

Math H113 - Spring 2014 Notes

April 22, 2014

Introduction

This is a sparse collection of facts taken from Dummit and Foote, 3e. The goal is to recap the most important things Prof. Vojta has covered, with an emphasis on non-obvious results.

Definitions

Normal Subgroup

Group such that... TODO

Chapter 0/1

1. gcd: $(m, n) = am + bn$.
2. The Euler function: $\varphi(p^a) = p^{a-1}(p-1)$, $\varphi(ab) = \varphi(a)\varphi(b)$ if $(a, b) = 1$.
3. The dihedral group: $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$.
4. Fields $(F, F^\times = F - \{0\})$: $(F, +)$ and (F^\times, \times) are abelian groups. Note: if $|F| < \infty$, $\exists p, m$ s.t. $|F| = p^m$.
5. Symmetries: $|S_n| = n!$. Disjoint cycles commute, but $S_{n \geq 3}$ is non-abelian.
6. Group actions on A by G : (i) $g_1(g_2a) = (g_1g_2)a$, (ii) $1a = a$. $\forall g_1, g_2, a$.
 - (a) Fixing $g \in G$ in the action gives $\sigma_g \in S_A$.
 - (b) $g \mapsto \sigma_g$ is a homomorphism ($G \rightarrow S_A$, the permutation representation).

Chapter 2

1. Subgroup criterion: $H \neq \phi$ and $xy^{-1} \in H \forall x, y \in H$.
2. Centralizer: $\leq G$, commutes with A . Center: $\leq G$, commutes with G itself.

3. Normalizer: $\leq G$ s.t. $gAg^{-1} = A \forall g$.
4. Stabilizer: fixing $a \in A$, $\leq G$ s.t. $ga = a$. Kernel: $\forall a \in A$, $\leq G$ s.t. $ga = a$.
5. If $x^m = 1$ and $x^n = 1$, $x^{(m,n)} = 1$ (in cyclic groups).
6. Let $x \in G$, $a \neq 0$: if $|x| = \infty$ then $|x^a| = \infty$. Else, if $|x| = n$, then $|x^a| = n/(n, a)$ (\star).
7. Every subgroup of a cyclic group is cyclic, and cyclic groups of the same order are isomorphic to each other.
8. Let $|x| = n$, $H = \langle x \rangle$. Only if $(n, a) = 1$, $H = \langle x^a \rangle$ (count these with $\varphi(n)$).
A general statement: $\langle x^m \rangle = \langle x^{(n,m)} \rangle$.
9. Let $A \neq \phi$ be a set of subgroups of G . Then their intersection $\langle A \rangle = \cap A \leq G$.

Chapter 3

1. Given $\varphi : G \rightarrow H$; $\varphi(1_G) = 1_H$, $\ker \varphi \leq G$, and $\text{im}(\varphi) \varphi \leq H$.
2. G/K is basically arithmetic on the fibers of φ , which are all cosets of $\ker \varphi$.
3. The set of left cosets of any $N \leq G$ partitions G . However, the operation $uN \cdot vN = (uv)N$ is only well defined if $N \trianglelefteq G$ (or equivalently $N_G(N) = G$, $gN = Ng \forall g$, or $gNg^{-1} \subseteq N \forall g$).¹
4. **Lagrange's Theorem:**
If $|G| < \infty$ and $H \leq G$, then $|H| \mid |G|$ and $|G : H| = |G|/|H|$
5. **Cauchy's Theorem:**
If $|G| < \infty$ and prime $p \mid |G|$, $\exists x \in G$ s.t. $|x| = p$.
6. **Sylow's Theorem:**
If $|G| = p^\alpha m$ ($p \nmid m$), $\exists H \leq G$ s.t. $|H| = p^\alpha$.
7. $\ker \varphi \trianglelefteq G$, $G/\ker \varphi \cong \varphi(G)$, φ is 1-1 iff $\ker \varphi = 1$, and $|G : \ker \varphi| = |\varphi(G)|$.
8. If finite $H, K \leq G$, then $|HK| = \frac{|H||K|}{|H \cap K|}$. $HK \leq G$ only if $KH \leq G$.
9. Let $A, B \leq G$ and $A \leq N_G(B)$. Then $AB \leq G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \cong A/(A \cap B)$.
10. Let $H \leq K$ and $H, K \trianglelefteq G$: then $K/H \trianglelefteq G/H$ so $(G/H)/(K/H) \cong G/K$.
11. For $N \trianglelefteq G$, there is a bijection between subgroups of G containing N and subgroups of G/N
12. To show that a homomorphism from $\varphi : G/N \rightarrow H$ is well-defined, one must prove $N \leq \ker \Phi$ (with $\Phi : G \rightarrow H$).

¹A useful theorem for later: $gH = H$ iff $g \in H$.

13. $(a_1 a_2 \dots a_m) = (a_1 a_m)(a_1 a_{m-1}) \dots (a_1 a_2)$. The sign of a permutation (i.e the parity of the number of 2-cycles $\epsilon(\sigma) \in \{\pm 1\}$ ²) is representation-independent.
14. $\epsilon : S_n \rightarrow \{\pm 1\}$ is a surjective homomorphism. $\ker \epsilon = A_n$, the group of even permutations. Note $S_n/A_n \cong \epsilon(S_n) = \{\pm 1\}$ and $|A_n| = \frac{n!}{2}$.
15. If $G = MN$ with $M, N \trianglelefteq G$ then: $G/(M \cap N) \cong (G/M) \times (G/N)$
16. For $H \trianglelefteq G$ and $|G : H| = p$ prime, then for all $K \leq G$, either $K \leq H$ or $(G = HK$ and $|K : K \cap H| = p)$
17. The commutator subgroup $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$ is normal, and G/N is always abelian.
18. “Normal in” relation NOT transitive: $A \trianglelefteq B \wedge B \trianglelefteq C \not\Rightarrow A \trianglelefteq C$
19. In general, $H \times (G/H) \not\cong G$

Chapter 4

1. $\sigma_g : A \rightarrow A$ ($a \mapsto ga$), and $\varphi : G \rightarrow S_A$ ($g \mapsto \sigma_g$). Note: the kernel of the action $\cap_{a \in A} G_a = \ker \varphi$.³
2. For $A \neq \emptyset$, each action $G \times A \rightarrow A$ is isomorphic to a homomorphism $G \rightarrow S_A$. Let $a \sim b$ iff $a = gb$ for some $g \in G$: then \sim partitions G , and the order of the equivalence class (i.e orbit) containing a is $|G : G_a|$.⁴
3. Elements in G effect the same permutation on A iff they’re in the same coset of the kernel of the action.
4. Let $H \leq G$, A be the set of left cosets of H in G , and G act on A (with $\pi_H : G \rightarrow S_A$). Then the action is transitive, $G_{1H} = H$, and $\ker \pi_H = \cap_{x \in G} xHx^{-1}$ (giving the largest normal subgroup of G in H).
5. If $|G| = n$, $G \cong H$ for some $H \leq S_n$. If p is the smallest prime s.t $p|n$, then any subgroup $H \leq G$ s.t $|G : H| = p$ is normal.

Chapter 5

1. Given a direct product $G_1 \times G_2 \times \dots \times G_n$, $G_i \cong \langle (1, \dots, g_i, \dots, 1) \mid g_i \in G_i \rangle$
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2. **Invariant Factor Decomposition:** Let $G = \langle A \rangle$ ($A \subseteq G$, finite). Then $G \cong \mathbb{Z}^r \times Z_{n_1} \times Z_{n_2} \times \dots \times Z_{n_s}$ s.t $r \geq 0$, $n_j \geq 2 \forall j$, and $n_{i+1} | n_i$ for $1 \leq i < s$ uniquely (up to isomorphism).

²An m -cycle is composed of $m-1$ transpositions, immediately giving $\epsilon(\sigma) = \text{Parity}(m-1)$.

³‘Faithful’ actions have kernels equal to $\{1_G\}$

⁴‘Transitive’ actions induce only one orbit in A .

⁵The projection $\pi : G \rightarrow G_i$ is $g \mapsto g[i]$.

3. **Elementary Divisor Decomposition:** Let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Then $G \cong A_1 \times \dots \times A_k$ where $|A_i| = p_i^{\alpha_i}$. Each $A_i \cong Z_{p_i^{\beta_1}} \times \dots \times Z_{p_i^{\beta_t}}$ where $\beta_i \geq \beta_{i+1}$ and $\sum_i^t \beta_i = \alpha_i$.
4. Let $n = \prod n_i$: if $p|n$ then $p|n_1$. If n is a product of distinct primes, then Z_n is the only abelian group of order n (up to isomorphism).
5. Full converse to Lagrange's Theorem: For any abelian group G of order n , there exists a subgroup of order k for all $k|n$ (proof using decomposition theorems).
6. $Z_m \times Z_n \cong Z_{mn}$ iff $(m, n) = 1$, so $Z_n \cong Z_{p_1^{\alpha_1}} \times \dots \times Z_{p_k^{\alpha_k}}$.
7. The group exponent is the smallest positive integer s.t. $x^n = 1 \ \forall x \in G$.
8. The elementary abelian group of order p^n : $E_{p^n} = (Z_p)^n$. Each non-identity element has order p , and there are $p+1$ subgroups of order p in E_{p^2} .

Chapter 6

1. Let $F(S)$ be the group of words formed from S . Given a map $\psi : S \rightarrow G$, there exists a unique homomorphism $\Phi : F(S) \rightarrow G$ s.t. $\Phi|_S = \psi$.
2. Because Φ is a homomorphism, $\Phi(s_i^{\epsilon_i} \dots) = \psi(s_i)^{\epsilon_i} \dots$.
3. Empty word: $(1, 1, \dots)$. Reduced word: $s_{i+1} \neq s_i^{-1}$ and $s_i = 1 \Rightarrow s_{k \geq i} = 1$.
4. Subgroups of free groups are also free groups.
5. Let $S \subseteq G$ s.t. $G = \langle S \rangle$. A presentation of G is some (S, R) s.t. $\ker \Phi$ is the smallest normal subgroup containing $\langle R \rangle$. G is finitely generated if S is finite, and finitely presented if R is also finite.
6. Any free abelian group of rank n is $\cong \mathbb{Z}^n$.

Chapter 7

1. Rings: $(R, +)$ is abelian, \times is associative/distributive. If \times commutes, so does R . If $1 \in R$, $1r = r1 \ \forall r$. If $1 \neq 0$ and every nonzero element has an inverse, R is a division ring. Commutative division rings are fields.
2. Let $a, b \in R$ be nonzero. Then $a0 = 0a = 0$. Zero divisors: $ab = 0$ or $ba = 0$. The set of units (i.e. $uv = vu = 1$) is R^\times .
3. Integral domains are commutative rings (with $1 \neq 0$) with no zero divisors. Zero divisors cannot be units, therefore fields have no zero divisors.
4. Finite integral domains are fields. Subrings are $\leq R$ and closed under \times .

5. $R[x]$: $(ab)x^k = \sum_{i=0}^k a_i b_{k-i} = \sum_{i=0}^k a_{k-i} b_i$. Note: $R \subset R[x]$ (as the constant polynomials) and $R[x]$ is commutative iff R is.
6. If R is an integral domain, $\deg(ab) = \deg(a) + \deg(b)$, $R[x]^\times = R^\times$, and $R[x]$ is an integral domain.
7. Square matrices: $(a_{ij}) \in M_n(R)$. Invertible: $GL_n(R)$.
8. Fix a commutative ring R with $1 \neq 0$ and let G be a finite group. Group rings RG contain all formal sums $\sum_i r_i g_i$ $r_i \in R$. Addition is done componentwise, and RG always has zero divisors.
9. Ring homomorphism: $\varphi : R \rightarrow S$ s.t $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$. $\ker \varphi = \{r \in R \mid \varphi(r) = 0_S\}$, as if φ were a group homomorphism.
10. $I = \ker \varphi$ is a subring of R (and an ideal/normal subgroup thereof), $\text{im}(\varphi)$ is a subring of S . If $\alpha \in \ker \varphi$, then $r\alpha, \alpha r \in R \forall r \in R$.
11. Ideals: if $rI \subseteq I$, $Ir \subseteq I$, and I subring of R . R/I is a quotient ring s.t $(r+I) + (s+I) = (r+s) + I$ and $(r+I) \times (s+I) = rs + I$. $R/\ker \varphi \cong \varphi(R)$. Note: every ideal is the kernel of a ring homomorphism and vice versa.
12. Let A be a subring and B an ideal of R . Then $A+B$ is a subring of R , $A \cap B$ is an ideal of A , and $(A+B)/B \cong A/(A \cap B)$.
13. Let $I \subseteq J$ be ideals of R , then $(R/I)/(J/I) \cong R/J$.
14. For ideals I of R , there is a bijection between subrings of R containing I and subrings of R/I .
15. Ideal math: $I + J = \{i + j \mid i \in I, j \in J\}$, IJ is the set of all finite sums of elements of the form ij , and I^n are all n -length products within I .
16. $I + J$ is the smallest ideal containing I and J , and $IJ \subseteq I \cap J$.
17. If R is a commutative ring with a 1, and $I + J = R$, then $IJ = I \cap J$.
18. If I is ideal and homomorphism ϕ is surjective, then $\phi(I)$ is also ideal.
19. If S is ideal, then $\phi^{-1}(S)$ (complete pre-image) is ideal.
20. Every maximal ideal is prime.

Chapter 8

1. In a PID, every prime ideal is maximal.