Counterexamples in (Introductory) Algebra

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Isomorphism of quotients does not imply isomorphism of quotient groups

ie: $H \cong K \not\Longrightarrow G/H \cong G/K$
Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, with $H =< (\bar{2}, \bar{0}) >$ and $K = (\bar{0}, \bar{1})$.
Then $H \cong K \cong Z_2$ but $G/K \cong Z_4 \ncong Z_2 \times Z_2 \cong G/H$

Isomorphism of quotient groups does not imply isomorphism of quotients

ie: $G/H \cong G/K \not\Longrightarrow H \cong K$
(D&F 3.3.8): For prime p, let G be the group of p-power roots of unity. And $\phi : G \to G$ be the surjective homomorphism $z \mapsto z^p$. Then $G/\mathrm{ker}\phi \cong G$.
So let $K = \mathrm{ker}\phi$ and H be trivial. Then $G/K \cong G \cong G/H$, but $H \ncong K$ (because $\mathrm{ker}\phi$ is non-trivial).

A group can be isomorphic to a proper quotient of itself

Same example as above.

Direct product of Hamiltonian Groups[1] may not be Hamiltonian

In $Q_8 \times Q_8$, the subgroup $< (i, j) >$ is not normal because $< (i, j) >= \{(1, 1), (i, j), (-1, -1), (-i, -j)\}$ but $(j, 1)(i, j)(j, 1)^{-1} = (-i, j) \notin < (i, j) >$

[1] non-abelian group where every subgroup is normal