MATHEMATICAL MODELS IN THE ECONOMICS OF RENEWABLE RESOURCES*

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Abstract. Three classes of models of renewable resource economics are discussed in this paper: profit-maximizing models, competitive equilibrium models, and cooperative equilibrium models. A variety of complexities, both biological and economic, are also discussed, including: irreversibility problems, disaggregated biological models, delay effects, and stochastic models. Questions deserving further research are raised.

1. Introduction. In 1931 the noted mathematical economist Harold Hotelling published an article on "The Economics of Exhaustible Resources" [31]. The mathematical technique underlying Hotelling's article consisted of several nontrivial applications of the calculus of variations. Perhaps anticipating some unfamiliarity with such methods, the author noted that "Problems of exhaustible assets cannot avoid the calculus of variations, including even the most recent researches in this branch of mathematics." At the time, natural resources was an unfashionable topic among economists, and Hotelling's article was largely ignored until the 1973 energy crisis, which led to an upsurge of interest in the field [59].

Renewable resources were, if anything, even more neglected than exhaustible resources, at least from the theoretical point of view. This may have been due again in part to unfamiliarity with the necessary variational methods. With the development of optimal control theory and dynamic programming, however, familiarity with such methods is nowadays widespread; economists for example have used these methods extensively in economic growth models and other dynamical problems regarding the accumulation (or dis-cumulation) of capital assets.

In this article I shall review a number of dynamic economic models of renewable resource exploitation, concentrating on the case of the fishery. The recent establishment of 200-mile "economic zones" by many coastal states has resulted in the national appropriation of a large fraction of the world's marine fishery resources. It might be hoped that this appropriation would lead to more rational management of these important resources, although experience suggests that *mismanagement* could actually be facilitated by resource appropriation. At any rate, a well-developed theory of renewable resource economics seems necessary, if not sufficient, for intelligent management of fisheries and other resources.

I shall begin with an exceedingly simplified model of the fishery, introducing complexities one at a time in a gradual approach towards realism. The simpler models have been described fully elsewhere [19], and I shall dwell upon some of the more recent developments.

2. A fishery model. The following model was originally developed by M. B. Schaefer [54] as a management tool for the Eastern Tropical Pacific Tuna Fishery:

$$(2.1) \qquad \frac{dx}{dt} = G(x) - qEx$$

where

$x =$ biomass of fish population,

$t =$ time, $\quad G(x) =$ biological net growth rate,

$q =$ catchability coefficient, $\quad E =$ fishing effort.

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Some of these terms may require explanation. "Biomass" is clear enough. The growth function $G(x)$ incorporates certain generalized assumptions of density dependent rates of birth, growth, and mortality in the (unexploited) population. In the original Schaefer model, $G(x)$ was specified in "logistic" form

$$(2.2) \qquad\qquad G(x) = rx(1 - x/K)$$

where r, K are positive parameters called the "intrinsic growth rate" and the "carrying capacity" respectively.

The term

$$(2.3) \qquad\qquad h = qEx$$

in (2.1) represents the rate of mortality imposed by the fishery, i.e. the rate of catch, corresponding to a given input of fishing "effort" E. In the case of the tuna fishery, effort is defined as a certain standardized measure of the number of vessels operating at a given time (standardization being necessary to account for variations in effectiveness of different sizes and types of vessels). It is clear that the form of (2.3) is predicated on an assumption of random (and independent) search for randomly distributed fish. In actual fisheries these assumptions are always more or less violated, and fishery scientists have been greatly concerned with problems of bias in catch-effort data [27].

The Schaefer model depends on three parameters (r, K, q); statistical estimation techniques were devised by Schaefer [55]; see also Schnute [56]. The Schaefer model has since been employed in the management of many other commercial fisheries.

The above model can be summarized by the simple system diagram shown in Fig. 1. The fish population is represented by a single state variable, x; effort E represents the input, or control variable, while catch rate h represents the output of the fishery system. In real life, of course, these variables are usually highly disaggregated, and this leads to severe problems of interpretation and application of the Schaefer model. Some of these problems will be taken up in later sections of the paper.
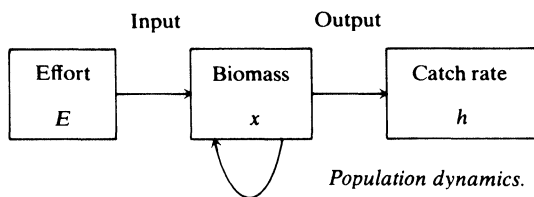

FIG. 1.

Assume for the moment that $G(x)$ is given by (2.2). Then for any fixed level of effort E, (2.1) has a unique stable equilibrium $x \geq 0$ given by

$$(2.4) \qquad\qquad x = \begin{cases} K(1 - qE/r) & \text{if } qE < r, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding equilibrium catch rate is

$$(2.5) \qquad\qquad h = qKE(1 - qE/r) \quad \text{if } qE < r.$$

Note that this has a maximum at $qE = r/2$, with $x = K/2$; this is the famous "maximum sustained yield" (MSY) solution for the Schaefer model. Although much of fisheries management has been—and still is—based on this objective, the MSY criterion has

been strongly criticized by economists [26], and also by some biologists [36]. In the next section we will investigate an economically oriented management objective.

Regarding stability, let us note that the equilibrium solution given by (2.4) is (globally) stable if effort is assumed constant. However, if instead the catch rate h is kept constant,

$$\frac{dx}{dt} = G(x) - h, \qquad h = \text{constant},$$

then the system undergoes a bifurcation exactly at the MSY solution $\hat{h} = \max G(x)$. Thus, for $h < \hat{h}$ there are two steady states, one stable and the other unstable. These states coalesce for $h = \hat{h}$ and disappear for $h > \hat{h}$, the population then being driven to extinction [9]. Management policies that specify an MSY objective (as most have done in the past) may thus ironically have a built-in predilection towards instability.

It is of some interest to study the behavior of the model

$$(2.6) \qquad \frac{dx}{dt} = G(x) - h(x, E)$$

under alternative forms of the growth function $G(x)$ and of the "catch equation" $h = h(x, E)$. For example, a detailed model of the tuna purse-seine fishery [22] led to a catch equation of the form

$$h(x, E) = \frac{qEx}{aE + bx}$$

with $a, b, q > 0$. With the usual logistic $G(x)$, this model leads (under suitable parameter combinations) to "catastrophic" fishery collapses as effort E increases. This is also the case for the traditional catch equation (2.3) if the growth function $G(x)$ is nonconcave [19, Ch. 2].

3. Maximization of profit in fishery management. Suppose now that full rights to exploitation of the fish population are allocated in perpetuity to a single owner. If this owner is a private firm (the objectives of government ownership may be less clear), traditional economic theory asserts that the objective of exploitation will be the maximization of the present value of discounted net economic revenue. Letting $E = E(t)$ denote the effort input (which we allow to vary over time), we can express this present value as

$$(3.1) \qquad PV = \int_0^\infty \alpha(t)\pi(x, E, t)\, dt$$

where

$$\alpha(t) = \text{discount factor}, \qquad \pi(x, E, t) = \text{net revenue flow}$$

and PV stands for "present value." Maximization of (3.1) subject to the differential equation (2.1) and additional constraints (including $x \geq 0, E \geq 0$) is a variational, or "optimal control" problem [51]. Although necessary conditions (Euler–Lagrange, Pontryagin, etc.) are easily written down, analytic determination of the optimal policy is not possible in general unless additional simplifying assumptions are made.

A simple but instructive specialization [15], [19], [23] arises if

$$(3.2) \qquad \alpha(t) = \exp(-\delta t)$$

and

$$\pi(x, E, t) = ph - cE$$
$$= pqEx - cE$$

(3.3)

where δ, p, c are positive constants representing the discount rate, the price of fish, and the cost of effort, respectively. Equation (3.1) now becomes

(3.4)
$$PV = \int_0^\infty e^{-\delta t}(pqx - c)E \, dt.$$

Substituting for E from (2.1) and integrating by parts, we obtain

(3.5)
$$PV = \int_0^\infty e^{-\delta t}\left\{\left(p - \frac{c}{qx}\right)G(x) - \delta z(x)\right\} dt + z(x(0))$$

where

(3.6)
$$z(x) = \int_{x_\infty}^x \left(p - \frac{c}{qu}\right) du.$$

Here x_∞ is a fixed biomass level, which we specify below.

Solution of the maximization problem for (3.5) is now almost self-evident: the biomass level x should be adjusted (via E) so that the integrand

(3.7)
$$\left(p - \frac{c}{qx}\right)G(x) - \delta z(x) \quad \text{is maximized.}$$

Moreover, the adjustment from $x(0)$ to this optimal x should take place as rapidly as possible. (See [19, Ch. 2] or [60] for the simple rigorous proof of this assertion; cases can arise in which the solution is more complicated than described here.)

Condition (3.7) has a clear economic interpretation. Note that for $dx/dt = 0$ we have for the first term:

$$\left(p - \frac{c}{qx}\right)G(x) = ph - cE,$$

which is simply the sustained economic revenue (or economic "rent") when harvesting maintains the biomass level x. If we specify x_∞ by the condition

(3.8)
$$p - \frac{c}{qx_\infty} = 0,$$

the quantity $z(x)$ in (3.6) is seen to represent the "capitalized value" of the fishery resource at biomass level x. The second term $\delta z(x)$ in (3.7) is then the interest on this "capital." The optimization criterion (3.7) thus calls for a stock level x that maximizes the sustained economic yield net of interest on "invested capital."

To an economist, the marginal form of (3.7) may be more meaningful:

(3.9)
$$\frac{1}{\delta} \frac{d}{dx} R_s(x) = p - \frac{c}{qx},$$

where $R_s(x)$ denotes sustainable economic yield. This equation involves economic concepts that go back (at least) to J. M. Keynes, for whom the left side would represent "marginal user cost." Equation (3.9) may also be written in the form

(3.10)
$$G'(x) - \frac{c'(x)G(x)}{p - c(x)} = \delta \quad \left(\text{with } c(x) = \frac{c}{qx}\right).$$

The left side of this equation represents the "own rate of interest" of the resource asset x; see [19] and [23].

Some special cases of our optimization rule may be noted. First if $\delta = 0$ the rule calls simply for the maximization of sustained economic rent; this criterion had long been recommended by fisheries economists [26], apparently without recognizing the implicit assumption of zero discounting. Secondly, if $c = 0$ the rule becomes

$$G'(x) = \delta$$

which is known as the Ramsey–Robinson–Cass–Samuelson–Solow modified golden rule of capital accumulation. Finally, if both $c = \delta = 0$ the rule reduces to the MSY criterion $G'(x) = 0$.

Inclusion of the costs of exploitation in criteria for the optimal management of resources is by now a widely accepted principle. The same cannot be said for the discounting of future revenues; economists themselves have argued extensively about the social justification of time discounting (see [29], [49] for the resources setting). What often comes as a surprise to the uninitiated is the extreme sensitivity of many resources-oriented decisions to the choice of discount rate. To quote an example outside the fishery setting, it has been estimated that raising the U.S. Army Corps of Engineers' discount rate from $2\frac{5}{8}\%$ to 8% would have resulted in rejection of 80% of its projects in 1962 [49].

For the logistic model (2.2), (3.9) can be solved for x:

$$(3.11) \qquad x = \frac{K}{4}\left\{ \left(\frac{c}{pK} + 1 - \frac{\delta}{r} \right) + \left[\left(\frac{c}{pK} + 1 - \frac{\delta}{r} \right)^2 + \frac{8c\delta}{pKr} \right]^{1/2} \right\}.$$

This formula shows that the optimal biomass level x is a function of two ratios, the cost/price ratio c/p, and the "bionomic growth ratio" r/δ. The smaller is either of these ratios, the more heavily will the resource be exploited. Also, if the bionomic growth ratio is small ($r < \delta$), then the optimal level of exploitation will be highly sensitive to the discount rate. Slow-growing biological resources are particularly subject to "overexploitation" by profit-maximizing agencies [15]. For example, using available biological and economic data, I have estimated that the Antarctic blue whales ($r \approx 4\%$) would be "optimally" depleted to approximately 20% of their original abundance, by a profit maximizing whaling industry using a rather conservative 10% discount rate [17]. (See § 6 below for further remarks on the whale "fishery.")

Could extinction ever be an "optimal" policy under the profit maximization criterion? Although this cannot happen for the Schaefer-logistic model with economic revenue given by (3.3) (since the cost of extinction is infinite under these assumptions), it definitely can occur under other conditions [6], [13], [14], [15], [57]. Obviously, however, a host of other issues should be raised before any realistic conclusion can be accepted as to the optimality of an extinction policy [24]. All we can say here is that if such issues are neglected, or considered irrelevant, then the exploiter of a renewable resource might deliberately choose to exhaust the resource completely. The chances of extinction may be much greater if the resource is not "owned" by a single agent, as we shall see in the next section.

4. The common-property fishery. In the discussion so far we have imagined a single "owner" who possesses complete rights to the resource in question. Although such a situation may be realistic for certain renewable resource stocks (privately owned forests, cattle herds, etc.), it is wildly unrealistic for most marine fishery resources, which have virtually always been exploited by numerous competing fishermen, often from several nations. This fact was fully recognized in the first study

of fishery economics, by H. S. Gordon [26]; see also the famous article by G. Hardin [28] on "The tragedy of the commons."

Gordon's principal result can be stated briefly as follows. In an unregulated common-property fishery, effort will always expand to such a level that economic rent becomes zero. In other words, the economic rent that the fishery is capable of yielding becomes "dissipated" as a consequence of an excessive application of effort. The reasoning behind this conclusion is simple enough: if rent is positive it will pay additional fishermen to enter the fishery; conversely if rent becomes negative, it will pay some fishermen to leave the fishery.

To formulate this mathematically, let us continue with the model of the previous section. Economic rent per unit of effort is given by

$$\pi(x, E)/E = pqx - c.$$

The rent-dissipating biomass level is thus $x = x_\infty$, as given in (3.8). Gordon referred to this as the "bionomic equilibrium" of the common-property fishery, and contrasted it to the "economic optimum" given by the rent-maximization criterion $R_s'(x_0) = 0$. It is evident that $x_\infty < x_0$ (and $E_\infty > E_0$). If positive discounting is used, then we have $x_\infty < x_\delta < x_0$ where x_δ is given by (3.10).

Economists have put forward various recommendations towards overcoming the rent-dissipation problem in common-property resources. For example, the government might impose a tax on the catch (or on effort), and otherwise allow free entry. If τ represents the tax per unit catch, then the perceived rent to the fishermen, per unit effort, equals

$$(p - \tau)qx - c.$$

The government can select the tax τ so that the modified bionomic equilibrium occurs at any desired biomass level x. (It could even subsidize the fishery, if desired, by selecting a negative τ). Thus the government can act as a sole owner, taxing the fishery to benefit the public coffers according to whatever criterion it wishes to employ.

Since fishermen would continue to receive nothing (and less than nothing if they refuse to quit) from such a taxation scheme, it is not surprising that fishermen have shown little actual enthusiasm for it. No government seems as yet to have had the nerve to impose significant "rent-collecting" taxes on its fishing industry. Indeed, the Fishery Conservation and Management Act of 1976 (U.S. Public Law 94–265) expressly prohibits the collection of revenues from domestic fishermen. Some economists have seen this act as a social disaster [12].

An alternative suggestion now receiving considerable support is that entry to the fishery should be restricted. This has the opposite distributional implications to the taxation proposal, since any rent generated will now be retained by the licensed fishermen. The question arises, will it work? More precisely, what other requirements, if any, are necessary in order that rent be generated?

Let n denote the number of fishermen licensed to exploit the resource. The case $n = 1$ has been dealt with in the previous section. Next let $n = 2$; we suppose, to start with, that two agents are awarded unlimited (and equal) rights to the resource. The problem can then be formulated as a nonzero-sum two-person differential game.

Namely, let $E_i(t)$ denote the effort expended by agent i ($i = 1, 2$), and assume that each agent has some finite effort capacity:

$$0 \le E_i \le E_i^{max} \quad (i = 1, 2).$$

The differential equation for the fish population becomes

(4.1) $$\frac{dx}{dt} = G(x) - q_1 E_1 x - q_2 E_2 x.$$

Each agent attempts to maximize his discounted economic yield:

(4.2) $$PV_i = \int_0^\infty \exp(-\delta_i t) \cdot (p_i q_i x - c_i) E_i(t)\, dt.$$

In general, the two agents may have different prices, costs, and discount rates.

Of course the outcome depends on whether the game is assumed to be played under competitive or cooperative conditions. First let us consider the Nash competitive solution [48], which is defined as a pair of feedback control policies $E_i^* = E_i^*(x)$ satisfying

(4.3) $$PV_1(E_1^*, E_2^*) \geqq PV_1(E_1, E_2^*) \quad \text{for all } E_1,$$

and similarly for PV_2. Thus if player 2 uses policy E_2^*, then the first player's optimal policy is E_1^* and vice versa. (In most cases there will exist effort policies $E_i^\dagger$ such that

$$PV_i(E_1^\dagger, E_2^\dagger) > PV_i(E_1^*, E_2^*), \qquad i = 1, 2,$$

but such policies will only be employed if both players agree to cooperate in limiting their effort levels—see below.)

Let $x_i^\infty = c_i/p_i q_i$ denote the zero-rent biomass level for agent i. In a sense, x_i^∞ is an inverse measure of agent i's efficiency, since lower costs, or higher prices, or greater technological effectiveness, lead to lower values of x_i^∞.

THEOREM. *Assume for example that $x_1^\infty < x_2^\infty$ (i.e. agent 1 is more efficient than agent 2). Then, provided $E_i^{\max}$ ($i = 1, 2$) are sufficiently large, a Nash competitive solution exists in which agent 2 is ultimately eliminated from the fishery, and agent 1 maintains equilibrium at $x = \min(x_1^*, x_2^\infty)$, where x_1^* is the optimal biomass level for agent 1 alone.*

Unless the efficiency advantage enjoyed by agent 1 is large, we will have $x_2^\infty < x_1^*$. The presence of a competitor thus forces agent 1 to exploit the resource more intensively than he would otherwise desire. (In fact the mere *threat* of competition can be shown to have the same effect.) If the two agents are equally efficient then the biomass equilibrium $x_1^\infty = x_2^\infty$ will result in zero economic rent to both agents. The full tragedy of the commons thus arises with as few as two competitors.

The above result extends in an obvious way to the case of n agents: the most efficient agent will tend to eliminate the rest, but then will be forced to exploit the resource at a level that may be far below the optimum, in order to ensure that his competitors do not return.

In economics texts it is usually taught that free competition is a good thing for society, partly because it ensures that the inefficient producers do not survive. But when significant externalities are present, as in most common-property resources, the benefits of free competition are by no means clear. Nevertheless, responsible critics seldom suggest that renewable resource assets should actually be turned over to monopolistic owners (the monopolist may have his own view on this!). Unfortunately, the above results throw considerable doubt also on the possible efficacy of a "limited entry" scheme (with $n \geqq 1$) in achieving optimal results. Unless the scheme is somehow devised to prevent each entrant from expanding his input of effort beyond the optimum, most of the potential economic rent will probably continue to be dissipated.

Of course, it is conceivable that the licensed fishermen could themselves foresee advantages in a cooperative approach, and would then call upon the government for help in achieving an efficient fishery. For example, suppose that a limited-entry scheme is accompanied by a quota system assigning a certain catch quota h_i to each agent. (Effort quotas could also be considered, but would probably be much more difficult to control in practice.) If the h_i are suitably chosen, each agent will harvest in perpetuity at his assigned rate, and economic rent should thus be generated. Who receives the rent will obviously depend on the method used to allocate quotas. Such allocated catch quotas have been recommended by a leading fishery economist, F. T. Christy [11]. On an international scale, they have been used for several decades by the whaling industry (which has always been aware of its best economic interests), and were used from 1973–1977 in the ICNAF (International Commission for Northwest Atlantic Fisheries) area.

In this respect, it would be of interest to study cooperative games based on the above model. In situations where the number of competing firms is relatively small, and where substantial economic benefits would result from cooperative agreements (both of which conditions seem to have been valid for the whaling industry), it can be expected that such agreements will be favored. Another example would be a fish population that crosses the boundaries of the fishing zones of two adjacent countries. A recent paper of Munro [46] applies the Nash theory of cooperative two-person games to this situation.

As an example of several types of problem considered by Munro, let us consider the case of two neighboring countries exploiting a common stock. For simplicity, imagine that the countries have the same prices, technologies, and costs, but take different views of the future as reflected by different discount rates δ_i. Thus the two countries have objective functions

$$PV_i = \int_0^\infty e^{-\delta_i t}\, (pqx - c)E_i(t)\, dt, \qquad i = 1, 2.$$

The state equation is still (4.1)

Let us suppose first that the countries have already agreed, by treaty perhaps, to a fixed division of the catch, so that

$$E_1(t) = \alpha E(t), \qquad E_2(t) = (1 - \alpha)E(t)$$

where $0 \leqq \alpha \leqq 1$ and $E(t)$ is the total effort. (This assumption will be relaxed later.) An effort policy $E(t)$ is said to be (weakly) Pareto optimal if, for any other feasible policy $E_1(t)$ we have either $PV_1(E_1) \leqq PV_1(E)$ or $PV_2(E_1) \leqq PV_2(E)$—i.e. if no other policy can improve both countries' return from the fishery.

For $0 \leqq \beta \leqq 1$ consider the weighted total PV:

(4.4) $$PV(E; \beta) = \beta PV_1(E) + (1 - \beta)PV_2(E).$$

In a sense β represents a "bargaining parameter"—if $\beta = 1$ the views of country 1 completely dominate the outcome, whereas if $\beta = 0$ those of country 2 do so. If $E_\beta(t)$ is a maximizing policy for $PV(E; \beta)$, it is clear that E_β is also Pareto optimal. The converse would be true provided that the set $\{(PV_1(E), PV_2(E))\}$ is a convex subset of the plane. I expect this to be the case, but have not succeeded in proving it. But even if there do exist some Pareto optima not represented by policies E_β, it seems reasonable to restrict attention to these latter policies.

Writing out (4.4), we have

(4.5)
$$PV(E; \beta) = \int_0^\infty \{\alpha\beta e^{-\delta_1 t} + (1-\alpha)(1-\beta) e^{-\delta_2 t}\}(pqx - c)E(t)\, dt.$$

$$= \int_0^\infty \psi(t)(pqx - c)E\, dt.$$

Expression (4.5) is a slight generalization of (3.4), with the discount factor $\exp(-\delta t)$ replaced by a more complicated function $\psi(t)$. It is easy to show that the optimal "equilibrium" biomass now becomes a function of time $x_\beta^*(t)$, which is determined by the following generalization of (3.10):

(4.6)
$$G'(x^*) - \frac{c'(x^*)G(x^*)}{p - c(x^*)} = -\frac{\dot{\psi}(t)}{\psi(t)}.$$

As in the simpler model, the adjustment from $x(0)$ to the so-called "singular path" $x_\beta^*(t)$ should be carried out as rapidly as possible.

Suppose, for example, that $\delta_1 < \delta_2$, so that country 1 is more "conservationist-minded" than country 2 (at least as far as fisheries go). It can be seen from (4.5) that, for $0 < \alpha, \beta < 1$, we have

$$\delta_1 < -\dot{\psi}(t)/\psi(t) < \delta_2,$$

and moreover, $-\dot{\psi}/\psi \downarrow \delta_1$ as $t \to \infty$. Hence

$$x_\beta^*(t) \uparrow x_1^* \quad \text{as } t \to \infty,$$

where x_1^* is the optimal biomass level as perceived by country 1. The compromise reached thus has the property that exploitation will initially be more intensive than desired by country 1 (reflecting the higher discount rate preferred by country 2), but in the long run country 1's conservationist views will prevail (because country 2 is less concerned with the future than country 1). Of course there is a slight fly-in-the-ointment here: country 2 may have an incentive to renegotiate the contract later on! (Could this model also be telling us something about the never-ending conflict between exploiters and conservationists?)

Finally, suppose α is not fixed in advance, but is treated as an additional control variable. According to the maximum principle [51], $\alpha = \alpha(t)$ and $E = E(t)$ must at each time t maximize the Hamiltonian

$$H = \{\alpha\beta e^{-\delta_1 t} + (1-\alpha)(1-\beta) e^{-\delta_2 t}\}(pqx - c)E + \lambda\{G(x) - qEx\}.$$

By linearity, we have $\alpha(t) = 0$ or 1, for all t. Munro [46] shows that in fact

$$\alpha(t) = \begin{cases} 0 & \text{for } 0 \leq t < t_1, \\ 1 & \text{for } t > t_1, \end{cases}$$

where of course t_1 depends on β. Thus country 1 agrees to let country 2 exploit the resource exclusively for a while, provided country 2 eventually gives up all claims to it.

A third possibility is that transfer payments may be agreed upon as part of the original bargaining process. If this is allowed, it can be seen that, since country 1 places a higher value on the asset, it will be able to completely buy out country 2's rights to the resource. Although such an outcome may seem unrealistic, let it be noted that in fact a treaty of this sort—cash payments in lieu of exploitation rights—was signed between the U.S., Canada, Japan, and Russia in 1911, relating to fur seal stocks in the Northern Pacific. This treaty is still in force today.

5. Fixed costs. From the biological viewpoint the Schaefer fishery model is seriously oversimplified in many respects. Likewise, the economic components introduced in the previous sections are also greatly oversimplified. In the remainder of the paper I shall consider a few of the many possible extensions of the basic model. These extensions have been selected either for their usefulness in applications, or because they seem to illustrate important principles.

One of the most unsatisfactory aspects of the analysis so far, it would seem, lies in the way that costs of fishing have been modeled. This becomes particularly clear if one attempts to employ the model to estimate optimal effort and yield in a particular fishery. Many of today's fishing vessels are extremely expensive (e.g. approximately $10 million for a modern tuna purse seiner). Such vessels have high fixed costs, which are incurred regardless of the actual amount of fishing undertaken. Moreover, these vessels are often highly specialized, and cannot be used effectively in other fisheries. (On the other hand, such vessels may be highly mobile. Many tuna seiners, for example, customarily exploit both Pacific and Atlantic tuna populations each year.)

Let $K(t)$ denote the number of vessels in the fleet at time t. Smith [58] proposed the following ad hoc model of the open-access fishery in which the movement of vessel capital is assumed to be "sticky:"

$$(5.1) \qquad \frac{dx}{dt} = G(x) - qKx,$$

$$(5.2) \qquad \frac{dK}{dt} = \theta(\pi(x, K)),$$

where $\theta(z) = \theta_1 z$ or $\theta_2 z$ for $z \geqq 0$ and $z \leqq 0$ respectively. With $\pi(x, K) = (pqx - c)K$, as in (3.3), this model clearly has an equilibrium at $x = x_\infty$, $K = K_\infty = G(x_\infty)/qx_\infty$—i.e. corresponding to Gordon's bionomic equilibrium. If G is convex the equilibrium is stable, although the approach may involve damped oscillations, with $K(t)$ being alternatively above and below the break-even level K_∞. If G is not convex, the equilibrium can be unstable, leading to limit-cycle oscillations [32]. In any event, Smith's model does describe a real phenomenon, inasmuch as many fisheries do appear to suffer, at least temporarily, from levels of capitalization well in excess of the break-even level (not to mention the optimal level).

What can be said regarding the movement of vessel capital in and out of the fishery for the case of profit maximization? To address this question we consider the following model [20]:

$$(5.3) \qquad \frac{dx}{dt} = G(x) - qEx,$$

$$(5.4) \qquad 0 \leqq E \leqq K,$$

$$(5.5) \qquad \frac{dK}{dt} = I(t) - \gamma K,$$

$$(5.6) \qquad PV = \int_0^\infty e^{-\delta t}\{(pqx - c)E(t) - \pi I(t)\}\, dt.$$

Here K represents the number of vessels in existence at time t, while E denotes the number actually employed in the fishery. The rate of investment in vessels, $I(t)$, is considered as an additional control variable. The constant $\gamma \geqq 0$ represents deprecia-

tion rate, and $\pi > 0$ the cost of one vessel. Thus c now denotes variable cost of fishing, and fixed costs can be assumed to be included in the term $\pi I(t)$; see (5.7) below.

Suppose first that vessels can be freely shifted in and out of the fishery, so that $I(t)$ is unrestricted. We can then substitute from (5.5) into (5.6) and integrate by parts:

$$PV = \int_0^\infty e^{-\delta t}\{(pqx - c)E - \pi(\delta + \gamma)K\}\,dt + K(0).$$

Clearly an optimal policy will never have $E < K$, because the unused vessels can be disposed of for a cash payment. Hence we can assume that

$$(5.7) \qquad PV = \int_0^\infty e^{-\delta t}\{pqx - c - \pi(\delta + \gamma)\}E(t)\,dt.$$

This is now exactly the same as (3.4), except that costs of fishing have been broken down into variable costs c plus fixed costs (interest plus depreciation) $\pi(\delta + \gamma)$. As before, let x^* denote the optimal equilibrium biomass level, as determined from (3.9) but with c replaced by $c + \pi(\delta + \gamma)$.

At the opposite extreme, suppose that investment in vessels is completely irreversible (except for losses through depreciation), so that

$$(5.8) \qquad I(t) \geqq 0.$$

This apparently minor change in our optimization problem has a profound influence on the solution. For example, suppose we begin at time $t = 0$ with an underexploited stock: $x(0) > x^*$. (The Antarctic whale stocks in 1925, or the Peruvian anchovies in 1960, might be taken as representative examples.) Clearly the profit maximizing policy will involve the purchase of an appropriate fleet of vessels, so as to run down the "excess" biomass—presumably at a very profitable rate to begin with. Indeed, if vessels could simply be "rented" temporarily with no permanent capital cost, there would be no limit to the number that would be desired. But under our assumption that capital costs do have to be paid, and cannot be recovered, it becomes plain that some finite fleet size will prove optimal. It can be shown, moreover, that this optimal level of initial capitalization always exceeds the long run optimum K^*, even allowing for depreciation [20]. Consequently, when the optimal biomass level x^* is reached, there will exist an excess of vessel capacity. By assumption the excess vessels cannot be disposed of—at least not at a postitive price. One might suppose that, inasmuch as the extra vessels would otherwise lie around going to waste, they might as well be utilized. In fact this is correct, provided the stock is not depleted to a level where variable costs are no longer recovered.

Thus the initial "overcapitalization" (which is in fact optimal) gives rise to a situation where temporary depletion of the biomass is also optimal. Over the very long run, however, the fleet capacity will decline (owing to depreciation) to a level that is too small to harvest the depleted population at the sustainable yield level. But new capital investment is not warranted unless $x \geqq x^*$. Ultimately, therefore, equilibrium will be established at (x^*, K^*), with all biological and economic forces (population growth, discounting, depreciation, capital costs, and variable costs) in equilibrium.

This scenario (a rigorous proof appears in [20]) suggests that, under positive discounting, the *optimal* (profit-maximizing) development of a renewable resource industry bears some qualitative resemblance to the development expected to occur under open-access conditions. Quantitatively, however, the open-access development will probably involve more vessels, leading to a greater degree of depletion of the

resource, and to a lower long-run equilibrium stock level, than the optimal development requires.

In both the biological and economic literature pertaining to fisheries, excess capacity has often been proclaimed as an obvious sign of mismanagement and waste of resources. No doubt this assertion is valid in most cases. Our analysis, however, shows that the development of excess capacity will probably actually be a rational move from the viewpoint of the exploiting industry. This adds a new dimension to the difficulties inherent in fishery management.

Economics is a treacherous subject; bringing in one additional factor can often reverse the conclusions of an accepted theory. In this section I have tried to show how factors such as discounting and irreversibility of investment can affect the course of an optimal development policy. Obviously many other factors ought to be considered also—changing technology and demand, segregation of harvesting, processing, and marketing sectors, questions of distributional equity, uncertainty, and so on. Some of these have been addressed in the literature [10], [23], [38], but others await future research.

Let us now turn briefly to some of the biological complexities that have so far been swept under the rug.

6. Some biological considerations. The Schaefer model treats the fish population as if it were a single homogeneous entity. But real fish populations possess a great deal of complex structure, including spatial structure (fish swim), age structure (fish are born, reach maturity, spawn, grow old, and die), and even genetic structure. Moreover, these structures usually vary over time, as do the physical and biological environments in which the fish exist.

Besides all this, most commercial fisheries exploit more than a single population [1]—and conversely, most fish populations are exploited by more than a single fishery [3]. The resulting bio-economic structure may be one of overwhelming complexity.

With all this real-world complexity before us—and we are still pretending that the real world is deterministic and completely knowable!—it is hard to know where to begin looking for underlying general principles. It may be tempting to conclude that there are no general principles in such a complex field, and that each instance must be studied completely on its own. But in the long run, this is clearly a self-defeating attitude. Science always has been, and always will be involved with the search for general principles that simplify and unify our perception of nature in all its complexity.

Be that as it may, a danger does exist that academic theorists will go to extremes in developing ever more refined models in complete isolation from reality. This is particularly problematic when mathematicians invade another field. Whenever authors seem to be more impressed by technique than by the insights and practical applications of their results, it is a clear sign of theoretical overdevelopment of their field.

What questions, then, should we ask in a more general framework of renewable resource economics? First, it should be determined to what extent the principles enunciated for the simple models extend to more complex cases. Secondly, we should search for new general phenomena in complex models, particularly models associated with practical situations. There is still much work worth doing in addressing these problems in the area of natural resource management.

As an example of the first type of consideration, let us consider the following discrete-time version of our biological model [13], [14]:

(6.1) $$x_{k+1} = G(x_k)$$

or, with periodic harvesting

(6.2) $$x_{k+1} = G(x_k - h_k).$$

In the fisheries literature these are called spawner-recruit models; x_k represents recruitment to the fishery in year k (in numbers of fish), and $e_k = x_k - h_k$ represents escapement following the harvest h_k. Escapement e_k is assumed to constitute the spawning population, which then determines the subsequent recruitment $x_{k+1} = G(e_k)$. Note that the model does not allow for overlapping generations. In fisheries its main application has been in the management of the Pacific salmon fisheries [53], since Pacific salmon die upon completion of the spawning cycle. (The cycle period varies from 3 to 5 years, depending on species, and k thus represents an appropriate cycle index in this case.) The spawner-recruit function $G(x)$ is usually taken to be of the Ricker form [53]:

(6.3) $$G(x) = x \, e^{a(1-x/K)},$$

where K = carrying capacity and a is a growth parameter.

The continuous-time growth model $dx/dt = G(x)$ and its discrete-time analog (6.3) are of course similar in many respects. One difference worth noting, however, is the possibility of oscillatory behavior of the discrete-time model, which of course involves a built-in delay mechanism. For example, the Ricker model (6.3) gives rise to "chaotic" fluctuations in x_k if a is large [44]. Also, whereas one usually imagines that harvesting is likely to destabilize a biological population, the discrete-time model can obviously be stabilized by harvesting—even by sustained-yield harvesting. Indeed, this is sometimes put forward as a reason why game-animal populations "ought" to be hunted!

The following generalization of (6.2) allows for a maturation period between birth and recruitment to the adult population [18]:

(6.4) $$x_{k+1} = \sigma \cdot (x_k - h_k) + G(x_{k-n} - h_{k-n}).$$

Here x_k denotes the total adult breeding population prior to the annual harvest h_k, while $\sigma \in (0, 1)$ represents the survival rate of adults. As before $G(e)$ represents the number of young fish produced by escapement e, but these fish do not recruit to the breeding stock until $n+1$ years later. (If maturity is reached in one year, the model reduces to (6.2).)

Stability of (6.4) for the case of sustained-yield harvesting (h_k independent of k) can be analyzed locally by linearization [18], or globally by means of a suitable Lyapunov function, if known [25]. (See Brauer [8] for stability theorems for continuous-time delay models of harvested populations.)

To obtain an economic model for profit maximization, let us introduce the objective function

(6.5) $$PV = \sum_{n=0}^{\infty} \alpha^n \pi(x_n, h_n),$$

where $\alpha = 1/(1+i)$ is the annual discount factor, and where $\pi(x_n, h_n)$ denotes net economic return from harvest h_n and recruitment x_n. For the general model of (6.4), the necessary condition for an optimal equilibrium solution (x^*, h^*), $h^* = G(x^*) - (1-\sigma)x^*$, can be shown formally to be [14], [18]:

(6.6) $$\{\sigma + \alpha^n G'(x^*)\} \frac{\pi_h + \pi_x}{\pi_h} = \frac{1}{\alpha}.$$

With some care, this can be identified as the appropriate analog to (3.10) for the Schaefer model.

Equation (6.4) has been employed as a model of harvested baleen whale populations [33]. In [17] I employed biological and economic data to estimate the solution to (6.6) for various interest rates. These results were compared with those obtained from the Schaefer model using the same data—see Table 1. Note that, for both models, the optimal (profit-maximizing) policy is extremely sensitive to the interest rate; the interest rates shown in the table are within "normal" rates ordinarily employed, and may even understate the case for highly risky ventures such as whaling. Note also that the results for the two models are practically indistinguishable, suggesting that the use of simple models may be appropriate even when they seem seriously to misrepresent nature. (See Brauer and Sanchez [9] for a similar instance involving sandhill cranes.) This latter possibility is one that needs to be stressed by applied mathematicians, since many contemporary complex (and expensive) ecosystem models may actually be quite unnecessary for the purposes at hand [42]. An example is the current trend towards the development of incredibly complex multi-species fishery models, in most cases without any hope of achieving reasonable predictability. Since the qualitative behavior of these models is also usually incomprehensible, their value seems subject to serious question.

TABLE 1

Optimal Antarctic fin whale escapement levels: (a) delay model; (b) Schaefer model. (See [17].)

Annual interest rate	Optimal escapement level	
	(a)	(b)
0	219,302 whales	220,000 whales
0.01	195,249 whales	190,229 whales
0.03	148,752 whales	140,008 whales
0.05	111,752 whales	105,342 whales
0.10	70,335 whales	68,811 whales
0.15	58,183 whales	57,682 whales
0.20	52,958 whales	52,777 whales

The delayed-recruitment model (6.4) is a special case of the following nonlinear age-structured model:

(6.7)
$$
\begin{aligned}
x_k^1 &= F(x_{k-1}^2, \cdots, x_{k-1}^n), \\
x_k^2 &= \psi^2(x_{k-1}^1), \\
&\vdots \\
x_k^n &= \psi^n(x_{k-1}^{n-1}),
\end{aligned}
$$

where x_k^i represents the number of animals of age i present in year k. This is the so-called nonlinear Leslie model of population dynamics [37]; the linear version has been extensively studied, and has found applications in ecology and demography. Harvesting strategies for the linear model have been described by Beddington and Taylor [5]; the general case has recently been resolved by W. Reed (unpublished).

A special form of the above model was employed by Beverton and Holt [7] in connection with North Sea demersal (bottom-dwelling) species. In its simplest version,

the Beverton–Holt model can be expressed as:

$$(6.8) \qquad x_k^1 = R_k,$$

$$(6.9) \qquad x_k^j = \exp\left(-Z_{k-1}^{j-1}\right)x_{k-1}^{j-1}, \qquad j \geqq 2,$$

where Z_k^j denotes total mortality, $Z_k^j = M_k^j + F_k^j$, the sum of natural and fishing mortality, and where R_k denotes recruitment of 1-year fish in year k. If $w(t)$ denotes the average weight of fish of age t, then the total biomass of the k-cohort (stock of fish recruiting in year k) is given by

$$(6.10) \qquad B_k^j = x_k^j \, w(j).$$

Biologically speaking, recruitment R_k must of course depend on past levels of the breeding stock. In practice, however, since almost nothing of a quantitative nature is known about stock-recruitment relationships for most cohort fisheries, R_k is treated as a constant (average) value, or possibly as a random variable. Because of the extremely high fecundity of many species (a single female cod lays up to 15 million eggs per year!), recruitment may not decline significantly except under very intensive exploitation, so the above assumption may not be wholly unrealistic.

Even so, heavy fishing can, and often does lead to severe losses of productivity in cohort fisheries. The average size of fish in the catch declines as the level of fishing effort increases. Older cohorts may almost completely disappear from the population. This leads not only to declining catches, but also usually to increased variability of the catch. In many fisheries, therefore, regulations are imposed to restrict the capture of small fish. In some cases, sub-legal specimens must be returned to the sea, while in other cases the size of nets or hooks are restricted in an attempt to allow small fish to escape.

The profit-maximizing fishery manager would presumably adjust both effort and the size restriction in an optimal manner over time. The corresponding dynamic optimization model is obviously a complex one, and analytic solutions are available only for special cases [19, Ch. 8]. In the unregulated open-access fishery, overexploitation results both from excessive fishing effort and from the capture of immature fish. In these respects, the major predictions derived from the Schaefer model are seen to extend to the Beverton–Holt cohort model. Also, in the latter model, time discounting can have a significant influence on the optimal size restriction, as well as on optimal sustained effort and yield [19].

An important characteristic of many cohort fisheries is that the size distribution of fish caught is very difficult to control. The same holds true for the mix of species caught in many multi-species fisheries. For example, in the Northwest Atlantic one of the most valuable species is the haddock, which at present is severely depleted. Yet rehabilitation of the haddock stocks is now extremely difficult to achieve, because of the fact that haddock are frequently taken as a bi-catch in fisheries directed primarily at other species. Reducing catches in these other fisheries to allow the haddock to recover is neither economically nor politically feasible. Thus we get the apparently paradoxical result that optimal fishing may result in the depletion (even possibly the extinction) of the most valuable species [19, Ch. 9]. This is a novel bioeconomic "principle" of considerable practical importance.

We have not yet discussed phenomena pertaining explicitly to the nonlinearity of complex ecosystems. As ecologists are now coming to realize [30], [45], these nonlinear phenomena can be extremely important, but also extremely difficult to predict with any degree of confidence. According to the current view [61], bifurcations

and "catastrophes" are to be considered the rule rather than the exception in nonlinear systems, particularly when exposed to some form of external stress. Many explicit bifurcation-inducing biological mechanisms have recently been noted [16], [22], [47], and doubtlessly more will be discovered. The economic implications of all this have yet to be studied in any detail.

The development of useful biological (and bioeconomic) models of an intermediate degree of complexity seems to be potentially one of the most fruitful areas for collaboration between applied mathematicians and other scientists.

7. Stochastic models. With the exception of an occasional remark, our analysis so far has been entirely of a deterministic nature. The theory becomes much more difficult, of course, when stochastic elements are introduced.

Stochastic models of population dynamics are fairly well established in mathematical biology [39], [43], [50], and a few authors have explored the implications of simple harvesting strategies in this setting [4]. Our ambition is somewhat greater, inasmuch as we wish to discuss questions of optimization in the stochastic setting. Unfortunately stochastic optimization theory is presently largely restricted to systems with linear dynamics [35]. However, some work has been done on discrete-time Markov-process optimization models of fishery exploitation [34], [38], [52]. For the continuous-time case, a recent paper of Ludwig [40] applies perturbation methods to a stochastic version of the Schaefer model:

$$(7.1) \qquad \frac{dx}{dt} = G(x) - H.$$

Writing $Y = \log x$, we transform (7.1) to:

$$(7.2) \qquad \frac{dY}{dt} = g(Y) - h,$$

where $g(Y) = G(x)/x$ and $h = H/x$. Noise is introduced by modifying (7.2) to:

$$(7.3) \qquad dY = [g(Y) - h]\, dt + dB,$$

where dB denotes the increment of Gaussian white noise (Brownian motion), with

$$(7.4) \qquad \mathscr{E}(dB) = 0, \qquad \mathscr{E}(dB^2) = \sigma^2.$$

(See Ludwig [40] for a heuristic justification for this form of the model.) In terms of the transformed equations, the profit-maximization objective becomes

$$(7.5) \qquad J = \mathscr{E}\left\{ \int_0^\infty e^{-\delta t} n(Y) h\, dt \right\},$$

where $n(Y) = x[p - c/qx]$.

What is the effect of introducing noise, as far as optimal harvest policies are concerned? In order to address this problem, Ludwig assumes that the noise level σ^2 is "small," and uses asymptotic methods. (Direct solution by numerical integration is used in [41].) Briefly stated, the results are as follows. First, the optimal policy is again a "bang-bang" control policy of the form

$$(7.6) \qquad H = H(Y; Y_\sigma^*) = \begin{cases} H_+ & \text{for } Y > Y_\sigma^*, \\ H_- & \text{for } Y < Y_\sigma^*, \\ g(Y_\sigma^*) & \text{for } Y = Y_\sigma^*. \end{cases}$$

Thus the optimal policy attempts to achieve an equilibrium at $Y = Y_\sigma^*$, but must adjust to stochastic changes in Y. The level Y_σ^* constitutes a switching point for the harvest policy. The value of Y_σ^* is equal to the value Y_0^* deduced from (3.9), which now becomes

$$(7.7) \qquad \frac{1}{\delta} \frac{d}{dY} \{n(Y)g(Y)\} = n(Y),$$

plus a "correction" dY_σ^* resulting from the noise. Ludwig derives an approximate expression

$$(7.8) \qquad dY_\sigma^* \sim \frac{\sigma^2}{2} \left\{ \frac{1}{g(Y_0^*) - H_+} + \frac{1}{g(Y_0^*) - H_-} \right\}.$$

What is the intuitive content of (7.8)? The simplest case arises when $H_- = 0$ and H_+ is very large:

$$(7.9) \qquad dY_\sigma^* \sim \sigma^2 / (2g(Y_0^*)).$$

In this case, the presence of noise implies a larger stock level Y_σ^* than would otherwise be optimal, the magnitude of the increase being larger as the noise level increases, and smaller as the biological growth rate increases. Because of the constraint $H \geq H_- = 0$, fluctuations of Y below Y_σ^* induce undesirable effects (no harvesting) which persist for some time. Fluctuations above Y_σ^*, on the other hand, are immediately correctible $(H_+ = \infty)$. To reduce the influence of the undesirable effects, a higher stock level Y is required.

The general case, (7.8), is similarly explained.

Note that the model treats the maximum and minimum harvest rates H_+, H_-, as given. Assuming $H_- = 0$, an interesting problem would be to extend the model to allow H_+ to be optimized. The question is, how much excess capacity (above $H = g(Y_0^*)$) should be built in order to take advantage of higher-than-average levels of recruitment? Deterministic models completely ignore such questions, which are, however, of considerable importance in practice.

8. Conclusion. As I have argued above, theoretical work is essential to the development of any scientific field, and "bioeconomics" is no exception to this rule. The problems discussed in this article by no means exhaust the variety of important questions that can arise. Economic aspects, for example, are worthy of much greater emphasis, particularly concerning the efficacy of various techniques for the management of common property resources (taxes, licenses, quotas, legal constraints, etc.). Decisions under uncertainty, and the cost-effectiveness of data collection and research, are additional problems deserving attention.

In this article I have not attempted to describe economic models pertaining in a quantitative sense to any specific renewable resource. Several such models have been developed; a few examples are listed in the bibliography [21], [22]. The study of specific practical models is an important adjunct to theoretical work; it indicates shortcomings of the theory and points to directions for fruitful research. Conversely, experience with practical problems tends to inhibit the pursuit of worthless abstractions. Unfortunately, bona fide practical problems have a tendency to be extremely difficult, especially in a field such as resource management, where science, economics, and politics become inextricably intermingled. Perhaps mathematical analysis can help to clarify some of the complexity inherent in such problems.

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