MTH5105 Differential and Integral Analysis 2009-2010

Solutions 5

1 Exercise for Feedback/Assessment

1) Let $f(x) = \exp(-1/\sqrt{x})$, $g(x) = \cos(\pi x/2)$, and $P = \{1, 4, 9, 16\}$.

   (a) Find the upper and lower sums $U(f, P)$ and $L(f, P)$ of f for the partition P. Use these sums to give bounds for $\int_1^{16} f(x)\, dx$. [10 marks]

   (b) Find the upper and lower sums $U(g, P)$ and $L(g, P)$ of g for the partition P. Use these sums to give bounds for $\int_1^{16} g(x)\, dx$. [10 marks]

   Solution:

   (a) Recall that $I_i = [x_i - x_{i-1}]$, $\Delta x_i = x_i - x_{i-1}$, $M_i = \sup_{x \in I_i} f(x)$, and $m_i = \inf_{x \in I_i} f(x)$. We have

$$\begin{aligned}
I_1 &= [1,4]\ , & \Delta_1 &= 3\ , & M_1 &= \exp(-1/2)\ , & m_1 &= \exp(-1/1)\ ,\\
I_2 &= [4,9]\ , & \Delta_2 &= 5\ , & M_2 &= \exp(-1/3)\ , & m_2 &= \exp(-1/2)\ ,\\
I_3 &= [9,16]\ , & \Delta_3 &= 7\ , & M_3 &= \exp(-1/4)\ , & m_3 &= \exp(-1/3)\ .
\end{aligned}$$

   [4 marks]

   Therefore

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{3} M_i \Delta x_i = 3\exp(-1/2) + 5\exp(-1/3) + 7\exp(-1/4)\ ,\\
L(f,P) &= \sum_{i=1}^{3} m_i \Delta x_i = 3\exp(-1) + 5\exp(-1/2) + 7\exp(-1/3)\ .
\end{aligned}$$

   [4 marks]

   Hence we have

$$3\exp(-1) + 5\exp(-1/2) + 7\exp(-1/3) \le \int_1^{16} f(x)\, dx \le 3\exp(-1/2) + 5\exp(-1/3) + 7\exp(-1/4)\ .$$

   [2 marks]

   (In fact, the integral evaluates to about 10.17, while the lower and upper sums are approximately 9.15 and 10.85.)

   (b) We have now

$$M_1 = 1\ , \quad m_1 = -1\ , \quad M_2 = 1\ , \quad m_2 = -1\ , \quad M_3 = 1\ , \quad m_3 = -1\ .$$

   [4 marks]

   Therefore

$$U(g,P) = 3\cdot 1 + 5\cdot 1 + 7\cdot 1\ , \qquad L(g,P) = 3\cdot(-1) + 5\cdot(-1) + 7\cdot(-1)\ .$$

[4 marks]

Hence we have

$$-15 \le \int_1^{16} g(x)\,dx \le 15 \,.$$

[2 marks]

(In fact, the integral evaluates to $-2/\pi \approx -0.637$.)

2 Extra Exercises

2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \,. \end{cases}$$

(a) Given a partition P of $[-1, 1]$, what is $L(f, P)$?
What is $\int_{*-1}^{1} f(x)\,dx$?

(b) For fixed $\epsilon > 0$, find a partition P of $[-1, 1]$ such that $U(f, P) < \epsilon$.
What is $\int_{-1}^{*1} f(x)\,dx$?

(c) Is f integrable on $[-1, 1]$? If so, what is its integral?

Solution:

(a) Given a partition P of $[-1, 1]$, the function f has infimum 0 in any subinterval. Therefore $L(f, P) = 0$ for any partition P.
Hence $\int_{*-1}^{1} f(x)\,dx = 0$.

(b) For $0 < \delta < 1$, choose $P = \{-1, -\delta, \delta, 1\}$. On the intervals $[-1, -\delta]$ and $[\delta, 1]$ the function f has maximum value 0. On the interval $[-\delta, \delta]$ it has maximum value 1. Therefore

$$U(f, P) = ((-\delta) - (-1)) \cdot 0 + (\delta - (-\delta)) \cdot 1 + (1 - \delta) \cdot 0 = 2\delta \,,‘$$

and if we choose $\delta < \epsilon/2$, we have $U(f, P) < \epsilon$.
Hence $\int_{-1}^{*1} f(x)\,dx \le 0$. Using (a), we have

$$0 = \int_{*-1}^{1} f(x)\,dx \le \int_{-1}^{*1} f(x)\,dx \le 0 \,,$$

so that $\int_{*-1}^{1} f(x)\,dx = 0$.

(c) As

$$\int_{-1}^{*1} f(x)\,dx = \int_{-1}^{*1} f(x)\,dx = 0 \,,$$

f is integrable and $\int_{-1}^{1} f(x)\,dx = 0$.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the equidistant partitions P_n of $[0, 1]$ into n subintervals.

(a) Find $U(f, P_n)$. What can you say about $\int_0^{*1} f(x)\,dx$?

(b) Find $L(f, P_n)$. What can you say about $\int_{*0}^{1} f(x)\,dx$?

(c) Is f integrable on $[0, 1]$? If so, what is its integral?

[Hint: $\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1)$.]

Solution:

We have

$$P_n = \{0/n, 1/n, \ldots, n/n\} \ ,$$

or $x_i = i/n$ for $i = 0, \ldots, n$. Thus, $I_i = [(i-1)/n, i/n]$ and $\Delta x_i = 1/n$.

(a) We have $M_i = (i/n)^2$ and thus

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \ . \end{aligned}$$

Hence, $\int_0^{*1} f(x)\,dx \le 1/3$.

(b) Similarly we have $m_i = ((i-1)/n)^2$ and thus

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \ . \end{aligned}$$

Hence, $\int_{*0}^{1} f(x)\,dx \ge 1/3$.

(c) Combining these we see that $\int_0^1 x^2\,dx$ exists and equals $1/3$.

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