will skip the obstacle and ignore it. However, if the time span has a suitable value, the vehicle avoids the obstacle, as seen in Fig.(20). Note that as the prediction horizon increases, the way the obstacle is avoided is smoother. Additionally, an increase in the prediction horizon leads to a closer approximation of the vehicle to the final point of hte trajectory: $(x_{1final}, x_{2final}) = (60, 60)$.

3 Quadcopter Dynamics

3.1 Example 1 - Quadcopter Model considering only Translation

In this example a quadcopter that only moves in a direction will be analysed. No rotation is considered.

3.1.1 Frame Definitions

To determine the Quadcopter dynamics model, it is first necessary to define two different frames of reference. The first one is the inertial frame of reference (I), that is fixed and connected to the Earth. In this frame, a body is at rest or moving at a constant velocity. The second frame of reference is a moving one (B), that is connected to the quadcopter body, with its origin in the quadcopter gravity center. It is called the body frame. Z_B and Z have the same direction. Both

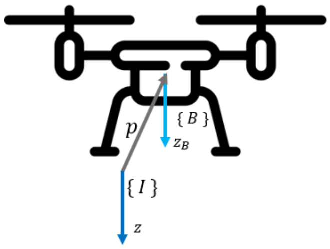

Figure 21: Example 1 - The Quadcopter Model (adapted from 3.3.4)

are shown in Fig.(21). Note that when the quadcopter is stabilized, the two frames of reference coincide.

3.1.2 Dynamics of the Quadcopter

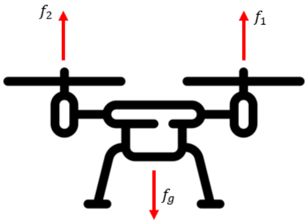

Figure 22: Example 1 - The Quadcopter forces Diagram (adapted from 3.3.4)

As it can be seen in Fig.(22), each one of the propellers applies a force f_1 and f_2, respectively, in the opposite direction of the force of gravity f_g. Let u be defined as the total propulsive force,

$$u = f_1 + f_2. \tag{32}$$

The module of the sum of the propellers force is equal to the module of the force f_g.

$$\left| \vec{f_1} + \vec{f_2} \right| = \left| \vec{f_g} \right|. \tag{33}$$

3.1.12 Conclusions

The transfer function of the proportional derivative controller (with the low-pass filter) for the best case analysed before is given by

$$C(s) = -100(s + 2.47), \tag{40}$$

In this equation, the degree of the numerator is bigger than the degree of the denominator. Therefore, this controller is non-causal.
The proportional derivative controller analysed in the first place, which had no filter action, has another practical constraint: this controller is a pure one. Thus, it is not feasible. If a situation where a reference or a perturbation to the system are noisy or have an abrupt transition occurs, the results achieved will not be satisfying. These situations are amplified by the derivative part of the controller. Thus, a more realist approach should be implemented, considering at least the insertion of one pole. That solution it is going to be explored in the next problem.

3.2 Example 2 - Quadcopter Model considering only Rotation

In this example a quadcopter that only rotates will be analysed. No translation is considered.

3.2.1 Dynamics of the Quadcopter

In this problem, the frames of reference that were considered were the same as the ones defined in section (3.1.1).

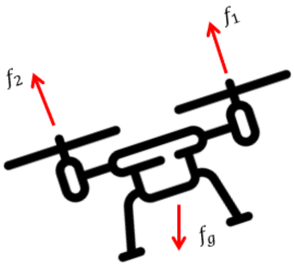

Figure 32: Example 2 - The Quadcopter forces Diagram (adapted from 3.3.4)

The forces applied to the previous model will be considered for this problem, too. In Fig.(32), the forces that are applied to the system are represented. f_1 and f_2 are the forces that the propellers apply to the quadcopter, with an opposite direction of the force of gravity f_g. Let u be defined as the total propulsive force,

$$u = f_1 - f_2. \tag{41}$$

Let the module of the sum of the propellers force be equal to the module of the force f_g.

$$\left|\vec{f_1} + \vec{f_2}\right| = \left|\vec{f_g}\right|. \tag{42}$$

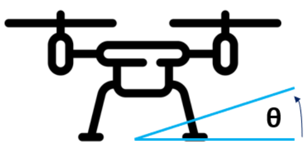

Figure 33: Example 2 - The Quadcopter Model (adapted from 3.3.4)

In this example it is assumed that the quadcopter only rotates.

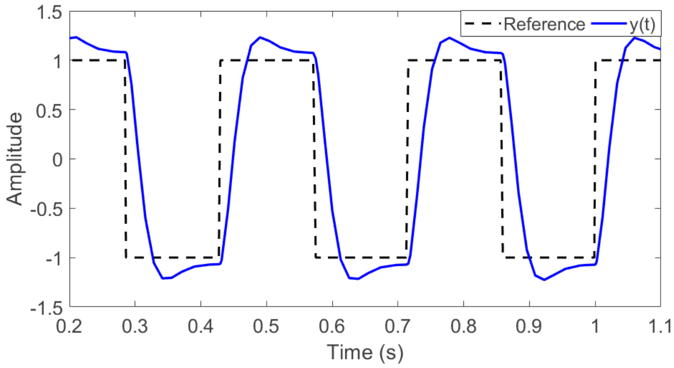

Figure 40: Example 2 - $y(t)$ response to a square wave input, with PID.

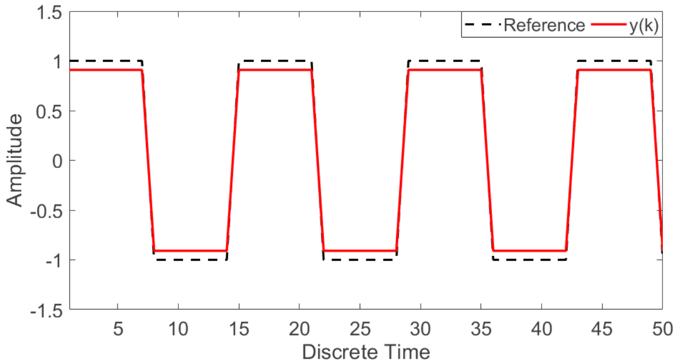

Figure 41: Example 2 - $y(k)$ response to a square wave input, with MPC.

and the reference. For that, it uses the integrative, derivative and the proportional gains that in this case were set to K_p= 250, K_I= 50, K_d= 50, N= 100. For this example, the two methods are shown in Fig.(40) and Fig.(41). Like it was expectable, the MPC approach achieves better results than the PID controller.

3.3 Example 3 - Quadcopter Model considering Rotation and Translation

Example 3 is a combination of examples 1 and 2, extrapolated to two directions. In example 1, the quadcopter could move upwards and downwards. Now, it can move to the sides as well. The quadcopter can rotate, too.

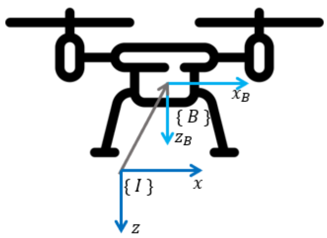

Figure 42: Example 3 - The Quadcopter Model (adapted from 3.3.4)

The frames of reference that were considered are shown in Fig.(42). Now, the quadcopter rotates and translates in two different directions : x and z. The inertial frame of reference has now two components

$$I = \begin{bmatrix} x_I \\ z_I \end{bmatrix} \tag{50}$$

as well as the body frame of reference

$$B = \begin{bmatrix} x_B \\ z_B \end{bmatrix} \tag{51}$$

Let $R(\theta)$ be the rotation matrix

$$R(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \tag{52}$$

A rotation of the body expressed in the inertial frame of reference is given by

$$I = R(\theta)B \Leftrightarrow \tag{53}$$

$$\Leftrightarrow \begin{bmatrix} x_I \\ z_I \end{bmatrix} = R(\theta) \begin{bmatrix} x_B \\ z_B \end{bmatrix} \tag{54}$$

3.3.1 Dynamics of the Quadcopter

The forces applied to the previous model were the same as before —the forces each one of the propellers applies, f_1 and f_2, and the force of gravity f_g.

Thus, the dynamics of this model can be written as

$$J\ddot{\theta} = b(f_1 - f_2) \tag{55}$$

$$m \begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ mg \end{bmatrix} - R(\theta) \begin{bmatrix} 0 \\ f_1 + f_2. \end{bmatrix} \tag{56}$$

where the states

$$\begin{bmatrix} x, & \dot{x}, & z, & \dot{z}, & \dot{\theta}, & \theta \end{bmatrix}.^T \tag{57}$$

describe the position of the quadcopter and the control inputs

$$\begin{bmatrix} u_1, & u_2 \end{bmatrix}.^T \tag{58}$$

correspond to the differential and common modes.

The differential mode corresponds to the signals that have opposite direction and is defined by

$$u_1 = f_1 - f_2, \tag{59}$$

while the common mode corresponds to the signals that follow the same direction and is given by

$$u_2 = f_1 + f_2. \tag{60}$$

3.3.2 Discretization of the Model

To discretize the model, the Direct Multiple Shooting method will be implemented, where both the control inputs and the states are the optimization variables.

Considering the states,

$$X = \begin{bmatrix} x, & \dot{x}, & z, & \dot{z}, & \dot{\theta}, & \theta \end{bmatrix}.^T \tag{61}$$

and the control inputs

$$U = \begin{bmatrix} u_1, & u_2 \end{bmatrix}.^T \tag{62}$$

$$\dot{X}(t) = f(t, X, U) = \begin{pmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ \dot{z}(t) \\ \ddot{z}(t) \end{pmatrix} = \begin{pmatrix} \dot{\theta}(t) \\ \frac{b}{J}u_1(t) \\ \dot{x}(t) \\ \frac{1}{m}sin(\theta(t))u_2(t) \\ \dot{z} \\ g - \frac{1}{m}cos(\theta(t))u_2(t) \end{pmatrix} \tag{63}$$

Hence, the discretized equations are

$$\begin{aligned} \theta(k+1) &= \theta(k) + T_s\dot{\theta}(k), \\ \dot{\theta}(k+1) &= \dot{\theta}(k) + T_s\frac{b}{J}u_1(k), \\ x(k+1) &= x(k) + T_s\dot{x}(k), \\ \dot{x}(k+1) &= \dot{x}(k) + T_s\frac{1}{m}sin(\theta(k))u_2(k), \\ z(k+1) &= z(k) + T_s\dot{z}(k), \\ \dot{z}(k+1) &= z(k) + T_s[g - \frac{1}{m}cos(\theta(k))u_2(k)], \end{aligned} \tag{64}$$

3.3.3 Cost Function

$$J = \sum_{i}^{H}[(x(k+i) - x_{final})^2 + (\dot{x}(k+i) - x_{dfinal})^2 + (z(k+i) - z_{final})^2 + (\dot{z}(k+i) - \dot{z}_{final})^2 +$$
$$(\theta(k+i) - \theta_{final})^2 + (\dot{\theta}(k+i) - \dot{\theta}_{final})^2 + u_1^2(k+i-1) + u_2^2(k+i-1)]. \tag{65}$$

3.3.4 Formulation of the NMPC Problem

The optimization formulation for the non linear model predictive control (NMPC) problem is given by

$$
\begin{aligned}
&\underset{u_1,u_2,x,\dot{x},z,\dot{z},\theta,\dot{\theta}}{\text{minimize}} \quad J \\
&\text{subject to} \\
&\qquad x(k+i+1) = x(k+i) + T_s\dot{x}(k)), \\
&\qquad \dot{x}(k+i+1) = \dot{x}(k+i) + \frac{T_s}{m}sin(\theta(k))u_2(k), \\
&\qquad z(k+i+1) = z(k+i) + T_s\dot{z}(k), \\
&\qquad \dot{z}(k+i+1) = z(k+i) + T_s[g - \frac{1}{m}cos(\theta(k))u_2(k)], \\
&\qquad \theta(k+i+1) = \theta(k+i) + T_s\dot{\theta}(k), \\
&\qquad \dot{\theta}(k+i+1) = \dot{\theta}(k+i) + T_s\frac{b}{J}u_1(k), \qquad i = 1, \cdots, H
\end{aligned}
\tag{66}
$$