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A note on an extension of Schaefer's model

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ABSTRACT

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Schaefer's model on the exploitation of a single species fish community obeying the logistic law of growth has been modified by adopting an alternative functional form for the harvest rate. The stability of the nontrivial steady state is discussed. The existence of a bionomic equilibrium is proved. The problem for optimal exploitation of the fishery is studied by using variational calculus. Some results are numerically computed to make a comparative study of Schaefer's model and the present one.

INTRODUCTION

The model formulated and solved by the biologist M.B. Schaefer (1957) is still used extensively in commercial fisheries. He considered a fish population obeying the logistic law of growth and adopted the catch-per-unit effort hypothesis to represent the rate of harvesting. The particular functional form used by Schaefer to denote the catch per unit time is based on the assumptions that fishing consists of a random search for fish and that all fish in the stock are equally likely to be captured. These assumptions, however, are not realistic.

Clark (1979a) suggested an alternative and realistic functional form for the catch rate and used it in the model of the purse-seine-tuna fishery (1979b).

The present study seeks modification of the Schaefer's model in the light of the catch-rate-function suggested by Clark (1979a). The steady state is found out and its stability is examined. The existence of the bionomic equilibrium is proved. The problem of optimal harvesting is studied by using the Euler–Lagrange principle of calculus of variations. Taking four sets of values of the parameters involved in the problem, some results are numerically computed to compare the present model with the Schaefer's model.

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THE PROBLEM

We consider here exploitation of a biological resource consisting of a single species population governed by the differential equation:

$$\frac{dx}{dt} = F(x) - h(t) \tag{1}$$

where $x(t)$ is size of the population at time 't', $F(x)$ net natural growth rate of the population, and $h(t)$ the rate of harvesting the resource population.

It is now assumed that the population obeys the logistic law of growth (Verhulst, 1838) as in the Schaefer's model (Schaefer, 1957). Thus the appropriate form of $F(x)$ becomes:

$$F(x) = rx\left(1 - \frac{x}{K}\right) \tag{2}$$

where 'r' denotes the biotic potential, and 'K' is the environmental carrying capacity of the resource population. We may also interpret 'r/K' as the coefficient of intraspecific competition within the population.

Regarding the rate of harvesting, Schaefer (1957) made the *catch-per-unit-effort hypothesis* (Clark, 1976) and took $h(t) = qEx$, where 'E' denotes the harvesting effort and 'q' (constant) the *catchability coefficient*. This particular form of $h(t)$ implies random and independent search for harvesting as well as equal availability of all individuals (in the resource population) to the harvesting set-up. These restrictions are not generally compatible with situations existing in actual fisheries. An alternative form of $h(t)$ has already been tried by Clark (1979b) and it is adopted here. We thus seek modifications of the Schaefer's (1957) model with the following functional form of $h(t)$:

$$h(t) = \frac{qEx}{aE + bx} \tag{3}$$

where 'a' and 'b' are positive constants.

First we discuss the salient features of this function:

- For $0 < E \ll 1$ and fixed x:

$$h(t) = \frac{qEx}{bx}\left(1 + \frac{aE}{bx}\right)^{-1} = \frac{qEx}{bx}\left(1 - \frac{aE}{bx} + \cdots\right) \doteqdot \frac{qE}{b}$$

- For $E \gg 1$ and fixed x:

$$h(t) = \frac{qx}{a + \dfrac{bx}{E}} \to \frac{qx}{a}$$

– For fixed E and $x \to \infty$:

$$h(t) = \frac{qE}{\dfrac{aE}{x} + b} \to \frac{qE}{b}$$

Thus $h(t)$ possesses a saturation effect with respect to harvesting effort 'E' as well as resource abundance 'x'. For smaller efforts levels, the rate of harvesting is proportional to the effort only for a fixed size of the resource population.

The differential equation (1) now becomes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - \frac{qEx}{aE + bx}; \qquad x(0) = x_0 \tag{4}$$

STABILITY

The two equilibrium points of the system (4) are $x = 0$ and $x = \bar{x}$, where:

$$\bar{x} = \frac{-\left(\dfrac{aE}{b} - K\right) + \sqrt{\left(\dfrac{aE}{b} - K\right)^2 - \dfrac{4KE}{b}\left(\dfrac{q}{r} - a\right)}}{2} \tag{5}$$

In order that $\bar{x}$ may be real and positive, we must have:

$$\left(\frac{aE}{b} - K\right)^2 - \frac{4KE}{b}\left(\frac{q}{r} - a\right) > 0$$

i.e. .

$$\left(\frac{aE}{b} + K\right)^2 > \frac{4KEq}{br} \tag{6}$$

and

$$\left(\frac{aE}{b} + K\right)^2 - \frac{4KEq}{br} > \left(\frac{aE}{b} - K\right)^2$$

i.e.

$$a > \frac{q}{r} \quad \text{or} \quad \frac{r}{q} > \frac{1}{a} \tag{7}$$

Condition (7) implies that reciprocal of the parameter 'a' must be less than the biotechnical productivity BTP[1] (Clark, 1976) of the resource population. Subsequent discussions are based on this basic condition.

[1] BTP (biotechnical productivity) of biological species is defined as the ratio of its intrinsic growth rate (biotic potential) to its catchability coefficient.

A little manipulation with (7) leads to the inequality:

$$\frac{aE}{b} + K > \frac{qE}{br} + K$$

Comparing this inequality with (6), we have either:

$$\left(\frac{qE}{br} + K\right)^2 \geqslant \text{ or } < \frac{4KEq}{br}$$

The first possibility yields:

$$\left(\frac{qE}{br} - K\right)^2 \geqslant 0$$

which is always true.

The above inequality leads in turn, to the two possibilities:

(a) $\frac{1}{a} < \frac{r}{q} \leqslant E/(Kb)$

or

(b) $\frac{1}{a} < \frac{r}{q} \geqslant E/(Kb)$

In (a), the BTP has nonzero upper and lower bounds. In (b), the lower bound of the BTP is $E/(Kb)$.

Now we examine the behaviour of the nontrivial equilibrium point '$\bar{x}$' by using a perturbation technique. Let us put $x = \bar{x}(1+u)$ (where $u \ll 1$) in equation (4).

After a little simplification (neglecting powers of u higher than the first), equation (4) becomes:

$$\frac{du}{dt} = r\lambda u \tag{8}$$

where 'λ' is the quadratic expression:

$$1 - \frac{2\bar{x}}{K} - \left(1 - \frac{\bar{x}}{K}\right)^2 \frac{ar}{q} \tag{9}$$

The solution of equation (8) is given by:

$$u = u_0 \, e^{\lambda t} \tag{10}$$

where

$$u_0 = u(0)$$

Now, the roots of the quadratic equation:

$$1 - \frac{2\bar{x}}{K} - \left(1 - \frac{\bar{x}}{K}\right)^2 \frac{ar}{q} = 0$$

are given by:

$$\frac{(1 - ar/q) \pm \sqrt{1 - ar/q}}{ar/(qK)}$$

These roots are complex by virtue of (7). Hence the quadratic expression 'λ' given by (9) has the same sign as that of '$\bar{x}^2$' in 'λ'. Thus 'λ' is negative so that $u(t) \to 0$ as $t \to \infty$.

Hence the equilibrium point '$\bar{x}$' (given by equation 2) is asymptotically stable.

BIONOMIC EQUILIBRIUM

The levels of population and effort at which bionomic equilibrium occurs are jointly given by the equations:

$$\left.\begin{aligned} &\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{qEx}{aE + bx} = 0 \\ &\text{and} \\ &\frac{pqEx}{aE + bx} - cE = 0 \end{aligned}\right\} \qquad (11)$$

where 'p' is the price per unit biomass and 'c' is the expenditure per unit effort of fishing. We consider 'p' and 'c' to be constants.

Let the solution of equation (11) be (x_∞, E_∞). The second equation of (11) yields:

$$E_\infty = \frac{pq - bc}{ac} x_\infty \qquad (12)$$

Again, using the second equation in the first, we get:

$$E_\infty = \frac{prx_\infty}{c}\left(1 - \frac{x_\infty}{K}\right)$$

Thus:

$$\frac{pq - bc}{ac} x_\infty = \frac{prx_\infty}{c}\left(1 - \frac{x_\infty}{K}\right)$$

and this leads to the solution:

$$x_\infty = \frac{K}{pra}[p(ar - q) + bc] > 0 \quad \text{by} \quad (7) \qquad (13)$$

This shows that the open-access exploitation never leads to the extinction of the fish population.

Using (13) in (12), we find:

$$E_\infty = \frac{K(pq-bc)}{a^2cpr}[p(ar-q)+bc] > 0 \qquad \text{always by} \quad (7) \tag{14}$$

provided $pq > bc$.

We know that the natural productivity of the resource is maximized at $\tilde{x} = \frac{1}{2}K$.

Now:

$$x_\infty - \tilde{x} = \frac{K}{pra}[p(ar-q)+bc] - \frac{K}{2} = \frac{K}{2pra}[p(ar-q)+2bc-pq]$$

or

$$x_\infty - \tilde{x} = \frac{x_\infty}{2} - \frac{(pq-bc)K}{2pra}$$

i.e.

$$x_\infty - \tilde{x} < \frac{x_\infty}{2}$$

provided $pq > bc$.

Thus:

$$x_\infty < 2\tilde{x} = K \qquad \text{if} \quad pq > bc$$

or

$$x_\infty > K \qquad \text{if} \quad pq < bc$$

Hence the bionomic equilibrium is established at a population level higher (or lower) than the natural equalibrium level 'K' according as $pq <$ (or $>$) bc. However, 'E_∞' becomes negative when $pq < bc$. Thus the bionomic equilibrium occurs only at a population level lower than 'K' and for this to happen, the cost–price ratio (c/p) must be less than (q/b). This establishes the relative dependence of the parameter 'b' on the cost–price ratio. It may further be proved that the bionomic equilibrium is established at a level higher (lower) than '$\tilde{x}$' according as c/p is greater (less) than $(2q-ar)/(2b)$.

OPTIMAL HARVEST POLICY

Now we want to determine the harvest policy that maximizes the present value of the total revenue, which is given by equation (14):

$$\text{PV} = \int_0^\infty e^{-\delta t}\left[\frac{pqEx}{aE+bx} - cE\right] dt \tag{14}$$

where 'δ' is the discount rate. We have from (4):

$$E = \frac{bx(F(x) - \dot{x})}{a\dot{x} - aF(x) + qx} \tag{15}$$

Eliminating 'E' from (14) with the help of (15), we get:

$$\mathrm{PV} = \int_0^\infty \mathrm{e}^{-\delta t}\left[p\{F(x) - \dot{x}\} - \frac{bcx\{F(x) - \dot{x}\}}{a\dot{x} - aF(x) + qx}\right]\mathrm{d}t$$

$$= \int_0^\infty \phi(x, \dot{x}, t)\,\mathrm{d}t, \text{ say}$$

Now:

$$\frac{\partial\phi}{\partial x} = \mathrm{e}^{-\delta t}\left[pF'(x) + \frac{abc\{F(x) - \dot{x}\}^2 - bcqx^2F'(x)}{\{a\dot{x} - aF(x) + qx\}^2}\right] \tag{16}$$

$$\frac{\partial\phi}{\partial\dot{x}} = \mathrm{e}^{-\delta t}\left[-p + \frac{bcqx^2}{\{a\dot{x} - aF(x) + qx\}^2}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi}{\partial\dot{x}}\right)$$

$$= \mathrm{e}^{-\delta t}\left[p\delta - \frac{bcq\delta x^2}{(a\dot{x} - aF + qx)^2} + \frac{2abcqx\{\dot{x}^2 - \dot{x}F - x\ddot{x} + x\dot{x}F'\}}{(a\dot{x} - aF + qx)^3}\right] \tag{17}$$

By virtue of the Euler–Lagrange equation:

$$\frac{\partial\phi}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi}{\partial\dot{z}}\right)$$

for maximum, we have from (16) and (17):

$$\begin{aligned} &p[F'(x) - \delta][a\dot{x} - aF(x) + qx]^3 \\ &\quad + \left[abc\{F(x) - \dot{x}\}^2\right. \\ &\quad \left.- bcqx^2\{F'(x) - \delta\}\right][a\dot{x} - aF(x) + qx] \\ &\quad - 2abcqx\left[\dot{x}^2 - \dot{x}F(x) + x\dot{x}F'(x) - x\ddot{x}\right] = 0 \end{aligned} \tag{18}$$

Equation (18) is a nonlinear differential equation in x. Solution of this equation gives the optimal population level $x = x^*$. However it is not easy to solve equation (18) by analytical means. If we look for the optimal equilibrium population level, equation (18) reduces to the form:

$$F'(x) + \frac{abc\{F(x)\}^2}{p\{qx - aF(x)\}^2 - bcqx^2} = \delta \tag{19}$$

This has some resemblence, so far as its form is concerned, with the corresponding equation deduced by Clark (1976, §2.5) in the case of the harvest rate:

$$h = EG(x)$$

where $G(x)$ is a nondecreasing function of x.

Equation (19) is not amenable to any straightforward biological, economic or bionomic interpretation. We, therefore, resort to numerical computation using (19) and other equations in order to make a comparative study of the present model and the Schaefer's model.

NUMERICAL COMPUTATIONS

Taking four sets of values of the parameters, the results for the Schaefer's model and the present model are shown side by side in Table 1. The symbols used in Table 1 are as follows:

$\bar{x}_S$ steady-state population for the Schaefer's model
$\bar{x}_N$ steady-state population for the new model
$x_{\infty S}$ bionomic equilibrium level of population for the Schaefer's model
$x_{\infty N}$ bionomic equilibrium level of population for the new model
$E_{\infty S}$ bionomic equilibrium level of effort for the Schaefer's model
$E_{\infty N}$ bionomic equilibrium level of effort for the new model
x_S^* optimal equilibrium level of population for the Schaefer's model
x_N^* optimal equilibrium level of population for the new model.

The x-values have all been rounded up to the nearest whole number. As the computations show, the relative difference between any two values of $\bar{x}_N$ or $x_{\infty N}$ is always much less than unity.

The following points emerge out of the results in Table 1:

(a) The level of steady-state population in the new model is comparatively higher than that for the Schaefer's model.

(b) The bionomic steady-state population of the new model is much higher than that of the Schaefer's model.

(c) The bionomic steady-state level of effort in the new model is much less than that in the Schaefer's model.

(d) The optimal equilibrium level of population is the same in both the models.

However, the validity of these results stands on the assumed parameter values. In reality, the biological parameters 'K', 'r' will depend on the fish species considered. The technical parameters 'a', 'b', 'q' have to be properly estimated for the fishing instruments to be used. The economic parameters 'c', 'p', 'δ' have to be estimated from the current market conditions.

TABLE 1

K	r	q	a	b	δ	c	p	$\bar{x}_S$	$\bar{x}_N$	$x_{\infty S}$	$x_{\infty N}$	$E_{\infty S}$	$E_{\infty N}$	x^*_S	x^*_N
500	0.1	0.005	21	0.005	0.04	1	13	375	499	15	499	19.38	1.43	150	150
500	0.2	0.005	21	0.005	0.05	2	13	438	499	31	499	37.53	0.65	188	188
500	0.15	0.005	20	0.005	0.04	3	16	417	499	38	499	27.75	0.54	183	183
500	0.15	0.007	18	0.005	0.04	1	13	383	499	11	499	20.95	2.38	183	183

DISCUSSION

We have considered the Schaefer's (1957) model for the exploitation of a single species fish community which obeys the logistic law of growth and which is subjected to the catch-rate $h = qEx$. The underlying assumptions for this functional form of the catch-rate are that harvesting consists of a random search for fish and that all fish in the stock are equally likely to be captured. These assumptions seem to be unrealistic. We have modified the Schaefer's model by using the alternative catch-rate function suggested by Clark (1979a).

It is found that the modified model possesses a unique nontrivial steady state $\bar{x}$ iff the reciprocal of the parameter 'a' is less than the biotechnical productivity (BTP) of the fish species. Moreover, BTP possesses finite bounds. It is then proved that $\bar{x}$ is asymptotically stable. The existence of a bionomic equilibrium is investigated. It is found that a bionomic equilibrium level of population is always lower than the carrying capacity of the unexploited population. Finally, the problem of determining the optimal (equilibrium as well as dynamic) population is discussed by using the Euler–Lagrange principle of the variational calculus. Lastly, some results for the Schaefer's model and the proposed model are computed using four artificial sets of values of the parameters involved in the problem.

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