

Roots $x_k(y)$ of a formal power series

$$f(x, y) = \sum_{n=0}^{\infty} a_n(y) x^n$$

with applications to graph enumeration
and q -series

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LECTURE #3

The leading root of the partial theta function

The basic set-up, reviewed

- Start from a formal power series

$$f(x, y) = \sum_{n=0}^{\infty} a_n(y) x^n$$

where

- (a) $a_0(0) = a_1(0) = 1$
- (b) $a_n(0) = 0$ for $n \geq 2$
- (c) $a_n(y) = O(y^{\nu_n})$ with $\lim_{n \rightarrow \infty} \nu_n = \infty$

and coefficients lie in a commutative ring-with-identity-element R .

- There exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$. We call $x_0(y)$ the **leading root** of f .
- Since $x_0(y)$ has constant term -1 , we write $x_0(y) = -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.
- We saw in Lecture #2 that $\xi_0(y)$ can be computed by
 - An elementary method.
 - A method based on the explicit implicit function formula.
 - A method based on the exponential formula.

Method based on the explicit implicit function formula

- In Lecture #2 we derived the formula

$$\frac{\xi_0(y)^\beta - 1}{\beta} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n_1, \dots, n_m \geq 0} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^m (-1)^{n_i} \widehat{a}_{n_i}(y)$$

where

$$\widehat{a}_n(y) = \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1 \\ a_n(y) & \text{for } n \geq 2 \end{cases}$$

- Can this formula be used for proofs of nonnegativity???
- Recall the definition: $\xi_0(y) \in \mathcal{S}_\beta$ in case $\frac{\xi_0(y)^\beta - 1}{\beta} \succeq 0$
(coefficientwise nonnegativity)
- *Empirically* I know that $\xi_0(y) \in \mathcal{S}_\beta$ when $a_n(y) = \alpha_n y^{n(n-1)/2}$ and
 - (a) $\beta \geq -2$ with $\alpha_n = 1$ (partial theta function)
 - (b) $\beta \geq -1$ with $\alpha_n = 1/n!$ (deformed exponential function)
 - (c) $\beta \geq -1$ with $\alpha_n = (1-q)^n/(q; q)_n$ and $q > -1$
- **How can we see these facts from this formula???**
[open combinatorial problem]
- All these examples have $\widehat{a}_n(y) \succeq 0$. The factors $(-1)^{n_i}$ then seem to cause trouble.

A very simple case: Alternating signs

Proposition. Suppose that

$$(-1)^n \widehat{a}_n(y) \succeq 0 \quad \text{for all } n \geq 0$$

where

$$\widehat{a}_n(y) = \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1 \\ a_n(y) & \text{for } n \geq 2 \end{cases}$$

Then $\xi_0(y) \in \mathcal{S}_\beta$ and in fact

$$\frac{\xi_0(y)^\beta - 1}{\beta} \succeq \sum_{n=0}^{\infty} (-1)^n \widehat{a}_n(y)$$

in the following cases:

- (a) $\beta = 1$
- (b) $\beta = -1$ whenever $a_0(y) = 1$
- (c) $\beta = -3$ whenever $a_0(y) = a_1(y) = 1$
- (d) $\beta = -(2k - 1)$ whenever $a_0(y) = a_1(y) = 1$ and $a_2(y) = \dots = a_{k-1}(y) = 0$

Proof. Follows almost immediately from

$$\frac{\xi_0(y)^\beta - 1}{\beta} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n_1, \dots, n_m \geq 0} \binom{\beta - 1 + \sum n_i}{m - 1} \prod_{i=1}^m (-1)^{n_i} \widehat{a}_{n_i}(y)$$

- (a) Set $\beta = 1$. Then the RHS of the Proposition comes from the term $m = 1$. All the other terms are $\succeq 0$ since $(-1)^n \widehat{a}_n(y) \succeq 0$ and $\binom{\beta - 1 + \sum n_i}{m - 1} \geq 0$.

- (b) Set $\beta = -1$ and observe that the sum can be restricted to $n_1, \dots, n_m \geq 1$. If $m = 1$ we have $\binom{\beta - 1 + \sum n_i}{m - 1} = 1$ and we get the RHS of the Proposition. If $m \geq 2$ we have $\sum n_i \geq 2$, so that $\beta - 1 + \sum n_i$ is a nonnegative integer and hence $\binom{\beta - 1 + \sum n_i}{m - 1} \geq 0$.
- (c) is analogous to (b), but using $\beta = -3$ and observing that the sum can be restricted to $n_1, \dots, n_m \geq 2$, so that $m \geq 2$ implies $\sum n_i \geq 4$.
- (d) is analogous to (b), but using $\beta = -(2k - 1)$ and observing that the sum can be restricted to $n_1, \dots, n_m \geq k$, so that $m \geq 2$ implies $\sum n_i \geq 2k$.

A slight strengthening (by rescaling of f)

Corollary.

(a) If $(-1)^n \frac{a_n(y)}{a_1(y)} \succeq 0$ for all $n \neq 1$, then $\xi_0(y) \in \mathcal{S}_1$ and satisfies

$$\xi_0(y) \succeq \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)}.$$

(b) If $1 - \frac{a_1(y)}{a_0(y)} \succeq 0$ and $(-1)^n \frac{a_n(y)}{a_0(y)} \succeq 0$ for all $n \geq 2$, then $\xi_0(y) \in \mathcal{S}_{-1}$ and satisfies

$$\xi_0(y)^{-1} \preceq \frac{a_1(y)}{a_0(y)} - \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_0(y)}.$$

Proof.

- (a) Apply part (a) of the Proposition to $f(x, y)/a_1(y)$.
- (b) Apply part (b) of the Proposition to $f(x, y)/a_0(y)$.

Alternative (elementary) proof of the Corollary

- No need to use explicit implicit function formula. Just bare hands!
- **Proof of part (a):** Start from the equation $f(-\xi_0(y), y) = 0$, divide by $a_1(y)$, and add $\xi_0(y)$ to both sides:

$$\xi_0(y) = \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)} \xi_0(y)^n$$

- The unique solution to this equation can be found iteratively as follows: Define a map $\mathcal{F}: R[[y]] \rightarrow R[[y]]$ by

$$(\mathcal{F}\xi)(y) = \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)} \xi(y)^n$$

and define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \dots \in R[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$. I then claim that

$$\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \xi_0^{(2)} \preceq \dots \preceq \xi_0$$

and that

$$\xi_0^{(k)}(y) = \xi_0(y) + O(y^{k+1}).$$

Proof of claim:

- If $f(y)$ and $g(y)$ are formal power series satisfying $0 \preceq f \preceq g$, then the hypotheses of the Corollary [part (a)] guarantee that $0 \preceq \mathcal{F}f \preceq \mathcal{F}g$.
- Applying this repeatedly to the obvious inequality $0 \preceq \xi_0^{(0)} \preceq \xi_0^{(1)}$, we obtain $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \xi_0^{(2)} \preceq \dots$.

- Likewise, if $f(y)$ and $g(y)$ are formal power series satisfying $f(y) - g(y) = O(y^\ell)$ for some $\ell \geq 0$, then it is easy to see that $(\mathcal{F}f)(y) - (\mathcal{F}g)(y) = O(y^{\ell+1})$ [since $a_n(y)/a_1(y) = O(y)$ for all $n \geq 2$].
 - Applying this repeatedly to the obvious fact $\xi_0^{(1)}(y) - \xi_0^{(0)}(y) = O(y)$, we obtain $\xi_0^{(k+1)}(y) - \xi_0^{(k)}(y) = O(y^{k+1})$.
 - It follows that $\xi_0^{(k)}(y)$ converges as $k \rightarrow \infty$ (in the topology of formal power series) to a limiting series $\xi_0^{(\infty)}(y)$, and that this limiting series satisfies $\mathcal{F}\xi_0^{(\infty)} = \xi_0^{(\infty)}$. But this means that $\xi_0^{(\infty)}(y) = \xi_0(y)$. It also follows that $\xi_0^{(k)}(y) = \xi_0(y) + O(y^{k+1})$. The inequality of the Corollary is precisely the statement $\xi_0 \succeq \xi_0^{(1)}$.
- The proof of part (b) is similar.
 - Can parts (c) and (d) of the Proposition be given a similarly elementary proof?
 - Can results analogous to the Proposition be proven for the spaces \mathcal{S}_β with $\beta \neq 1, -1, -3, -5, \dots$?

But isn't the case of alternating signs *too trivial*?

- After all, the most interesting examples have *constant signs*.
- Then the irritating factors $(-1)^{n_i}$ cannot be avoided.

The partial theta function $\Theta_0(x, y) = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2}$
 (which has *constant signs*!)

It seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned} \xi_0(y) = & 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ & + 948y^9 + 2610y^{10} + \dots + \text{terms through order } y^{6999} \end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned} \xi_0(y)^{-1} = & 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ & - 178y^9 - 490y^{10} - \dots - \text{terms through order } y^{6999} \end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-2}$:

$$\begin{aligned} \xi_0(y)^{-2} = & 1 - 2y - y^2 - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ & - 138y^9 - 386y^{10} - \dots - \text{terms through order } y^{6999} \end{aligned}$$

Can we prove any of this???

Yes!!! (for the first two properties; the third is still open)

Proof for the partial theta function

- Use standard notation for q -shifted factorials:

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j) \quad \text{for } |q| < 1$$

- A pair of identities for the partial theta function:

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (y; y)_\infty (-x; y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y; y)_n (-x; y)_n}$$

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (-x; y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n y^{n^2}}{(y; y)_n (-x; y)_n}$$

as formal power series and as analytic functions on $(x, y) \in \mathbb{C} \times \mathbb{D}$

- Rewrite these as

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (y; y)_\infty (-xy; y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y; y)_n (-xy; y)_{n-1}} \right]$$

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = (-xy; y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{(-x)^n y^{n^2}}{(y; y)_n (-xy; y)_{n-1}} \right]$$

- The brackets on the RHS have alternating signs in x (i.e. have nonnegative coefficients as a series in $-x$ and y)
- So we have reduced to the easy case of alternating signs!
- The second identity has $a_0(y) = 1$, so we prove also $\xi_0(y) \in \mathcal{S}_{-1}$.

The preceding proof, written more explicitly

- Let's say we use the first identity:

$$\Theta_0(x, y) = (y; y)_\infty (-xy; y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y; y)_n (-xy; y)_{n-1}} \right]$$

- So $\Theta_0(x, y) = 0$ is equivalent to “brackets = 0”.
- Insert $x = -\xi_0(y)$ and bring $\xi_0(y)$ to the LHS:

$$\xi_0(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^n (1 - y^j) \prod_{j=1}^{n-1} [1 - y^j \xi_0(y)]}$$

- This formula can be used iteratively to determine $\xi_0(y)$, and in particular to prove the strict positivity of its coefficients:
- Define the map $\mathcal{F}: \mathbb{Z}[[y]] \rightarrow \mathbb{Z}[[y]]$ by

$$(\mathcal{F}\xi)(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^n (1 - y^j) \prod_{j=1}^{n-1} [1 - y^j \xi(y)]}$$

- Define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \dots \in \mathbb{Z}[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$.
- Then $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \dots \preceq \xi_0$ and $\xi_0^{(k)}(y) = \xi_0(y) + O(y^{3k+1})$.
- In particular, $\lim_{k \rightarrow \infty} \xi_0^{(k)}(y) = \xi_0(y)$, and $\xi_0(y)$ has strictly positive coefficients.
- Thomas Prellberg has a combinatorial interpretation of $\xi_0(y)$ and $\xi_0^{(k)}(y)$.

Elementary proof of the first identity

- Proof uses nothing more than Euler's first and second identities

$$\frac{1}{(t; q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n}$$

$$(t; q)_\infty = \sum_{n=0}^{\infty} \frac{(-t)^n q^{n(n-1)/2}}{(q; q)_n}$$

valid for $(t, q) \in \mathbb{C} \times \mathbb{D}$.

- Write

$$\sum_{n=0}^{\infty} x^n y^{n(n-1)/2} = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2} \frac{(y; y)_\infty}{(y; y)_n (y^{n+1}; y)_\infty}$$

- Insert Euler's first identity for $1/(y^{n+1}; y)_\infty$:

$$\begin{aligned} \Theta_0(x, y) &= (y; y)_\infty \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(y; y)_n} \sum_{k=0}^{\infty} \frac{y^{(n+1)k}}{(y; y)_k} \\ &= (y; y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y; y)_k} \sum_{n=0}^{\infty} \frac{(xy^k)^n y^{n(n-1)/2}}{(y; y)_n} \\ &= (y; y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y; y)_k} (-xy^k; y)_\infty \quad \text{by Euler's second identity} \\ &= (y; y)_\infty (-x; y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y; y)_k (-x; y)_k} \end{aligned}$$

- This identity goes back to Heine (1847), but does not seem to be very well known.
- It can be found in Fine (1988) and Andrews and Warnaar (2007).
- **Did anyone know it between 1847 and 1988???**

Proof of the first and second identities

- A simple limiting case of Heine's first and second transformations

$${}_2\phi_1(a, b; c; q, z) = \frac{(b; q)_\infty (az; q)_\infty}{(c; q)_\infty (z; q)_\infty} {}_2\phi_1(c/b, z; az; q, b)$$

$${}_2\phi_1(a, b; c; q, z) = \frac{(c/a; q)_\infty (az; q)_\infty}{(c; q)_\infty (z; q)_\infty} {}_2\phi_1(abz/c, a; az; q, c/a)$$

for the basic hypergeometric function

$${}_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n$$

- Just set $b = q$ and $z = -x/a$, then take $a \rightarrow \infty$ and $c \rightarrow 0$.
- This is how Heine (1847) proved the first identity.
- Heine didn't know his second transformation, which is apparently due to Rogers (1893).
- **Who first wrote the second identity for the partial theta function???**
- Surely it must have been known before Andrews and Warnaar (2007)!?!

Can any of this be generalized?

- Recall our friend

$$\tilde{R}(x, y, q) = \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(1+q) \cdots (1+q+\dots+q^{n-1})}$$

- Can this proof be extended to cases $q \neq 0$?
- Here is a general identity:

$$\sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q; q)_n} = \frac{1}{(q; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} q^{\ell(\ell+1)/2}}{(q; q)_{\ell}} \Theta_0(xq^{\ell}, y)$$

- Can deduce generalizations of the first and second identities for the partial theta function:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q; q)_n} &= \\ \frac{(y; y)_{\infty}}{(q; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} q^{\ell(\ell+1)/2}}{(q; q)_{\ell}} (-xq^{\ell}; y)_{\infty} \sum_{n=0}^{\infty} \frac{y^n}{(y; y)_n (-xq^{\ell}; y)_n} \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q; q)_n} &= \\ \frac{1}{(q; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} q^{\ell(\ell+1)/2}}{(q; q)_{\ell}} (-xq^{\ell}; y)_{\infty} \sum_{n=0}^{\infty} \frac{(-xq^{\ell})^n y^{n^2}}{(y; y)_n (-xq^{\ell}; y)_n} \end{aligned}$$

- But I don't know what to do with these formulae, because of the factors $(-1)^{\ell}$.