

A SIMPLE NONLINEAR MODEL FOR THE EXPLOITATION OF RENEWABLE ECONOMIC RESOURCES

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Abstract—We examine the generalized L. von Bertalanffy growth model as a basis of bioeconomic models of renewable resource exploitation. It is shown that the model has a built-in protective mechanism that may be utilized to enhance conservation of the resource population in question. The model is also used to predict the dynamics of harvesting process involving common-property renewable resources. It is found that a much stronger economic condition than that obtained with the standard logistic is needed for biological overexploitation to occur.

1. INTRODUCTION

For decades, the quadratic growth curve, of which the standard logistic is a special case, has been proposed and used to derive harvesting policies for the management of renewable economic resources. The reasons usually given for the use of such growth curves include simplicity of form and the ease with which they can be fitted to growth data. However, as pointed out in [1, p. 171], mere fit of collected data does not prove a quadratic curve to be a law of growth in every case. In the case of the standard logistic, there are several transparent reasons why it is inappropriate for describing growth in animal or fish population biomass. For example, growth in biomass is a net of innumerable physiological processes but the parameters in logistic growth curve lack any physiological meaning. In addition, the logistic growth curve always predicts symmetry in growth which is quite incorrect for the large body of skewed growth data for fish and whale populations [2]. Indeed, it appears that the logistic growth equation has been used quite indiscriminately in the literature. It is often the case that the course of growth is different in different biological species. This calls for a growth model which is not only a realistic one for describing growth in animal or fish biomass but in addition possesses some degree of freedom that makes it applicable to different biological species.

In this paper we examine the problem of optimally harvesting a population using the generalized Bertalanffy growth equation because it is physiologically well-founded and therefore better suited for describing growth in animal or fish population biomass than any other known growth equation in the literature. Besides, it is well-known in the biology literature that the Bertalanffy growth curve fits a wide range of age–size growth data [1, 3]. It also possesses a degree of freedom (namely the surface parameter m , $0 < m < 1$) which makes it suitable for describing biomass growth in different animal, fish or whale populations and therefore useful for application to renewable resource management. Finally, the logistic equation of growth actually turns out to be a special case of the Bertalanffy one when certain restrictions are relaxed. Section 2 presents the generalized Bertalanffy model of growth and its variants. Section 3 examines two types of harvesting strategies in the generalized Bertalanffy model and discusses the implications of these applications. Section 4 examines the issue of stability of the harvested and unharvested population in the generalized Bertalanffy model. Section 5 discusses economic properties of density dependent harvesting in the generalized Bertalanffy growth model. Section 6 examines open-access harvesting in the generalized Bertalanffy growth model.

2. POPULATION BIOMASS DYNAMICS OF RENEWABLE ECONOMIC RESOURCES

Following L. von Bertalanffy [1], the equation

$$\dot{x} = \alpha x^m - \beta x^n \quad (1)$$

will be assumed to describe the growth of population biomass of a renewable economic resource over time, where $x(t)$ = aggregate biomass of animal, fish or whale population at time t , $\dot{x} = dx/dt$, the absolute growth rate and α , β , m , and n are system parameters representing the dependence of absolute growth rate on the two broad physiological processes of anabolism and catabolism. For population species whose growth is surface dependent, m has the value of two-thirds and for population species whose growth is mass dependent, m has the value of one. Population species whose growth rates are intermediate between surface and mass dependence will clearly have m values which satisfy $0 < m < 1$.

The limited growth or final size thesis implies the existence of a steady-state value \hat{x} which is approached in the growth process, and the growth rate \dot{x} is positive between $x = 0$ and $x = \hat{x}$. The steady-state value \hat{x} is obtained by setting the right-hand side of equation (1) to zero, yielding,

$$\hat{x} = \left(\frac{\alpha}{\beta} \right)^{\frac{1}{n-m}}. \quad (2)$$

By introducing the new variables

$$y = \frac{x}{\hat{x}}, \quad \tau = \beta t \quad (3)$$

and assuming $n = 1$, equation (1) now assumes the form

$$\dot{y} = y^m - y. \quad (4)$$

Equation (4) is a convenient reformulation of (1) for the purpose of making a comparison with other models of growth in the literature. It is also a convenient form to use if one is not concerned with empirical determination of the physiological parameters α and β .

Equation (4) possesses steady-state solutions $y = 0$, which is of no biological significance, and $y = 1$, which corresponds to the environmental carrying capacity of the resource in question. For $0 < y < 1$, $\dot{y} > 0$ and for $y > 1$, $\dot{y} < 0$. Thus the equilibrium solution $y = 1$ is globally stable. The graph of (4) is given in Fig. 1.

Equation (4) is Bernoulli-type with parameter m . Its unique solution is

$$y(t) = [1 - (1 - y_0^{1-m}) e^{-(1-m)(t-t_0)}]^{1/1-m}, \quad (5)$$

where y_0 is the initial aggregate biomass size in dimensionless units.

Equation (5) predicts the size of the population biomass at any time t . It is transparent from (3) and (5) that the resource population biomass approaches one exponentially as t approaches infinity. Hence the claim that the equilibrium biomass $y_e = 1$ is globally stable is justified.

Table 1 summarizes some well-known variants of equation (1). For $n = 1$ and $m > 1$, the predicted biomass level in (5) declines to zero exponentially and the resource population is doomed to extinction. Table 2 gives some population species and their approximate m values.

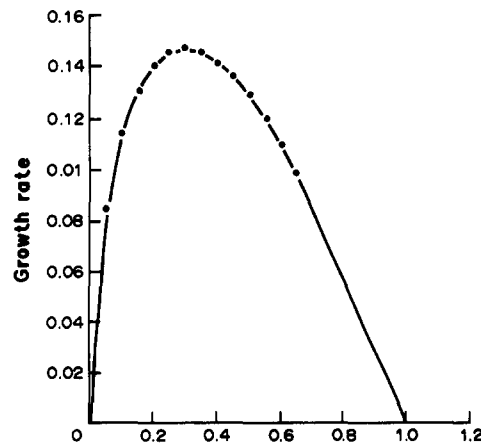


Fig. 1. Bertalanffy growth curve with $m = 2/3$; (●) biomass.

Table 1. Some variants of Bertalanffy equation and solutions

Exponents	Equation	Name	Solution
$m = 0, n = 1$	$\dot{x} = \alpha - \beta x$ $\alpha > 0, \beta > 0$	Schoener	$x(t) = \frac{\alpha}{\beta} + \left(x_0 - \frac{\alpha}{\beta}\right) \exp[-\beta(t - t_0)]$
$m = 1, n = 1$	$\dot{x} = (\alpha - \beta)x$	Malthusian	$x(t) = x_0 \exp[(\alpha - \beta)(t - t_0)]$
$m = 1, n = 2$	$\dot{x} = \alpha x - \beta x^2$	Logistic	$x(t) = \frac{\alpha x_0}{\beta x_0 + (\alpha + \beta x_0) \exp[-\alpha(t - t_0)]}$

3. HARVESTING MODELS

Suppose the biomass system described by equation (4) is subject to exploitation by man. Then the net growth rate is given by

$$\dot{y} = y^m - y - h(t), \quad (6)$$

where $h(t) > 0$, is the total biomass, in appropriate unit, removed by the harvester at time t and represents the control variable in system theoretic language.

In practice, the rate of removal $h(t)$ can assume many different forms, but in this article, only two cases are examined.

3.1. Sustainable yield harvesting

Under this scheme $\dot{y} = y^m - y - h = 0$, so that $h = y^m - y$. The maximum sustainable yield occurs at the level of biomass $y_{\text{MSY}} = m^{1/(1-m)}$ with the behavior

$$\lim_{m \rightarrow 1} y_{\text{MSY}} = e^{-1}.$$

The maximum sustainable yield is given by

$$h_{\text{MSY}} = (1 - m)m^{m/(1-m)}$$

with the behavior

$$\lim_{m \rightarrow 1} h_{\text{MSY}} = 0.$$

Thus with the Bertalanffy growth law, the maximum sustainable yield and the biomass level at which it is attained and functions of the population parameter m and can be predicted, given the knowledge of this parameter.

3.2. Constant effort harvesting

Constant effort or density-dependent harvesting strategy introduces the term Ey into equation (4). The resulting growth equation is

$$\dot{y} = y^m - y - Ey. \quad (7)$$

This equation is depicted in Fig. 2, where the asymmetric growth curve $f(y) = y^m - y$ rises and falls parabolically as the biomass y increases from zero to the carrying capacity, while losses due to harvesting increases linearly.

Table 2. Some population species and their approximate m values

Population	m value	Source of data
North sea plaice (<i>Pleuronectes platessa</i>)	2/3	[8]
<i>Serranus alexandrinus</i>	5/6	[9]
<i>Hippiglossus stenolepis</i>	10/11	[9]
<i>Sardinella jussieu</i>	16/25	[9]
New Zealand gekkonid	2/3	[3]
Elephant	3/4	[1]
Gulf of Carpentaria prawn	2/3	[10]

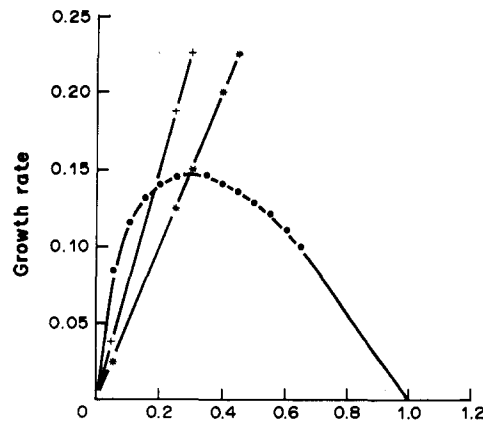


Fig. 2. Bertalanffy model with constant effort harvesting: (●) biomass, (*) moderate harvest, (+) large harvest.

Equation (7) has a unique nonzero equilibrium solution

$$y^*(m; E) = \left(\frac{1}{1 + E} \right)^{\frac{1}{1-m}}. \quad (8)$$

As expected and as has been revealed by studies of other models, (8) indicates that the effect of harvesting is to reduce the aggregate biomass level at equilibrium. The corresponding equilibrium yield is given by

$$Y(m; E) = E \left(\frac{1}{1 + E} \right)^{\frac{1}{1-m}}. \quad (9)$$

A numerical study of (8) indicates that the equilibrium biomass level is a decreasing function of both effort level and the population parameter m . A similar study of (9) indicates that the equilibrium yield in a Bertalanffy model is a nonsymmetric increasing function of the harvesting effort and a decreasing function of the population parameter m . The maximum sustainable yield is

$$Y_{\text{MSY}} = (1 - m)m^{m/(1-m)} \quad (10)$$

and is attained when the effort level is

$$E_{\text{MSY}} = (1 - m)m^{-1}. \quad (11)$$

These results emphasize the unique advantage of the Bertalanffy growth equation as a harvesting model. Some numerical aspects of these results are provided in Table 3.

We observe from Table 3 that E_{MSY} is a decreasing function of the population parameter m . Thus, the Bertalanffy model, unlike any other standard model in the literature possesses a built-in protective mechanism for the resource population under exploitation. The knowledge that a population under exploitation has a small (close to zero) or large (close to one) m value

Table 3. MSY values for various Bertalanffy growth curves

Growth curve with	MSY effort (E_{MSY})	MSY level (y_{MSY})	MSY yield (Y_{MSY})
$m = 1/4$	3.000	0.158	0.474
$m = 1/3$	2.000	0.192	0.384
$m = 1/2$	1.000	0.250	0.250
$m = 2/3$	0.500	0.296	0.148
$m = 3/4$	0.333	0.317	0.105
$m = 4/5$	0.250	0.328	0.082
$m = 5/6$	0.200	0.335	0.067
$m = 6/7$	0.167	0.340	0.057
$m = 7/8$	0.143	0.344	0.049
$m = 8/9$	0.125	0.346	0.043
$m = 9/10$	0.111	0.349	0.039

automatically determines the range of harvesting intensity that could be applied in harvesting the population in order to enhance conservation.

In general, populations with m values close to zero can withstand large harvesting pressures while populations with m values close to one are too sensitive to harvesting pressures and therefore need to be harvested cautiously. The findings here seem to support a conjecture that the E_{MSY} , given by (11), should be used as a constraint on rather than as a desideratum of exploitation [4]. In open-access situations where the Bertalanffy model is more reasonable to use, the information provided by the analysis of this section may be useful to a regulatory agency in the planning of allocated catch quotas to be given to prospective harvesters during a particular harvesting season.

4. STABILITY OF THE RESOURCE POPULATION

The stability of the resource population (unharvested and harvested) is an important concern in the management of renewable natural resources since the lack of it may lead to ultimate collapse of such populations. This realization motivates us to quantitatively examine the dynamic behavior of equations (4) and (7). A generally accepted quantitative measure of a system's stability is the system's time constant or characteristic return time in the sense that a small time constant or characteristic return time corresponds to a very stable system [5].

The linearized form of equation (4) about the asymptotically stable point $y_e = 1$ is

$$\dot{z} = -(1 - m)z, \quad (12)$$

with the speed of approach of a trajectory to an epsilon neighborhood of the asymptotically stable equilibrium state $z^* = 0$ equal $(1 - m)$. Hence the time constant or characteristic return time for the unharvested biomass system is

$$\text{CRT}|_{E=0} = \text{CRT}(0) = \frac{1}{1 - m}. \quad (13)$$

By a similar approach, the time constant or characteristic return time for the harvested biomass system described by equation (7) is

$$\text{CRT}(E) = \frac{1}{(1 - m)(1 + E)}. \quad (14)$$

It is transparent from (13) and (14) that the time constants for the unharvested and harvested population are increasing functions of the population parameter m . That is, populations with m values close to one are naturally unstable but may be stabilized through cautious harvesting. The time constant for a biomass system subject to a constant harvesting intensity E , relative to that of the unharvested biomass system is given by

$$\frac{\text{CRT}(E)}{\text{CRT}(0)} = \frac{1}{1 + E}. \quad (15)$$

Thus for a population biomass system whose growth follows the Bertalanffy growth law, its time constant or characteristic return time decreases steadily as the harvesting effort increases. At the MSY level, the characteristic return time for the harvested population is only m times as long as for the unharvested population. In contrast the logistic growth model leads to a characteristic return time that increases steadily with increasing harvesting effort. It can be shown that, as the harvesting effort approaches a certain critical value E^c , the population becomes sluggish taking longer time to return to a neighborhood of its asymptotically stable state. At the MSY level, the characteristic return time for the harvested population is twice as long as for the unharvested population. It follows that the stability of a population which follows a Bertalanffy growth equation can be enhanced through cautious harvesting while a population which follows the logistic growth equation can be destabilized by means of intensive harvesting. These results suggest that the logistic model of growth might be more appropriate to use in pest control problems than in the control of commercially exploited populations. Some numerical details of the preceding analysis for the standard logistic and Bertalanffy models are given in Table 4.

Table 4. Characteristic return times for unharvested and harvested populations which follow logistic and Bertalanffy growth laws

Growth curve	Characteristic return time for unharvested population CRT(0)	Characteristic return time for harvested population CRT(E)
Logistic	1.00	$\frac{1}{1-E}$
Bertalanffy with $m = 1/4$	1.33	$\frac{1.33}{1+E}$
$m = 1/3$	1.50	$\frac{1.50}{1+E}$
$m = 1/2$	2.00	$\frac{2.00}{1+E}$
$m = 2/3$	3.00	$\frac{3.00}{1+E}$
$m = 3/4$	4.00	$\frac{4.00}{1+E}$
$m = 4/5$	5.00	$\frac{5.00}{1+E}$
$m = 5/6$	6.00	$\frac{6.00}{1+E}$

We would like to point out that our approach to deriving equation (15) is equivalent to that of May *et al.* [6]. For, according to equation (15) on p. 230 of [6],

$$\frac{\text{CRT}(E)}{\text{CRT}(0)} = \frac{1}{\left(\frac{\partial}{\partial y} (y^m - y - Ey)\right) \Big|_{y=y^*(E)}} \div \frac{1}{\left(\frac{\partial}{\partial y} (y^m - y)\right) \Big|_{y=y^*(0)}} = \frac{1}{1+E},$$

where $y^*(E)$ is the equilibrium biomass defined by equation (8) and $y^*(0) = 1$.

5. ECONOMIC PROPERTIES OF DENSITY-DEPENDENT HARVESTING IN A BERTALANFFY GROWTH MODEL

In general the cost of production from a renewable resource population will depend on a number of factors such as the specific population under exploitation, its level of abundance and availability of labor.

To simplify the arithmetic involved in the analysis in this section, we assume that all factors of production are incorporated in the harvesting effort E and that total cost of harvesting is directly proportional to the level of expended effort. That is,

$$C = \gamma E, \quad (16)$$

where γ is the cost of a unit of effort. The further assumption that the demand for the renewable resource in question is perfectly elastic will be made. That is, the worth of a unit biomass of the resource in the market is constant.

From the preceding analysis, we have the yield at equilibrium is given by

$$Y = E \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}}. \quad (17)$$

This yield represents the long-run annual production level of the sole owner for a given level of effort. The monetary value of this harvest is given by

$$V = pY = pE \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}}, \quad (18)$$

where p is the price of a unit biomass of the harvested resource. It is transparent that under

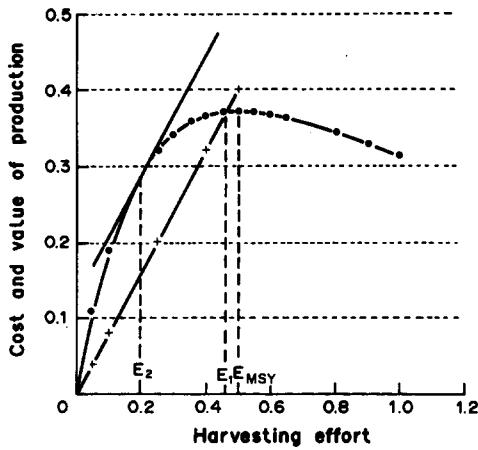


Fig. 3. Cost (+) and value (●) of production as functions of harvesting effort in a Bertalanffy model when $\gamma > pm^{1/(1-m)}$.

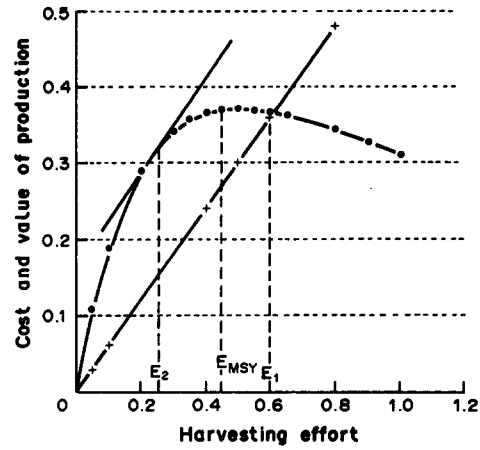


Fig. 4. Cost (+) and value (●) of production as functions of harvesting effort in a Bertalanffy model when $\gamma < pm^{1/(1-m)}$.

equilibrium conditions, both the physical yield and its monetary value are differentiable nonlinear functions of the harvesting effort. They both fall to zero as expected only when the resource is not exploited. As noted previously in this article, the yield-effort function is a monotone increasing function of the effort up to the instant when it attains its maximum. After this instant, the yield grows at a diminishing rate. The economic value of harvesting (18) has these same properties too. The relationships between the economic value of harvesting and harvesting effort are shown in Figs 3 and 4 which also include the graph of the total cost function as the line C through the origin.

In Figs 3 and 4, we have depicted two cases. Figure 3 illustrates the case where the unit cost of harvesting is "large" ($\gamma > pm^{1/(1-m)}$) and the level of harvesting effort E_1 for which the total harvesting cost is equal to the total revenue from harvesting is smaller than E_{MSY} , while Fig. 4 illustrates the case where the unit cost of harvesting is "small" ($\gamma < pm^{1/(1-m)}$) and E_1 is greater than E_{MSY} .

Equation (16) indicates that marginal cost and average cost of production are identical and constant as represented by the curve MC in Figs 5 and 6. The marginal and average revenue function are given by

$$\frac{dv}{dE} = p \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}} - \frac{pE}{1-m} \left(\frac{1}{1+E} \right)^{\frac{2-m}{1-m}} \quad (19)$$

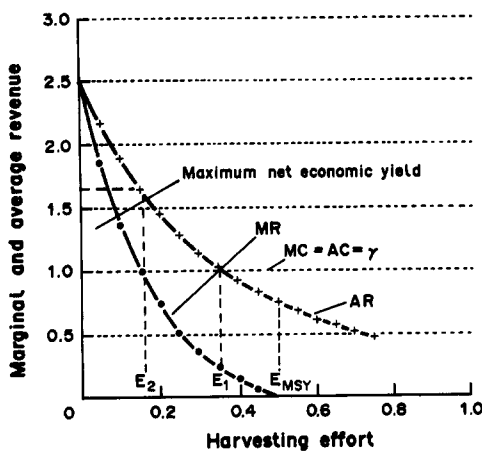


Fig. 5. Optimum intensity of utilization of a renewable resource in a Bertalanffy model when $\gamma > pm^{1/(1-m)}$; (●) marginal revenue, (+) average revenue.

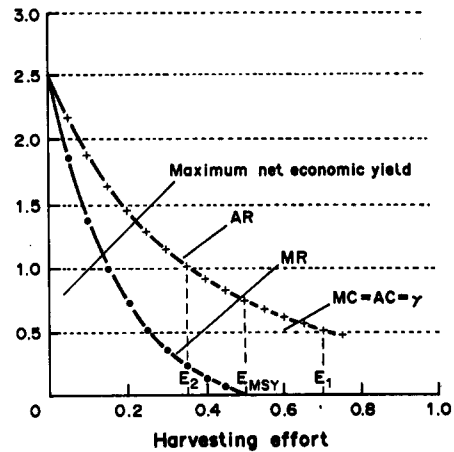


Fig. 6. Optimum intensity of utilization of a renewable resource in a Bertalanffy model with $\gamma < pm^{1/(1-m)}$; (●) marginal revenue, (+) average revenue.

and

$$\frac{v}{E} = p \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}} \quad (20)$$

For our model, both the average and marginal revenue are monotonically decreasing nonlinear functions of the harvesting effort. Also, the marginal revenue (19) falls to zero at the effort level which maximizes the sustainable physical yield.

The net economic yield is the difference between the total revenue and the total cost of harvesting and is given by

$$N = pE \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}} - \gamma E. \quad (21)$$

The net economic yield is maximum for that value of effort for which

$$\frac{dN}{dE} = p \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}} - \frac{pE}{1-m} \left(\frac{1}{1+E} \right)^{\frac{2-m}{1-m}} - \gamma = 0. \quad (22)$$

The solution of equation (22) is denoted by E_2 in Figs 3–6. It is the optimum intensity of effort on the resource domain and the resource will, at this level of exploitation, provide the maximum net economic yield represented by the shaded regions in Figs 5 and 6.

As expected, Fig. 6 indicates a smaller unit cost of harvesting leads to a larger maximum net economic yield, everything else remaining equal. It may also be observed that the optimum economic harvesting intensity is lower than the harvesting intensity which produces the maximum sustainable yield.

In the case of uncontrolled harvesting of common-property renewable resources where anyone who wishes to harvest is free to “jump in”, new harvesters will be attracted to the resource domain if and only if the average cost is less than the average revenue. Given that this condition holds, effort on the resource domain will grow until it reaches the level E_1 , where average cost equals average revenue and the net economic yield becomes completely dissipated.

Of course, average cost equals average revenue for our model at the effort level for which

$$p \left(\frac{1}{1+E} \right)^{\frac{1}{1-m}} = \gamma \quad \text{or} \quad E_1 = \left(\frac{p}{\gamma} \right)^{1-m} - 1. \quad (23)$$

Clearly, from (23), if the unit cost is “high” relative to the unit price, the level of harvesting intensity E_1 is not reachable. In this circumstance, the resource has lower opportunity income flow and is therefore economically unattractive to harvesters. This should cause harvesters to desert the resource domain and look for other forms of employment with higher opportunity income flow. In effect, the level of harvesting effort on the resource ground will be reduced. The departure of some or all harvesters will no doubt lead to decrease in total landings. In other words, to increase yield, unit price must increase or unit cost must decrease through federal or local government support programs such as price support or subsidies.

If on the other hand, unit price is sufficiently high relative to unit cost, E_1 may exceed the level E_{MSY} where maximum sustainable yield is attained. In this case, harvesting such a resource may have high opportunity income flow and more harvesters will be attracted to the resource ground leading possibly to biological and economic overexploitation and a consequent reduction in physical yield. Yield may be increased in this situation by restricting entry to the resource domain and hence the amount of harvesting effort reduced to the level E_2 . Since from the preceding analysis E_2 is always less than E_1 and E_{MSY} , and since physical yield depends on the amount of expended effort it follows that economic optimization in renewable resources entails some sacrifice of total physical yield.

6. OPEN-ACCESS HARVESTING AND MAXIMIZATION OF PRESENT VALUE OF NET ECONOMIC YIELD

Suppose the underlying biological model of growth for an open-access renewable resource is the Bertalanffy one. Suppose further that the objective of a regulatory agency is to control the

harvesting process so as to maximize the present value of net economic yield. Then the renewable resource management problem is

$$\text{maximize}_{y, E} \int_{\Omega} \exp(-rt)(TR - TC) dt, \quad (24)$$

subject to $\dot{y} = y^m - y - E(t)y$, where $0 \leq E(t) \leq E_{\max}$, $y(t) > 0$, $y(t_0) = y_0$, $t_0, t \in \Omega \subset \mathbb{R}^1$ and Ω is compact, represents the harvesting season, TR = total revenue derived from harvesting, TC = total cost of harvesting, r = rate of discount.

To gain a better prediction of the dynamics of the harvesting process, a more detailed economic component is required. In response to this need, economist V. L. Smith [7] suggested the introduction of an industry reaction equation of the form

$$\dot{E} = s(py - \gamma)E(t) \quad (25)$$

to compliment the resource dynamics where p and γ represent the unit price and cost of harvesting respectively, and $s > 0$ represents a behavioral parameter for the harvesting industry. If the economic yield $(py - \gamma)$ per unit of expended effort is positive, then $\dot{E} > 0$ implying a flow of more harvesters into the resource domain while $(py - \gamma)$ negative indicates harvesting the resource is economically unattractive and harvesters will be forced out of the resource domain thereby reducing the amount of effort on the resource domain. Thus equation (25) describes the growth or decline in harvesting effort on the resource domain.

The dynamics of the harvesting process can be predicted by studying the two-dimensional system

$$\begin{aligned} \dot{y} &= y^m - y - E(t)y \\ \dot{E} &= s(py - \gamma)E(t) \end{aligned} \quad (26)$$

with $y > 0$, $E(t) > 0$.

At bionomic equilibrium, $\dot{y} = \dot{E} = 0$ and occurs at the point (y^*, E^*) , where

$$y^* = \frac{\gamma}{p}, \quad E^* = \left(\frac{p}{\gamma}\right)^{1-m} - 1. \quad (27)$$

Equation (27) indicates the bionomic equilibrium effort level to be a function of the price-cost ratio p/γ . As noted in Section 5, if unit harvesting cost is high relative to the unit price of harvesting, namely if $p/\gamma < 1$, then the resource in question will not be exploited at all. At some higher unit price levels or lower unit cost levels, it may become profitable to harvest the resource and the bionomic equilibrium effort level becomes established at a higher level. The resource becomes biologically overexploited when $y^* < y_{\text{MSY}}$. The degree of the economic overexploitation associated with the bionomic equilibrium (27) is determined by the inequality $E^* > E_{\text{MSY}}$. That is, the harvesting industry will expand beyond the MSY level if and only if

$$p/\gamma > \left(\frac{1}{m}\right)^{\frac{1}{1-m}}.$$

Since $(1/m)^{1/(1-m)}$ approaches e (Euler's constant) from above as m approaches 1, it follows that the harvesting industry will expand beyond the MSY level whenever the price-to-cost ratio exceeds e . This is a more conservative result than that obtained by Clark [2] who utilized the standard logistic growth model in his analysis. Thus one may make the conjecture that the dynamics of the harvesting process in open-access resources depends critically on the underlying biologically model of growth.

The Jacobian matrix of (26) at the bionomic equilibrium (27) is

$$J = \begin{pmatrix} -(1-m)(p/\gamma)^{1-m} & -\gamma/p \\ sp((p/\gamma)^{1-m} - 1) & 0 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_i = -\frac{1}{2}(1-m) \left[\left(\frac{p}{\gamma}\right)^{1-m} \pm \frac{1}{2} \left\{ (1-m)^2 \left(\frac{p}{\gamma}\right)^{2-2m} - 4s\gamma \left[\left(\frac{p}{\gamma}\right)^{1-m} - 1 \right] \right\}^{1/2} \right]. \quad (28)$$

If the industry behavioral parameter s is "small", the eigenvalues in (28) are both real and negative, and, we know from system theory that the equilibrium point (y^*, E^*) is a stable node. On the other hand, if s is "large", the eigenvalues are complex with negative real parts and (y^*, E^*) becomes a stable focus.

To determine if the system (26) has a closed trajectory in the first quadrant, we introduce the smooth function

$$B(y, E) = y^{-m} E^{1/sp} \quad (29)$$

and conduct a Bendixon-du-Lac test [4]. With $F(y, E)$ and $G(y, E)$ the right-hand sides of (26), we have

$$\frac{\partial}{\partial y} (BF) + \frac{\partial}{\partial E} (BG) = y^{-m} E^{1/sp} \left[s \left(1 + \frac{1}{sp} \right) (py - \gamma) - (1 - m)(1 + E) \right], \quad (30)$$

which is positive in the first quadrant.

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