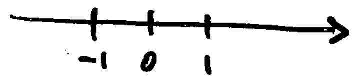
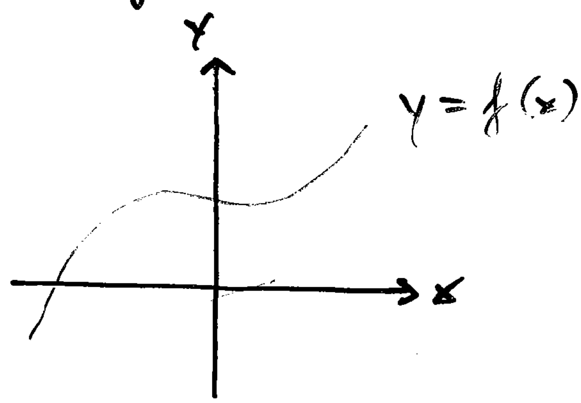
Calculus I

What is calculus?

- Study of functions of real variables
  - one real variable
  - many variables (Calculus II)

- Fundamental: real numbers 

- Geometric view: graph of a function



  - slope $\hat{=}$ derivative
  - area $\hat{=}$ integral

- many applications

Real numbers and the real line

Properties of real numbers $\mathbb{R}$

- algebraic (rules of calculation)
- order (geometric picture : line)
- completeness (no "gaps")

a) algebraic properties $a, b, c \in \mathbb{R}$

A1 $a + (b+c) = (a+b) + c$

A2 $a+b = b+a$

A3 there is a "0" such that $a + 0 = a$

A4 there is an x such that $a + x = 0$
$x = $ "$-a$"

M1 $a(bc) = (ab)c$

M2 $ab = ba$

M3 there is a "1" such that $a1 = a$

M4 there is an x such that $ax = 1$ (for $a \neq 0$)
$x = a^{-1} = \frac{1}{a}$

D $a(b+c) = ab + ac$

b) order: the real line

O1 for any a, b $a \le b$ or $b \le a$

O2 if $a \le b$ and $b \le a$ then $a = b$

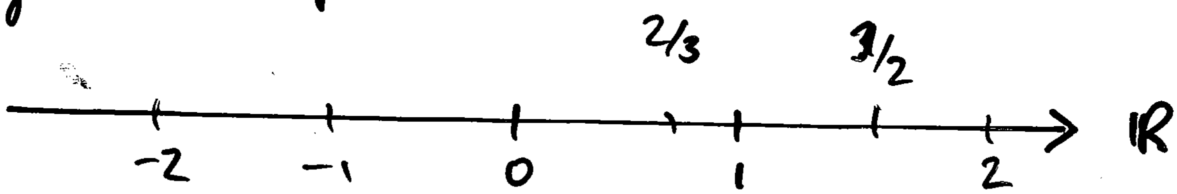
O3 if $a \le b$ and $b \le c$ then $a \le c$.

O4 if $a \le b$ then $a + c \le b + c$

O5 if $a \le b$ and $0 \le c$ then $ac \le bc$

(consequences see slide 1.4)

geometric interpretation:



c) completeness:

the real numbers correspond to all points on the line, there are no "holes" or "gaps".

Subsets of the real numbers $\mathbb{R}$

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$ natural numbers

solve $a + x = b$ for x

$\mathbb{Z} = \{\dots, -2, -1, 0, +1, +2, \dots\}$ integers

solve $ax = b$

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$

rational numbers

solve $x^2 = 2$

$\mathbb{R}$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\mathbb{N}$ and $\mathbb{Z}$ clearly have gaps, but

$\mathbb{Q}$ is "dense", so are there "holes"?

between any two rationals there is another one

Yes, there are "holes" (well, sort of):

Irrational numbers such as $\sqrt{2}$ or $\pi = 3.14159\ldots$

$\sqrt{2}$ is the positive solution to $x^2 = 2$

Theorem $x^2 = 2$ has no solution $x \in \mathbb{Q}$

Proof
- Assume there is an $x \in \mathbb{Q}$ with $x^2 = 2$. This must be of the form $x = \frac{p}{q}$ with p, q integers with no common factor.
- $x^2 = 2$ implies that $\left(\frac{p}{q}\right)^2 = 2$, or $\underline{p^2 = 2q^2}$, so that p is even.*
- Write $p = 2p_1$, so that $\overset{2}{\cancel{4}} p_1^2 = \overset{1}{\cancel{2}} q^2$, or $2p_1^2 = q^2$, so that q is even*
- We have shown that both p and q are even. This is a contradiction!

You have just seen a "theorem with proof"

University mathematics is built upon

- Basic properties (Axioms, Definitions)
- Statements deduced from these

  (Lemma, Theorem, Corollary, ...)
- <u>and</u> their proofs!

You have just seen one such proof,

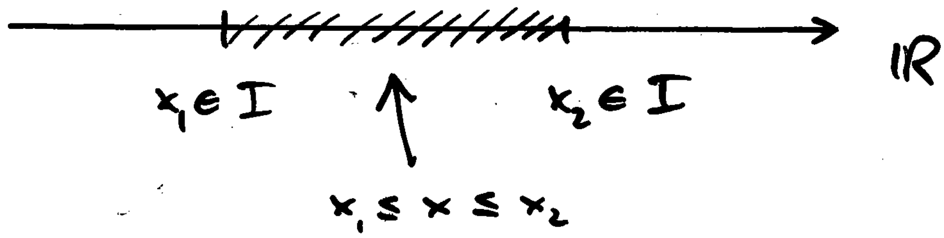
called "proof by contradiction".

There will be many more to come!

- And of course there are also

  examples, exercises, applications, ...

Intervals: A subset of the real line is called an <u>interval</u> if it contains at least two numbers and all the real numbers between any two of its elements.



(slide 1.5)

Examples:

(a) $2x - 1 < x + 3$

(b) $-\frac{x}{3} < 2x + 1$

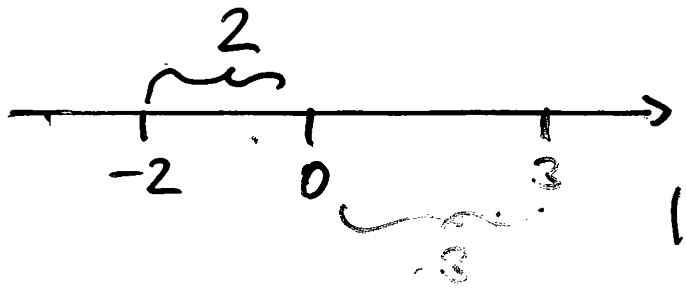
(c) $\frac{6}{x-1} \geq 5$

(slide 1.6)

Absolute value $|x|$:

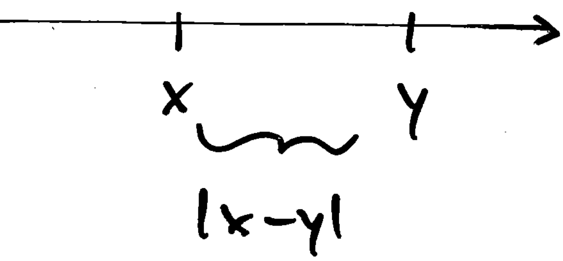
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

geometrically: $|x|$ distance from x to zero 0



$$|x| = |x-0|$$

$|x-y|$ distance between x and y



(slide ~~..~~, 1.7)

alternatively

$$|x| = \sqrt{x^2}$$

(square root is always non-negative!)

Inequality with $|x|$:

$$|x| < a \iff -a < x < a$$

(need $a > 0$, otherwise no solution)

(Slide 1.8, 1.9)

Properties of $|x|$:

1. $|-x| = |x|$

2. $|xy| = |x|\,|y|$

3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad (y \neq 0)$

4. $|x+y| \leq |x| + |y|$.

The last one is called <u>triangle inequality</u>

Examples:

(a) $|2x-3| \leq 1$ (b) $|2x-3| \geq 1$

(slide 1.10)

<u>Proof of properties 1. - 4.</u>:

1. use $|x| = \sqrt{x^2}$

$$|-x| = \sqrt{(-x)^2} = \sqrt{x^2} = |x|$$

2. use $|x| = \sqrt{x^2}$

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2 y^2} =$$

$$= \sqrt{x^2}\sqrt{y^2} = |x|\,|y|$$

3. as 2.

4. blackboard

Important inequalities:

- Triangle inequality $\underline{|a+b| \leq |a| + |b|}$

- Arithmetic - geometric mean inequality

  - arithmetic mean $\frac{1}{2}(a+b)$

  - geometric mean $\sqrt{ab}$

$$\boxed{\sqrt{ab} \leq \frac{1}{2}(a+b)} \qquad a,b \geq 0$$

Cauchy - Schwarz

- Cauchy - Schwarz inequality

$$\boxed{(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)}$$

a, b, c, d real

Proof of: $\sqrt{ab} \leq \frac{1}{2}(a+b)$

- multiply by 2 and square (why allowed ?)

$\Longleftrightarrow \quad 4ab \leq (a+b)^2$

- use <u>direct proof</u>: start on one side and transform until done ...

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$+ 2ab - 2ab$$

need 4ab

$$= 4ab + a^2 - 2ab + b^2$$

$$= 4ab + \underbrace{(a-b)^2}_{\geq 0.}$$

$$\geq 4ab$$

$\square$

symbol meaning "end of proof"

Proof of: Cauchy - Schwarz inequality

- use direct proof: start on one side and transform until done ...

rhs: $(a^2+b^2)(c^2+d^2) = a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2$

lhs: $(ac+bd)^2 = a^2c^2 + \boxed{2abcd} + b^2d^2$

start on rhs and work it out:

$$(a^2+b^2)(c^2+d^2) = a^2c^2 + \boxed{2abcd} + b^2d^2 + b^2c^2 - \boxed{2abcd} + a^2d^2$$

$$= (ac+bd)^2 + \underbrace{(bc-ad)^2}_{\geq 0}$$

$$\geq (ac+bd)^2$$

□

Second proof (using a "trick"):

Consider $(ax+c)^2 + (bx+d)^2$

$a^2\underline{x^2} + 2ac\underset{\sim}{x} + c^2 + b^2\underline{x^2} + 2bd\underset{\sim}{x} + d^2$

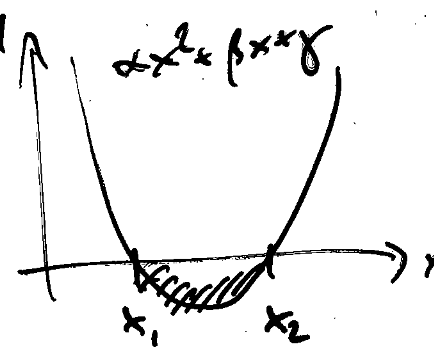
This is ≥ 0 as it is the sum of squares.

Multiplying out, we have also

$$0 \leq (a^2+b^2)\,x^2 + 2\,(ac+bd)\,x + (c^2+d^2) \qquad (*)$$

The right-hand-side is a quadratic equation in x (parabola, see 1.2)

$\alpha x^2 + \beta x + \gamma$



for $(*)$ to be true,

the <u>discriminant</u> $D = \beta^2 - 4\alpha\gamma$

must be non-positive $\left(x_{1,2} = \dfrac{-\beta \mp \sqrt{D}}{2\alpha}\right)$

$$\left.\begin{aligned}\alpha &= a^2+b^2\\ \beta &= 2(ac+bd)\\ \gamma &= c^2+d^2\end{aligned}\right\}\quad \text{compute } D$$

$$D = \beta^2 - 4\alpha\gamma = 4(ac+bd)^2 - 4(a^2+b^2)(c^2+d^2)$$

Now $\quad D \leq 0 \iff$ Cauchy - Schwarz. $\square$

<u>Generalisation</u>

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$
$$\geq (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$$

Proof: start with

$$0 \leq (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \dots + (a_n x + b_n)^2$$

Assigned reading: Chapter 1.2

[A-levels review: Lines, Circles, Parabolas]

Functions and Their Graphs

"y is a function of x"

$y = f(x)$

x independent variable (input value)

y dependent variable (output values)

f function (rule that assigns)

(Slide 1.34)

$y = f(x) = \sqrt{x}$: only $x \geq 0$

Important:
- what values of x are allowed
- what values of y are possible
- rule is <u>unique</u>: only one y for each x

Notation :

$\mathcal{D}$ domain of f

R range of f

$$f : \mathcal{D} \to R$$

$$f : x \mapsto y = f(x)$$

$$x \in \mathcal{D} \quad , \quad y \in R$$

(Slide 1.36)

Examples for Domain and Range:

$$y = \sqrt{1-x^2}$$

Domain $\quad 1-x^2 \geq 0 \quad , \quad \mathcal{D} = [-1,1]$

Range $\quad 0 \leq y \leq 1 \quad , \quad R = [0,1]$

(Slide 1.37)

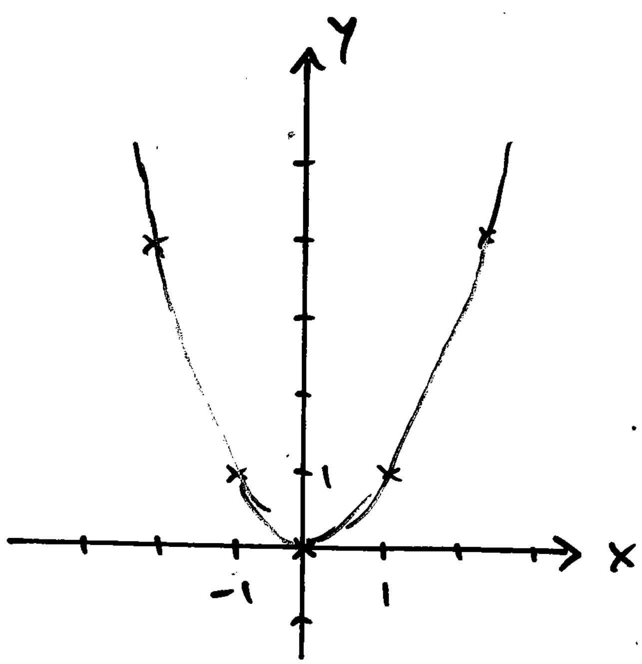
Graphs of Functions

If f is a function with domain D, its **graph** consists of the points (x,y) whose coordinates are the input-output pairs for f:

$$\{ (x, f(x)) \mid x \in D \}$$

(Slide 1.38)

Example: $y = x^2$, $D = [-2,2]$



$x=0 \rightsquigarrow y=0$

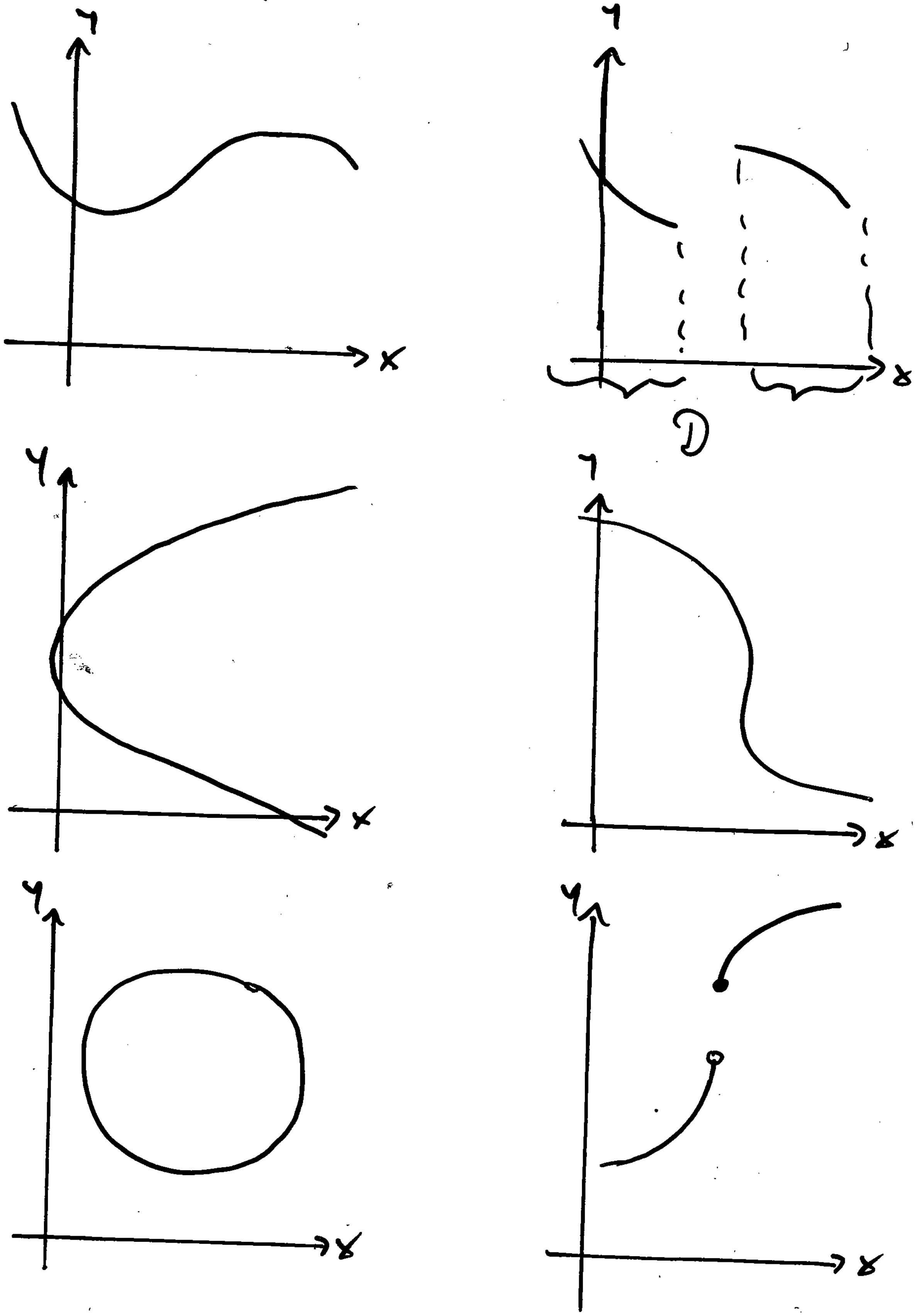
$x=1 \rightsquigarrow y=1$

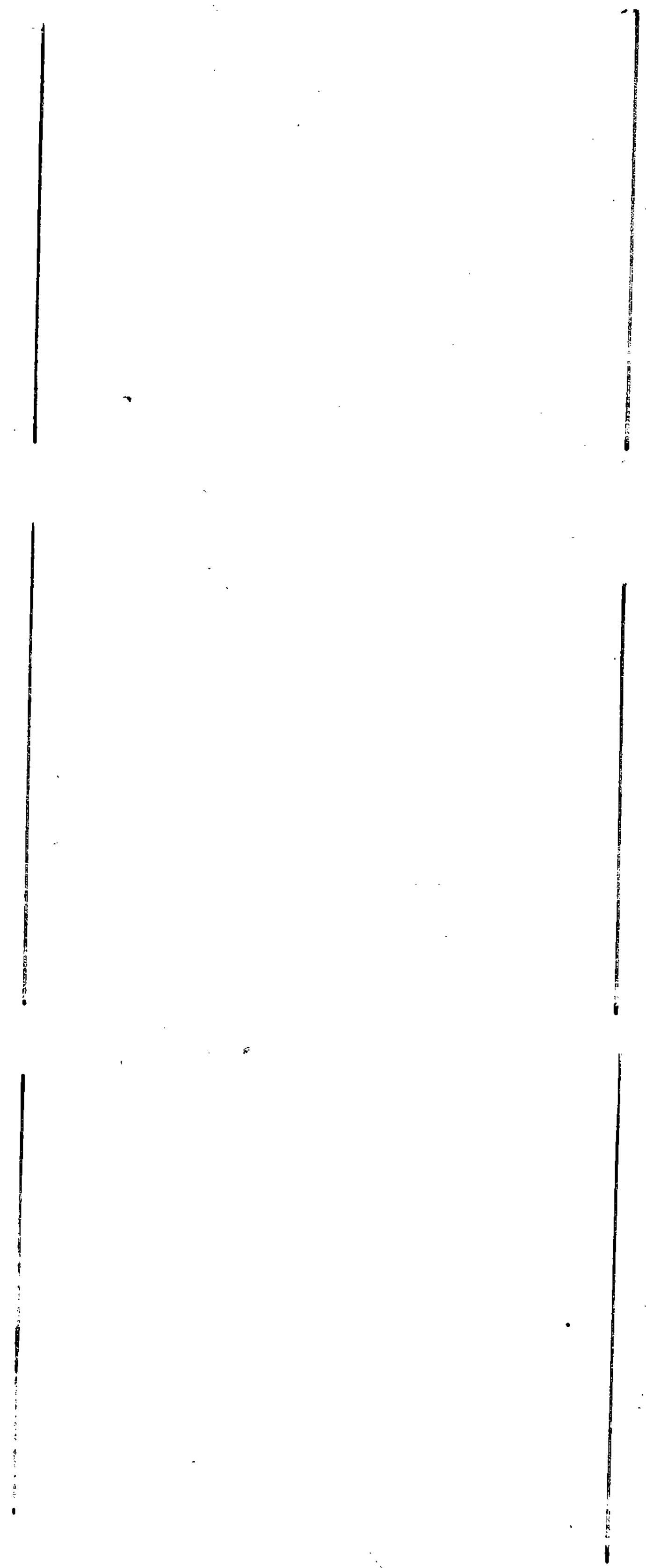
$x=-1 \rightsquigarrow y=-1$

$x=2 \rightarrow y=4$

$x=-2 \rightarrow y=4$

Which of the following curves are graphs of functions?





The vertical line test

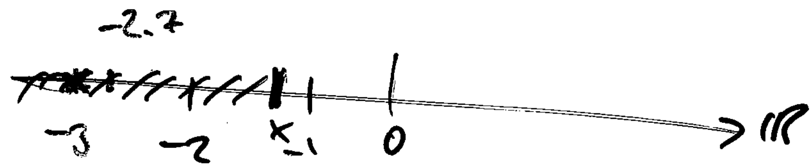
Piecewise defined functions

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

(slide 1.44)

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(slide 1.45)



$f(x) = \lfloor x \rfloor$ greatest integer $\leq x$

$\lfloor 1.3 \rfloor = 1$

(slide 1.46)

$\lfloor -2.7 \rfloor = -3$

$f(x) = \lceil x \rceil$ least integer $\geq x$

$\lceil 3.5 \rceil = 4$

(slide 1.47)

$\lceil -1.8 \rceil = -1$

Identifying Functions

- Linear functions

$$\underline{f(x) = mx + b}$$ [slide 1-50]

special case: constant function

$$\underline{f(x) = b}$$ [slide 1-51]

- Power functions

$$\underline{f(x) = x^a}$$

(a) $a = 1, 2, 3, \ldots$ [slide 1-52]

(b) $a = -1, -2, \ldots$ [slide 1-53]

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$ [slide 1-54]

$f(x) = \sqrt{x}$ Domain: $\underline{[0, \infty)}$

$f(x) = \sqrt[3]{x}$ Domain: $\underline{(\infty, \infty)}$

- Polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$a_n \neq 0$, $a_0, \dots, a_n \in \mathbb{R}$

n <u>degree</u> of the polynomial [slide 1-55]

Domain ______

- Rational functions

$$f(x) = \frac{p(x)}{q(x)}$$ p, q polynomials

Domain __________

- Algebraic functions
- Trigonometric functions
- Exponential functions
- Logarithmic functions

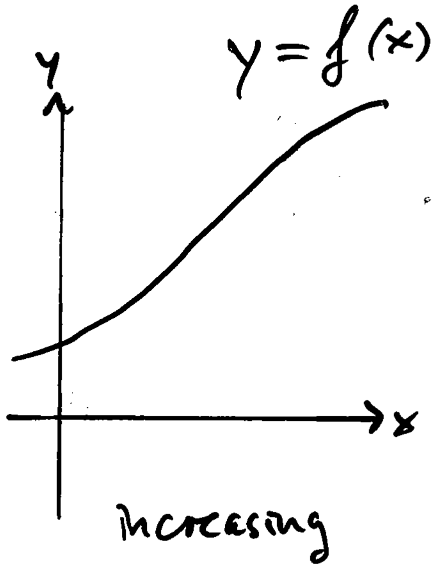
$\Bigg\}$ other classes (later)

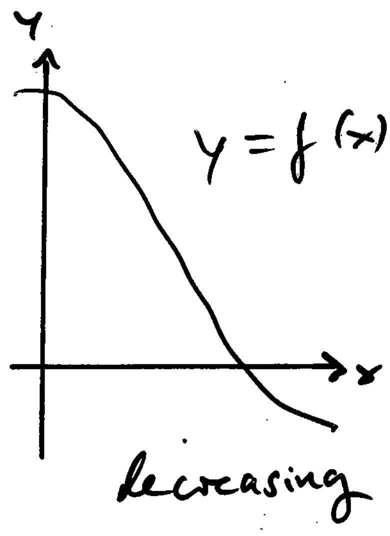
Increasing / decreasing functions

f is called <u>increasing</u>, if the graph of f "climbs" or "rises" as you move from left to right.

f is called <u>decreasing</u>, if the graph of f "descends" or "falls" as you move from left to right.

[Slide 1-62]





[precise definition later]

Even / odd functions

f is called <u>even</u>, if

$$f(x) = f(-x)$$

for all x in the domain of f

f is called <u>odd</u>, if

$$f(x) = -f(-x)$$

for all x in the domain of f

[Slide 1-64]

It follows that for f <u>even</u>,

the graph of f is symmetric

with respect to the <u>y-axis</u>,

and that for f <u>odd</u>, the

graph of f is symmetric

with respect to the <u>origin</u> $O = (0,0)$

Examples (algebraic):

- $f(x) = x^2$ — even

  $f(-x) = (-x)^2 = x^2 = f(x)$

- $f(x) = x^2+1$ — even

  $f(-x) = (-x)^2+1 = x^2+1 = f(x)$

- $f(x) = x$ — odd

  $f(-x) = -x = -f(x)$

- $f(x) = x+1$ — neither

  $f(-x) = -x+1 \neq \begin{cases} f(x) \\ f(-x) \end{cases}$

[Slide 1-65]

Sums, Differences, Products and Quotients of functions

If f and g are functions, then

for every $x \in D(f) \cap D(g)$

[that belongs to the domains of f and g]

we define

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\, g(x)$$

and if $g(x) \neq 0$,

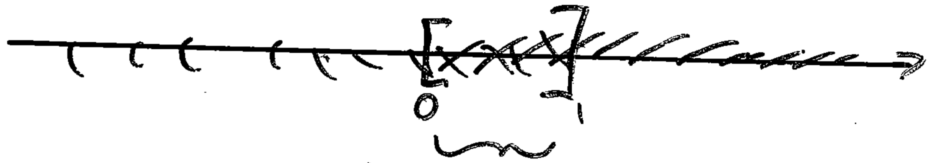
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Special case:

Multiplication by a constant c

$$(c\,f)(x) = c\,f(x)$$

[For this, take $g(x) = c$ constant function]



Example: take

$$f(x) = \sqrt{x}\,, \quad D(f) = \underline{[0, \infty)}$$

$$g(x) = \sqrt{1-x}\,, \quad D(g) = \underline{(-\infty, 1]}$$

$$D(f) \cap D(g) = \underline{[0, 1]}$$

Then

$$(f+g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{1-x}$$

$$(f\,g)(x) = f(x)\,g(x) = \sqrt{x}\,\sqrt{1-x}$$

[Slide 1-71-73

Composition of Functions

If f and g are functions,

the <u>composite</u> function $f \circ g$

(" f composed with g ")

is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$

consists of the numbers x

in the domain of g for which

$g(x)$ lies in the domain of f

i.e.

$$\mathcal{D}(f \circ g) = \{x \mid x \in \mathcal{D}(g) \text{ and } g(x) \in \mathcal{D}(f)\}$$

[Slides 1-75, 76]

Example

$f(x) = \sqrt{x}$ $\qquad$ $g(x) = x+1$

$D(f) = [0, \infty)$ $\qquad$ $D(g) = (-\infty, \infty)$

Domain:

- $f \circ g\,(x) =$

- $g \circ f\,(x) =$

- $f \circ f\,(x) =$

- $g \circ g\,(x) =$

<u>Example</u>

$$f(x) = \sqrt{x} \qquad g(x) = x^2$$

$$D(f) = [0,\infty) \qquad D(g) = (-\infty,\infty)$$

<u>Domain</u>

- $f \circ g(x) =$

- $g \circ f(x) =$

<u>Example</u>

$$f(x) = \frac{1}{x} \qquad D(f) = (-\infty,0) \cup (0,\infty)$$

- $f \circ f(x) =$

Domain :

Shifting a graph of a function

$$y = f(x) + k$$

$$y = f(x + h)$$

Slides [1-77, 78, 79]

Scaling and reflecting a graph of a function

$$y = c\,f(x)$$

$$y = f(cx)$$

Slides [1-81, 82, 83, 84, 85]

The effect of the size of c:

- $y = c f(x)$

  $c > 1$: stretches the graph <u>vertically</u>

  $c = 1$: (stays the same)

  $0 < c < 1$: compresses the graph <u>vertically</u>

  $c = -1$: <u>flips the graph</u> <u>vertically</u>

- $y = f(cx)$

  $c > 1$: compresses the graph <u>horizontally</u>

  $c = 1$:

  $0 < c < 1$: stretches the graph <u>horizontally</u>

  $c = -1$: flips the graph <u>horizontally</u>

Trigonometric functions

Radian measure [slide 1-89]

- angle measured in length of arc cut from a circle with radius 1:
  - full angle 360° corresponds to circumference: 2π
  - half angle 180° corresponds to  : π

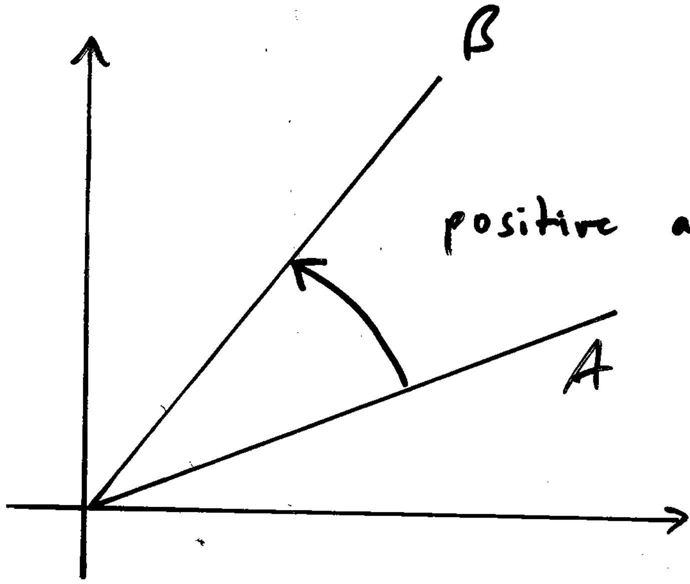
Degrees to radians:

multiply by $\dfrac{\pi}{180^\circ}$

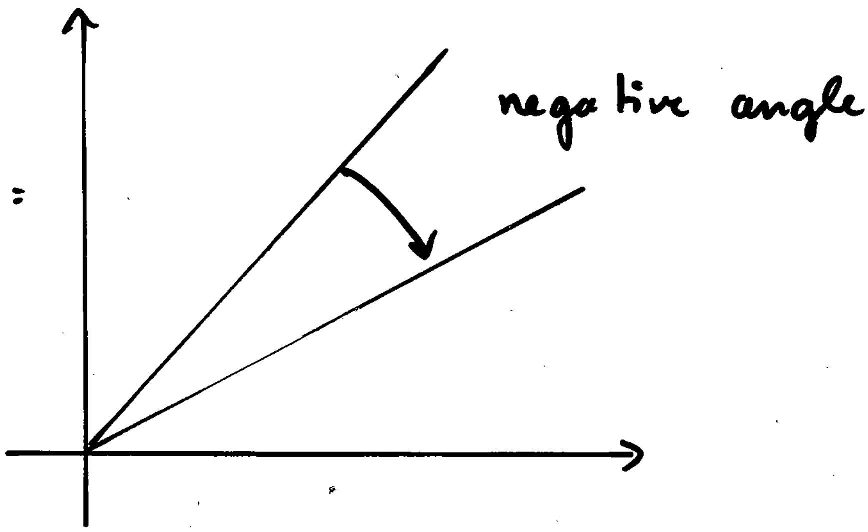
[1-90, 91]

Signed angles

anti-
clockwise:
counter-

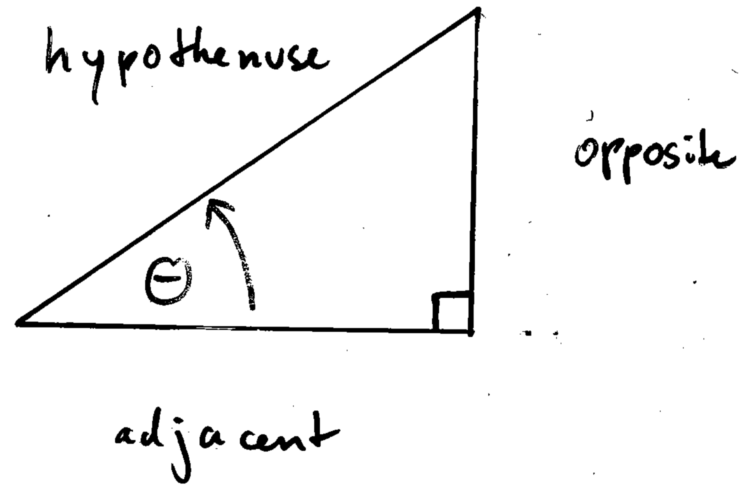


~~[struck]~~ clockwise:



[1-92, 93]

Trigonometric functions

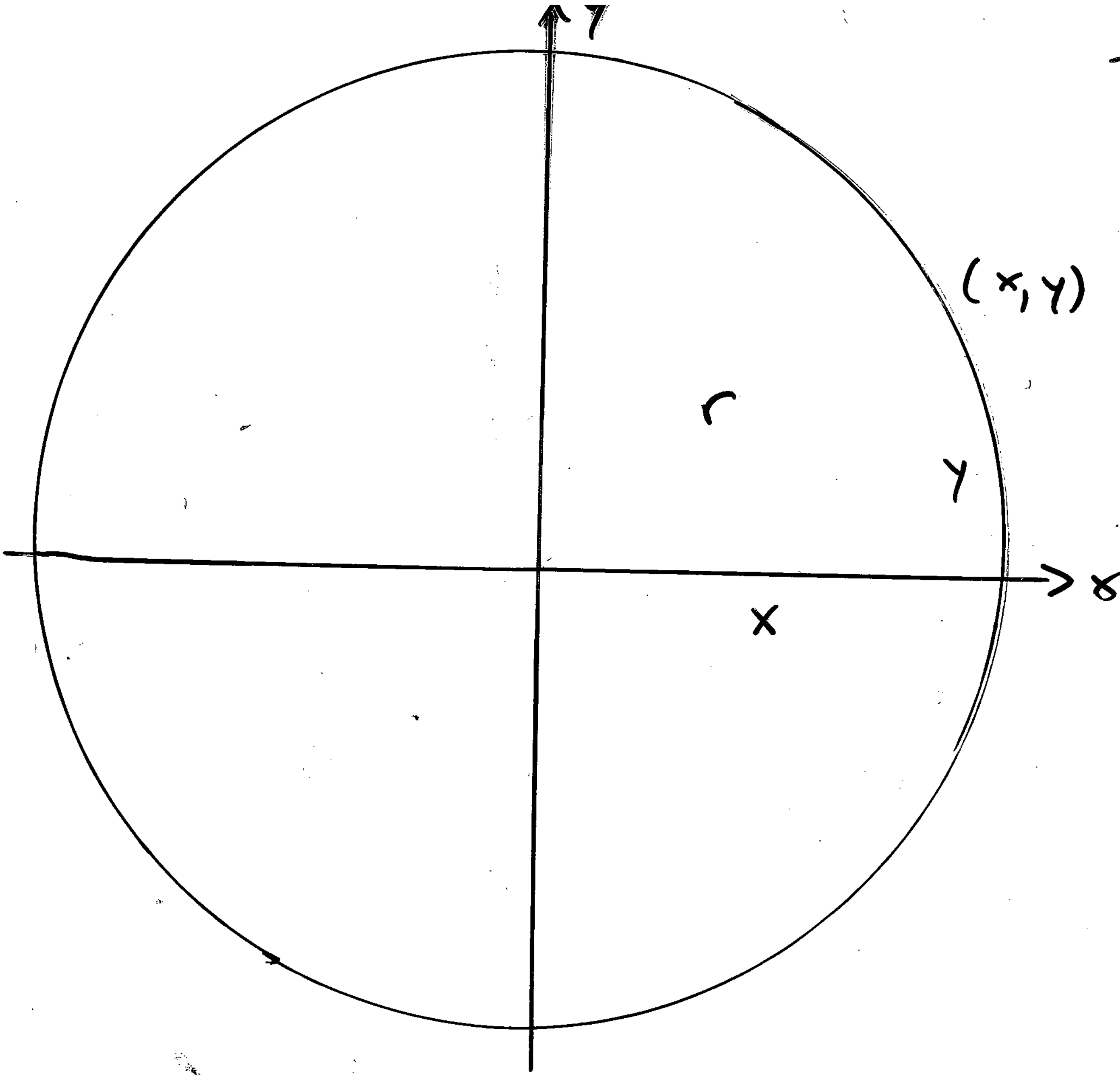


$$\sin\Theta = \frac{opp}{hyp}$$

$$\cos\Theta = \frac{adj}{hyp}$$

$$\tan\Theta = \frac{opp}{adj}$$

[slide 1-34



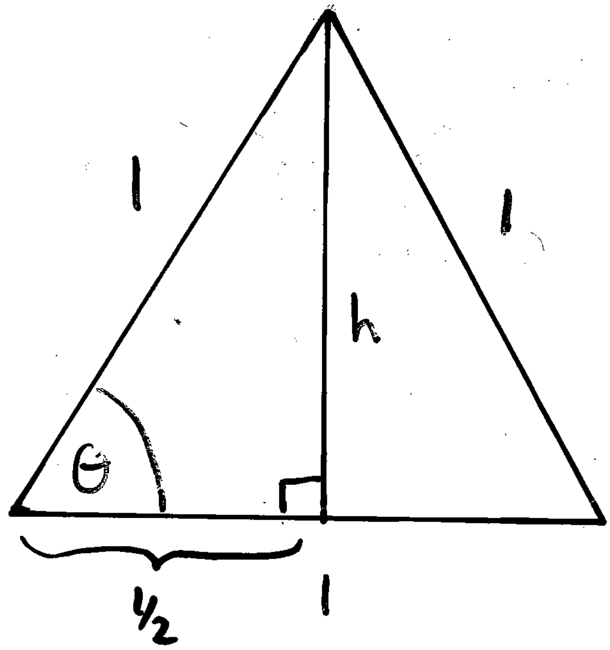
$$= \frac{y}{r}$$

$$= \frac{x}{r}$$

$$= \frac{y}{x}$$

Values at selected angles

for example, $\theta = \frac{\pi}{3}$:



$$1^2 = \left(\tfrac{1}{2}\right)^2 + h^2$$

$$\Rightarrow h = \frac{\sqrt{3}}{2}$$

$$\sin\theta = \frac{h}{1} = h = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{1/2}{1} = \frac{1}{2} = \frac{1}{2}$$

$$\tan\theta = \frac{h}{1/2} = 2h = \sqrt{3}$$

for more values, see [slide 1-99]

Graphs of trigonometric functions

Definition: a function f is *periodic* if there is a positive number p such that

$$f(x+p) = f(x)$$

for all values of x. The smallest value of p is called the *period* of f

$$\sin(\theta + 2\pi) = \sin\theta \quad : \text{period } 2\pi$$

$$\cos(\theta + 2\pi) = \cos\theta \quad : \text{period } 2\pi$$

$$\tan(\theta + \pi) = \tan\theta \quad : \text{period } \pi$$

Graphs see [slide 1-102]

Identities

$$\cos^2\theta + \sin^2\theta = 1$$

Proof by Pythagorean Theorem [slide 1-103]

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

special cases:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Revise on your own:

law of cosines, law of sines

Examples:

compute $\underline{\sin \frac{\pi}{12}}$:

use

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \overbrace{1-\sin^2\alpha} - \sin^2\alpha$$

so that $\sin^2\alpha = \frac{1}{2}\left(1-\cos 2\alpha\right)$

$$\sin^2\frac{\pi}{12} = \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)$$

$\Rightarrow$ $\underline{\sin\frac{\pi}{12} = \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)}}$ [sign?]

also $\cos\frac{\pi}{12} = \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)}$

Examples :

compute $\underline{\cos\frac{\pi}{8}}$:

$$\cos^2\alpha = \frac{1}{2}\left(1+\cos 2\alpha\right)$$

$$\cos^2\frac{\pi}{8} = \frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \underline{\cos\frac{\pi}{8} = \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)}}$$

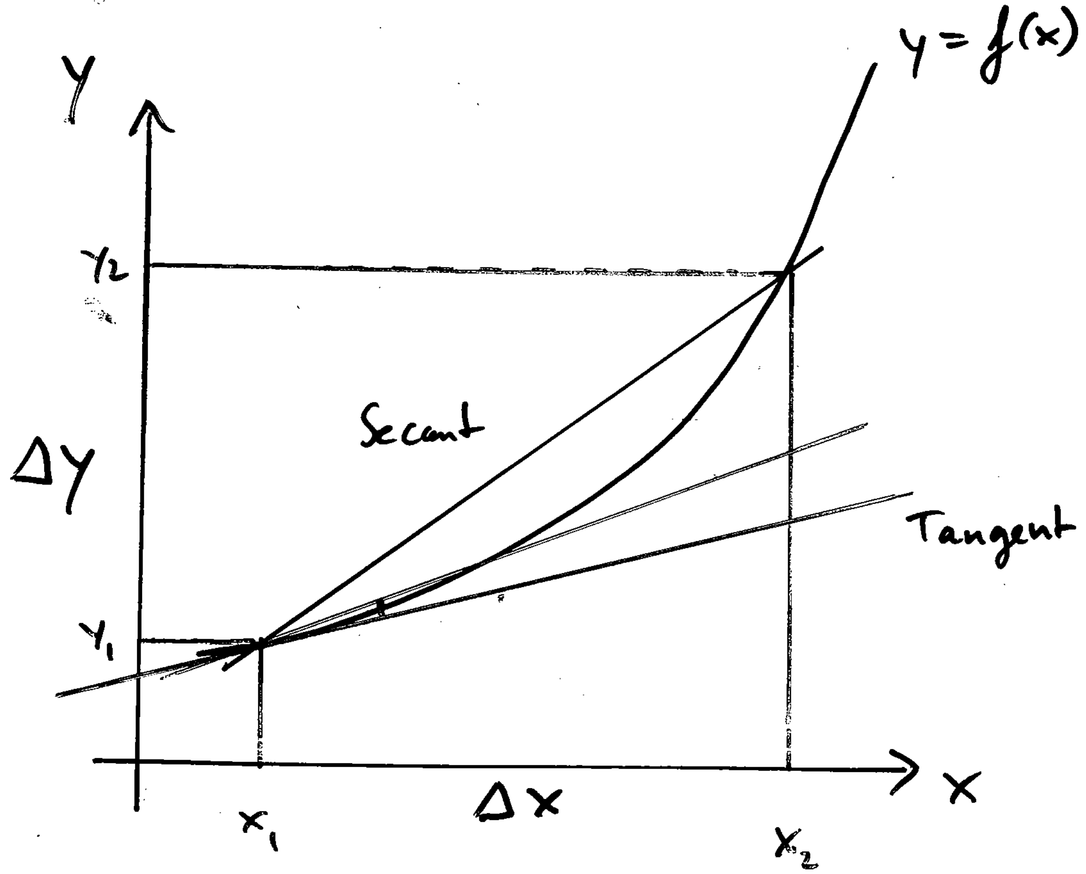
Limits

Motivation:

- average rate of change $\dfrac{\Delta Y}{\Delta X}$

versus

- instantaneous rate of change



[Slides 2-5, 6, 7, 8]

To move from

- average rates of change

to

- instantaneous rates of change

we need to consider <u>limits</u>

<u>Definition</u> Let $f(x)$ be defined on an open interval about x_0, except possibly x_0 itself. | If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the <u>limit</u> L as x approaches x_0,

$$\lim_{x \to x_0} f(x) = L$$

Example

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$$x_0 = 1$$

- $f(x)$ is not defined for $x_0 = 1$

- we can simplify for $x \neq 1$

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1$$

- this suggests that

$$\lim_{x \to 1} f(x) = 1 + 1 = 2$$

[Slides 2-9,10,11]

Examples of functions having limits at every point:

- $f(x) = x$ identity function
- $g(x) = k$ constant function

$$\lim_{x \to x_0} f(x) = x_0 \quad , \quad \lim_{x \to x_0} g(x) = k$$

[2-12]

Examples of functions having no limit at $x = 0$:

- $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ jump

- $g(x) = \begin{cases} 0 & x = 0 \\ 1/x & x \neq 0 \end{cases}$ values that become larger

- $f(x) = \begin{cases} 0 & x \leq 0 \\ \sin \frac{1}{x} & x > 0 \end{cases}$ rapid oscillations

[2-13]

Limit Laws: slide [2-15]

Examples

(a) $\lim_{x \to c} (x^3 + 4x - 2) = c^3 + 4c - 2$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = \dfrac{c^4 + c^2 - 1}{c^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\underbrace{4(-2)^2 - 3}_{\geq 0}}$

[Blackboard]

Consequences of limit laws: slides [2-16, 17]

Elimination of zero denominators

Example
$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$$

- substitution of $x=1$?
- algebraic simplification:

$$\frac{x^2+x-2}{x^2-x} = \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

$$= \frac{x+2}{x}$$

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

We have eliminated a common factor!

[2-18, 19]

Example: $$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

- substitution of $x=0$?

- algebraic simplification ("trick"):

$$\frac{\overset{(a\ -b)}{\left(\sqrt{x^2+100}-10\right)}\overset{(a\ +b)}{\left(\sqrt{x^2+100}+10\right)}}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{\left(\sqrt{x^2+100}\right)^2-10^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2+\cancel{100}-\cancel{100}}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{1}{\sqrt{x^2+100}+10}$$

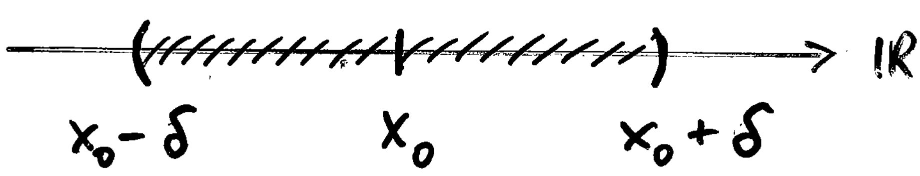
now "just plug in" and get $\frac{1}{20}$.

Assigned reading: the sandwich theorem

The precise definition of a limit

- we used informal phrases such as "sufficiently close" – what does this mean?

- be precise! x sufficiently close to x_0 means:



... there is a $\delta > 0$ such that for all $0 < |x - x_0| < \delta$ ...

Definition

Let $f(x)$ be defined on an open interval about x_0, except possibly x_0 itself.

We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

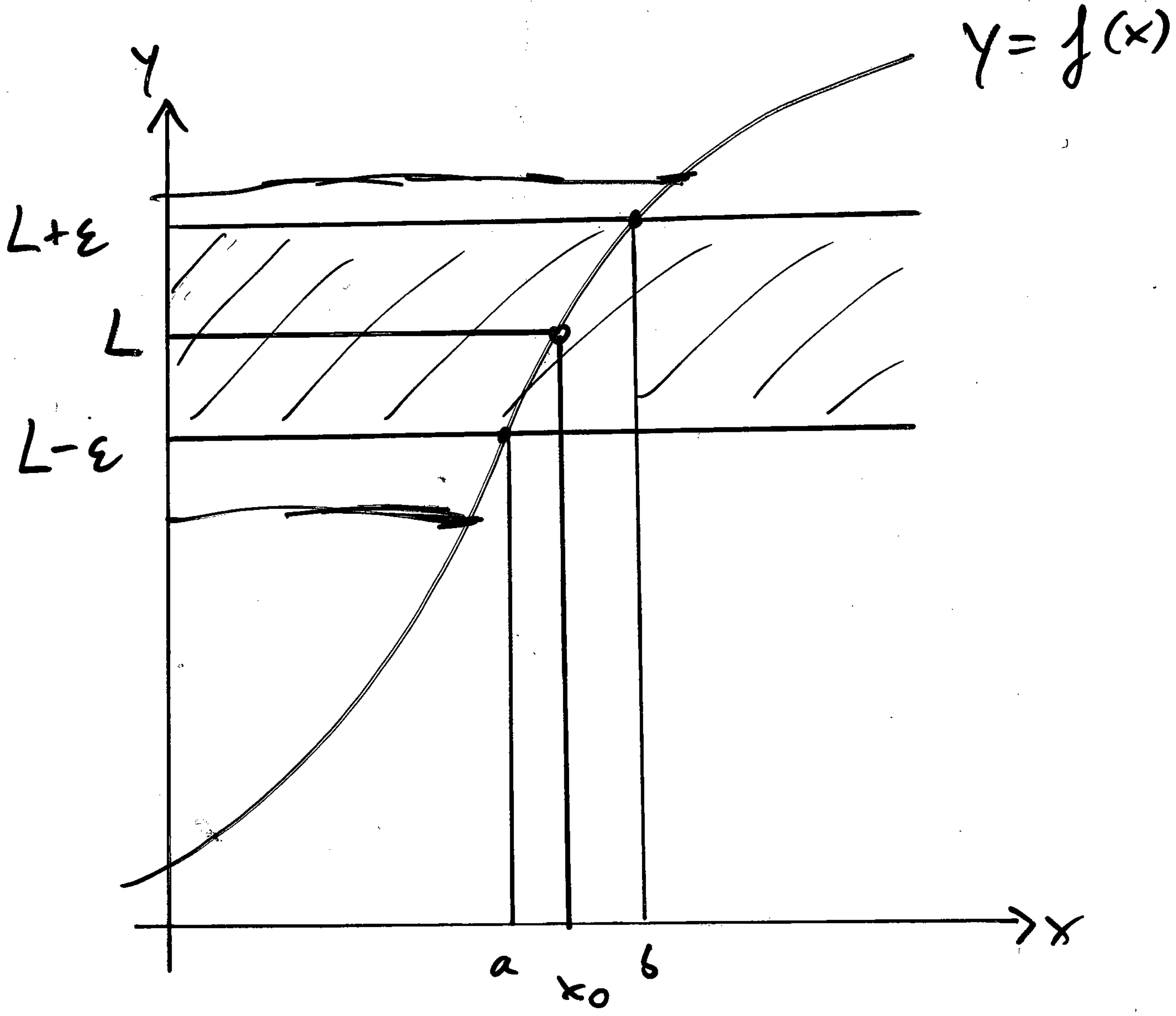
$$\lim_{x \to x_0} f(x) = L,$$

if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x,

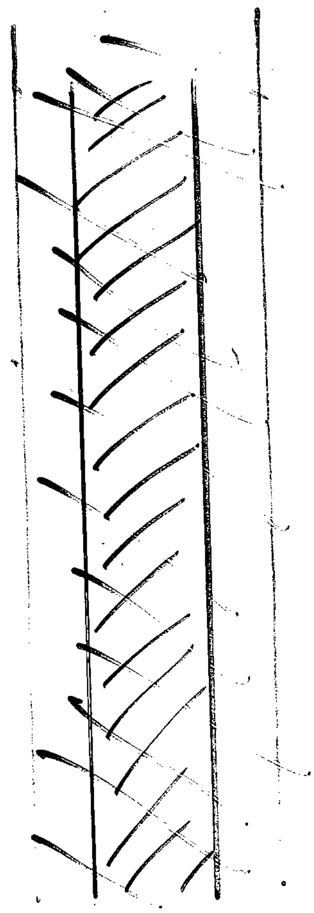
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

[2-29, 28]

[Visualisation: use animation]



for every $\varepsilon > 0$ ...



$x_0 - \delta$ $x_0 + \delta$

... there is a $\delta > 0$ ...

Finding δ for given ε

[2-30]

Example

$$f(x) = 5x - 3$$

$$x_0 = 1$$

$$L = 2$$

- Solve $|f(x) - L| < \varepsilon$ and get an interval (a, b)
- Find δ so that

$$(x_0 - \delta, x_0 + \delta) \subset (a, b)$$

Then $|f(x) - L| < \varepsilon$ will hold for all $0 < |x - x_0| < \delta$

[Blackboard]

[2-31]

- $|f(x) - L| < \varepsilon$:

$|5x - 3 - 2| < \varepsilon \iff |5x - 5| < \varepsilon$

$\iff |x - 1| < \frac{\varepsilon}{5} \iff 1 - \frac{\varepsilon}{5} < x < 1 + \frac{\varepsilon}{5}$

$(a, b) = \left(1 - \frac{\varepsilon}{5}, 1 + \frac{\varepsilon}{5}\right)$

- Find δ :

choose $\delta = \frac{\varepsilon}{5}$. Then

$(1 - \delta, 1 + \delta) = \left(1 - \frac{\varepsilon}{5}, 1 + \frac{\varepsilon}{5}\right)$

Example

$$f(x) = \sqrt{x-1}$$

$$x_0 = 5 \qquad\qquad L = 2$$

$$\varepsilon = 1$$

- $$\left| \sqrt{x-1} - 2 \right| < 1$$

$$\Longleftrightarrow \quad 2 < x < 10$$

$$(a, b) = \underline{(2, 10)}$$

- $$(5-\delta,\ 5+\delta) \subset (a, b)$$

$$\delta = \underline{\ 3\ }$$

One-sided limits

- To have a <u>limit</u> as $x \to c$, a function f must be defined on both sides of c (<u>two-sided</u> limit)

- If f fails to have a limit as $x \to c$, it may still have a <u>one-sided limit</u> if the approach is only from the right (<u>right-sided</u> limit) or from the left (<u>left-sided</u> limit).

- We write

$$\lim_{x \to c^+} f(x) = L \quad \text{or} \quad \lim_{x \to c^-} f(x) = L$$

Example

$$f(x) = \sqrt{4-x^2} \quad \text{for } x \in [-2,2]$$

$$\lim_{x \to 2^-} \sqrt{4-x^2} = \underline{0}$$

$$\lim_{x \to 2^+} \sqrt{4-x^2} = \underline{\text{does not}} \text{ exist!}$$

[2-40]

but $\lim_{x \to 2} \sqrt{4-x^2}$ does <u>not</u> exist!

Connection between limib

and one-sided limits see [2-41]

<u>Example</u> of slide [2-42]

c	$\lim_{x \to c} f(x)$	$\lim_{x \to c^-} f(x)$	$\lim_{x \to c^+} f(x)$
0	X	X	1
1	X	0	1
2	1	1	1
3	2	2	2
4	X	1	X

Precise definitions see [2-43]

Limits need not exist:

$$f(x) = \sin \frac{1}{x} \quad \text{for} \quad x \neq 0$$

$$\lim_{x \to 0} f(x), \quad \lim_{x \to 0^-} f(x), \quad \lim_{x \to 0^+} f(x)$$

don't exist [2-47]

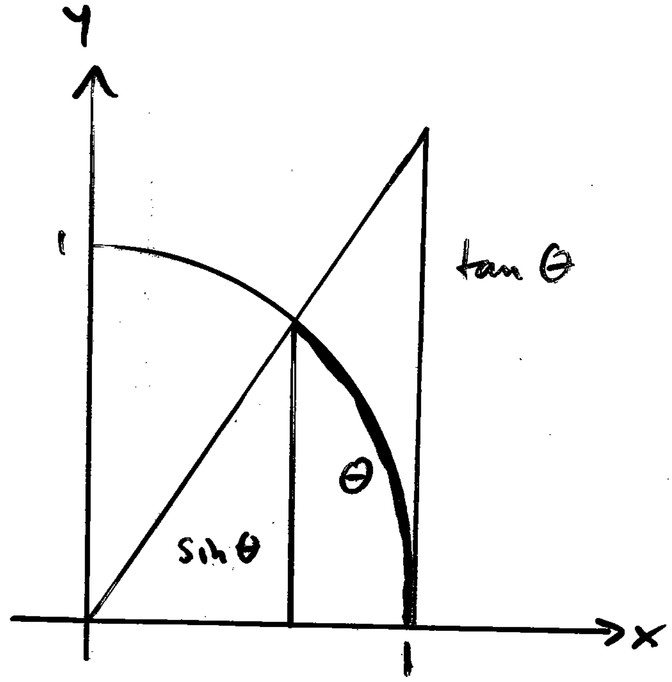
Limits involving $\frac{\sin x}{x}$:

Theorem: $$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

[2-48]

Proof

$\theta > 0$:



$$\sin\theta < \theta < \tan\theta \quad \Big| \cdot \frac{1}{\sin\theta}$$

$$\Longleftrightarrow \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Longleftrightarrow \quad 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$\theta \to 0$

$\theta \to 0^+$: $\quad \lim_{\theta \to 0^+} \cos\theta = 1$

$$\Longrightarrow \quad 1 \geq \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \geq 1$$

We have shown:

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

Similarly, we can show

$$\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1$$

Therefore

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$\square$

<u>Example</u>

Compute

(a) $$\lim_{h\to 0} \frac{\cos(2h)-1}{h} =$$

$\cos 2h = 1 - 2\sin^2 h$

$$= \lim_{h\to 0} \frac{\cancel{1} - 2\sin^2 h \cancel{-1}}{h} = \lim_{h\to 0}\left(\frac{\sin h}{h}(-2\sin h)\right)$$

$$= \lim_{h\to 0} \frac{\sin h}{h} (-2) \lim_{h\to 0} \sin h =$$

$$= 1 \cdot (-2) \cdot 0 = 0$$

(b) $$\lim_{x\to 0} \frac{\sin 2x}{5x} =$$

$$= \lim_{x\to 0} \left(\frac{\sin 2x}{2x} \frac{2}{5}\right) = \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x} =$$

$$= \frac{2}{5} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{2}{5}\cdot 1 = \frac{2}{5}$$

Limit as x approaches infinity:

- Observation:

  x approaching $\left\{ \begin{array}{l} \text{positive} \\ \text{negative} \end{array} \right\}$ infinity

  is like

  $\frac{1}{x}$ approaching 0 from the $\left\{ \begin{array}{l} \text{right} \\ \text{left} \end{array} \right.$

- Heuristics:

  ... there is a $\delta > 0$ such that for all $0 < \frac{1}{x} < \delta$ ...

  translates to

  ... there is an $M > 0$ such that for all $x > M$ ...

- Formal definition [2-51]
- Limit laws [2-54]

Examples

(a) $$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x}$$

$$= \underline{5+0}$$

(b) $$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{\cancel{x^2}\left(5+\frac{8}{x}-\frac{3}{x^2}\right)}{\cancel{x^2}\left(3+\frac{2}{x^2}\right)}$$

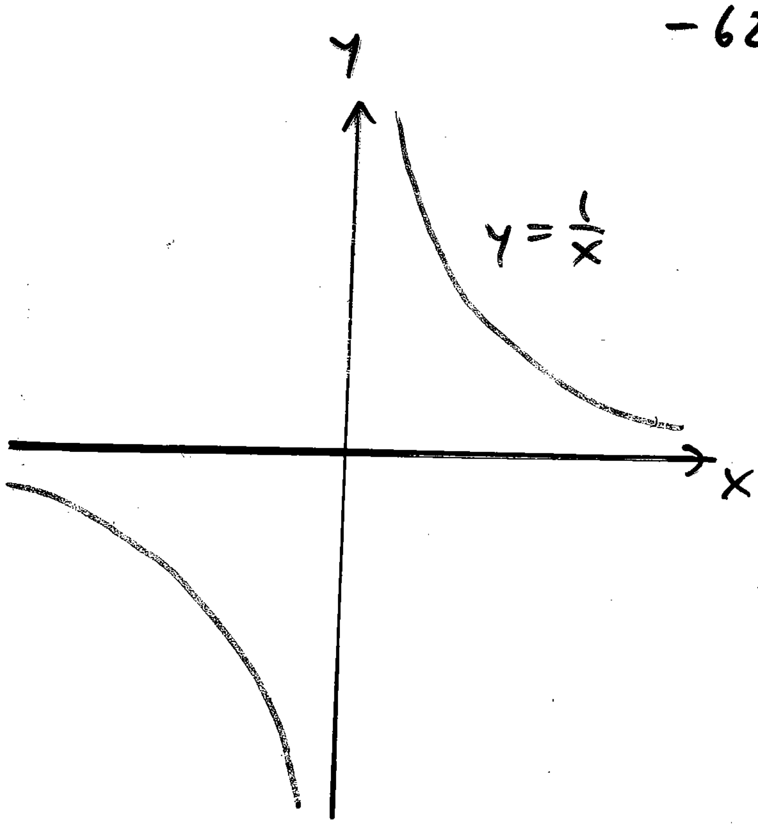
$$= \underline{\frac{5+\overset{0}{\cancel{\lim_{x\to\infty}\frac{8}{x}}} - \overset{0}{\cancel{\lim_{x\to\infty}\frac{3}{x^2}}}}{3+\overset{0}{\cancel{\lim_{x\to\infty}\frac{2}{x^2}}}} = \frac{5}{3}} \qquad [2\text{-}55]$$

(c) $$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}$$

$$= \underline{\frac{\overset{0}{\cancel{\lim_{x\to\infty}\frac{11}{x^2}}}+\overset{0}{\cancel{\lim_{x\to-\infty}\frac{2}{x^3}}}}{2-\overset{0}{\cancel{\lim_{x\to-\infty}\frac{1}{x^3}}}} = 0} \qquad [2\text{-}56]$$

Asymptotes

$$f(x) = \frac{1}{x}$$



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

The graph approaches the line

$\underline{y = 0}$

asymptotically, the line is

called an **asymptote** of the graph.

[Definition on 2-57]

Examples : [2-55, 56, 59]

Oblique Asymptotes (slanted)

Example: $f(x) = \dfrac{2x^2 - 3}{7x+4}$ [2-59]

Use polynomial division [Ess. Maths !]

to write

$$f(x) = \underbrace{\frac{2}{7}x - \frac{8}{49}}_{\text{linear function}} \underbrace{- \frac{115}{49(7x+4)}}_{\text{remainder}}$$

$$\lim_{x \to \pm\infty} \left(- \frac{115}{49(7x+4)} \right) = 0$$

so $g(x) = \frac{2}{7}x - \frac{8}{49}$ is a

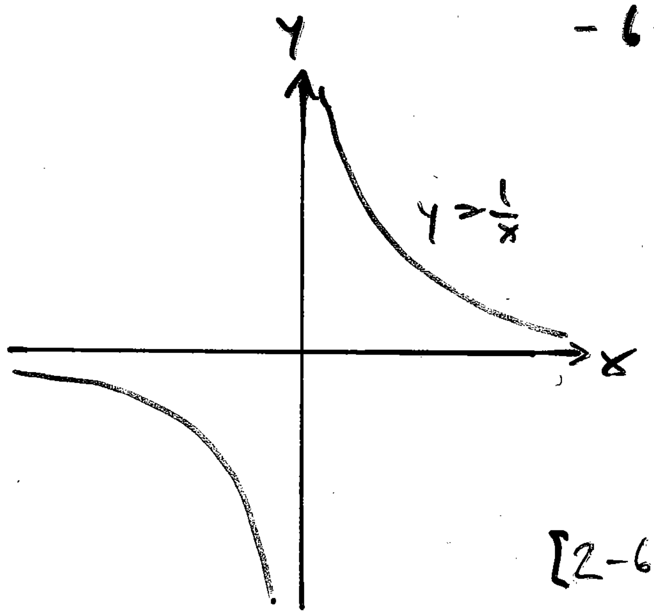
<u>slanted asymptote</u> of the graph of $f(x)$

$$f(x) = \frac{p(x)}{q(x)}$$, degree of p = degree of q + 1

①

Infinite Limits

$$f(x) = \frac{1}{x}$$



[2-61]

As $x \to 0^+$, $f(x)$ grows without bound

f has no limit $L \in \mathbb{R}$, but it is

convenient to write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Careful: We are not saying that the limit exists, and ∞ is not a real number!

$\lim_{x \to 0^+} \frac{1}{x} = \infty$ means that the limit does not exist as $\frac{1}{x}$ becomes arbitrarily large

Example One-Sided Infinite Limits

$$f(x) = \frac{1}{x-1} : \quad \lim_{x \to 1^+} f(x) , \lim_{x \to 1^-} f(x) \,?$$

- geometric solution see [2-62]
- analytic solution :

1) as $x \to 1^+$, $\underbrace{x-1}_{t} \to 0^+$

2) as $t \to 0^+$, $\frac{1}{t} \to \infty$

therefore as $x \to 1^+$, $\frac{1}{x-1} \to \infty$

similarly, as $x \to 1^-$, $\frac{1}{x-1} \to -\infty$

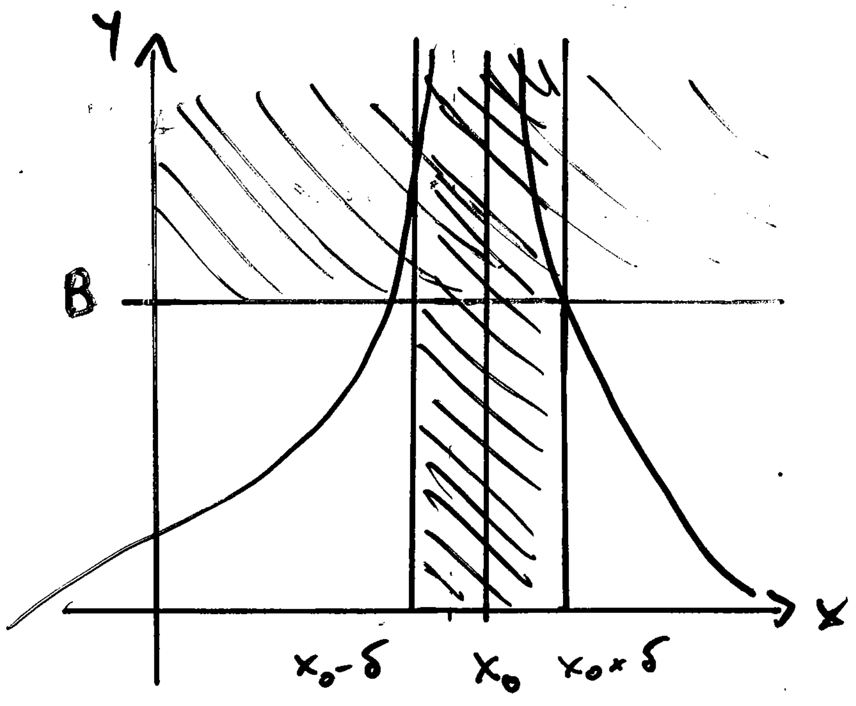
<u>Example</u> Two-sided infinite limits

$$f(x) = \frac{1}{x^2} \qquad \lim_{x \to 0} f(x) = \underline{\infty}$$

$$g(x) = \frac{1}{(x+3)^2} \qquad \lim_{x \to -3} g(x) = \underline{\infty}$$

[2-63]

<u>Precise definitions</u> see slide [2-64]



Example: Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$

- given $B>0$, find $\delta>0$ such that

$$0<|x-0|<\delta \Rightarrow \frac{1}{x^2} > B$$

- $\frac{1}{x^2} > B \iff |x| < \frac{1}{\sqrt{B}}$

- choose $\delta = \frac{1}{\sqrt{B}}$. Then

$$0<|x|<\delta \Rightarrow \frac{1}{x^2} > B$$

Therefore, by definition,

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

(back to 62)

Vertical

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$\underline{x = 0}$

[2-67]

[2-69, 70]

<u>Example:</u> Find the horizontal and vertical asymptotes of the graph of $f(x) = -\frac{8}{x^2-4}$

- $\lim\limits_{x \to \pm\infty} f(x) = \underline{\quad 0 \quad}$

- division by zero for $x = \underline{\pm 2}$

[trick: $-\frac{8}{x^2-4} = \frac{\overset{2}{\cancel{4}}}{x+2} - \frac{\overset{2}{\cancel{4}}}{x-2}$]

- $\lim\limits_{x \to -2^-} f(x) = \underline{-\infty}$ $\quad \lim\limits_{x \to 2^-} f(x) = \underline{\infty}$

  $\lim\limits_{x \to -2^+} f(x) = \underline{\infty}$ $\quad \lim\limits_{x \to 2^+} f(x) = \underline{-\infty}$

- Asymptotes are $y = 0$
  $x = -2$, $x = 2$ [2-70]

<u>Example</u>

$$f(x) = \frac{x^2-3}{2x-4} \qquad [2-73]$$

rewrite [polynomial division !]

$$f(x) = \underbrace{\frac{x}{2}+1}_{\text{linear term}} + \underbrace{\frac{1}{2x-4}}_{\text{remainder}}$$

Asymptotes are $\quad y = \underbrace{\frac{x}{2}+1} \quad$ and $\quad x = 2$

we say that this term <u>dominates</u> $f(x)$ as $x \to \pm\infty$

<u>Example</u> $\quad f(x) = \underbrace{3x^4} - 2x^3 + 3x^2 - 5x + 6$

dominant term $\quad g(x) = 3x^4$

$g(x)$ dominates $f(x)$ as $x \to \pm\infty \qquad [2-74]$

Continuity

- Informally, any function whose graph can be sketched over its domain in one continuous motion without lifting the pen is an example of a continuous function.

- Formally, a function $y = f(x)$ is continuous at an interior point c of its domain if

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

Whole definition see [2-79].

Examples: [2-80, 81], Continuity Test [2-82]

If a function is not continuous at a point c, we say that f is <u>discontinuous</u> at c.

Examples of discontinuities : [2-84]

A function is <u>continuous on an interval</u> if and only if it is continuous at every point of the interval.

<u>Examples:</u>

$f(x) = k$ continuous for all x

$f(x) = x$ continuous for all x

$f(x) = \frac{1}{x}$ continuous for all $x \neq 0$

Limit Laws imply <u>Properties of Continuous Functions</u>

[2-86]

$\left(f+g,\ fg,\ f/g,\ \text{etc}\right)$

<u>Example</u> Polynomials are continuous

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

satisfies $\lim\limits_{x\to c} p(x) = p(c)$

[Remember, we computed limits by substitution]

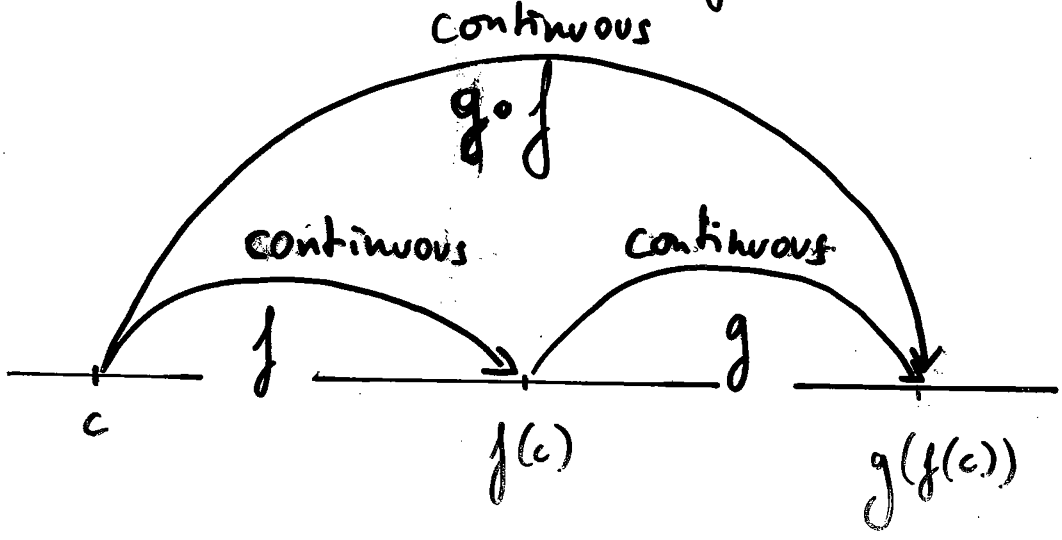
<u>Example</u> Rational functions are continuous

whenever they are defined:

$$\frac{P(x)}{Q(x)} \text{ is continuous for } Q(x) \neq 0$$

<u>Example</u> $f(x) = |x|$ is continuous

Composition of continuous functions



<u>Theorem:</u> If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

<u>Example</u>

$$y = \left| \frac{x \sin x}{x^2+1} \right|$$ [2-89]

$$f(x) = \underline{\frac{x \sin x}{x^2+1}}$$

$$g(x) = \underline{|x|}$$

Continuous extension to a point

$$f(x) = \frac{\sin x}{x} \quad \text{for } x \neq 0$$

is defined and continuous for all $x \neq 0$.

As $\lim_{x \to 0} \frac{\sin x}{x} = 1$, it makes sense to define

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \qquad [2-90]$$

We call $F(x)$ the <u>continuous extension</u> of $f(x)$.

<u>Example:</u> $$f(x) = \frac{x^2+x-6}{x^2-4} \qquad [2-91]$$

$$F(x) = \frac{x+3}{x+2} \qquad (c=2)$$

Intermediate Value Theorem for Continuous Functions

Whenever f takes on two values, it also takes on all the values in between. [2-92]

<u>Geometrically</u>: any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$

Consequences of the Intermediate Value Theorem

- The graph of a continuous function over an interval is connected (i.e. no jumps, no separate branches)

- Root-finding : A solution of the equation $f(x)=0$ is called a root of the equation or a zero of f.

> If $f(x)$ is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite sign, then $f(x)=0$ has roots on $[a,b]$

[2-94]

Tangents and Derivatives

- Construct a <u>tangent</u> to a curve using a <u>limit of secants</u> [2-98]

- Compute the <u>slope</u> of the tangent as a limit of slopes of tangents

---

<u>Example:</u> Tangent line to a parabola

[2-99]

$y = x^2$

Point $P = (2,4)$

choose Point at "distance" h :

$$Q = (2+h\,,\ (2+h)^2)$$

Secant through $P, Q \xrightarrow[h \to 0]{}$ Tangent through P

- Secant slope:

(Note: h can be negative)

$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{(2+h) - 2}$$

$$= \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{\cancel{2} + h - \cancel{2}}$$

$$= \frac{4h + h^2}{h} = 4 + h$$

- Tangent slope:

$$m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = 4$$

- Equation of tangent through $P = (2,4)$:

$$y = 4 + 4(x-2) = -4 \pm 4x$$

$4+h$, $-4+4x$

Definition

The <u>slope of the curve</u> $y = f(x)$ at the point $P = (x_0, f(x_0))$ is the number

$$m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

[2-101]

[Also used : $x = x_0 + h$; $h \to 0$]

Slope of a straight line

$$f(x) = mx + b$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{mx + \cancel{b} - mx_0 - \cancel{b}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{m\,\cancel{(x - x_0)}}{\underset{1}{\cancel{x - x_0}}} = m$$

Recipe for finding the tangent to a curve

[2-102]

<u>Example</u> Slope and tangent to

$$y = \frac{1}{x} \quad \text{at} \quad x_0 = a \neq 0$$

- $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$, $f(a+h) = \frac{1}{a+h}$

- $$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

- tangent line at $\left(a, \frac{1}{a}\right)$:

$$y = \frac{1}{a} + m(x - a) \qquad m = -\frac{1}{a^2}$$

$$\underline{\underline{y = \frac{2}{a} - \frac{x}{a^2}}}$$

[2-103, 104]

The derivative as a function

Definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad [3\text{-}4,5]$$

is called the <u>derivative</u> of $f(x)$,

if it exists, f is called <u>differentiable</u>

Notation: $\qquad y = f(x)$

$$f'(x) = \frac{d}{dx} f(x) \qquad \text{"dee-by-dee-ek}$$

also used: $\quad y' = \dfrac{dy}{dx}$

Computing the derivative is called

<u>differentiation</u> (and NOT derivation!)

Example

$$f(x) = \frac{x}{x-1}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x-1) - (x+h-1)x}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{hx} - \cancel{x} - h - \cancel{x^2} - \cancel{hx} + \cancel{x}}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= -\frac{1}{(x-1)^2}$$

<u>Example</u> $f(x) = \sqrt{x}$ for $x > 0$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$ use trick

$$= \lim_{z \to x} \frac{\cancel{\sqrt{z} - \sqrt{x}}^{\,1}}{(\cancel{\sqrt{z} - \sqrt{x}})(\sqrt{z} + \sqrt{x})}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

[3-7]

For example, tangent line at $x_0 = 4$:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$= 2 + \frac{1}{2 \cdot 2}(x - 4)$$

$$= 1 + \frac{1}{4}x$$

Just as with left and right limits,

we define <u>left and right derivatives</u>:

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \qquad \text{(right)}$$

and

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \qquad \text{(left)}$$

<u>Example</u>: $f(x)=|x|$ is not differentiable at 0

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1$$

so the right and left derivatives differ!

[3-11]

Differentiability implies continuity

Theorem If f has a derivative at $x=c$, then f is continuous at $x=c$.

Proof For $h \neq 0$, we write

$$f(c+h) = f(c) + \underbrace{\frac{f(c+h)-f(c)}{h}}_{\substack{\to\ f'(c) \text{ as } h\to 0 \\ \text{exists by assumption}}} h$$

Therefore we have

$$\lim_{h\to 0} f(c+h) = f(c) + f'(c)\cdot 0$$

which says that f is continuous at $x=c$

$\square$

Caution: The converse is wrong!

Differentiation Rules

$$f(x) = c \; : \qquad [3\text{-}18]$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0$$

$$f(x) = x^n \; , \quad n \in \mathbb{N} \; : \qquad [3\text{-}19]$$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

$$= \lim_{z \to x} \big(\underbrace{z^{n-1} + z^{n-2}x + \dots + z\,x^{n-2} + x^{n-1}}_{n \text{ terms}} \big)$$

$$= n\,x^{n-1}$$

$$f(x) = c\,g(x) : \qquad [3\text{-}21]$$

$$f'(x) = \lim_{h\to 0} \frac{c\,g(x+h) - c\,g(x)}{h}$$

$$= c \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= c\,g'(x)$$

$$f(x) = u(x) + v(x) : \qquad [3\text{-}22]$$

$$f'(x) = \lim_{h\to 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h}$$

$$= u'(x) + v'(x)$$

<u>Example</u> $y = x^4 - 2x^2 + 2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 - 2x^2 + 2\right)$$

Rule 4: $= \frac{d}{dx}(x^4) + \frac{d}{dx}(-2x^2) + \frac{d}{dx}(2)$

Rule 3: $= \frac{d}{dx}(x^4) + (-2)\,\frac{d}{dx}(x^2) + \frac{d}{dx}(2)$

Rule 2: $= 4x^3 + (-2)\,2x + \frac{d}{dx}(2)$

Rule 1: $= 4x^3 - 4x + 0$

$$y' = 4x^3 - 4x$$

Now, find for example, horizontal tangents

$$y' = 0 \Rightarrow 4x(x^2 - 1) = 0 \Rightarrow x \in \{0, 1, -1\}$$

[3-24]

$$f(x) = u(x)\, v(x) : \qquad [3\text{-}25]$$

$$f'(x) = \lim_{h \to 0} \frac{u(x+h)\,v(x+h) - u(x)\,v(x)}{h}$$

$$= \lim_{h \to 0} \frac{\overbrace{u(x+h)\,v(x+h) - u(x)\,v(x+h)}^{(u(x+h)-u(x))\,v(x+h)} + \overbrace{u(x)\,v(x+h) - u(x)\,v(x)}^{u(x)\,(v(x+h)-v(x))}}{h}$$

$$= \lim_{h \to 0} \left(\frac{u(x+h) - u(x)}{h}\, v(x+h) + u(x)\, \frac{v(x+h) - v(x)}{h} \right)$$

$$= \underbrace{\lim_{h \to 0} \frac{u(x+h) - u(x)}{h}}\ \underbrace{\lim_{h \to 0} v(x+h)} + u(x) \underbrace{\lim_{h \to 0} \frac{v(x+h) - v(x)}{h}}$$

$$= \qquad u'(x) \qquad v(x) \quad + u(x) \quad v'(x)$$

Shown:

$$(uv)'(x) = u'(x)\, v(x) + u(x)\, v'(x)$$

$$f(x) = \frac{u(x)}{v(x)} \qquad \text{very similar} \qquad [3\text{-}26]$$

Note: Generally

$$(uv)' \neq u'v'$$

$$\left(\frac{u}{v}\right)' \neq \frac{u'}{v'}$$

WRONG

Examples:

$$y = (x^2+1)(x^3+3)$$

$$u = x^2+1 \quad , \quad v = x^3+3$$

$$u' = 2x \quad , \quad v' = 3x^2$$

$$y' = u'v + uv' = 2x(x^3+3) + (x^2+1)\,3x^2$$

$$y = \frac{t^2-1}{t^2+1}$$

$$u = t^2-1 \quad , \quad v = t^2+1$$

$$u' = 2t \quad , \quad v' = 2t$$

$$y' = \frac{u'v - uv'}{v^2} = \frac{2t(t^2+1) - (t^2-1)\,2t}{(t^2+1)^2}$$

$$f(x) = x^n \quad , x \neq 0, \quad n \in \mathbb{Z}^-$$

[3-27]

$$n = -m, \quad m \in \mathbb{N}$$

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{\frac{d}{dx}(1)\; x^m - 1\; \frac{d}{dx}(x^m)}{(x^m)^2}$$

$$= \frac{0 \cdot x^m - 1 \cdot m\, x^{m-1}}{x^{2m}}$$

$$= \frac{-m}{x^{m+1}} = n\, x^{n-1}$$

Examples: $\frac{d}{dx}(x^{10}) = 10\, x^9$

$$\frac{d}{dx}(x^{-11}) = -11\, x^{-12}$$

Higher Derivatives

If f' is differentiable,

we write the <u>second derivative</u>

$$f'' = (f')'$$

Notation: $$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

Similarly, we write the <u>third derivative</u>

$$f''' = (f'')'$$

and the <u>n-th derivative</u> (for $n \in \mathbb{N}_0$)

$$f^{(n)} = (f^{(n-1)})'$$

with

$$f^{(0)} = f$$

$$f(x) = x^5$$

$$f'(x) = 5x^4$$

$$f''(x) = 20x^3$$

$$f'''(x) = 60x^2$$

$$f^{(4)}(x) = f^{\text{iv}}(x) = 120x$$

$$f^{(5)}(x) = 120$$

$$f^{(6)}(x) = 0$$

$$f^{(7)}(x) = 0$$

$$\vdots$$

$$= 0$$

$$g(x) = x^{-5}$$

$$g'(x) = -5x^{-6}$$

$$g''(x) = 30x^{-7}$$

$$g'''(x) = -210x^{-8}$$

$$g^{(4)}(x) = 1680x^{-}$$

$$\vdots$$