Helmholtz equation in 2D using ADI

We want to solve the two-dimensional Helmholtz equation numerically:

$$(1 - c\partial_{xx} - c\partial_{yy})u = f, \tag{1}$$

where u and f are spectral coefficients and c is a scalar constant.

ADI

Our intent is to decouble the x and y dimension, which allows the use of efficient solvers. This can be achieved by the eigendecomposition method as descibred in *doc_poisson2d*. However, here we use the alternating direction implicit (ADI) method, whose accuracy depends on the size of the parameter c (c smaller $\to$ more accurate). The idea of the ADI method is to approximate the multidimensional Helmholtz operator by a product of one-dimensional operators, i.e.

$$\left(I - c\nabla^2\right) = \left(I - c\nabla_x^2\right)\left(I - c\nabla_y^2\right) + c^2\nabla_x^2\nabla_y^2 \overset{c \text{ small}}{\approx} \left(I - c\nabla_x^2\right)\left(I - c\nabla_y^2\right) \tag{2}$$

which can be solved successively along x and y, i.e.

$$\begin{aligned}\left(\mathbf{I} - c\nabla_x^2\right) g &= f,\\ \left(\mathbf{I} - c\nabla_y^2\right) u &= g\end{aligned} \tag{3}$$

Implementation

Multivariate problems can be described well with Kronecker notation, i.e.

$$\left[I \otimes I - c\left(D_2 \otimes I - I \otimes D_2\right)\right] u = f, \tag{4}$$

where D_2 is the discrete differentiation operator, I the identity matrix and u and f are the flattened one-dimensional vectors. We impose boundary conditions by using basis recombinations $u = (S \otimes S)v$, where v are spectral coefficients in a function base that already satisfies the boundary conditions. Here we will impose Dirichlet BCs on all sides. In this case $u \in \mathbb{R}^{N\times N}$, $v \in \mathbb{R}^{N-2\times N-2}$ and $S \in \mathbb{R}^{N\times N-2}$. We get

$$\left[S \otimes S - c\left(D_2S \otimes S - S \otimes D_2S\right)\right] u = f. \tag{5}$$

Using the ADI approximation eq. (2) or eq. (3), we get a set of one dimensional helmholtz equations

$$\begin{aligned}\left(S - cD_2S\right) g_i &= f_i,\\ \left(S - cD_2S\right) v_i &= g_i.\end{aligned} \tag{6}$$

Those can be solved more efficiently by preconditioning with the inverse of the D_2 operator, i.e. D_2^{-1}, thus we get

$$\begin{aligned}\left(D_2^{-1}S - cS\right) g_i &= D_2^{-1}f_i,\\ \left(D_2^{-1}S - cS\right) v_i &= D_2^{-1}g_i.\end{aligned} \tag{7}$$

where the lhs $(D_2^{-1}S - cS)$ and rhs (D_2^{-1}) operators are banded with diagonals on $-2, 0, 2, 4$, respectively $-2, 0, 2$.

Example

See *examples/hholtz2d_cheb_dirichlet.rs.*