Helmholtz equation in 2D using ADI

We want to solve the two-dimensional Helmholtz equation numerically:

$$(1 - c\partial_{xx} - c\partial_{yy})u = f, \tag{1}$$

where u and f are spectral coefficients and c is a scalar constant.

ADI

Our intent is to decouble the x and y dimension, which allows the use of efficient solvers. This can be achieved by the eigendecomposition method as descibred in *doc_poisson2d*. However, here we use the alternating direction implicit (ADI) method, whose accuracy depends on the size of the parameter k. The idea of the ADI method is to approximate the multidimensional Helmholtz operator by aproduct of one-dimensional operators, i.e.

$$\left(\mathbf{I} - c\nabla^2\right) \approx \left(\mathbf{I} - c\nabla_x^2\right)\left(\mathbf{I} - c\nabla_y^2\right) \tag{2}$$

which can be solved successively along x and y, i.e.

$$\begin{aligned} \left(\mathbf{I} - c\nabla_x^2\right) g &= f, \\ \left(\mathbf{I} - c\nabla_y^2\right) u &= g \end{aligned} \tag{3}$$

Implementation

Multivariate problems can be described well with Kronecker notation, i.e.

$$\left[I \otimes I - c\left(D_2 \otimes I - I \otimes D_2\right)\right] u = f, \tag{4}$$

where D_2 is the discrete differentiation operator, I the identity matrix and u and f are the flattened one-dimensional vectors. We impose boundary conditions by using basis recombinations $u = (S \otimes S)v$, where v are spectral coefficients in a function base that already satisfies the boundary conditions. Here we will impose Dirichlet BCs on all sides. In this case $u \in \mathbb{R}^{N\times N}$, $v \in \mathbb{R}^{N-2\times N-2}$ and $S \in \mathbb{R}^{N\times N-2}$. We get

$$\left[S \otimes S - c\left(D_2 S \otimes S - S \otimes D_2 S\right)\right] u = f. \tag{5}$$

Using the ADI approximation eq. (2) or eq. (3), we get a set of one dimensional helmholtz equations

$$\begin{aligned} \left(S - cD_2 S\right) g_i &= f_i, \\ \left(S - cD_2 S\right) v_i &= g_i. \end{aligned} \tag{6}$$

Those can be solved more efficiently by preconditioning with the inverse of the D_2 operator, i.e. D_2^{-1}, thus we get

$$\begin{aligned} \left(D_2^{-1} S - cS\right) g_i &= D_2^{-1} f_i, \\ \left(D_2^{-1} S - cS\right) v_i &= D_2^{-1} g_i. \end{aligned} \tag{7}$$

where the lhs $(D_2^{-1}S - cS)$ and rhs (D_2^{-1}) operators are banded with diagonals on $-2, 0, 2, 4$, respectively $-2, 0, 2$.

Example

See *examples/hholtz2d_cheb_dirichlet.rs*.