.3.8): For prime p, let G be the group of p-power roots of unity. And $\phi : G \to G$ be the surjective homomorphism $z \mapsto z^p$. Then $G/\ker\phi \cong G$.

So let $K = \ker\phi$ and H be trivial. Then $G/K \cong G \cong G/H$, but $H \not\cong K$ (because $\ker\phi$ is non-trivial).

A group can be isomorphic to a proper quotient of itself

Same example as above.

The image of an ideal may not be an ideal

ie: I **ideal** $\not\Longrightarrow \phi(I)$ **ideal for homomorphism** ϕ

Let $\phi : \mathbb{Z} \to \mathbb{Z}[x]$ by inclusion. Then $2\mathbb{Z}$ is ideal in $\mathbb{Z}$, but not in $\mathbb{Z}[x]$ ($2x \notin 2\mathbb{Z}$).

An infinite group in which every element has finite order but for each positive integer n there is an element of order n

$\prod_{n \in \mathbb{N}} Z_n$

A group such that every finite group is isomorphic to some subgroup

1) The direct product of all finite groups, or 2) The group of all bijections $\mathbb{N} \to \mathbb{N}$ (then applying Cayley's Theorem)

A nontrivial group G s.t. $G \cong G \times G$

$G = Z_2 \times Z_2 \times \cdots$, with isomorphism $(g_1, g_2, g_3, \ldots) \mapsto ((g_1, g_3, g_5, \ldots), (g_2, g_4, g_6, \ldots))$

A group of order n may not have a subgroup of order k for all $k|n$

The alternating group A_4 has order 12, but no element of order 6 (all elements have order 1, 2, or 3).

Direct product of Hamiltonian Groups[1] may not be Hamiltonian

In $Q_8 \times Q_8$, the subgroup $< (i, j) >$ is not normal because $< (i, j) >= \{(1, 1), (i, j), (-1, -1), (-i, -j)\}$ but $(j, 1)(i, j)(j, 1)^{-1} = (-i, j) \notin< (i, j) >$

Subgroups of finitely-generated groups may not be finitely generated

The commutator subgroup of the free group on two elements $F(\{x, y\})$ cannot be finitely generated (proof omitted).

[1] non-abelian group where every subgroup is normal

Elementary Examples

A non-trivial ring homomorphism with $\phi(1) \neq 1$

$\phi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ given by $n \mapsto \overline{3n}$

Ideals I, J **where** $IJ \subsetneq I \cap J$

Let $I = 2\mathbb{Z}$, $J = 4\mathbb{Z}$. Then $IJ = 8\mathbb{Z}$ but $I \cap J = J = 4\mathbb{Z}$.