Queen Mary
University of London
BSc Examination 2005

MAS205 Complex Variables

`19th May 2005   14.30`

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.

(a) Find all solutions $z \in \mathbb{C}$ of the equation

$$z^4 + 2i = 0 \ .$$

(b) Find all solutions $z \in \mathbb{C}$ of the equation

$$e^{2iz} = -1$$

and hence find all solutions $z \in \mathbb{C}$ of the equation

$$\cos(z) = 0 \ .$$

(c) How many roots of the polynomial $11z^{1001} + 101z$ lie in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$?

A2.
Consider the transformation $z \mapsto w = z^2 + i$.

(i) Show that the images of the lines $\Im(z) = c$ (for $c \in \mathbb{R}$ constant) are parabolas by computing their equations.

(ii) Sketch the image of the strip $\{z \in \mathbb{C} : -1 \leq \Im(z) \leq 1\}$.

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A3.
Let $f = u + iv$ be a complex-valued function of a complex variable $z = x + iy$.

(i) Write down the definition of the *derivative* of f at z_0. What are the *Cauchy-Riemann equations* and when do they hold?

(ii) Give a sufficient condition for $f(z)$ to be differentiable at z_0.

(iii) Show that $f(z) = |z|^2$ is only differentiable at the point $z = 0$.

A4.
Let

$$f(z) = \frac{1}{(z^2 + 1)} \ .$$

(i) Determine the Laurent series

$$\sum_{n=0}^{\infty} a_n (z - i)^n + \sum_{n=1}^{\infty} b_n (z - i)^{-n}$$

of $f(z)$ on a punctured disk centred at $z_0 = i$.

(ii) For which values of z is this Laurent series valid?

(iii) What type of singularity does f have at the point $z_0 = i$?

(iv) Determine the residue of f at the point $z_0 = i$.

A5.

(a) Let $\mathcal{C}$ be a *contour* parametrised by a piecewise smooth function $\gamma : [a, b] \to \mathbb{C}$. Define what is meant by the *contour integral*

$$\int_{\mathcal{C}} f(z)\, dz$$

of the complex function f along the contour $\mathcal{C}$.

(b) Evaluate $\int_{\mathcal{C}} f(z)\, dz$ for $f(z) = z + \bar{z}$ (where $\bar{z}$ is the complex conjugate of z) and for the following contours:

(i) $\mathcal{C}$ is the straight line segment from $-2i$ to $2i$;

(ii) $\mathcal{C}$ is the semi-circle from $-2i$ to $2i$ in the right half plane.

(c) Is it possible that the function $f(z) = z + \bar{z}$ has an *anti-derivative* on $\mathbb{C}$? Find one or else give a reason why such an anti-derivative cannot exist.

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SECTION B *Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.

(a) Define what is meant by an *isolated singularity* of a complex function f, define the three *types* of isolated singularity, and define the *residue* of f at an isolated singularity.

(b) Determine the singularities and their residues for each of the following functions:

(i)
$$f(z) = z^{-4}e^{z} \ ,$$

(ii)
$$f(z) = \frac{\cos(z)}{z} - \frac{1}{z} \ ,$$

(iii)
$$f(z) = z^{3}(e^{-\frac{1}{z}} - 1) \ .$$

B7.

(a) State the *Cauchy Integral Formula* for a function f holomorphic on and inside a simple closed contour.

(b) State the *Extended Cauchy Integral Formula* for f and show how it follows from (a) by formal differentiation.

(c) Compute
$$\int_{\mathcal{C}} \frac{e^{z}}{(z-1)^{n}} dz$$
where $\mathcal{C}$ is a positively oriented contour encircling $z = 1$ and n is a non-negative integer.

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B8.

Let $p(z)$ and $q(z)$ be holomorphic at $z = z_0$ with $p(z_0) \neq 0$ and let $q(z)$ have a simple zero at z_0.

(i) Prove that p/q has a simple pole at z_0.

(ii) Using the Taylor series of $q(z)$ at z_0 or otherwise, prove that

$$\mathrm{R}es_{z_0}\left(\frac{p(z)}{q(z)}\right) = \frac{p(z_0)}{q'(z_0)} .$$

B9.

(a) State the *Residue Theorem.*

(b) Use the Residue Theorem to calculate

$$\int_{\mathcal{C}} \frac{1}{\sin z} dz$$

where $\mathcal{C}$ is the positively oriented circle with centre i and radius 2.

(c) Use the Residue Theorem to prove that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx = \frac{\pi}{e} .$$

End of examination paper