We want to solve the two-dimensional Poisson equation numerically:

$$(\partial_{xx} + \partial_{yy})u = f, \tag{1}$$

where u and f are spectral coefficients. Multivariate problems can be described well with Kronecker notation, i.e.

$$(D_2 \otimes I + I \otimes D_2)u = f, \tag{2}$$

where D_2 is the discrete differentiation operator, I the identity matrix and u and f are the flattened one-dimensional vectors. We impose boundary conditions by using basis recombinations $u = (S \otimes S)v$, where v are spectral coefficients in a function base that already satisfies the boundary conditions. Here we will impose Dirichlet BCs on all sides. In this case $u \in \mathbb{R}^{N\times N}$, $v \in \mathbb{R}^{N-2\times N-2}$ and $S \in \mathbb{R}^{N\times N-2}$. We get

$$(D_2S \otimes S + S \otimes D_2S)v = f. \tag{3}$$

Our intent is to decouble the x and y dimension, which allows the use of efficient solvers. First, we multiply (3) with S^{-1} from the left

$$(S^{-1}D_2S \otimes S + I \otimes D_2S)v = (S^{-1} \otimes I)f. \tag{4}$$

Applying the eigendecomposition $S^{-1}D_2S = Q\Lambda Q^{-1}$

$$(Q\Lambda Q^{-1} \otimes S + I \otimes D_2S)v = (S^{-1} \otimes I)f. \tag{5}$$

and multiplying with Q^{-1} from the left

$$(\Lambda Q^{-1} \otimes S + Q^{-1} \otimes D_2S)v = (Q^{-1}S^{-1} \otimes I)f \tag{6}$$

or simplified

$$(\Lambda \otimes S + I \otimes D_2S)\widehat{v} = \widehat{f} \tag{7}$$

with $(Q^{-1} \otimes I)v = \widehat{v}$ and $(Q^{-1}S^{-1} \otimes I)f = \widehat{f}$. Since I and Λ are both diagonal matrices, eq. (7) decoubles in x and y, such that we get a couple of one-dimensional equations

$$(\lambda_i S + D_2S)\widehat{v}_i = \widehat{f}_i, \tag{8}$$

where i runs from 0 to $N - 3$.

In short:

Preprocessing

- Eigendecomposition $S^{-1}D_2S = Q\Lambda Q^{-1}$

Main

- (i) Transform $\widehat{f} = (Q^{-1}S^{-1} \otimes I)f$
- (ii) Solve $(\lambda_i S + D_2S)\widehat{v}_i = \widehat{f}_i$, where $i = 0, .., N - 3$.
- (iii) Transform $v = (Q \otimes I)\widehat{v}$.

Notes:

Step (ii) can be preconditioned with D_2^{-1}, i.e.

- (ii) Solve $(\lambda_i D_2^{-1}S + S)\widehat{v}_i = D_2^{-1}\widehat{f}_i$

which is banded, instead of upper triangular.