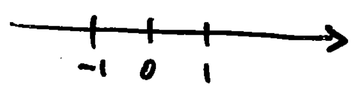
Calculus I

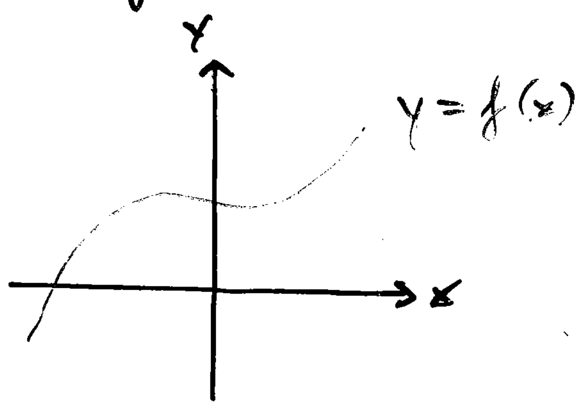
What is calculus ?

- Study of functions of real variables
  - one real variable
  - many variables (Calculus II)

- Fundamental : real numbers



- Geometric view : graph of a function



  - slope ≙ derivative

  - area ≙ integral

- many applications

Real numbers and the real line

Properties of real numbers $\mathbb{R}$

- algebraic (rules of calculation)
- order (geometric picture : line)
- completeness (no "gaps")

a) algebraic properties $a, b, c \in \mathbb{R}$

A1 $a + (b+c) = (a+b)+c$

A2 $a+b = b+a$

A3 there is a "0" such that $a+0 = a$

A4 there is an x such that $a+x = 0$

$x = $ "$-a$"

M1 $a(bc) = (ab)c$

M2 $ab = ba$

M3 there is a "1" such that $a1 = a$

$x = a^{-1} = \frac{1}{a}$

M4 there is an x such that $ax = 1$

(for $a \neq 0$)

D $a(b+c) = ab + ac$

b) order : the real line

O1 for any a, b $a \leq b$ or $b \leq a$

O2 if $a \leq b$ and $b \leq a$ then $a = b$

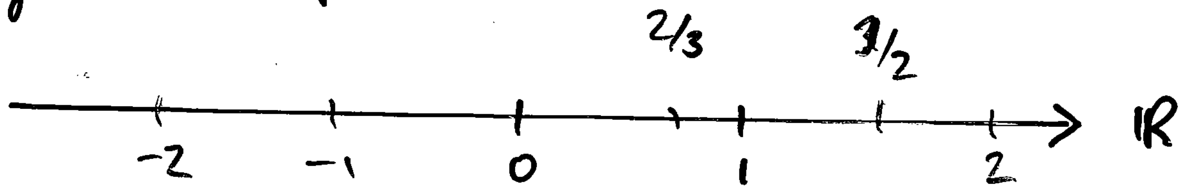
O3 if $a \leq b$ and $b \leq c$ then $a \leq c$

O4 if $a \leq b$ then $a + c \leq b + c$

O5 if $a \leq b$ and $0 \leq c$ then $ac \leq bc$

(consequences see slide)

geometric interpretation:



c) completeness :

the real numbers correspond to all points on the line, there are no "holes" or "gaps".

Subsets of the real numbers $\mathbb{R}$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ natural numbers

solve $a + x = b$ for x

$\mathbb{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$ integers

solve $ax = b$

$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\}$

rational numbers

solve $x^2 = 2$

$\mathbb{R}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

$\mathbb{N}$ and $\mathbb{Z}$ clearly have gaps, but

$\mathbb{Q}$ is "dense", so are there "holes"?

between any two rationals there is another one

Yes, there are "holes" (well, sort of):

Irrational numbers such as $\sqrt{2}$ or $\pi = 3.14159\ldots$

$\sqrt{2}$ is the positive solution to $x^2 = 2$

<u>Theorem</u> $x^2 = 2$ has no solution $x \in \mathbb{Q}$

<u>Proof</u>
- Assume there is an $x \in \mathbb{Q}$ with $x^2 = 2$. This must be of the form $x = \frac{p}{q}$ with p, q integers with no common factor.
- $x^2 = 2$ implies then $\left(\frac{p}{q}\right)^2 = 2$,

  or $\underline{p^2 = 2q^2}$, so that p is even.*
- Write $p = 2p_1$, so that $\overset{2}{\cancel{4}} p_1^2 = \overset{1}{\cancel{8}} q^2$,

  or $2p_1^2 = q^2$, so that q is even*
- We have shown that both p and q are even. This is a contradiction!

You have just seen a <u>"theorem with proof"</u>.

University mathematics is built upon

- Basic properties (Axioms, Definitions)
- Statements deduced from these
  (Lemma, Theorem, Corollary, ...)
- <u>and</u> their proofs!

You have just seen one such proof, called "proof by contradiction".

There will be many more to come!

- And of course there are also
  examples, exercises, applications, ...