MAS115 Calculus I

Week 11

Thomas Prellberg

School of Mathematical Sciences
Queen Mary, University of London

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Revision

- Derivatives and Integrals of Hyperbolic Functions
- Inverse Hyperbolic Functions
- Derivatives and Integrals of Inverse Hyperbolic Functions
- Techniques of Integration
- Integration by Parts

Repeated Integration by Parts

Evaluate

$$\int x^2 e^x dx \ :$$

Choose $u = x^2$ and $dv = e^x dx$, so that $du = 2x dx$ and $v = e^x$. Integrate by parts:

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx \ .$$

Next, choose $u = x$ and $dv = e^x dx$, so that $du = dx$ and $v = e^x$. Integrate by parts:

$$\begin{aligned} \int x e^x dx &= x e^x - \int e^x dx \\ &= x e^x - e^x + C_1 \ . \end{aligned}$$

Together,

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C \ .$$

Repeated Integration by Parts with a "Twist"

Evaluate

$$\int e^x \cos x \, dx \ :$$

Choose $u = e^x$ and $dv = \cos x \, dx$, so that $du = e^x dx$ and $v = \sin x$. Integrate by parts:

$$\int e^x \cos x \, dx \quad = \quad e^x \sin x - \int e^x \sin x \, dx \ .$$

Next, choose $u = e^x$ and $dv = \sin x \, dx$, so that $du = e^x dx$ and $v = -\cos x$. Integrate by parts:

$$\int e^x \sin x \, dx \quad = \quad -e^x \cos x + \int e^x \cos x \, dx$$

Together,

$$\int e^x \cos x \, dx \quad = \quad e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \ .$$

What now? Solve for $\int e^x \cos x \, dx$

Repeated Integration by Parts with a “Twist”

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \; .$$

Solve for $\int e^x \cos x \, dx$ (and add a constant of integration):

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1 \; .$$

The final answer is

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C \; .$$

It is easy to forget the constant of integration here.

A Reduction Formula

Evaluate

$$\int \cos^n x \, dx \ :$$

Choose $u = \cos^{n-1} x$ and $dv = \cos x \, dx$, so that

$$du = (n-1)\cos^{n-2} x(-\sin x)dx \quad \text{and} \quad v = \sin x \, .$$

Integrate by parts:

$$\begin{aligned}
\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x \, dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx \\
&\quad -(n-1)\int \cos^n x \, dx \, .
\end{aligned}$$

In the last step, we have replaced $\sin^2 x = 1 - \cos^2 x$.
Solve for $\int \cos^n x \, dx$:

$$n\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx$$

A Reduction Formula

Integration by parts reduces the power from n to $n-2$:

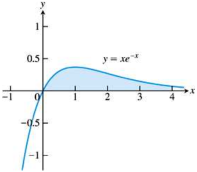
$$\int \cos^n x \, dx \quad = \quad \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Application:

$$\begin{aligned} \int \cos^3 x \, dx \quad &= \quad \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx \\ &= \quad \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C \end{aligned}$$

Example

Find the area between the x-axis and $y = xe^{-x}$ from $x = 0$ to $x = 4$:



$$\int_0^4 xe^{-x}dx$$

Choose $u = x$ and $dv = e^{-x}dx$, so that $du = dx$ and $v = -e^{-x}$. Integrate by parts:

$$\begin{aligned}
\int_0^4 xe^{-x}dx &= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}dx \\
&= -xe^{-x}\Big|_0^4 - e^{-x}\Big|_0^4 \\
&= -4e^{-4} + 0e^{-0} - e^{-4} + e^{-0} = 1 - 5e^{-4}
\end{aligned}$$

What about the area between $x = 0$ and $x = \infty$?

$$\int_0^\infty xe^{-x}dx = \left(-xe^{-x} - e^{-x}\right)\Big|_0^\infty = 1 \ .$$

A more careful treatment will follow shortly.

The Method of Partial Fractions

If you know that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

then you can integrate easily

$$\begin{aligned} \int \frac{5x-3}{x^2-2x-3}dx &= \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx \\ &= 2\ln|x+1| + 3\ln|x-3| + C \end{aligned}$$

To obtain such simplifications, we use the method of partial fractions.

The Method of Partial Fractions

Let $f(x)/g(x)$ be a rational function, for example

$$\frac{f(x)}{g(x)} = \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}$$

- If $\deg(f) \geq \deg(g)$, we first use polynomial division:

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

  and consider the remainder term.
- We also have to know the factors of $g(x)$:

$$x^2 - 2x - 3 = (x+1)(x-3)$$

- Now we can write

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x+1} + \frac{B}{x-3}$$

  and obtain $A = 2$ and $B = 3$

The Method of Partial Fractions

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example for Distinct Linear Factors

Find
$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx \ :$$

- Write
$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

- Multiply by $(x-1)(x+1)(x+3)$ to get
$$\begin{aligned} x^2+4x+1 &= A(x+1)(x+3)+B(x-1)(x+3) \\ &\quad +C(x-1)(x+1) \\ &= (A+B+C)x^2+(4A+2B)x \\ &\quad +(3A-3B-C) \end{aligned}$$

- Equate coefficients of equal powers of x:
$$A+B+C=1\,, \quad 4A+2B=4\,, \quad 3A-3B-C=1$$

Example for Distinct Linear Factors

- Solve

$$A + B + C = 1\,, \quad 4A + 2B = 4\,, \quad 3A - 3B - C = 1$$

to get

$$A = \frac{3}{4}\,, \quad B = \frac{1}{2}\,, \quad C = -\frac{1}{4}$$

- Now integrate

$$\begin{aligned}
&\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx \\
&= \quad \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+3} \\
&= \quad \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C
\end{aligned}$$

Example for a Quadratic Factor

Find
$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx \quad :$$
[x^2+1 is *irreducible* in $\mathbb{R}$, i.e. cannot be factored]

- Write
$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$
- Set of linear equations:
$$\begin{aligned} 0 &= A + C & 0 &= -2A + B - C + D \\ -2 &= A - 2B + C & 4 &= B - C + D \end{aligned}$$
- We find
$$A = 2\,, \quad B = 1\,, \quad C = -2\,, \quad D = 1$$

  so that
$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

Example for a Quadratic Factor

- Now integrate

$$
\begin{aligned}
&\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx \\
&\quad = \int \frac{2x+1}{x^2+1}dx - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \\
&\quad = \int \frac{2x\,dx}{x^2+1} + \int \frac{dx}{x^2+1} - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \\
&\quad = \ln(x^2+1) + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C
\end{aligned}
$$

Revision

- Repeated Integration by Parts
- The Method of Partial Fractions

Example for a Repeated Quadratic Factor

Find
$$\int \frac{dx}{x(x^2+1)^2} \quad :$$

- Write
$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

- Solve a set of linear equations to get
$$A = 1\,, \quad B = -1\,, \quad C = 0\,, \quad D = -1\,, \quad E = 0$$

- Now integrate
$$\int \frac{dx}{x(x^2+1)^2} \quad = \quad \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

Example for a Repeated Quadratic Factor

$$
\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \frac{1}{2}\int \frac{2x\,dx}{(x^2+1)^2} \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(x^2+1)^{-1} + C
\end{aligned}
$$

The method of partial fractions is conceptually easy, but it gets quickly cumbersome!

Trigonometric Integrals

$$\int \sin^m x \cos^n x \, dx \quad \text{for } m, n \text{ non-negative integers}$$

Case 1: $m = 2k+1$ odd: use $\sin^{2k+1} x = (1-\cos^2 x)^k \sin x$

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (1-\cos^2 x)^k \cos^n x \sin x \, dx$$
$$= -\int (1-u^2)^k u^n du \quad \text{where } u = \cos x$$

Example:

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int (1-\cos^2 x) \cos^2 x \sin x \, dx \\
&= -\int (1-u^2) u^2 du = \int (u^4 - u^2) du \\
&= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\
&= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C
\end{aligned}$$

Trigonometric Integrals

$$\int \sin^m x \cos^n x \, dx \quad \text{for } m, n \text{ non-negative integers}$$

Case 2: $n = 2k+1$ odd: use $\cos^{2k+1} x = (1-\sin^2 x)^k \cos x$

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (1-\sin^2 x)^k \cos x \, dx$$

$$= \int u^m (1-u^2)^k du \quad \text{where } u = \sin x$$

Example:

$$\begin{aligned} \int \cos^5 x \, dx &= \int (1-\sin^2 x)^2 \cos x \, dx \\ &= \int (1-u^2)^2 du = \int (u^4 - 2u^2 + 1) du \\ &= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C \\ &= \frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C \end{aligned}$$

Trigonometric Integrals

$$\int \sin^m x \cos^n x \, dx \quad \text{for } m, n \text{ non-negative integers}$$

Case 3: both $m = 2k$ and $m = 2l$ even: use

$$\sin^{2k} x = \left(\frac{1 - \cos 2x}{2}\right)^k \quad \text{and} \quad \cos^{2l} x = \left(\frac{1 + \cos 2x}{2}\right)^l$$

Example:

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx \\
&= \frac{1}{8} \left(x + \frac{1}{2} \sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right)
\end{aligned}$$

Trigonometric Integrals

So far we simplified $\int \sin^2 x \cos^4 x \, dx$

$$= \frac{1}{8}\left(x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right)$$

$\int \cos^2 2x \, dx$ is again Case 3:

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x) dx = \frac{1}{2}(x + \frac{1}{4}\sin 4x)$$

$\int \cos^3 2x \, dx$ is Case 2:

$$\begin{aligned} \int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\ &= \frac{1}{2}\int (1 - u^2) du = \frac{1}{2}(\sin 2x - \frac{1}{3}\sin^3 2x) \end{aligned}$$

Putting it all together, we find

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x + \frac{1}{3}\sin^3 2x - \frac{1}{4}\sin 4x\right) + C$$

Trigonometric Integrals

$$\int \sin mx \cos nx \, dx \,, \quad \int \sin mx \sin nx \, dx \,, \quad \int \cos mx \cos nx \, dx$$

Write products of sin and cos as a sum of sin and cos:

$$\sin mx \cos nx = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$
$$\sin mx \sin nx = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x)$$
$$\cos mx \cos nx = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$

Example:

$$\begin{aligned}
\int \sin 3x \cos 5x \, dx &= \int \frac{1}{2}(\sin(3-5)x + \sin(3+5)x) dx \\
&= -\frac{1}{2}\int \sin 2x \, dx + \frac{1}{2}\int \sin 8x \, dx \\
&= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C
\end{aligned}$$

Trigonometric Substitutions

Integrals containing $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$

Use an appropriate substitution:

- $x = a \tan\theta$:

$$a^2 + x^2 = a^2 + a^2 \tan^2\theta = a^2 \sec^2\theta$$

- $x = a \sin\theta$:

$$a^2 - x^2 = a^2 - a^2 \sin^2\theta = a^2 \cos^2\theta$$

- $x = a \sec\theta$:

$$x^2 - a^2 = a^2 \sec^2\theta - a^2 = a^2 \tan^2\theta$$

Careful with signs when taking the square-root!

Example

Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In the first quadrant, $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and thus

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$$

Substitute $x = a\sin\theta$: $dx = a\cos\theta\,d\theta$ and

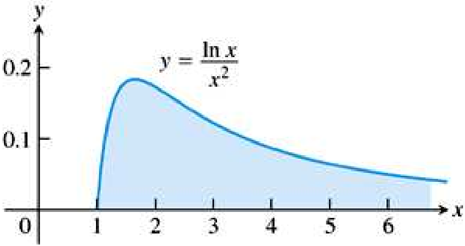
$$a^2 - x^2 = a^2\cos^2\theta$$

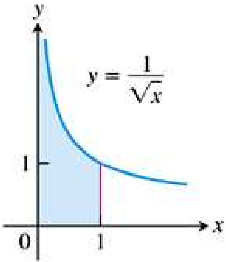
$$A = 4\int_0^{\pi/2} \frac{b}{a}\sqrt{a^2\cos^2\theta}a\cos\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta = \pi ab$$

Improper Integrals

Can we compute areas under infinitely extended curves?

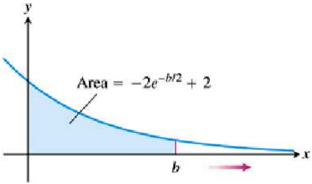




The two cases are slightly different:

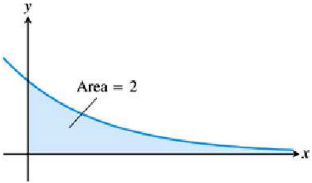
- Type I: on the left, the area extends from $x = 0$ to "$x = \infty$"
- Type 2: on the right, the area extends from $x = 0$ to $x = 1$, but $f(x)$ diverges at $x = 0$.

Type I Improper Integrals



$$A(b) = \int_0^b e^{-x/2} dx$$

$$= \quad -2e^{-x/2}\Big|_0^b = 2 - 2e^{-b/2}$$



Take the limit

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (2 - 2e^{-b/2}) = 2$$

Assign to the infinitely extended area the value 2:

$$\int_0^\infty e^{-x/2} dx = 2$$

Type I Improper Integrals

DEFINITION **Type I Improper Integrals**

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

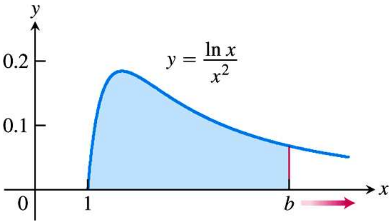
where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Revision

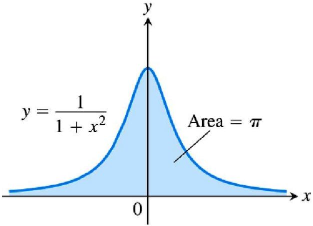
- The Method of Partial Fractions
- Trigonometric Integrals
- Trigonometric Substitutions
- Improper Integrals

Example



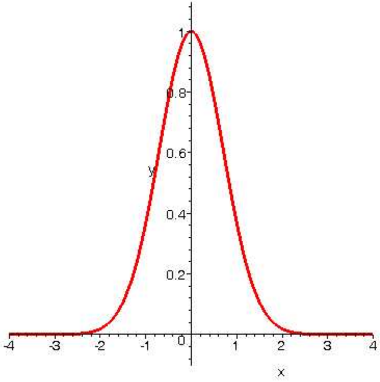
$$
\begin{aligned}
\int_0^\infty \frac{\ln x}{x^2} dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} dx \\
&= \lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^b \\
&= \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1 \right) = 1
\end{aligned}
$$

Example



$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \arctan x\Big|_a^0 + \lim_{b\to\infty} \arctan x\Big|_0^b \\
&= -\lim_{a\to-\infty} \arctan a + \lim_{b\to\infty} \arctan b \\
&= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi
\end{aligned}
$$

The Bell Curve



$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof in Calculus II
(involves trick using 2-dimensional integration)

Convergence of $\int_1^\infty \frac{dx}{x^p}$

For which values of p does $\int_1^\infty \frac{dx}{x^p}$ converge?

- $p = 1$:

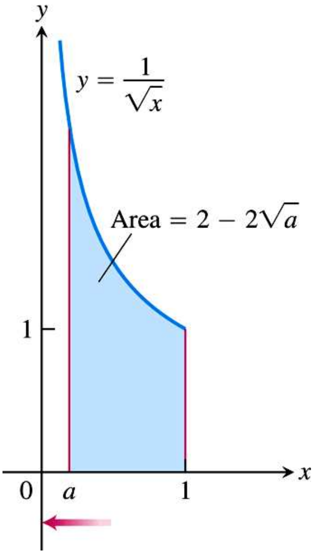
$$\begin{aligned}\int_1^\infty \frac{dx}{x} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x}\\ &= \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty} \ln b = \infty\end{aligned}$$

- $p \neq 1$:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}\\ &= \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Bigg|_1^b = \begin{cases} 1/(p-1) & p > 1\\ \infty & p < 1\end{cases}\end{aligned}$$

Thus, the integral converges if and only if $p > 1$.

Type II Improper Integrals



$$
\begin{aligned}
A(a) &= \int_a^1 \frac{dx}{\sqrt{x}} \\
&= 2\sqrt{x}\Big|_a^1 = 2 - \sqrt{a}
\end{aligned}
$$

Take the limit

$$\lim_{a \to 0} A(a) = \lim_{a \to 0} (2 - 2\sqrt{a}) = 2$$

Assign to the infinitely extended area the value 2:

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2$$

Type II Improper Integrals

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

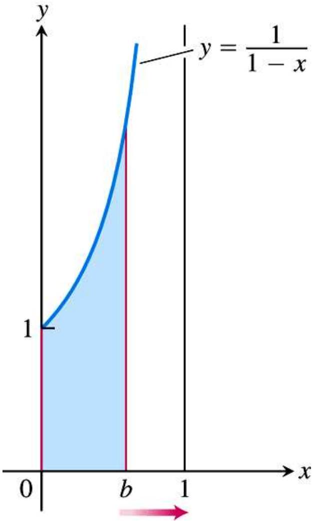
$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

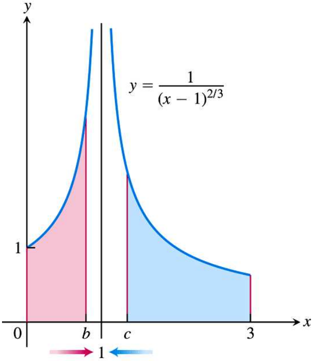
In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example



$$
\begin{aligned}
\int_0^1 \frac{dx}{1-x} &= \lim_{b\to 1^-} \int_0^b \frac{dx}{1-x} \\
&= -\lim_{b\to 1^-} \ln|1-x|\big|_0^b \\
&= -\lim_{b\to 1^-} \ln(1-b) \\
&= \infty
\end{aligned}
$$

Example



$$\int_0^3 \frac{dx}{(1-x)^{2/3}} = \int_0^1 \frac{dx}{(1-x)^{2/3}} + \int_1^3 \frac{dx}{(1-x)^{2/3}}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{(1-x)^{2/3}} + \lim_{c\to 1^+} \int_c^3 \frac{dx}{(1-x)^{2/3}}$$

We compute

$$
\begin{aligned}
\int_0^1 \frac{dx}{(1-x)^{2/3}} &= \lim_{b\to 1^-} \int_0^b \frac{dx}{(1-x)^{2/3}} \\
&= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b \\
&= \lim_{b\to 1^-} \left(3(b-1)^{1/3} - 3(0-1)^{1/3}\right) = 3
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\int_1^3 \frac{dx}{(1-x)^{2/3}} &= \lim_{c\to 1^+} \int_c^3 \frac{dx}{(1-x)^{2/3}} \\
&= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big|_c^3 \\
&= \lim_{c\to 1^+} \left(3(3-1)^{1/3} - 3(c-1)^{1/3}\right) = 3\cdot 2^{1/3}
\end{aligned}
$$

Therefore

$$
\int_0^3 \frac{dx}{(1-x)^{2/3}} = 3 + 3\sqrt[3]{2}
$$

A Wrong Calculation

Where is the mistake?

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big|_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is not a proper integral, as there is a rather "bad" discontinuity at $x = 1$.
If we split up the integral

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

then, for example

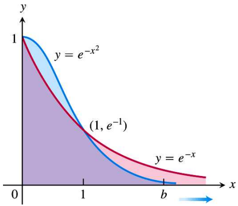
$$\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-} \ln|b-1| = -\infty$$

diverges.

Tests for Convergence/Divergence

Example: does $\int_1^\infty e^{-x^2}\,dx$ converge?

Idea: find a function that dominates e^{-x^2} and is easier to integrate:



$$e^{-x^2} \leq e^{-x} \quad \text{for } x \geq 1$$

Therefore

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = e^{-1} - e^{-b} < e^{-1}$$

for all $b \geq 1$ and

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx \leq e^{-1}$$

First Comparison Test

Theorem (Direct Comparison Test)

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

- $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges*
- $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

Example:

$$\int_1^\infty \frac{\sin^2 x}{x^2} dx \ :$$

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \text{and} \quad \int_1^\infty \frac{dx}{x^2} \text{ converges}$$

so $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges.

First Comparison Test

Theorem (Direct Comparison Test)

Let f and g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

- $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges*
- $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

Example:

$$\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}} :$$

$$\frac{1}{x} \le \frac{1}{\sqrt{x^2-0.1}} \quad \text{and} \quad \int_1^\infty \frac{dx}{x} \text{ diverges}$$

so $\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}}$ diverges.

Second Comparison Test

Theorem (Limit Comparison Test)

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \quad 0 < L < \infty\ ,$$

then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

Example: $$\int_1^\infty \frac{1}{1+x^2}dx\ :$$

$$\lim_{x\to\infty} \frac{1/x^2}{1/(x^2+1)} = 1 \quad \text{and} \quad \int_1^\infty \frac{dx}{x^2} = -\left.\frac{1}{x}\right|_1^\infty = 1$$

so $\int_1^\infty \frac{1}{x^2+1}dx$ converges.

Second Comparison Test

Theorem (Limit Comparison Test)

If the positive functions f and g are continuous on $[a,\infty)$ and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L \quad 0 < L < \infty\ ,$$

then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

Example: $$\int_1^\infty \frac{3}{e^x+5}dx\ :$$

$$\lim_{x\to\infty}\frac{1/e^x}{3/(e^x+5)} = \frac{1}{3} \quad \text{and} \quad \int_1^\infty \frac{dx}{e^x} = -\left.\frac{1}{e^x}\right|_1^\infty = \frac{1}{e}$$

so $\int_1^\infty \frac{3}{e^x+5}dx$ converges.

The End