Math H113 - Spring 2014 Notes

April 8, 2014

Introduction

This is a sparse collection of facts taken from Dummit and Foote, 3e. The goal is to recap the most important things Prof. Vojta has covered, with an emphasis on non-obvious results.

Chapter 0/1

1. gcd: $(m, n) = am + bn$.
2. The Euler function: $\varphi(p^a) = p^{a-1}(p-1)$, $\varphi(ab) = \varphi(a)\varphi(b)$ if $(a, b) = 1$.
3. The dihedral group: $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$.
4. Fields $(F,\ F^\times = F - \{0\})$: $(F, +)$ and $(F^\times, \times)$ are abelian groups. Note: if $|F| < \infty$, $\exists p, m$ s.t $|F| = p^m$.
5. Symmetry groups: $|S^n| = n!$. Disjoint cycles commute, but $S_{n\geq 3}$ is non-abelian.
6. Group actions on A by G: (i) $g_1(g_2a) = (g_1g_2)a$, (ii) $1a = a$. $\forall g_1, g_2, a$.
   (a) Fixing $g \in G$ in the action gives $\sigma_g \in S_A$.
   (b) $g \mapsto \sigma_g$ is a homomorphism ($G \to S_A$, the permutation representation).

Chapter 2

1. Subgroup criterion: $H \neq \phi$ and $xy^{-1} \in H$ $\forall x, y \in H$.
2. Centralizer: $\leq G$, commutes with A. Center: $\leq G$, commutes with G itself.
3. Normalizer: $\leq G$ s.t $gAg^{-1} = A$ $\forall g$.
4. Stabilizer: fixing $a \in A$, $\leq G$ s.t $ga = a$. Kernel: $\forall a \in A$, $\leq G$ s.t $ga = a$.
5. If $x^m = 1$ and $x^n = 1$, $x^{(m,n)} = 1$.

6. Let $x \in G$, $a \neq 0$: if $|x| = \infty$ then $|x^a| = \infty$. Else, if $|x| = n$, then $|x^a| = n/(n,a)$ $(\star)$.

7. Every subgroup of a cyclic group is cyclic, and cyclic groups of the same order are isomorphic to each other.

8. Let $|x| = n$, $H = \langle x \rangle$. Only if $(n,a) = 1$, $H = \langle x^a \rangle$ (count these with $\varphi(n)$). A general statement: $\langle x^m \rangle = \langle x^{(n,m)} \rangle$.

9. Let $A \neq \phi$ be a set of subgroups of G. Then their intersection $\langle A \rangle = \cap A \leq G$.

Chapter 3

1. Given $\varphi : G \to H$; $\varphi(1_G) = 1_H$, $\ker \varphi \leq G$, and $\text{im}(\varphi)$ $\varphi \leq H$.

2. G/K is basically arithmetic on the fibers of φ, which are all cosets of $\ker \varphi$.

3. The set of left cosets of any $N \leq G$ partitions G. However, the operation $uN \cdot vN = (uv)N$ is only well defined if $N \trianglelefteq G$ (or equivalently $N_G(N) = G$, $gN = Ng$ $\forall g$, or $gNg^{-1} \subseteq N$ $\forall g$).

4. If $|G| < \infty$ and $H \leq G$, then $|H| \mid |G|$ and $|G : H| = |G|/|H|$ $(\star)$.

5. If $|G| < \infty$ and $p \mid |G|$, $\exists x \in G$ s.t $|x| = p$.

6. If $|G| = p^\alpha m$ $(p \not| m)$, $\exists H \leq G$ s.t $|H| = p^\alpha$.

7. $\ker \varphi \trianglelefteq G$, $G/\ker \varphi \cong \varphi(G)$, φ is 1-1 iff $\ker \varphi = 1$, and $|G : \ker \varphi| = |\varphi(G)|$.

8. If finite $H, K \leq G$, then $|HK| = \frac{|H||K|}{|H \cap K|}$. $HK \leq G$ only if $KH \leq G$.

9. Let $A, B \leq G$ and $A \leq N_G(B)$. Then $AB \leq G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \cong A/(A \cap B)$.

10. Let $H \leq K$ and $H, K \trianglelefteq G$: then $K/H \trianglelefteq G/H$ so $(G/H)/(K/H) \cong G/K$.

11. Let $\pi : G \to G/N$ be the natural projection $g \mapsto gN$, and (stopped at page 100)