Equation for Estimating Poisson SPKD

Inter-Spike Interval (ISI) Distribution

For a homogeneous Poisson process with rate r (in Hz), the inter-spike intervals (ISIs) between successive spikes are exponentially distributed. This distribution characterizes the time between consecutive spikes.

Probability Density Function (PDF)

The PDF of the exponential distribution is given by:

$$f(t) = re^{-rt}, \quad t \geq 0 \tag{1}$$

where:

- t is the time interval between spikes,
- r is the spike rate (mean number of spikes per second),
- $f(t)$ gives the probability density of an interval of duration t.

This distribution allows us to model the relative timing between spikes in one train and potential matches in the other.

Distribution of Nearest-Spike Differences

To compute the expected cost of matching spikes across two spike trains, we must understand the distribution of the time difference Δt between a spike in one train and its nearest spike in the other. Suppose both spike trains are independently generated from a homogeneous Poisson process with the same rate r.

Fix a spike at time $t = 0$ in spike train 1. The key quantity of interest is the distance to the *nearest* spike in spike train 2, which is also a Poisson process of rate r. Because spike train 2 is memoryless and translation-invariant, the probability of finding a spike within any time window depends only on the size of that window.

For a Poisson process of rate r, the probability that there are no spikes in an interval of length x is e^{-rx}. Therefore, the probability that there are no spikes in spike train 2 within a distance Δt on either side of 0 is $\mathrm{e}^{-2r\Delta t}$. Consequently, the probability that the nearest spike occurs within time Δt (either before or after the reference spike) is the complement:

$$F(\Delta t) = 1 - \mathrm{e}^{-2r\Delta t}.$$

Differentiating this cumulative distribution function yields the probability density function:

$$f(\Delta t) = \frac{\mathrm{d}}{\mathrm{d}\Delta t} F(\Delta t) = 2r\,\mathrm{e}^{-2r\Delta t}, \qquad \Delta t \geq 0.$$

This is the exponential distribution with rate $2r$, reflecting that the minimum distance to events in a Poisson process from a fixed point in time is governed by the spacing of points in both directions.

Expected Cost of Matching Two Spikes

The cost of aligning two spikes separated by a time difference Δt in the Victor–Purpura metric is

$$\text{cost}(\Delta t) \;=\; \min\bigl(2,\, q\,\Delta t\bigr),$$

which reflects a linear shift cost up to a maximum threshold of 2 (the combined cost of deleting one spike and inserting another).

Expected Cost over Nearest–Spike Time Differences

When evaluating the Victor–Purpura cost of aligning spikes across two trains, we must account for the temporal distance between a spike in one train and its nearest match in the other. Since both spike trains are independent realizations of homogeneous Poisson processes with rate r, the time Δt to the nearest spike in the opposing train follows an exponential distribution with rate $2r$, as derived previously. This distribution captures the likelihood of encountering a nearby match and serves as the weighting function in the expected cost computation. Integrating the Victor–Purpura cost over this distribution yields a closed-form estimate of the average matching cost. $F(\Delta t) = 1 - \mathrm{e}^{-2r\Delta t}$, so its probability density function is

$$f(\Delta t) \;=\; 2r\,\mathrm{e}^{-2r\Delta t}, \qquad \Delta t \geq 0.$$

The expected cost of matching a spike is therefore

$$\mathbb{E}[\text{cost}_{\text{match}}] \;=\; \int_0^\infty \min\bigl(2,\, q\,\Delta t\bigr)\, 2r\,\mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t. \tag{2}$$

We split the integral at the transition point where $q\,\Delta t = 2$, i.e. $\Delta t = 2/q$, to handle the linear and constant portions separately:

$$\mathbb{E}[\text{cost}_{\text{match}}] \;=\; 2q\,r \int_0^{2/q} \Delta t\,\mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t \;+\; 4r \int_{2/q}^\infty \mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t.$$

Evaluating the Integrals

Integral 1. We evaluate $2q\,r \int_0^{2/q} \Delta t\,\mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t$ via the substitution $u = 2r\,\Delta t$. Then $\Delta t = u/(2r)$ and $\mathrm{d}\Delta t = \mathrm{d}u/(2r)$, and the limits transform as $u = 0$ when $\Delta t = 0$ and $u = 4r/q$ when $\Delta t = 2/q$. Hence

$$2q\,r \int_0^{2/q} \Delta t\,\mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t = 2q\,r \int_0^{4r/q} \frac{u}{2r}\,\mathrm{e}^{-u}\,\frac{\mathrm{d}u}{2r} = \frac{q}{2r} \int_0^{4r/q} u\,\mathrm{e}^{-u}\,\mathrm{d}u.$$

Using the identity $\int_0^a u\,\mathrm{e}^{-u}\,\mathrm{d}u = 1 - (a+1)\mathrm{e}^{-a}$, we obtain

$$2q\,r \int_0^{2/q} \Delta t\,\mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t = \frac{q}{2r}\Bigl[1 - \Bigl(\tfrac{4r}{q} + 1\Bigr)\mathrm{e}^{-4r/q}\Bigr].$$

Integral 2. The tail integral is a simple exponential:

$$4r \int_{2/q}^\infty \mathrm{e}^{-2r\,\Delta t}\,\mathrm{d}\Delta t = 4r \cdot \frac{\mathrm{e}^{-4r/q}}{2r} = 2\,\mathrm{e}^{-4r/q}.$$

Final result. Combining the two parts gives

$$\boxed{\mathbb{E}[\text{cost}_{\text{match}}] = \frac{q}{2r}\Bigl[1 - \Bigl(\tfrac{4r}{q} + 1\Bigr)\mathrm{e}^{-4r/q}\Bigr] + 2\,\mathrm{e}^{-4r/q}}$$

which simplifies to the more compact form

$$\mathbb{E}[\text{cost}_{\text{match}}] = \frac{q}{2r}\Bigl(1 - \mathrm{e}^{-4r/q}\Bigr).$$

From Pairwise Cost to Total Victor–Purpura Distance

The expression derived earlier gives the expected Victor–Purpura cost of matching a single spike to its nearest spike in another Poisson train:

$$\text{match\_cost} = \frac{q}{2r}\left(1 - \left(\frac{4r}{q} + 1\right) \mathrm{e}^{-4r/q}\right) + 2\,\mathrm{e}^{-4r/q},$$

which we abbreviate by defining $b = 4r/q$, giving:

$$\text{match\_cost} = \frac{q}{2r}(1 - (b+1)\mathrm{e}^{-b}) + 2\mathrm{e}^{-b}.$$

To estimate the total Victor–Purpura distance between two independent Poisson spike trains of duration T, we combine this expected match cost with statistical estimates of how many spikes are matched versus unmatched. On average, each spike train contains rT spikes. However, due to random timing differences, not all spikes can be matched. To compute how many spikes are likely to be matched versus unmatched, we begin by recognizing that the total number of spikes in each train, N_1 and N_2, are random variables:

$$N_1 \sim \text{Poisson}(rT), \qquad N_2 \sim \text{Poisson}(rT).$$

The number of matches is at most $\min(N_1, N_2)$, since each match is one-to-one. Therefore, the number of *unmatched* spikes across both trains is $|N_1 - N_2|$. That is, any excess spikes in one train over the other must remain unmatched.

The difference $D = N_1 - N_2$ follows a Skellam distribution with mean zero and variance $2rT$, because it is the difference of two independent Poisson variables. A known property of this distribution gives the expected absolute difference as:

$$\mathbb{E}[|N_1 - N_2|] = \sqrt{\frac{4rT}{\pi}}.$$

This expression tells us that, on average, $\sqrt{4rT/\pi}$ spikes remain unmatched between the two spike trains.

Each unmatched spike contributes a cost of 1 (for insertion or deletion), while each matched pair contributes the expected match cost derived earlier. The number of matched pairs is therefore:

$$\mathbb{E}[\text{matched pairs}] = \mathbb{E}\left[\min(N_1, N_2)\right] = rT - \frac{1}{2}\sqrt{\frac{4rT}{\pi}}.$$

Final Expression for Total Expected Cost

Combining the expected cost of matched and unmatched spikes, the total expected Victor–Purpura distance is:

$$\mathbb{E}[\text{VP total}] = \left(rT - \tfrac{1}{2}\sqrt{\tfrac{4rT}{\pi}}\right) \cdot \text{match\_cost} + \sqrt{\tfrac{4rT}{\pi}}.$$

This gives the expected VP distance in absolute terms. It can be normalized per spike or per second depending on the application.