MTH5105 Differential and Integral Analysis 2010-2011

Solutions 5

1 Exercises for Feedback

1) Let $f(x) = \exp(\sqrt{x})$, $g(x) = \sin(\pi x)$, and $P = \{0, 1, 4, 9\}$.

   (a) Find the upper and lower sums $U(f, P)$ and $L(f, P)$ of f for the partition P. Use these sums to give bounds for $\int_0^9 f(x)\,dx$.

   (b) Find the upper and lower sums $U(g, P)$ and $L(g, P)$ of g for the partition P. Use these sums to give bounds for $\int_0^9 g(x)\,dx$.

Solution:

(a) Recall that $I_i = [x_i - x_{i-1}]$, $\Delta x_i = x_i - x_{i-1}$, $M_i = \sup_{x \in I_i} f(x)$, and $m_i = \inf_{x \in I_i} f(x)$. We have

$$
\begin{array}{llll}
I_1 = [0,1]\ , & \Delta_1 = 1\ , & M_1 = \exp(1)\ , & m_1 = \exp(0)\ , \\
I_2 = [1,4]\ , & \Delta_2 = 3\ , & M_2 = \exp(2)\ , & m_2 = \exp(1)\ , \\
I_3 = [4,9]\ , & \Delta_3 = 5\ , & M_3 = \exp(3)\ , & m_3 = \exp(2)\ .
\end{array}
$$

Therefore

$$
\begin{aligned}
U(f,P) &= \sum_{i=1}^{3} M_i \Delta x_i = 1\exp(1) + 3\exp(2) + 5\exp(3)\ , \\
L(f,P) &= \sum_{i=1}^{3} m_i \Delta x_i = 1\exp(0) + 3\exp(1) + 5\exp(2)\ .
\end{aligned}
$$

Hence we have

$$1 + 3e + 5e^2 \le \int_0^9 f(x)\,dx \le e + 3e^2 + 5e^3\ .$$

(In fact, the integral evaluates to $2 + 4e^3 \approx 82.3$, while the lower and upper sums are approximately 46.1 and 125.3.)

(b) We have now

$$M_1 = 1\ , \qquad m_1 = 0\ , \qquad M_2 = 1\ , \qquad m_2 = -1\ , \qquad M_3 = 1\ , \qquad m_3 = -1\ .$$

Therefore

$$U(g,P) = 1 \cdot 1 + 3 \cdot 1 + 5 \cdot 1\ , \qquad L(g,P) = 1 \cdot 0 + 3 \cdot (-1) + 5 \cdot (-1)\ .$$

Hence we have

$$-8 \le \int_0^9 g(x)\,dx \le 9\ .$$

2 Extra Exercises

2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0\,. \end{cases}$$

(a) Given a partition P of $[-1, 1]$, what is $L(f, P)$? What is $\int_{*-1}^{1} f(x)\,dx$?

(b) For fixed $\epsilon > 0$, find a partition P of $[-1, 1]$ such that $U(f, P) < \epsilon$. What is $\int_{-1}^{*1} f(x)\,dx$?

(c) Is f integrable on $[-1, 1]$? If so, what is its integral?

Solution:

(a) Given a partition P of $[-1, 1]$, the function f has infimum 0 in any subinterval. Therefore $L(f, P) = 0$ for any partition P.
Hence $\int_{*-1}^{1} f(x)\,dx = 0$.

(b) For $0 < \delta < 1$, choose $P = \{-1, -\delta, \delta, 1\}$. On the intervals $[-1, -\delta]$ and $[\delta, 1]$ the function f has maximum value 0. On the interval $[-\delta, \delta]$ it has maximum value 1. Therefore

$$U(f, P) = ((-\delta) - (-1)) \cdot 0 + (\delta - (-\delta)) \cdot 1 + (1 - \delta) \cdot 0 = 2\delta\,,$$

and if we choose $\delta < \epsilon/2$, we have $U(f, P) < \epsilon$.
Hence $\int_{-1}^{*1} f(x)\,dx \leq 0$. Using (a), we have

$$0 = \int_{*-1}^{1} f(x)\,dx \leq \int_{-1}^{*1} f(x)\,dx \leq 0\,,$$

so that $\int_{*-1}^{1} f(x)\,dx = 0$.

(c) As

$$\int_{-1}^{*1} f(x)\,dx = \int_{-1}^{*1} f(x)\,dx = 0\,,$$

f is integrable and $\int_{-1}^{1} f(x)\,dx = 0$.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the equidistant partitions P_n of $[0, 1]$ into n subintervals.

(a) Find $U(f, P_n)$. What can you say about $\int_{0}^{*1} f(x)\,dx$?

(b) Find $L(f, P_n)$. What can you say about $\int_{*0}^{1} f(x)\,dx$?

(c) Is f integrable on $[0, 1]$? If so, what is its integral?

[Hint: $\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n + 1)(2n + 1)$.]

Solution:

We have

$$P_n = \{0/n, 1/n, \ldots, n/n\}\,,$$

or $x_i = i/n$ for $i = 0, \ldots, n$. Thus, $I_i = [(i - 1)/n, i/n]$ and $\Delta x_i = 1/n$.

(a) We have $M_i = (i/n)^2$ and thus

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\,. \end{aligned}$$

Hence, $\int_{0}^{*1} f(x)\,dx \leq 1/3$.

(b) Similarly we have $m_i = ((i-1)/n)^2$ and thus

$$L(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right)$$
$$= \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\ .$$

Hence, $\int_{*0}^{1} f(x)\,dx \geq 1/3$.

(c) Combining these we see that $\int_0^1 x^2\,dx$ exists and equals 1/3.

4*) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1/q & x = p/q \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0, \text{ and} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Prove that f is Riemann-integrable on $[0,1]$ and that $\int_0^1 f(x)dx = 0$. If you want to practice old material, show also that f is discontinuous at $x \in \mathbb{Q}$ and continuous at $x \notin \mathbb{Q}$ (easy), and that it is nowhere differentiable (hard).

Solution:

Clearly, for any partition P of $[0,1]$, $m_i = 0$ (can you explain why this is obvious?) and thus $L(f,P) = 0$.

If we can show that for any $\varepsilon > 0$ there exists a partition P of $[0,1]$ such that $U(f,P) < \varepsilon$, then it follows that $U(f,P) - L(f,P) < \varepsilon$ and therefore that f is Riemann integrable. As $L(f,P) = 0$ for all partitions, it will then follow that $\int_0^1 f(x)dx = \int_{*0}^1 f(x)dx = 0$ as needed.

The key for estimating the upper sum is to observe that there are actually very few points at which $f(x)$ is not small. More precisely, given $\varepsilon' > 0$, there are only finitely many points $x \in [0,1]$ such that $f(x) \geq \varepsilon'$ (only those rational numbers with denominator not exceeding $1/\varepsilon'$), i.e.

$$N + 1 = |\{x \in [0,1] : f(x) > \varepsilon'\}|$$

is finite. Let's call these points $y_0 < y_1 < \ldots < y_N$. We now choose a partition

$$P = \{x_0, x_1, x_2, \ldots x_{2N+1}\}$$

such that

$$x_0 = y_0 < x_1 < x_2 < y_1 < x_3 < x_4 < y_2 < x_5 < x_6 < y_3 < \ldots < x_{2N} < y_N = x_{2N+1}\ ,$$

and such that $\Delta_{2j+1} = x_{2j+1} - x_{2j} < \varepsilon'/(N+1)$ for $j = 0, \ldots, N$. Then we can estimate $M_{2j+1} \leq 1$ for $j = 0, \ldots, N$ and $M_{2j} < \varepsilon'$ for $j = 1, \ldots, N$. Splitting the upper sum $U(f,P)$ into even and odd parts, we estimate

$$U(f,P) = \sum_{i=1}^{2N+1} M_i \Delta_i = \sum_{j=0}^{N} M_{2j+1}\Delta_{2j+1} + \sum_{j=1}^{N} M_{2j}\Delta_{2j}$$
$$< \sum_{j=0}^{N} 1 \cdot \Delta_{2j+1} + \sum_{j=1}^{N} \varepsilon' \Delta_{2j} < (N+1)\varepsilon'/(N+1) + \varepsilon' \cdot 1 = 2\varepsilon'\ .$$

Thus, by choosing $\varepsilon' = \varepsilon/2$ for a given $\varepsilon > 0$, $U(f,P) < \varepsilon$ as needed.

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