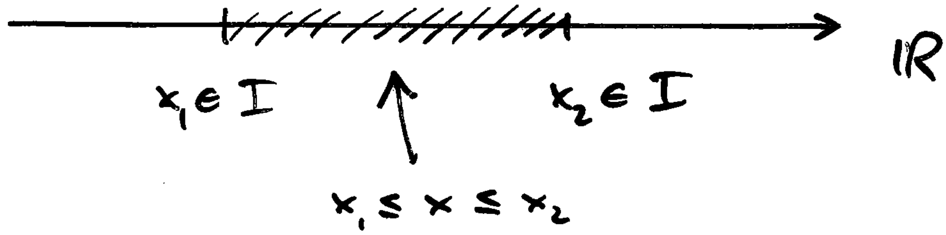
Intervals: A subset of the real line is called an <u>interval</u> if it contains at least two numbers and all the real numbers between any two of its elements.



(slide 1.5)

Examples:

(a) $2x - 1 < x + 3$

(b) $-\frac{x}{3} < 2x + 1$

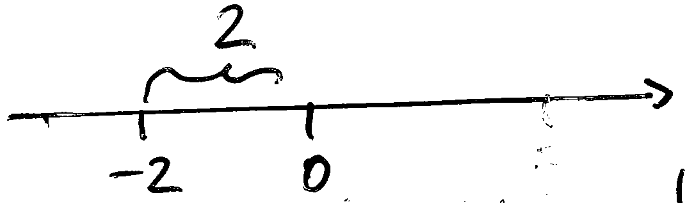
(c) $\frac{6}{x-1} \geq 5$

(slide 1.6)

Absolute value $|x|$:

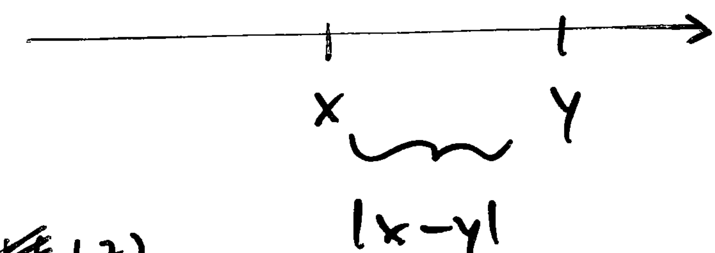
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

geometrically : $|x|$ distance from x to zero 0



$|x| = |x - 0|$

$|x-y|$ distance between x and y



(slide ~~..~~,1.7)

alternatively

$$|x| = \sqrt{x^2}$$

(square root is always non-negative!)

Inequality with $|x|$:

$$|x| < a \iff -a < x < a$$

(need $a > 0$, otherwise no solution)

(Slide 1-8, 1-9)

Properties of $|x|$:

1. $|-x| = |x|$

2. $|xy| = |x|\,|y|$

3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $\quad (y \neq 0)$

4. $|x+y| \leq |x| + |y|$

The last one is called <u>triangle inequality</u>

Examples:

(a) $|2x-3| \leq 1$ (b) $|2x-3| \geq 1$

(slide 1.10)

<u>Proof of properties 1.-4.</u>:

1. use $|x| = \sqrt{x^2}$

$$|-x| = \sqrt{(-x)^2} = \sqrt{x^2} = |x|$$

2. use $|x| = \sqrt{x^2}$

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2 y^2} =$$

$$= \sqrt{x^2}\sqrt{y^2} = |x|\,|y|$$

3. as 2.

4. blackboard

Important inequalities:

- Triangle inequality $\underline{|a+b| \leq |a| + |b|}$

- Arithmetic - geometric mean inequality

  - arithmetic mean $\frac{1}{2}(a+b)$

  - geometric mean $\sqrt{ab}$

$$\boxed{\sqrt{ab} \leq \frac{1}{2}(a+b)} \qquad a,b \geq 0$$

Cauchy - Schwarz

- Cauchy - Schwarz inequality

$$\boxed{(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)}$$

a, b, c, d real

Proof of: $\sqrt{ab} \leq \frac{1}{2}(a+b)$

- multiply by 2 and square (why allowed ?)

$\iff 4ab \leq (a+b)^2$

- use <u>direct proof</u>: start on one side

and transform until done ...

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$+ 2ab - 2ab$$

need 4ab $\rightsquigarrow$

$$= 4ab + a^2 - 2ab + b^2$$

$$= 4ab + \underbrace{(a-b)^2}_{\geq 0}$$

$$\geq 4ab$$

$\square$

symbol meaning "end of proof"

Proof of: Cauchy - Schwarz inequality

- use direct proof: start on one side

  and transform until done ...

rhs: $(a^2+b^2)(c^2+d^2) = a^2c^2 + b^2c^2$

$+ a^2d^2 + b^2d^2$

lhs: $(ac+bd)^2 = a^2c^2 + 2abcd + b^2d^2$

start on rhs and work it out:

$$(a^2+b^2)(c^2+d^2) = a^2c^2 + 2abcd + b^2d^2$$
$$+ b^2c^2 - 2abcd + a^2d^2$$
$$= (ac+bd)^2 + \underbrace{(bc-ad)^2}_{\geq 0}$$
$$\geq (ac+bd)^2$$

$\square$

Second proof (using a "trick"):

Consider $(ax+c)^2 + (bx+d)^2$

$$a^2x^2 + 2acx + c^2 + b^2x^2 + 2bdx + d^2$$

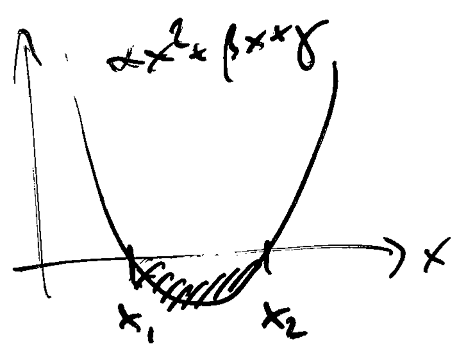
This is ≥ 0 as it is the sum of squares.

Multiplying out, we have also

$$0 \leq (a^2+b^2)\,x^2 + 2\,(ac+bd)\,x + (c^2+d^2) \qquad (*)$$

The right-hand-side is a quadratic equation in x (parabola, see 1.2)

$$\alpha x^2 + \beta x + \gamma$$



for $(*)$ to be true,

the <u>discriminant</u> $D = \beta^2 - 4\alpha\gamma$

must be non-positive $\left(x_{1,2} = \dfrac{-\beta \mp \sqrt{D}}{2\alpha}\right)$

$$\left.\begin{aligned} \alpha &= a^2+b^2 \\ \beta &= 2(ac+bd) \\ \gamma &= c^2+d^2 \end{aligned}\right\} \text{ compute } D$$

$$D = \beta^2 - 4\alpha\gamma = 4(ac+bd)^2 - 4(a^2+b^2)(c^2+d^2)$$

Now $\qquad D \leq 0 \iff$ Cauchy-Schwarz. $\square$

<u>Generalisation</u>

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$
$$\geq (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$$

Proof: start with

$$0 \leq (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \dots + (a_n x + b_n)^2$$