Table of Contents

Table of Figures

Executive Summary

Our report encompasses comprehensive portfolio optimization analyses leveraging Mean-Variance, Minimum-Variance, and Naïve models. Through meticulous data selection and exploratory data analysis, we curated insights into risk-return dynamics across portfolios formed based on firm size and Book-to-Market ratios. Employing rolling windows of 12, 36, and 60 months, we optimized portfolios and evaluated their performance using Mean-Standard Deviation diagrams and Sharpe Ratios. Our findings elucidate the effectiveness of sophisticated optimization techniques in maximizing risk-adjusted returns and empower investors with actionable insights for navigating volatile markets.

Data Selection

We utilized the "6 Developed Portfolios Formed on Size and Book-to-Market (2 x 3)" (Kenneth, 2024) dataset, providing monthly returns for portfolios categorized by market cap and Book-to-Market ratios. Monthly returns were chosen for their ability to smooth daily fluctuations, providing reliable estimates crucial for portfolio optimization and risk management (Brown & Warner, 1985). Monthly data reduces biases inherent in daily returns and facilitates pattern identification, enhancing computational efficiency for portfolio analysis (Dyckman, Philbrick and Stephan, 1984).

Spanning from July 1990 to April 2024, the dataset encompasses returns from 23 developed countries, ensuring stability, transparency, and liquidity, supported by well-regulated exchanges and sophisticated financial systems (Frank, n.d.). Portfolios were constructed based on size—Big Firms (top 90% by market cap) and Small Firms (bottom 10% by market cap)—and Book-to-Market ratios, delineating growth, value, and neutral stocks. Growth stocks, with B/M ratios below the 30th percentile, anticipate significant earnings growth, while value stocks, above the 70th percentile, are valued lower relative to their book value, offering promising returns. Neutral stocks fall between these thresholds. This detailed dataset provides a robust foundation for analyzing risk-return dynamics across diverse market segments, essential for informed portfolio decision-making.

Combining small-cap and large-cap stocks in a portfolio balances high growth potential with stability (Investopedia, n.d.). The Fama and French study offers empirical evidence supporting the advantages of integrating size and B/M as factors in portfolio formation. Such diversification helps in effectively managing the risk – return trade off to optimise investment performance (Fama and French, 1992).

Table 1: The terminology and interpretation of portfolio

Original Name	Revised Name	Interpretation
SMALL LoBM	Small_Growth	Small stock with low B/M
ME1 BM2	Small_Neutral	Small stock with medium B/M
SMALL HiBM	Small_Value	Small stock with high B/M
BIG LoBM	Big_Growth	Big stock with low B/M
ME2 BM2	Big_Neutral	Big stock with medium B/M
BIG HiBM	Big_Value	Big stock with high B/M

Exploratory Data Analysis

We have used Average Value Weighted Returns -Monthly for EDA and portfolio optimisation. Most major indices (e.g., S&P 500, FTSE 100) are value weighted. Using value-weighted returns allows for better comparison and benchmarking against these widely recognized indices. Also note each portfolio is constructed based on size the value-weighted returns might not vary much from equal weighted as similar market capitalization stocks are grouped in a portfolio.

Before any further analysis we did corrected the data types and check for no missing values.

1. Summary of Monthly returns

In Figure 1, the summary of mean and standard deviation of monthly returns shows that small value stocks offer the highest mean return, presenting a high-risk, high-reward opportunity. Small growth stocks have the highest volatility, unsuitable for risk-averse investors. Large-cap stocks provide stable returns of around 0.7% per month, with varying volatility levels.

Index	Small_Growth	Small_Neutral	Small_Value	Big_Growth	Big_Neutral	Big_Value
count	406	406	406	406	406	406
mean	0.413448	0.682365	0.903793	0.736232	0.75335	0.72697
std	5.50986	4.75531	4.46682	4.6024	4.30095	4.91755
min	-24.76	-22.57	-20.06	-17.5	-18.74	-22.41
25%	-2.525	-1.875	-1.275	-1.6675	-1.72	-2.0325
50%	0.93	1.09	1.12	1.05	1.09	1.135
75%	3.675	3.485	3.62	3.6725	3.33	3.6175
max	17.5	14.92	15.31	13.04	12.26	18.79

Figure 1 Summary of returns

2. Histogram Analysis

All the portfolios have monthly returns of around 0 as all of them peak around 0 which states the returns are more balanced. Some points extend beyond the central peak but there are not many extreme outliers. Small_Growth and Small_value seem a little left skewed means that they are more volatile and have some cases of occasional significant losses. Risk averse investors would prefer more symmetric (like small_neutral and big_neutral) or right skewed curve. (Figure 2)

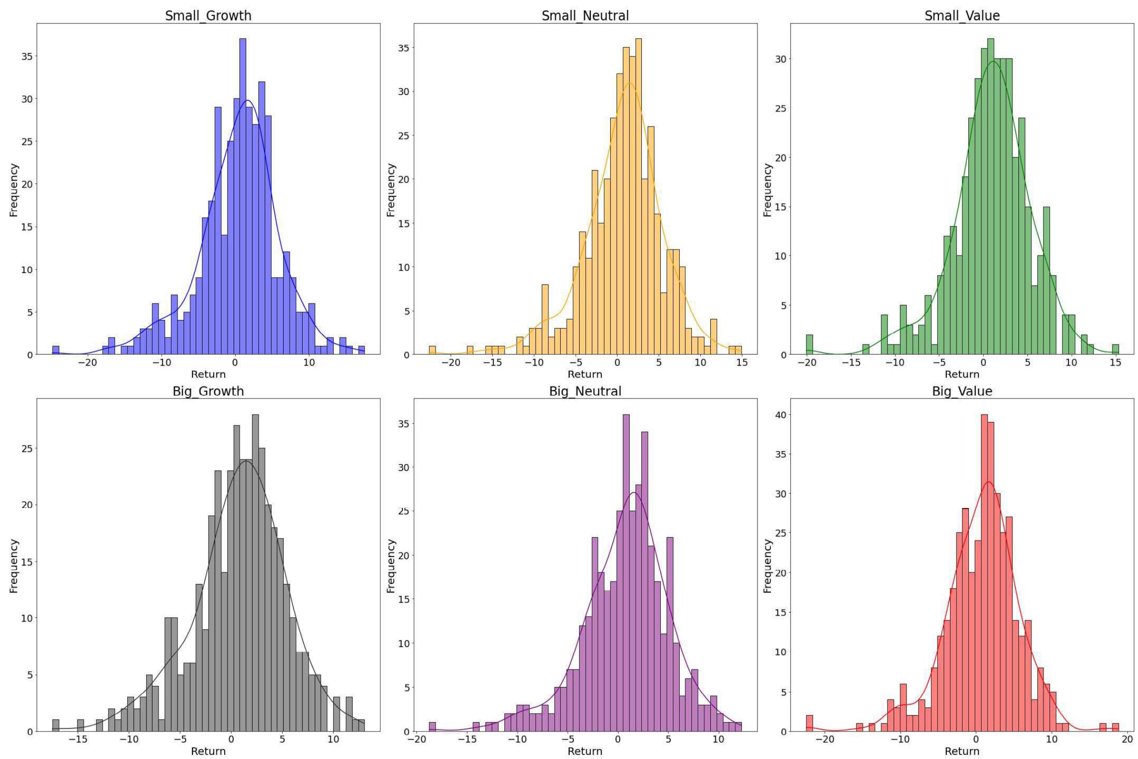

Figure 2 Returns distribution

3. Firm Size and No. of Firms

The significant expansion in firm size across small and big cap portfolios underscores market vitality. Small and Big Value segments, being undervalued, offer potential returns. The staggering 30X growth in Big Growth compared to 4X in small portfolios suggests diverse market forces at play, possibly driven by sectoral shifts, investment trends, and economic influences favouring larger firms.

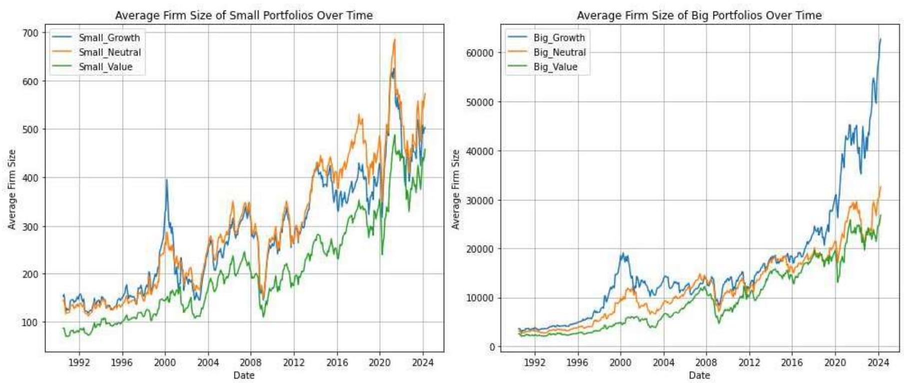

Figure 3 Comparison of average firm size

The observed increase in small firms, particularly notable in the small value segment, contrasts with a decline in big firms over time. Remarkably, both big growth and big value portfolios exhibit similar downward trends in firm numbers.

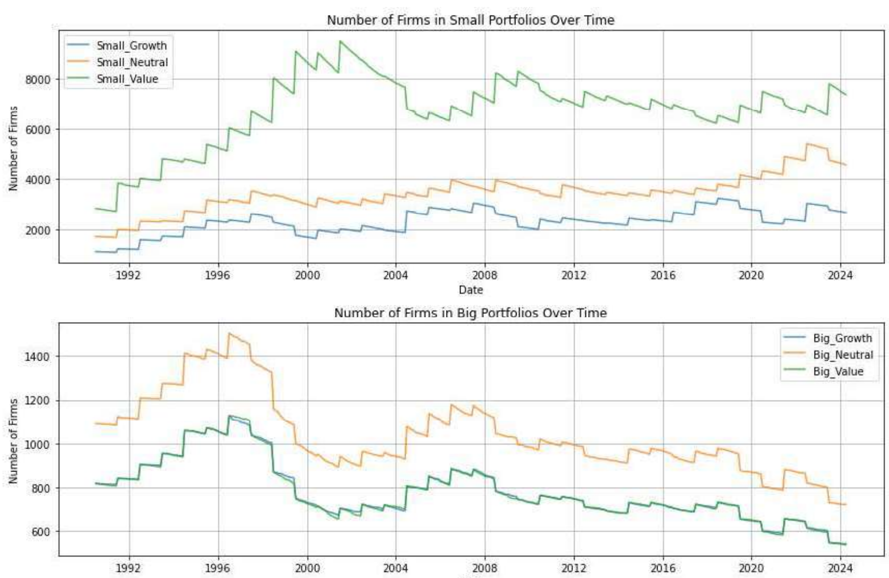

Figure 4 Comparison of average no. of firms

4. Correlation Analysis

Highly positively correlated assets, as observed in all portfolios' average monthly returns, which elevate systematic risk, making portfolios more sensitive to market movements. This heightened correlation diminishes diversification benefits, increasing vulnerability to significant losses. Incorporating systemic risk into portfolio optimization, as suggested by DAS and UPPAL (2004), underscores the importance of risk management strategies in volatile markets.

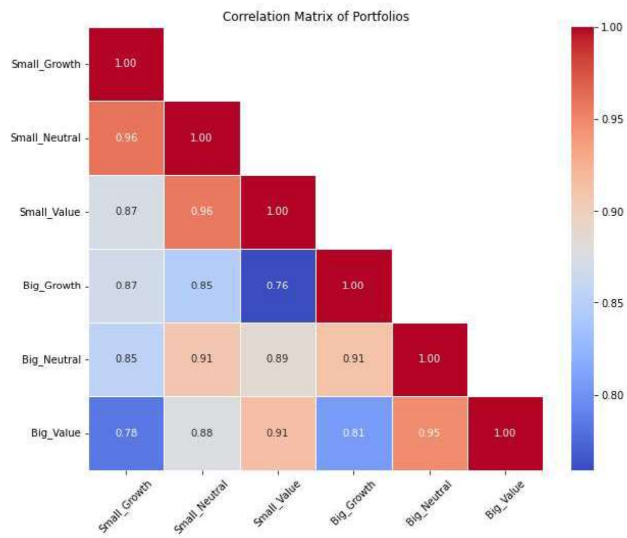

Figure 5 Correlation Heat Map

5. Trend Analysis of Return

Monthly returns show consistent trends with notable downturns during significant market events.

1. During the dot-com bubble of the late 1990s, heavy investments in internet companies led to a market crash in 2002, notably affecting tech firms listed on the Nasdaq (Finbold, n.d.). Identified as growth stocks, these tech companies saw significant declines, impacting portfolios like Big Growth and Small Growth (UPCOMINGTRADER, 2023).
2. The 2008 financial crisis resulted from issues like bad mortgages and weak regulations, causing significant stock market declines, with small companies recovering faster than larger ones (SuperMoney, n.d.).
3. Despite a general decline in stocks due to COVID-19, growth stocks in technology, healthcare, and e-commerce sectors proved more resilient, benefiting from increased demand for remote work and digital goods (UNCTAD, 2021).

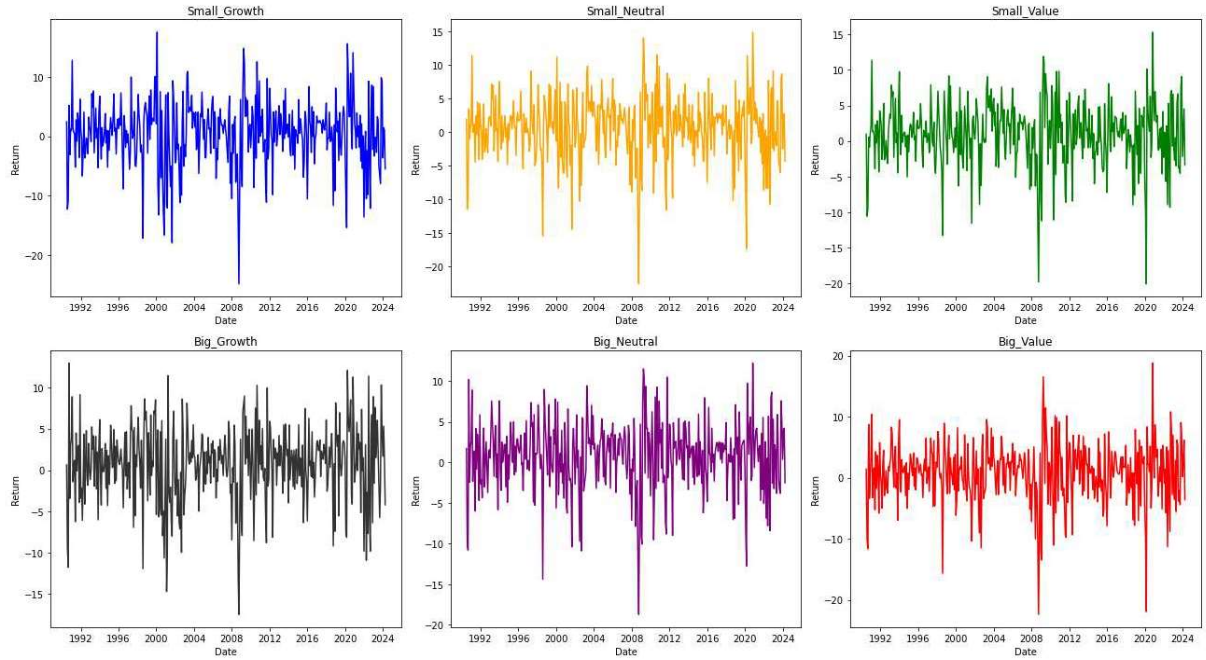

Figure 6 Returns over time

6. Efficient Frontier

The Efficient Frontier illustrates the risk-return relationship, plotting portfolio points based on six portfolios. Portfolios on this frontier offer optimal balance between risk and return. The minimum-variance point, around 0.775% monthly return and less than 4.3 volatility, signifies the lowest risk. Beyond this point, the efficient frontier depicts increasing expected return with higher risk. Portfolios with higher Sharpe Ratios, denoted by yellow points, are preferred as they yield better returns per unit of risk. Luenberger (2014) highlights the significance of this minimum-variance point in portfolio optimization.

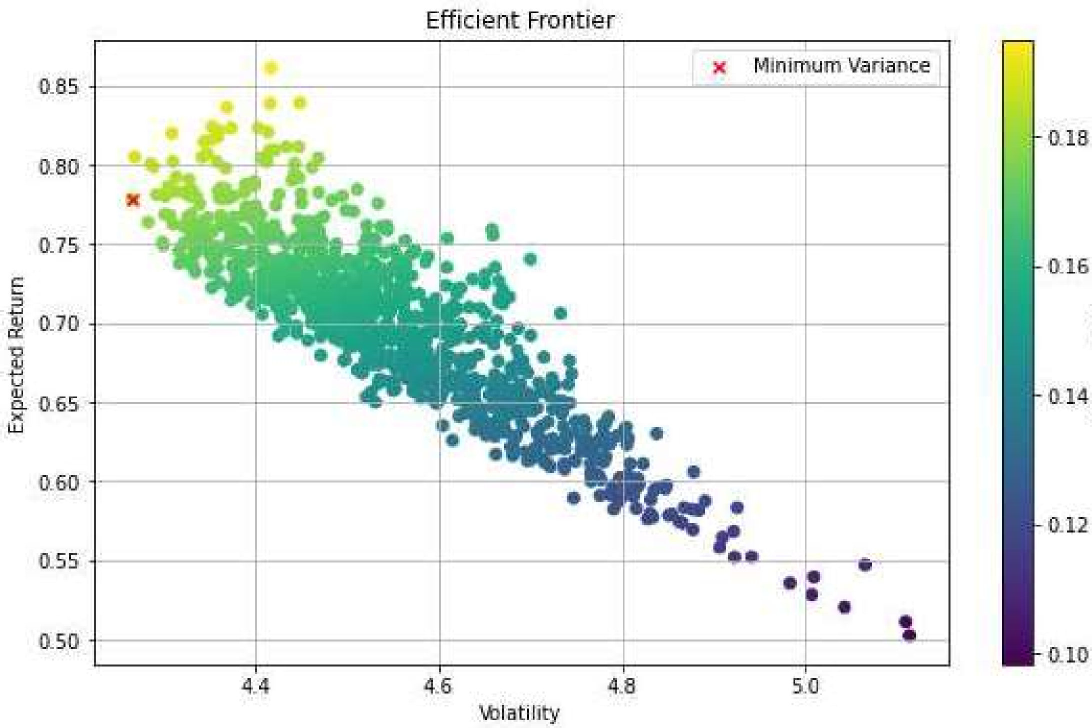

Figure 7 Efficient frontier and MVP

The CML is drawn from the risk-free rate of 4.13% p.a. The point where CML touches the efficient frontier offers highest Sharpe ratio (i.e., best risk-adjusted return) which is at over 0.85 % return per month at volatility around 4.5.

Investors can use the CML to determine the optimal mix of the risk-free asset and the tangency portfolio based on their risk tolerance. (Luenberger, 2014)

The risk free rate is based on average of 10 year US treasury yield from 1990 to 2024 ((Macrotrends, 2023).

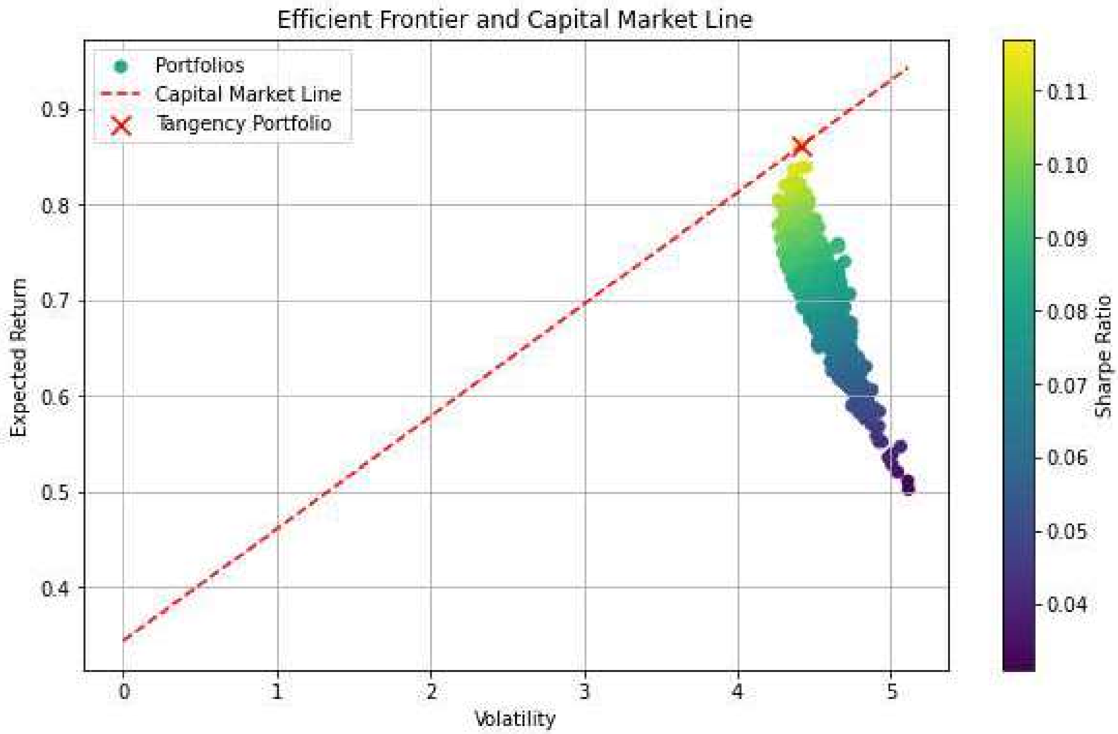

Figure 8Efficient frontier and CML

Portfolio Optimisation

1. Estimation of Rolling Window

Rolling window analysis can be used to access the performance of portfolio optimisation and smoothens short-term volatility (Kakko, 2023).

In our analysis, we've utilized three estimation windows to gauge mean and covariance matrix of returns, offering varied perspectives on portfolio performance.

A 12-month window captures short-term dynamics, promptly detecting market changes. The 36-month window balances short-term fluctuations with broader trends, aiding pattern identification. Meanwhile, a 60-month window provides a long-term view, smoothing short-term noise and evaluating strategic effectiveness. These windows—12 months for short-term, 36 months for medium-term, and 60 months for long-term—comprehensively inform portfolio decisions by encompassing diverse market behaviours over distinct time frames.

2. Estimation of Mean and Covariance Matrix

The rolling mean return per month is calculated for a given window size 'w' at time t. So, the mean return calculated at time 'i' corresponds to previous period which is equal to window size.

$$\mu_t^{(w)} = \frac{1}{w} \sum_{i=t-w+1}^{t} r_i$$

Similarly, The rolling covariance matrix of returns for a given window size 'w' at time t. So, the covariance matrix calculated at time 'i' corresponds to previous period which is equal to window size.

$$\Sigma_t^{(w)} = \mathrm{Cov}(r_{t-w+1}, r_{t-w+2}, \ldots, r_t)$$

We stored the rolling mean and covariance matrix of returns to be used as input for further single-period portfolio optimization.

3. Using CAPM for Target Return

CAPM is a tool to evaluate investment returns based on their risk and expected returns (www.morpher.com, 2024). We can set a target return which can be used to bound the constraint for expected returns from portfolio in Mean- variance portfolio optimization.

We take 2 assumptions for CAPM:

i. Risk free rate – 4.3/12 = 0.3443% per month (same as taken for CML above)
ii. Market_return – 9.71/12 = 0.8091% per month

The market return is taken as MSCI world index 10 year annualized return of 9.71% (MSCI WORLD INDEX (USD) n.d.). We have used MSCI world index as it also considers the same 23 developed countries and would be the most suitable rate to set market return.

Then beta for each portfolio is computed using the below formulation. Beta is a measurement of an assets risk compared to benchmark (market_return). The beta of market is set as 1 (Liberto, 2021).

Mathematical Formulation:

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_m)}{\mathrm{Var}(r_m)}$$

Figure 9 depicts that Beta of Big_value, Small_neutral and Small_growth is greater than 1 hence it is riskier than market. Big_neutral, Big_growth and Small_value are less riskier than market.

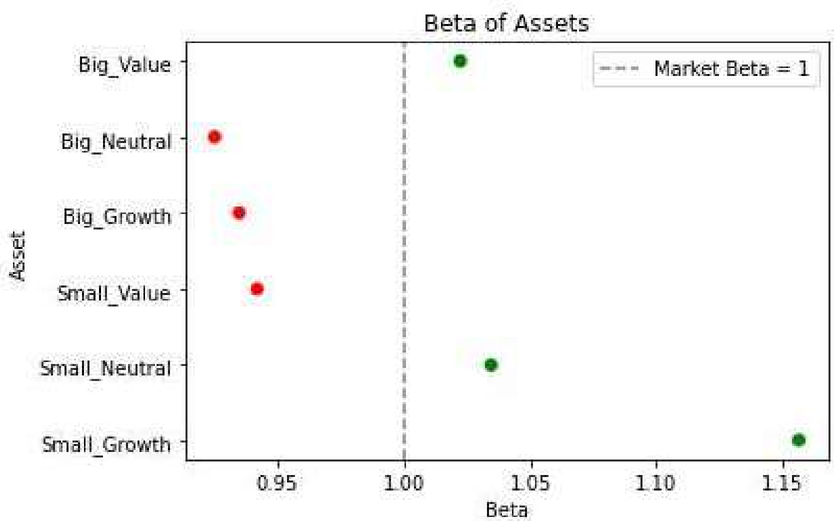

Figure 9 Beta comparison

Based on beta calculated we calculate target return (Expected return) for each portfolio.

CAPM mathematical formulation:

$$E(r_i) = r_f + \beta_i \times (E(r_m) - r_f)$$

Below are the target return for each portfolio and for assets which were more risky than market the target return is also set more.

Small_Growth	0.881813
Small_Neutral	0.825185
Small_Value	0.782144
Big_Growth	0.778867
Big_Neutral	0.774335
Big_Value	0.819544

Figure 10Target return of Portfolio

For the purposes of setting a target return constraint in mean-variance portfolio optimization we will take average return of all the portfolio i.e., 0.8103% per month.

4. Mean – Variance Portfolio optimization

Modern Portfolio Theory (MPT) is a mathematical framework for constructing efficient investment portfolios that maximize expected return for a given level of risk or minimize risk for a given level of expected return. Key fundamental assumptions for MPT are in appendix 1.2 (Markowitz, 1952).

Portfolio performance computes the expected return and standard deviation of the portfolio using the optimal weights, expected return and covariance matrix of asset return (computed rolling mean and cov mat for each window earlier)

$$\text{port\_return} = \sum_{i=1}^{n} w_i r_i$$

$$\text{port\_volatility} = \sqrt{\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}}$$

Optimize portfolio by minimizing the portfolio volatility subject to constraints on the sum of weights and the portfolio return being equal to the target return.

$$\begin{aligned} &\text{minimize} \quad \text{port\_volatility} \\ &\text{subject to:} \\ &\sum_{i=1}^{n} w_i = 1 \\ &\sum_{i=1}^{n} w_i r_i = \text{target\_return} \end{aligned}$$

The optimization is performed using an optimization algorithm such as Sequential Least Squares Programming (SLSQP) to find the optimal portfolio weights ‘w’

5. Minimum - Variance Portfolio Optimisation

It has same formulation as mean-variance but doesn't consider any expected return constraint. It focuses on majorly minimizing the return with weight sum equal to 1 constraint.

Mathematical formulation apart from a constraint of expected return remains same.

Minimum variance optimization focuses solely on minimizing risk, while mean variance optimization considers the trade-off between risk and return to identify portfolios that provide the most efficient risk-return profiles.

6. Naïve model

This is a simple formulation of calculating return of portfolio by taking weight of 1/n for each n available asset.

In our case there were 6 portfolios so we took weight 1/6 for each portfolio return and computed the optimal return as per naïve model.

Naive models, simple yet powerful, offer speed, sustainability, and interpretability, serving as benchmarks and solutions amidst complex optimization models (Algorithma, n.d.).

Performance Evaluation

1. Mean – standard deviation diagram

Based on the models , Table 2 contains summary of results mean Optimal Return and mean Optimal Volatility for each window.

Methods	Window	Mean_Optimal_Return	Mean_Optimal_Volatility
Mean_Variance	12	0.785319	3.75174
Mean_Variance	36	0.763668	4.03139
Mean_Variance	60	0.784605	4.11137
Minimum_Variance	12	0.811705	3.42343
Minimum_Variance	36	0.777205	3.73491
Minimum_Variance	60	0.722784	3.82282
Naive	NA	0.702693	4.50606

All optimization models worked better than naïve model, giving higher returns at low level of risk. This highlights the benefits of using more sophisticated optimization techniques, as the Naive approach fails to manage risk effectively or achieve higher returns.

For 12 months and 36 months window the minimum variance model is giving better returns at a lower risk compared to Mean variance model which suggest that opting for risk minimization can lead to better returns over short period.

For long term 60 months window even though mean_variance is better but compared to other windows there is decline in performance which indicates that there can be some issues with model stability.

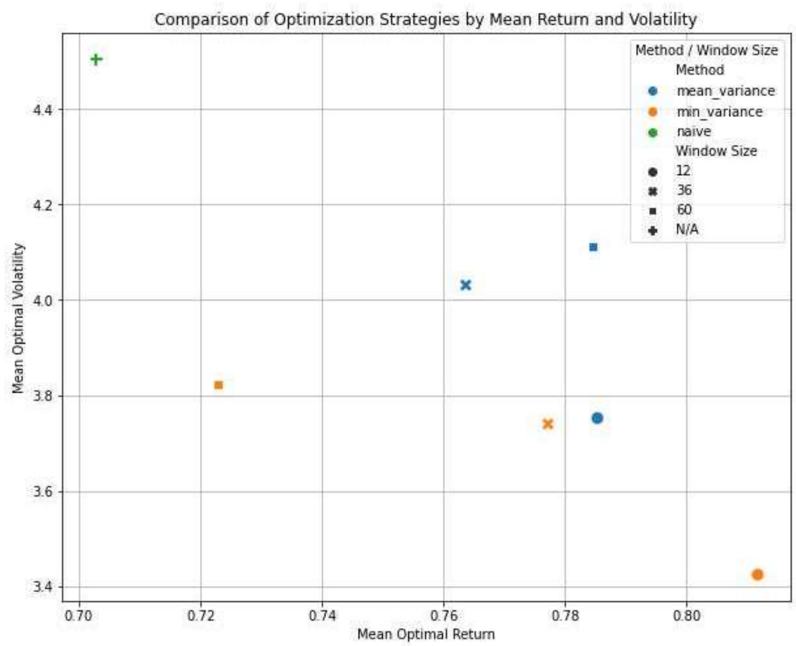

Figure 11 Mean- standard deviation diagram

2. Sharpe Ratio

The Mean-Variance method shows decreasing Sharpe Ratios as the window size increases. This decreasing trend suggests that the Mean-Variance method becomes less effective at providing risk-adjusted returns over longer periods.

The Minimum-Variance method shows a very high Sharpe Ratio for the 12-month window, which seems abnormally high (16.494279). This could indicate an anomaly and may skew the overall perception of Minimum-Variance performance.

The Naive portfolio has the lowest Sharpe Ratio (0.079565), which is consistent because it does not optimize for risk-adjusted return.

Methods	Window	Sharpe Ratio
Mean_Variance	12	0.200837
Mean_Variance	36	0.125805
Mean_Variance	60	0.120092
Minimum_Variance	12	16.494279
Minimum_Variance	36	0.150797
Minimum_Variance	60	0.117281
Naive	NA	0.079565

3. Optimal weights comparison

To further investigate the high sharpe ratio in the minimum variance in 12 months window by comparing it with mean variance optimal weight allocation.

The main aim of mean variance is to strike a balance between risk and return, hence it seemed to have more balanced weights and minimum variance is just focused on minimizing the portfolio volatility. Lower volatility leads to smaller fluctuations in returns, which can result in a higher Sharpe Ratio, particularly in periods of market volatility.

Minimum Variance has not considered diversifying much into small cap growth and neutral shares compared to mean variance.

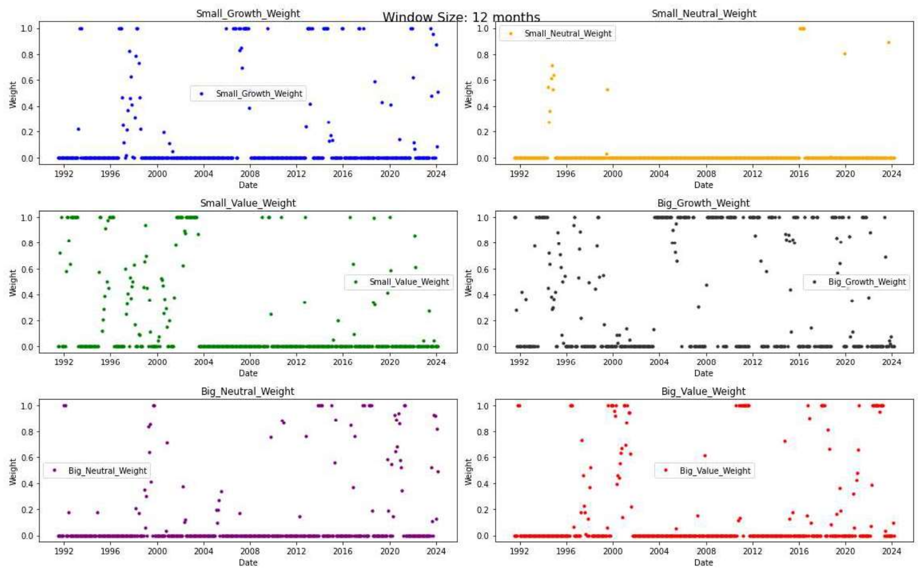

Figure 12 Optimal weights - mean_variance_12 months

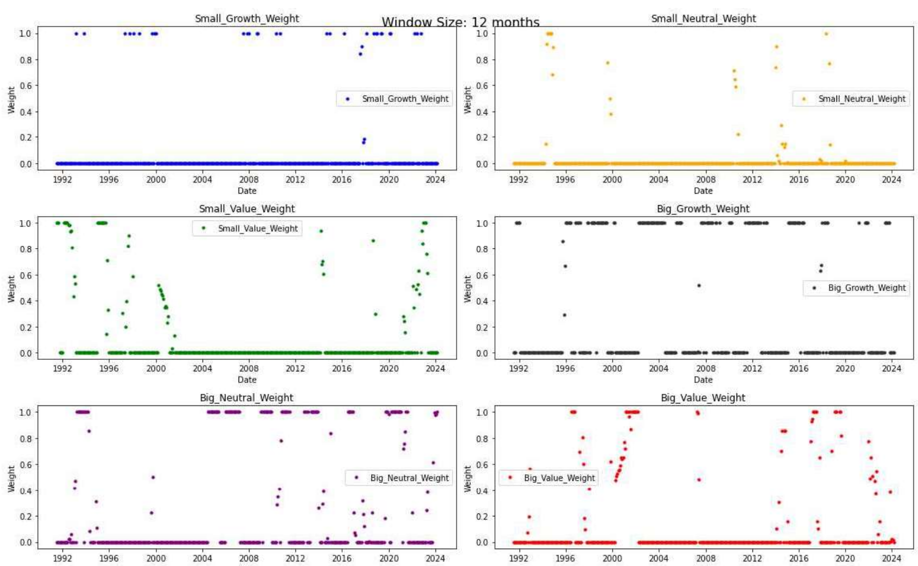

Figure 13 Optimal weight_min_variance_12 months

4. Naïve vs Optimization model returns (t+1)

This study examines the performance of optimization models applied to t+1 returns using optimal weights from time t, contrasting them with the naïve model. Our aim is to assess and compare returns from time t to t+1

Mean variance optimization demonstrates superior and less volatile returns compared to the naïve model across 12 and 36-month windows. However, in the 60-month window, optimization occasionally generates spikes in negative returns, suggesting the need for further model refinement to enhance robustness and stability.

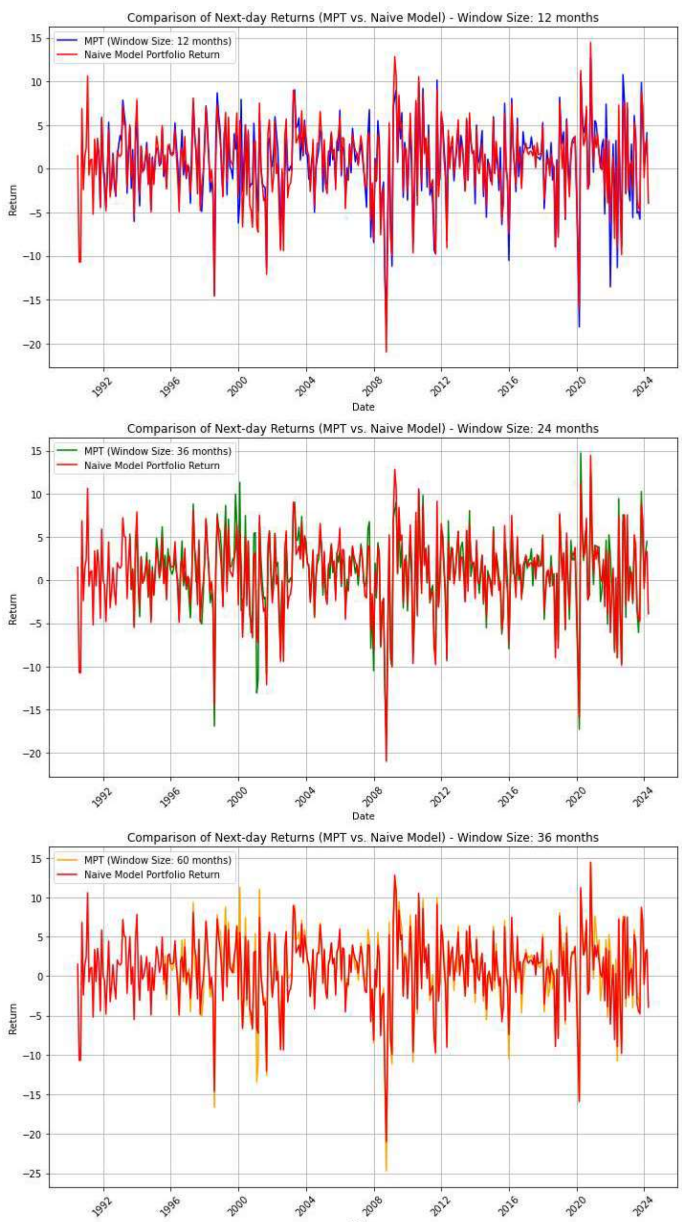

Figure 14 Comparison of naive and mean variance returns

In the 12-month minimum variance optimization, the allocation of optimal weights leads to notably higher positive returns compared to the naïve approach, potentially contributing to the observed high Sharpe ratio. However, in the 60-month window optimization, there is room for improvement, as the returns closely resemble those of the naïve model. Further refinement of the optimization strategy could enhance performance in this longer-term scenario.

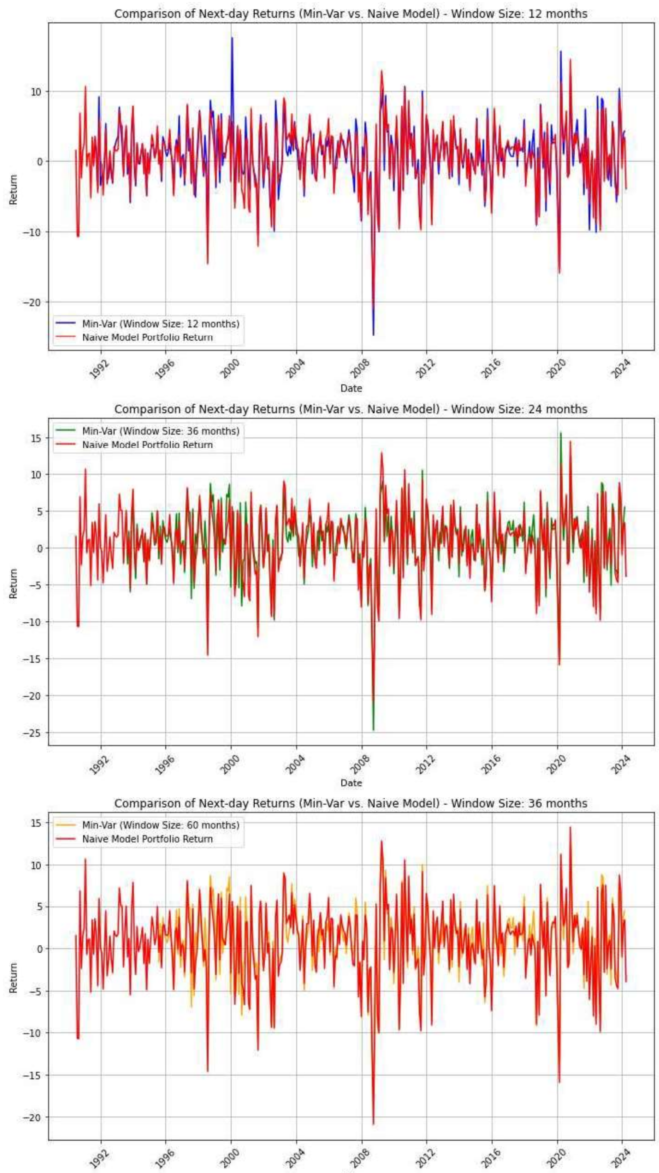

Figure 15 Comparison of naive and minimum variance returns

Conclusion

In conclusion, our analysis revealed that while the 12-month minimum variance model provided optimal returns at the lowest volatility. Mean variance optimization displayed good performance but couldn't achieve the target return at a minimum level of risk. However, the 60-month window indicated the need for refinement to enhance model stability. Our findings underscore the importance of continuous evaluation and refinement of optimization strategies to adapt to evolving market dynamics and ensure robust portfolio performance.

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Appendix

Appendix 1.1 – Other details about data selected

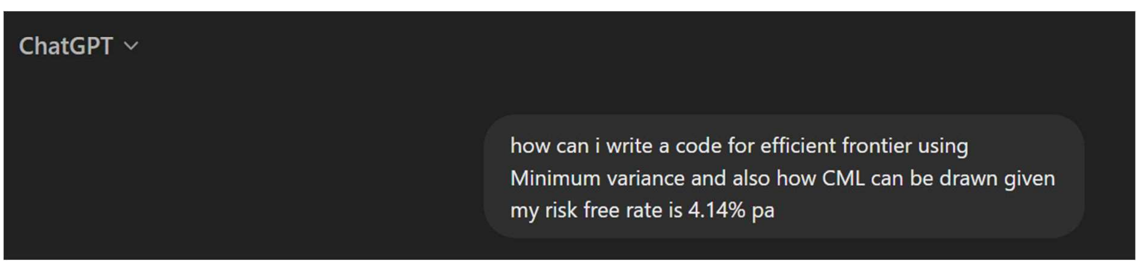
1. All returns are in US dollars which includes dividend and capital gain and are not continuously compounded.
2. The portfolios are constructed at the end of June. ME is market cap at the end of June. BE/ME is book equity at the last fiscal year end of the prior calendar year divided by ME as of 6 months before formation. Firms with negative BE are not included in any portfolio.
3. The portfolios include utilities and include financials.
4. Missing data are indicated by -99.99.
5. The provided dataset consists of two main tables showing monthly and annual returns (both value-weighted and equal-weighted) for different portfolios categorized by size and investment levels from July 1990 to April 2024. Additionally, it includes the number of firms in each portfolio over the same period.

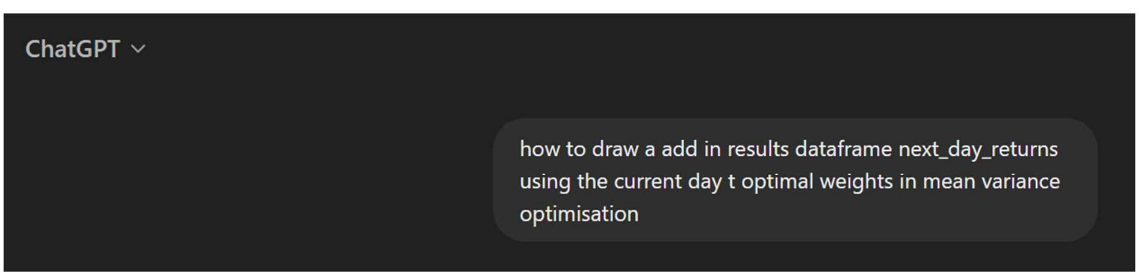
Appendix 1.2 – Fundamental assumptions of MPT

1. Investors are rational decision-makers who aim to maximize their utility or wealth.
2. Investors like high expected returns
3. Investors dislike volatility of returns
4. Investors cannot manipulate market prices through their trading activities.
5. Investors have a single-period investment horizon, meaning they make investment decisions based on their expectations for one future time period

Appendix 1.3 – Use of Chat GPT

Used Chat gpt in codes format and code understanding





ChatGPT

how to add a graph that compares the optimal weight of all 3 windows like 12, 36 and 60 giving the allocation in all 6 portflios in a single graph

ChatGPT

how to calculate target return using CAPM , risk free rate 4.14% pa , market return 9.17% p.a., calculate beta and then target return

ChatGPT

how to add the window size loop inside the result optimisation problem to run the below function and get the result

```
# Define the function to calculate portfolio performance
def portfolio_performance(weights, returns, cov_matrix):
    port_return = np.dot(weights, returns)
    port_volatility = np.sqrt(np.dot(weights.T,
np.dot(cov_matrix, weights)))
    return port_return, port_volatility

# Define the function to minimize (volatility)
def minimize_volatility(weights, returns, cov_matrix):
    return portfolio_performance(weights, returns,
cov_matrix)[1]

# Define the optimization function
def optimize_portfolio(mean_returns, cov_matrix,
target_return):
    num_assets [illegible]n(mean_returns)
    # Constraints: sum of weights is 1 and portfolio return
```