MTH5105 Differential and Integral Analysis

Revision Lecture

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General Information

- Two hours
- Four questions
- Each question counts 25%
- No calculators

Question 1

(a) Let $a, x \in \mathbb{R}$ with $a < x$. Let the real-valued function f be n times continuously differentiable on $[a, x]$ and $(n+1)$ times continuously differentiable on (a, x).

(i) Write down the n-th Taylor polynomial $T_{n,a}$ of f at a, and write down both integral and Lagrange forms of the remainder

$$R_{n,a} = f - T_{n,a} \ .$$

(ii) Find the Taylor polynomial $T_{2,1}$ of f at $a = 1$ for

$$f(x) = (1 + 2x)^{-1/2} \ ,$$

and find both integral and Lagrange forms of the remainder $R_{2,1}$.

(b) Let $g(x) = \log(1 - x)$.

(i) Write down the Taylor series at zero for g.

(ii) By factorising $1 - x^4$, or otherwise, determine the Taylor series at zero for $f(x) = \log(1 + x + x^2 + x^3)$ up to order x^7.

Question 2

Suppose that the function $f : [0,1] \to \mathbb{R}$ is decreasing.

(a) State why $\int_0^1 f(x)\,dx$ exists.

(b) Given the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$ of $[0,1]$, find the upper and lower sums $U(f, P_n)$ and $L(f, P_n)$.

(c) Let

$$S_n = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right)\right) .$$

Prove that

$$S_n \le \int_0^1 f(x)\,dx \le S_n + \frac{1}{n}\left(f(0) - f(1)\right) .$$

Hence deduce that $S_n \to \int_0^1 f(x)\,dx$ as $n \to \infty$.

(d) By considering the function $f(x) = (2+x)^{-2}$, prove that

$$n\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} + \dots + \frac{1}{(3n)^2}\right) \to \frac{1}{6}$$

as $n \to \infty$.

Question 3

We say that a real-valued function f defined on an interval I is *Lipschitz* if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all x and y in I.

(a) State the Boundedness Principle and the Mean Value Theorem.

(b) By using the Boundedness Principle and the Mean Value Theorem, or otherwise, prove that every continuously differentiable function on $[0, 1]$ is Lipschitz.

(c) What does it mean to say that a real-valued function f defined on an interval I is uniformly continuous?

(d) Show that a Lipschitz function is uniformly continuous.

Question 4

For $m \in \mathbb{N}$, define $f_m : \mathbb{R} \to \mathbb{R}$ by

$$f_m(x) = \frac{x}{m^2 + x^2} \, .$$

(a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{m=1}^{\infty} f_m(x)$ converges.

(b) Show that the sum $\sum_{m=1}^{\infty} f'_m(x)$ converges uniformly for all $x \in \mathbb{R}$.
[Hint: $|m^2 - x^2| \leq m^2 + x^2$]

(c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{m=1}^{\infty} \frac{x}{m^2 + x^2}$$

is differentiable. What is $f'(x)$?

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The n-th Taylor polynomial $T_{n,a}(x)$ equals

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n \; .$$

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The integral and Lagrange forms of the remainder $R_{n,a}(x)$ are

$$\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$$

for some $c \in (a,x)$, respectively.

Question 1 (a) (ii)

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(ii) Find the Taylor polynomial $T_{2,1}$ of f at $a = 1$ for

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We compute $f'(x) = -(1+2x)^{-3/2}$, $f''(x) = 3(1+2x)^{-5/2}$, and $f'''(x) = -15(1+2x)^{-7/2}$.

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$$T_{2,1}(x) = \frac{1}{3}\sqrt{3} - \frac{1}{9}\sqrt{3}(x-1) + \frac{1}{18}\sqrt{3}(x-1)^2 \ .$$

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The integral and Lagrange forms of the remainder are

$$R_{2,1}(x) = -\frac{15}{2}\int_1^x \frac{(x - t)^2}{(1 + 2t)^{7/2}}\, dt = -\frac{5}{2}\frac{(x - 1)^3}{(1 + 2c)^{7/2}}$$

for some $c \in (1, x)$, respectively.

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The series expansion for $\log(1-x)$ is known:

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k \, .$$

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We factorise $1 - x^4 = (1 + x + x^2 + x^3)(1 - x)$.

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We factorise $1 - x^4 = (1 + x + x^2 + x^3)(1 - x)$.

We find

$$\begin{aligned}\log(1 + x + x^2 + x^3) &= \log(1 - x^4) - \log(1 - x) \\ &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \dots .\end{aligned}$$

Question 2

Suppose that the function $f:[0,1]\to\mathbb{R}$ is decreasing.

(a) State why $\int_0^1 f(x)\,dx$ exists.

(b) Given the partition $P_n=\{0,\frac{1}{n},\frac{2}{n},\dots,\frac{n}{n}\}$ of $[0,1]$, find the upper and lower sums $U(f,P_n)$ and $L(f,P_n)$.

(c) Let

$$S_n=\frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right)\ .$$

Prove that

$$S_n\leq\int_0^1 f(x)\,dx\leq S_n+\frac{1}{n}\left(f(0)-f(1)\right)\ .$$

Hence deduce that $S_n\to\int_0^1 f(x)\,dx$ as $n\to\infty$.

(d) By considering the function $f(x)=(2+x)^{-2}$, prove that

$$n\left(\frac{1}{(2n+1)^2}+\frac{1}{(2n+2)^2}+\ldots+\frac{1}{(3n)^2}\right)\to\frac{1}{6}$$

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Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable.

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f is decreasing, so that on $I_i = [x_{i-1}, x_i]$ we have

$$\sup_{[x_{i-1},x_i]} f = f(x_{i-1}) \quad \text{and} \quad \inf_{[x_{i-1},x_i]} f = f(x_i)\ .$$

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For P_n, $x_i = i/n$ and $|I_i| = 1/n$, and we find

$$U(f, P_n) = \frac{1}{n}\left(f\left(\frac{0}{n}\right) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right)\right)$$

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and

$$L(f, P_n) = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right)\ .$$

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(c) Let $S_n = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right)$. Prove that

$$S_n \leq \int_0^1 f(x)\,dx \leq S_n + \frac{1}{n}\left(f(0) - f(1)\right) \ .$$

Hence deduce that $S_n \to \int_0^1 f(x)\,dx$ as $n \to \infty$.

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From $L(f, P_n) \leq \int_0^1 f(x)\,dx \leq U(f, P_n)$ and

$$L(f, P_n) = S_n \quad \text{and} \quad U(f, P_n) = S_n + (f(0) - f(1))/n \ ,$$

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we find $S_n \leq \int_0^1 f(x)\,dx \leq S_n + \frac{1}{n}(f(0) - f(1))$.

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From $L(f,P_n) \leq \int_0^1 f(x)\,dx \leq U(f,P_n)$ and

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we find $S_n \leq \int_0^1 f(x)\,dx \leq S_n + \frac{1}{n}(f(0) - f(1))$.
As $\frac{1}{n}(f(0) - f(1)) \to 0$ as $n \to \infty$, we find

$$\lim_{n\to\infty} S_n \leq \int_0^1 f(x)\,dx \leq \lim_{n\to\infty} S_n \ .$$

Question 2 (d)

Suppose that the function $f : [0,1] \to \mathbb{R}$ is decreasing.

(d) By considering the function $f(x) = (2+x)^{-2}$, prove that

$$n\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} + \ldots + \frac{1}{(3n)^2}\right) \to \frac{1}{6}$$

as $n \to \infty$.

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as $n \longrightarrow \infty$.

The function $f(x) = (2+x)^{-2}$ is decreasing, hence

$$\begin{aligned} S_n &= \frac{1}{n}\left(\frac{1}{(2+1/n)^2} + \frac{1}{(2+2/n)^2} + \ldots + \frac{1}{(2+n/n)^2}\right) \\ &= n\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} + \ldots + \frac{1}{(3n)^2}\right) . \end{aligned}$$

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By (c), $S_n \to \int_0^1 f(x)\,dx = \int_0^1 (2+x)^{-2} dx = 1/6$ as $n \to \infty$.

Question 3

We say that a real-valued function f defined on an interval I is *Lipschitz* if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all x and y in I.

(a) State the Boundedness Principle and the Mean Value Theorem.

(b) By using the Boundedness Principle and the Mean Value Theorem, or otherwise, prove that every continuously differentiable function on $[0, 1]$ is Lipschitz.

(c) What does it mean to say that a real-valued function f defined on an interval I is uniformly continuous?

(d) Show that a Lipschitz function is uniformly continuous.

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The boundedness principle states that a real-valued function continuous on $[a, b]$ attains its minimum and maximum.

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(a) State the Boundedness Principle and the Mean Value Theorem.

The boundedness principle states that a real-valued function continuous on $[a, b]$ attains its minimum and maximum.

The mean value theorem states that for a real-valued function continuous on $[a, b]$ and differentiable on (a, b) there exists a $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a)$.

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As f' is continuous on a closed interval, f' attains its minumum and maximum, hence f' is bounded, i.e. $|f'| \leq M$ for some $M \geq 0$.

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As f' is continuous on a closed interval, f' attains its minumum and maximum, hence f' is bounded, i.e. $|f'| \leq M$ for some $M \geq 0$. Now by the Mean Value Theorem, for all $x, y \in I$ with $x < y$ there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

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which implies $|f(y) - f(x)| \leq M|y - x|$ as needed.

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As f' is continuous on a closed interval, f' attains its minumum and maximum, hence f' is bounded, i.e. $|f'| \leq M$ for some $M \geq 0$. Now by the Mean Value Theorem, for all $x, y \in I$ with $x < y$ there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

which implies $|f(y) - f(x)| \leq M|y - x|$ as needed. The case $y < x$ is analogous, and the case $y = x$ is obvious.

Question 3 (c)

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for all x and y in I.

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(c) What does it mean to say that a real-valued function f defined on an interval I is uniformly continuous?

f is uniformly continuous on I if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x, y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \, .$$

Question 3 (d)

We say that a real-valued function f defined on an interval I is *Lipschitz* if there is a constant M such that

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(d) Show that a Lipschitz function is uniformly continuous.

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We say that a real-valued function f defined on an interval I is *Lipschitz* if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all x and y in I.

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Assume that f is Lipshitz, i.e.

$$\exists M > 0 \, \forall x, y \in I : |f(x) - f(y)| \leq M|x - y| \, .$$

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Now given $\epsilon > 0$ choose $\delta = \epsilon / M$. Then

$$\forall x, y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon \; .$$

Question 4

For $m \in \mathbb{N}$, define $f_m : \mathbb{R} \to \mathbb{R}$ by

$$f_m(x) = \frac{x}{m^2 + x^2} \; .$$

(a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{m=1}^{\infty} f_m(x)$ converges.

(b) Show that the sum $\sum_{m=1}^{\infty} f'_m(x)$ converges uniformly for all $x \in \mathbb{R}$.
[Hint: $|m^2 - x^2| \leq m^2 + x^2$]

(c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{m=1}^{\infty} \frac{x}{m^2 + x^2}$$

is differentiable. What is $f'(x)$?

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$\sum \frac{1}{m^2}$ converges, so that the sum converges absolutely (for fixed x).

Question 4 (b)

For $m \in \mathbb{N}$, define $f_m : \mathbb{R} \to \mathbb{R}$ by

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(b) Show that the sum $\sum_{m=1}^{\infty} f_m'(x)$ converges uniformly for all $x \in \mathbb{R}$.
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The bound $\sum \frac{1}{m^2}$ is independent of x.
By the Weierstraß criterion, the sum converges uniformly.

Question 4 (c)

For $m \in \mathbb{N}$, define $f_m : \mathbb{R} \to \mathbb{R}$ by

$$f_m(x) = \frac{x}{m^2 + x^2} \,.$$

(c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

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As $\sum f_m$ converges pointwise and $\sum f'_m$ converges uniformly, $f = \sum f_m$ is differentiable and $f' = \sum f'_m$. Therefore

$$f'(x) = \sum_{m=1}^{\infty} \frac{m^2 - x^2}{(m^2 + x^2)^2} .$$

The End