The fundamental theorem of calculus

Preparation: The mean value theorem for definite integrals [5-38]

> If f is continuous on $[a,b]$, then there is a $c \in [a,b]$ with
>
> $$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

- This means f assumes its average value somewhere on $[a,b]$
- Geometry: [5-39]
- Continuity is necessary: [5-40]

Proof: take max-min-inequality

$$\min f \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{w} \le \max f$$

and the intermediate value theorem for continuous functions implies that there is a $c \in [a,b]$ with $f(c) = w$ $\square$

Example: If f is continuous on $[a,b]$, $a \neq b$, and if $\int_a^b f(x)\,dx = 0$ then there is a c in $[a,b]$ with

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = 0$$

so that $f(x) = 0$ at least once in $[a,b]$

Towards the fundamental theorem:

For a continuous function f, define

$$F(x) = \int_a^x f(t)\,dt$$

Geometric interpretation: area [5-42]

Compute $F'(x)$: [5-43]

$$\underline{\underline{\frac{F(x+h)-F(x)}{h}}} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt = \underline{f(c)} \quad \text{for some } x \leq c \leq x+h$$

Mean Value Theorem for Integrals

Therefore, $\underline{F'(x) = f(x)}$ [5-44]

Examples

$f(t) = \cos t$ $\qquad f(x)$

$$\frac{d}{dx}\int_a^x \cos t \, dt = \cos x$$

$f(t) = \frac{1}{1+t^2}$ $\qquad f(x)$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\, dt = \frac{1}{1+x^2}$$

$$y = \int_x^5 3t \sin t \, dt, \qquad \frac{dy}{dx} = -3x \sin x$$

$$y = -\int_5^x 3t \sin t \, dt$$

$$y = \int_1^{x^2} \cos t \, dt, \qquad \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$y = \int_1^u \cos t \, dt \qquad = \cos(u)\,\frac{du}{dx}$$

$$u = x^2 \qquad = \cos(x^2)\, 2x$$

$2x \cos x^2$

So far, we know that

$$\int_a^x f(t)\,dt = G(x)$$

is an antiderivative of f (as $G'(x)=f(x)$)

The most general antiderivative is

$F(x) = G(x) + C$, and

$$F(b) - F(a) = \left(G(b)+C\right) - \left(G(a)+C\right)$$

$$= G(b) - G(a)$$

$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0} = \int_a^b f(t)\,dt$$

we have proved [5-45].

Recipe to calculate

$$\int_a^b f(x)\,dx$$

1) find any antiderivative F of f

2) Calculate $F(b) - F(a)$

Notation:

$$F(b) - F(a) = F(x)\Big|_a^b$$

The book uses

$$= F(x)\Big]_a^b = \Big[F(x)\Big]_a^b$$

Examples

$$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi} = \sin \pi - \sin 0$$

$$= 0$$

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx =$$

$$= \int_1^4 \frac{3}{2}\sqrt{x} \, dx - \int_1^4 \frac{4}{x^2} \, dx$$

$$= \frac{3}{2} \int_1^4 \sqrt{x} \, dx - 4 \int_1^4 \frac{1}{x^2} \, dx$$

$$= \frac{3}{2} \left[\frac{2}{3} x^{\frac{3}{2}}\right]_1^4 - 4 \left[-\frac{1}{x}\right]_1^4$$

$$= \frac{3}{2} \left[\frac{2}{3} 4^{\frac{3}{2}} - \frac{2}{3} 1^{\frac{3}{2}}\right] - 4 \left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right]$$

$$= 4$$

Finding total area

$f(c_k)$ positive $\rightsquigarrow$ $f(c_k)\Delta x$ is area

$f(c_k)$ negative $\rightsquigarrow$ $f(c_k)\Delta x$ is $-$ area

Example: Area between x-axis and

$y = \sin x$ between $x=0$ and $x=2\pi$:

[5.47] The total area is

$$\int_0^{\pi} \sin x \, dx + \left| \int_{\pi}^{2\pi} \sin x \, dx \right|$$

$$= -\cos x \Big|_0^{\pi} + \left| -\cos x \Big|_{\pi}^{2\pi} \right|$$

$$= -\cos\pi + \cos 0 + \left| -\cos 2\pi + \cos\pi \right|$$

$$= 1 + 1 + |-1-1| = 4$$

Recipe for finding area between $y = f(x)$ and x-axis over the interval $[a,b]$:

1) Subdivide $[a,b]$ at the zeros of f
2) Integrate over each subinterval
3) Add the <u>absolute</u> value of the integrals

<u>Example</u> $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$

1) $f(x) = x(x+1)(x-2)$

zeros are $-1, 0, 2$

2) $$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{0}^{2} = -\frac{8}{3}$$

3) $$\text{Area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

Substitution rule for indefinite integrals

- Recall the chain rule for $F(g(x))$:

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x)$$

- If F is the antiderivative of f,

$$\frac{d}{dx} F(g(x)) = f(g(x))\, g'(x)$$

- Now compute

$$\int f(g(x))\, g'(x)\, dx = \int \frac{d}{dx} F(g(x))\, dx$$

- continue with $u = g(x)$:

$$= F(g(x)) + C_1$$

$$= F(u) + C_1 = \int F'(u)\, du$$

$$= \int f(u)\, du$$

We have proved [5-51].

Method for $\int f(g(x))\, g'(x)\, dx$:

1) Substitute $u = g(x)$, $du = g'(x)\, dx$

to obtain $\int f(u)\, du$

2) Integrate with respect to u

3) Replace $u = g(x)$

<u>Example:</u>

$$\int \cos(\underbrace{7\theta+5}_{u})\, d\theta = \int \cos u \; \frac{1}{7}\, du$$

$\frac{du}{d\theta} = 7$, so $d\theta = \frac{1}{7}\, du$

$$= \frac{1}{7} \sin u + C_1 = \frac{1}{7} \sin(7\theta+5) + C_1$$

Example (different substitutions)

Evaluate $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

i) $u = z^2+1 \qquad du = 2z\,dz$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du$$

$$= \frac{3}{2}u^{\frac{2}{3}} + C_1 = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C_1$$

ii) $u = \sqrt[3]{z^2+1} \quad \rightsquigarrow \quad u^3 = z^2+1$

so that $\quad 3u^2\,du = 2z\,dz$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} = 3\int u\,du$$

$$= \frac{3}{2}u^2 + C = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C$$

Example (for using identities)

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C_1$$

[similarly $\int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + C_1$]

Area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$:

$$\int_0^{2\pi} \sin^2 x \, dx = \left(\frac{1}{2} x - \frac{1}{4} \sin 2x \right) \Big|_0^{2\pi}$$

$$= \left(\frac{2\pi}{2} - \frac{1}{4} \sin 4\pi \right) - \left(\frac{0}{2} - \frac{1}{4} \sin 0 \right)$$

$$= \pi$$

Graph [5-52] shows that the average value $= \frac{1}{2}$

Substitution in definite integrals

Theorem: If g is continuous on $[a,b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof: For F with $F' = f$ we have

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x)) \Big|_{x=a}^{x=b}$$

$$= F(u) \Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)\, du$$

$\square$

<u>Example</u>

$$\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\, dx$$

Substitute $u = x^3+1$, $du = 3x^2 dx$

$x=-1$ gives $u = (-1)^3+1 = 0$

$x=1$ gives $u = 1^3+1 = 2$

so that

$$\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\, dx = \int_0^2 \sqrt{u}\, du$$

$$= \frac{2}{3} u^{\frac{3}{2}} \Big|_0^2 = \frac{2}{3} 2^{\frac{3}{2}} - 0 = \frac{4\sqrt{2}}{3}$$

Integrals of symmetric functions

Let f be continuous on $[-a, a]$

- If $f(x)$ is even $(f(x) = f(-x))$

then $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

- If $f(x)$ is odd $(f(x) = -f(-x))$

then $$\int_{-a}^{a} f(x)\,dx = 0$$

[5-56]

This is a useful fact, for example

$$\int_{-997\pi}^{997\pi} \left(\sin x + x^3 - \frac{x}{1+x^2}\right) dx =$$

Areas between curves

Similarly to the area between $y = f(x)$ and the x-axis $(y=0)$, we define the area between $y = f(x)$ and $y = g(x)$ for $f(x) \geq g(x)$ as

$$A = \int_a^b \left(f(x) - g(x) \right) dx \qquad [5-58, 61]$$

For total area, find where $f(x) = g(x)$ and integrate over the subintervals, then add the absolute values (as above).

<u>Example</u> Area enclosed by $y = 2 - x^2$

[5-62] and $y = -x$

1) solve $2 - x^2 = -x$

$\rightsquigarrow$ $x = -1$, $x = 2$

2)
$$A = \int_{-1}^{2} \underbrace{(2-x^2)}_{\text{lies above}} - \underbrace{(-x)}_{\text{lies below}} \, dx$$

$$= \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{-1}^{2} = \frac{9}{2}$$

<u>Example</u> Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and $y = x-2$ [5-63]

$$A = \int_0^2 \sqrt{x}\, dx + \int_2^4 \sqrt{x} - (x-2)\, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 + \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{x^2}{2} + 2x \right) \Big|_2^4$$

$$= \frac{2}{3} 2^{\frac{3}{2}} - \frac{2}{3} 0^{\frac{3}{2}} + \frac{2}{3} 4^{\frac{3}{2}} - \frac{4^2}{2} + 2\cdot 4$$
$$- \frac{2}{3} 2^{\frac{3}{2}} + \frac{2^2}{2} - 2\cdot 2$$

$$= \frac{10}{3}$$

Geometrical Trick : [5-66]

Compute area A_1 below the parabola

$$A_1 = \int_0^4 \sqrt{x}\, dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^4 = \frac{16}{3}$$

Subtract the area of the triangle

$$A_2 = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

$$A = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3}$$

Inverse functions and their derivatives

Definition [7-4]: A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

These function take on any value in their range exactly once!

Examples: [7-5]

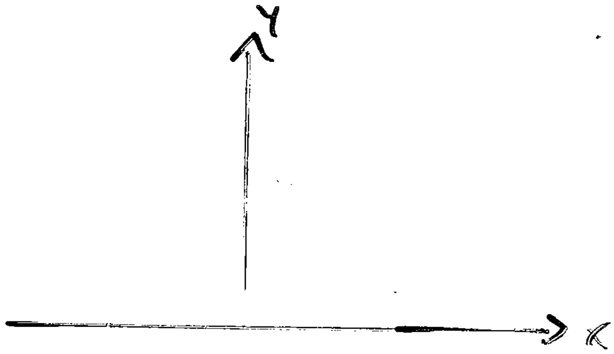
$y = x^3$ one-to-one on $\mathbb{R}$

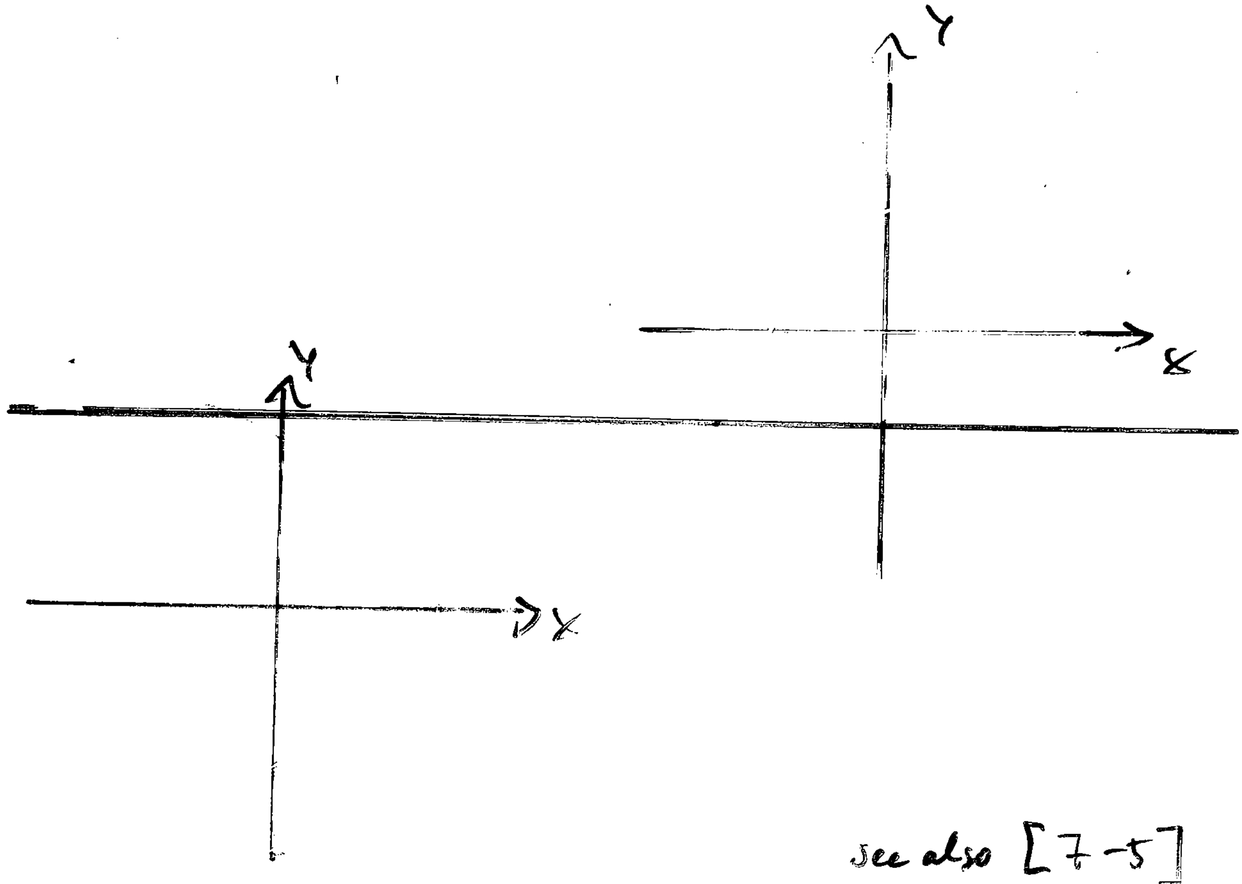
$y = \sqrt{x}$ one-to-one on $\mathbb{R}_0^+$

$y = x^2$ one-to-one on $\mathbb{R}_0^+$, but not $\mathbb{R}$

$y = \sin x$ one-to-one on $[0, \frac{\pi}{2}]$ not on $\mathbb{R}$

<u>The horizontal line test</u>: A function is <u>one-to-one</u> if and only if its graph intersects each horizontal line at most once





see also [7-5]

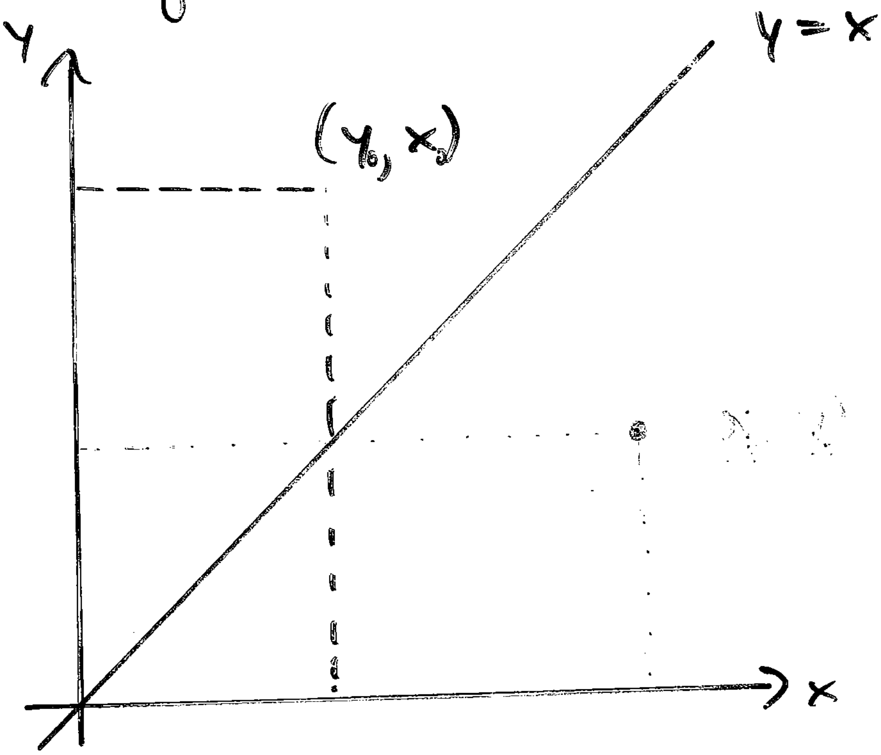
<u>Definition [7-7]:</u> Suppose that f is a one-to one function on a domain D with range R. The <u>inverse function</u> f^{-1} is defined by

$$f^{-1}(a) = b \quad \text{if} \quad f(b) = a$$

The domain of f^{-1} is R and the range of f^{-1} is D

- f^{-1} read "f inverse"
- $f^{-1}(x)$ is <u>not</u> $1/f(x)$, it is <u>not</u> an exponent!
- $(f^{-1} \circ f)(x) = x$ for all $x \in D(f)$
- $(f \circ f^{-1})(x) = x$ for all $x \in R(f)$

Finding inverses [7-8]



reflect the graph along the line $y = x$

Algebraically:

- solve $y = f(x)$ for x : $x = f^{-1}(y)$
- interchange x and y : $y = f^{-1}(x)$

Derivatives of inverses of differentiable functions

Use implicit differentiation for $y = f^{-1}(x)$:

$$x = f(y) \quad \Big| \ \frac{d}{dx}$$

$$1 = f'(y)\,\frac{dy}{dx}$$

Therefore $\dfrac{dy}{dx} = \dfrac{1}{f'(y)}$

Now $x = f(y)$ means $y = f^{-1}(x)$

So that

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$

Theorem on slide [7-13]

<u>Example</u> $f(x) = x^2$ on $\mathbb{R}_0^+$

$f^{-1}(x) = \sqrt{x}$ and $f'(x) = 2x$:

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{2\, f^{-1}(x)}$$

$$= \frac{1}{2\sqrt{x}}$$

[7-14]

Natural Logarithms

- For $a \in \mathbb{Q} \setminus \{-1\}$, we know

$$\int_1^x t^a \, dt = \frac{1}{a+1}\left(x^{a+1} - 1\right)$$

- What happens if $a = -1$?

$$\int_1^x \frac{1}{t} \, dt \quad \text{is well-defined}$$

[7-18]

for $x > 0$.

But what is it ? We <u>define</u>

$$\ln x = \int_1^x \frac{1}{t} \, dt \qquad [7\text{-}17]$$

A special value : the number e

$$\ln(e) = 1 \qquad [7\text{-}20]$$

$$e = 2.718281828459\ldots$$

Differentiating $\ln(x)$ is easy!

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

And for $u(x)$ with $u(x) > 0$,

$$\frac{d}{dx}\ln u(x) = \frac{1}{u(x)}\,u'(x)$$

by chain-rule. [7-21]

Properties of Logarithms : [7-22]

proof omitted

Examples

$$\log 6 = \log(2\cdot 3) = \log 2 + \log 3$$

$$\log 4 - \log 5 = \log \frac{4}{5} = \log 0.8$$

$$\log \frac{1}{8} = -\log 8 = -\log 2^3 = -3 \log 2$$

$$\log 4 + \log \sin x = \log (4 \sin x)$$

$$\log \frac{x+1}{2x+3} = \log (x+1) - \log (2x+3)$$

$$\log \sqrt[3]{x+1} = \log (x+1)^{1/3} = \frac{1}{3} \log (x+1)$$

$$\log \cot x = \log \frac{1}{\tan x} = -\log \tan x$$

The range of $\ln x$

- $\ln 2 > \frac{1}{2}$ [7_23]

- $\ln 2^n = n \ln 2$

  $\ln 2^{-n} = -n \ln 2$

- $\lim_{x \to \infty} \ln x = \infty$

  $\lim_{x \to 0^+} \ln x = -\infty$

- The range of $\ln x$ is $\mathbb{R}$

For $t>0$, $\int \frac{1}{t}\,dt = \ln t + C_1$

what about $t<0$?

then $-t$ is positive and

$$\int \frac{1}{t}\,dt = \int \frac{1}{(-t)}\,d(-t) = \ln(-t) + C_1$$

together $$\underline{\int \frac{1}{t}\,dt = \ln|t| + C_1}$$

Substitution $t = f(x)$, $dt = f'(x)\,dx$

leads to

$$\underline{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C_1}$$

This is very useful.

Examples

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$t = \cos x$, $dt = -\sin x \, dx$ gives

$$\int \tan x \, dx = -\int \frac{1}{t} \, dt =$$

$$= -\log |t| + C' = -\log |\cos x| + C$$

Similarly,

$$\int \cot x \, dx = \log |\sin x| + C$$

The exponential function exp(x)

- $\ln(x)$ has Domain $\mathbb{R}^+$ and Range $\mathbb{R}$
- $\ln(x)$ is increasing, therefore invertible

The inverse function of $\ln(x)$ is the exponential function with Domain $\mathbb{R}$:

$$\exp(x) = \ln^{-1}(x) \qquad [7\text{-}28]$$

We had defined e by $\ln(e) = 1$

Now we see that $\underline{e = \exp(1)}$

and we can write for all $x \in \mathbb{R}$

$$\exp(x) = e^x$$

[Technically, we had only been able to compute e^x for $x \in \mathbb{Q}$, so this is new]

Rules for simplification:

- $e^{\ln x} = x$ for $x > 0$
- $\ln(e^x) = x$ for $x \in \mathbb{R}$

General exponential function: $a > 0$

$$a = e^{\ln a}, \text{ so that}$$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

This finally defines exponentiation with irrational exponent x. [7-32]

We have the usual rules for exponents for

$$\exp(x) = e^x$$ [7-33]

Proofs (using $\ln x$) in textbook.

Differentiating and integrating e^x

use that $y = e^x = f^{-1}(x)$

with $f(x) = \ln x$, $f'(x) = \frac{1}{x}$:

$$\frac{dy}{dx} = \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} =$$

$$= f^{-1}(x) = e^x$$

Therefore

$$\frac{d}{dx} e^x = e^x$$

and, conversely

$$\int e^x dx = e^x + C_1$$

Example:

Solve the initial value problem

$$e^y \frac{dy}{dx} = 2x \quad , \quad x > \sqrt{3} \quad , \quad y(2) = 0 :$$

Note that $\displaystyle\int e^y \frac{dy}{dx}\, dx = \int e^y dy = e^y + C_0$

Therefore

$$e^y = x^2 + C_1$$

[check by differentiation!]

$y(2) = 0$: $\quad e^0 = 2^2 + C_1 \Rightarrow C_1 = -3$

Take logarithms to get

$$y = \ln(x^2 - 3)$$

which is valid for $x > \sqrt{3}$.

<u>Theorem:</u> $$e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}$$

<u>Proof:</u> Take logarithm of the right side:

ln is continuous

$$\ln\left(\lim_{x\to 0} (1+x)^{\frac{1}{x}}\right) = \lim_{x\to 0}\left(\ln (1+x)^{\frac{1}{x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right)$$

this is zero

$$= \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x}$$

$$= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \quad \text{with } f(t) = \ln t$$

$$= f'(1) = \frac{1}{1} = 1 = \ln(e) \qquad f'(t) = \frac{1}{t}$$

$\square$

Equivalently, $$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

General exponential functions and logarithms

$(a>0)$

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} =$$

$$= \ln a \; e^{x \ln a} = a^x$$

Thus, for $a>0$ and $x \in \mathbb{R}$,

$$\underline{\frac{d}{dx}a^x = a^x \ln a}$$

Similarly

$$\underline{\int a^x\,dx = \frac{a^x}{\ln a} + C}$$

<u>Example:</u> $\frac{d}{dx}2^x = 2^x \ln 2$

with $\ln 2 \approx 0.69$

The inverse of $y = a^x$ is the

<u>logarithm of x with base a: $\log_a x$</u>

This makes sense for $a > 0$ with

$a \neq 1$ (why?)

<u>Relation between $\log_a x$ and $\ln x$:</u>

From $x = a^{\log_a x}$, we get

$$\ln x = \ln\left(a^{\log_a x}\right)$$
$$= \log_a x \cdot \ln a$$

Therefore $$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

For calculations, always express $\log_a x$ in terms of $\ln$, and then integrate etc!

Inverse Trigonometric Functions

- sin, cos, sec, csc, tan, cot are not one-to-one <u>unless</u> the domain is restricted, see e.g. [7-59]

- once the domains are restricted [7-60], we can define

$$\arcsin x = \sin^{-1} x$$

$$\arccos x = \cos^{-1} x$$

$$\arctan x = \tan^{-1} x$$

etc.

<u>Caution</u>: $\sin^{-1} x \neq (\sin x)^{-1}$

Unfortunately, one does write: $\sin^2 x = (\sin x)^2$

Definition of arcsin, arccos : [7-61, 62, 63]

The "arc" in arcsin, arccos : [7-64]

Definition of arctan, arccot : [7-69, 70, 71]

As $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$

we define $\operatorname{arcsec} x = \arccos \dfrac{1}{x}$

and $\operatorname{arccsc} x = \arcsin \dfrac{1}{x}$

<u>Example</u> : [7-77]

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \frac{1}{3}\sqrt{x^2+9}$$

Differentiating arcsin x

$$\sin y = x \qquad \Big| \ \frac{d}{dx}$$

$$\cos y \, \frac{dy}{dx} = 1 \qquad \Big| \ \text{assume } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad \Big| \ \cos y = \sqrt{1 - \sin^2 y}$$

$$\boxed{\text{For } |x| < 1, \quad \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}}$$

Conversely, for $|x| < 1$,

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C}$$

Similarly $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ etc

[7-84, 85]

<u>Example</u> Find the line tangent to

$y = \operatorname{arccot} x$ at $x = -1$:

- $\operatorname{arccot}(-1) = \frac{\pi}{2} - \arctan(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{4}\right)$

  $= \frac{3\pi}{4}$

- $\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1+x^2}\right|_{x=-1} = -\frac{1}{1+(-1)^2} = -\frac{1}{2}$

equation of line is $\underline{y = \frac{3\pi}{4} - \frac{1}{2}(x+1)}$

<u>Example</u>

$$\int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1$$

$$= \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Examples (Completing the square)

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} =$$

[Trick: $4x-x^2 = 4-(x-2)^2$, $u = x-2$]

$$= \int \frac{du}{\sqrt{4-u^2}} = \arcsin \frac{u}{2} + C_1$$

$$= \arcsin \frac{x-2}{2} + C_1$$

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1} =$$

[Trick: $4x^2+4x+2 = (2x+1)^2+1$, $u = 2x+1$]

$$= \int \frac{\frac{1}{2}\,du}{u^2+1} = \frac{1}{2} \arctan u + C_1$$

$$= \frac{1}{2} \arctan (2x+1) + C_1$$

Hyperbolic Functions

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even function}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd function}}$$

We define $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

More definitions and graphs [7-87]

Identities [7-88]

Similarities between

sinh, cosh, tanh, ...

and sin, cos, tan, ...

no accident (but must wait for MAS 205)

Derivatives follow directly from definition:

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2}$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

Tables [7-89]

Example:

$$\int_0^1 \sinh^2 x \, dx = \int_0^1 \frac{\cosh 2x - 1}{2} \, dx$$

$$= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 = \frac{\sinh 2}{4} - \frac{1}{2}$$

$$= \frac{1}{8}e^2 - \frac{1}{2} - \frac{1}{8}e^{-2} \approx 0.40672$$

Inverse hyperbolic functions

As for trigonometric functions, restrict domain and invert [7-90, 91]

- Derivatives and integrals [7-93, 94]

This is useful for integration:

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \int_0^1 \frac{dx}{\sqrt{3/4 + x^2}} =$$

$a^2 = \frac{3}{4}$, $a = \frac{\sqrt{3}}{2}$

$$= \sinh\left(\frac{2x}{\sqrt{3}}\right)\Bigg|_0^1 = \sinh \frac{2}{\sqrt{3}} \approx 0.98665$$

Completing the square can help here, too.