MTH744 Dynamical Systems

Lecture Notes 2012-2013

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Contents

0 Introduction

Lecture 1/2: 27/09/12

Historical review

- **1666:** infinitesimal calculus, motion of planets (Newton)
- **18th century:** classical mechanics
- **19th century:** analytical studies of planetary motions
- **1890:** geometric approach, stability of solar system? (Poincaré)
- **1920-1950:** nonlinear oscillators, radio, radar, laser
- **1920-1960:** complexity in analytical mechanics (Birkhoff, Kolmogorov, Arnold, Moser)
- **1963:** strange attractor in a model of turbulence (Lorenz)
- **1970:** turbulence/chaos (Ruelle, Takens), chaos in logistic map (May), universality, renormalisation, transitions to chaos (Feigenbaum), experimental studies (Gollub, Libchaber, Swinney), nonlinear oscillators in biology (Winfree), fractals (Mandelbrot)
- **1980:** development of applications

When is a system "nonlinear"?

Dynamical systems are described by

(a) iterated maps ("discrete" time)

Example: logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

(b) Differential equations (ordinary or partial)

Example for ODE: dampened harmonic oscillator (time dependent $x(t)$)

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Example for PDE: heat equation (space and time dependent $u(x,t)$)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}$$

We shall consider almost exclusively ODEs. Remember also that we can always transform ODEs of arbitrary order to systems of ODSs of first order, i.e.

$$\begin{aligned}
\dot{x}_1 =& f_1(x_1, \ldots, x_n) \\
\dot{x}_2 =& f_2(x_1, \ldots, x_n) \\
&\vdots \\
\dot{x}_n =& f_n(x_1, \ldots, x_n)
\end{aligned}$$

where the dots denote differentiation with respect to time t. For example, in

$$m\ddot{x} + b\dot{x} + kx = 0$$

let $x_1 = x$, $x_2 = \dot{x}$. Then

$$\begin{aligned}
\dot{x}_1 =& x_2 \\
\dot{x}_2 =& -\frac{b}{m}x_2 - \frac{k}{m}x_1
\end{aligned}$$

This is a *linear* system, because $\dot{x}_i$ depends linearly on all x_j. Non-linear systems are systems that don't just contain linear terms, for example higher powers x_i^m, products $x_i x_j$, or more complicated functions such as $\sin(x_i)$.

Linear systems can be separated into simpler parts (diagonalisation) and these simpler parts be combined by linear superposition; a linear system is equivalent to the sum of its parts.

Example: Physical pendulum

$$\ddot{x} + c\sin x = 0 \quad \text{where} \quad c = g/L > 0$$

or, equivalently,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c \sin x_1\end{aligned}$$

and where x is the angle of the pendulum.

This is clearly non-linear and cannot be solved easily. The exact solution involves elliptic functions, which mainly just shifts the problem. For small values of x one can approximate $\sin x \approx x$ and linearise the system.

But maybe we can "understand" the equations without solving them? Let us shift to a **geometric viewpoint**:

$$(x_1(t), x_2(t)) \ , \quad t \in \mathbb{R}$$

describes a curve in a two-dimensional space, or, as it is properly called, a *trajectory* in the *phase space* of the system.

The idea is now to investigate the generic behaviour of the trajectories *without* solving the system of differential equations explicitly.

<u>Terminology:</u> A dynamical system is an n-th order system if the phase space of coordinates $(x_1, x_2, \ldots, x_n)$ is n-dimensional.

So far, the systems introduced do not depend on time explicitly, they are *autonomous*. For time-dependent, or *non-autonomous* n-th order systems we simply introduce an extra variable $x_{n+1} = t$ and augment the system with an additional equation $\dot{x}_{n+1} = 1$.

For example, for the forced harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F \cos t$$

we let $x_1 = x$, $x_2 = \dot{x}$, and additionally now $x_3 = t$. Then

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{F}{m}x_3 \\ \dot{x}_3 &= 1\end{aligned}$$

is a three-dimensional time-independent dynamical system.

The dimension of the system, together with the "degree of non-linearity" determines the phenomena that can occur (see Figure 1.3.1 in textbook).

1 One-dimensional flows

1.1 Flows on the line

We consider the case $n = 1$, i.e. the first-order system

$$\dot{x} = f(x)$$

where $x = x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of the position x.

Note that the system is not time-dependent. Time dependence leads to a two-dimensional dynamical system. These will be discussed later.

The geometric viewpoint:

$$\boxed{\text{Interpret a differential equation as a vector field}}$$

We will approach this viewpoint gently in an example. Consider

$$\dot{x} = \sin x$$

with initial condition $x(0) = x_0$.

We can solve this differential equation exactly (details in lecture) and get

$$t = \log \left| \frac{\csc x_0 + \cot x_0}{\csc x(t) + \cot x(t)} \right|$$

This is exact, but very impractical.

For example, consider the cases

1. $x_0 = \pi/4$. Describe $x(t)$ for all $t > 0$, in particular $t \to \infty$.

2. For x_0 arbitrary, what happens as $t \to \infty$?

Now turn to a graphical analysis. Turn $\dot{x} = \sin(x)$ into a vector field on the real line. $\dot{x}$ describes the length (and direction) of a vector at point x. It is convenient to think of the vector field as indicating a *flow* of the system. A trajectory moves with the flow. If $\dot{x} > 0$, the vector points to the right, and the flow is to the right. Conversely if $\dot{x} < 0$, the vector points to the left, and the flow is to the left. When $\dot{x} = 0$, the flow vanishes and these points are therefore fixed points of the flow. They can be stable (attractors) or instable (repellers). Lecture 3/4: 4/10/12

We can now easily answer the two questions posed above (see Figures 2.1.2, 2.1.3 in textbook).

Fixed Points and Stability: The ideas can be extended to any kind of smooth one-dimensional system $\dot{x} = f(x)$: simply plot the graph of $f(x)$ and use it to sketch the associated vector field on the real line.

The phase space is separated by fixed points, which correspond to stationary solutions. If the flow is directed toward the fixed point, the stationary solution is stable, and the fixed point is an attractor. If the flow is directed away from the fixed point, the stationary solution is unstable, and the fixed point is a repeller.

Example: $\dot{x} = x^2 - 1$

Example: $\dot{x} = x - \cos x$

Example: Logistic equation, $\dot{N} = rN(1 - N/K)$

Remark: $\dot{N} = rN$ leads to exponential growth. Assuming additonally that the growth rate decreases linearly with N down to zero when $N = K$ leads to the logistic equation.

Linear stability analysis:

Let x^* be a fixed point of $\dot{x} = f(x)$, i.e. $f(x^*) = 0$. To consider the effect of a small perturbation $\eta(t) = x(t) - x^*$ we calculate

$$\dot{\eta} = \dot{x} - \dot{x}^* = f(x) = f(x^* + \eta)$$

A Taylor-expansion of $f(x)$ about x^* gives

$$f(x^* + \eta) \approx f(x^*) + \eta f'(x^*) + O(\eta^2)$$

so that

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2)$$

If $f'(x^*) \neq 0$ then we have in good approximation the **linearization**

$$\dot{\eta} \approx f'(x^*)$$

This implies

- $f'(x^*) > 0$: $|\eta(t)|$ increases exponentially as t grows
- $f'(x^*) < 0$: $|\eta(t)|$ decreases exponentially as t grows

We see that the sign of $f'(x^*)$ determines the stability of the fixed point. A characteristic timescale is given by its magnitude, i.e. $\tau = 1/|f'(x^*)|$ is a measure for the time it takes for $\eta(t)$ to change significantly.

If $f'(x^*) = 0$, we need to consider higher derivatives.

Example: $\dot{x} = \sin x$

Example: Logistic equation, $\dot{N} = rN(1 - N/K)$

Example: $\dot{x} = x^2$

Exercises: 2.2.8,9; 2.3.1,3; 2.4.7,8 Lecture

Existence and Uniqueness 5/6:

So far, informal treatment of $\dot{x} = f(x)$. Can anything go wrong? 11/10/12

Example for non-uniqueness: $\dot{x} = x^{1/3}$, $x(0) = 0$

$x(t) = 0$ is a solution, but so is $x(t) = (2t/3)^{3/2}$.

The reason is that $x^* = 0$ is *very* unstable, $f'(0)$ is infinite.

Existence and Uniqueness Theorem: Let $f(x)$ and $f'(x)$ be continuous in an open interval I with $x_0 \in I$. Then there exists a unique solution of the initial value problem

$$\dot{x} = f(x) \quad x(0) = x_0$$

on some open interval $(-\tau, \tau)$.

In other words, if f is smooth enough, then solutions exist locally. This does not imply global existence for all times t, however.

Example: $\dot{x} = 1 + x^2$ with $x(0) = x_0$

The above theorem guarantees (local) existence and uniqueness for any choice of x_0.

If $x_0 = 0$ then the solution is $x(t) = \tan(t)$, i.e. we only have a solution for $|t| < \pi/2$. As t approaches $\pi/2$, the solution diverges (*blow-up*).

Impossibility of Oscillations

So far, we have seen stable and unstable fixed points, decay and growth. And this is about it. For example, there are no oscillations possible: the direction of a flow on the real line is always uniquely determined, there is no possibility of a reversal of direction of motion. Oscillations, or overshoot beyond a fixed point cannot happen.

Physical example: overdamped mechanical system

Consider $m\ddot{x} + b\dot{x} = F(x)$ where m is very small (or more accurately, where $m\ddot{x}$ is much smaller than $b\dot{x}$). We can then write $\dot{x} = f(x) = F(x)/b$ and no oscillations can occur (light particle on a spring in honey).

Movement in a Potential

If we write $\dot{x} = f(x)$ with $f(x) = -\frac{dV}{dx}$, then we can visualise the motion of $x(t)$ as downhill motion of a particle in the potential $V(x)$.

Claim: $V(x)$ never increases along a trajectory.

Proof: $\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\dot{x} = -\left(\frac{dV}{dx}\right)^2 \leq 0$.

We can also conclude that local minima of V correspond to stable fixed points and local maxima of V correspond to unstable fixed points.

Example: $\dot{x} = -x$

Example: $\dot{x} = x - x^3$

1.2 Bifurcations

One-dimensional dynamics is rather simple, but it gets more interesting if we consider *parameter dependence*. Upon changing a parameter in the dynamical system, we can create or destroy fixed points, change their stability, and so on.

Qualitative changes in the dynamical behaviour are called **bifurcations**, the parameters values at which these bifurcations happen are called *bifurcation points*.

Example: a beam buckling under a weight.

Saddle-Node Bifurcation: two fixed points collide and get destroyed.

Example: $\dot{x} = r + x^2$

$r < 0$: two fixed points; one stable, one unstable

$r = 0$: one half-stable fixed point

$r > 0$: no fixed point

The vector fields for $r < 0$ and $r > 0$ are qualitatively different, a bifurcation occurred at the bifurcation point $r = 0$.

Graphical conventions:

Change from a vertical "stack of vector fields" depending on r to a bifurcation diagram, in which the vector field is graphed as a function of the bifurcation parameter r. Note that the line of stable fixed points is drawn solid, and the line of unstable fixed points is drawn dashed.

Changing the sign of r, one can view the saddle-node bifurcation also as a bifurcation at which two fixed points appear "out of nowhere."

Example: linear stability analysis of $\dot{x} = r - x^2$

Solving $f(x^*) = 0$ gives two solutions $x^* = \pm\sqrt{r}$ for $r > 0$, one solution $x^* = 0$ for $r = 0$ and no solutions for $r < 0$.

For $r > 0$ we find $f'(\pm\sqrt{r}) = \mp 2\sqrt{r}$, so $x^* = \sqrt{r}$ is stable and $x^* = -\sqrt{r}$ is unstable. The characteristic time scale $\tau = 1/2\sqrt{r}$ diverges as $r \to 0$.

Example: does $\dot{x} = r - x - e^{-x}$ have a saddle-node bifurcation?

We can argue graphically that there must be a bifurcation point r_c such that there are two fixed points for $r > r_c$ and no fixed point for $r < r_c$.

At the bifurcation point r_c we must have $f(x) = 0$ and $f'(x) = 0$, i.e. $e^{-x} = r - x$ and $e^{-x} = 1$. This implies $r_c = 1$ and $x = 0$.

Exercises: 2.6.2; 2.7.7; 3.1.2

Lecture 7/8: 18/10/12

Normal Forms

$\dot{x} = r + x^2$ and $\dot{x} = r - x^2$ are representative of *all* saddle-node bifurcations, in that the dynamical system locally "looks like" $x = r \pm x^2$.

For example, an expansion of $\dot{x} = r - x - e^{-x}$ near the bifurcation at $x = 0$ and

$r = 1$ gives

$$\dot{x} = r - x - (1 - x + x^2/2 + \ldots) = (r - 1) - x^2/2 + \ldots$$

Rescaling with $\tilde{r} = r - 1$ and $\tilde{x} = x/2$ gives $\dot{\tilde{x}} = \tilde{r} - \tilde{x}^2 + \ldots$. In fact, one can show that by some non-linear rescaling one can get locally entirely rid of the $+\ldots$ to obtain

$$\dot{\tilde{x}} = \tilde{r} - \tilde{x}^2$$

Graphically, fixed points occur where the graph of $f(x)$ intersects the x-axis. If f is locally parabolic, a saddle-node bifurcation occurs if the local extremum of $f(x)$ crosses the x-axis upon changing a parameter.

Consider f as a function of *both* x and r and analyse

$$\dot{x} = f(x, r) \quad \text{near the bifurcation point } x = x^* \text{ and } r = r_c$$

An expansion gives

$$\dot{x} = f(x^*, r_c) + (x - x^*) \left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c) \left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \frac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \ldots$$

As $f(x^*, r_c) = 0$ and $\left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} = 0$, we are left with

$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \ldots$$

which again shows the normal form behaviour, provided $a, b \neq 0$.

<u>Transcritical Bifurcation:</u> Two fixed points change their stability

<u>Normal Form:</u> $\dot{x} = rx - x^2$

$r < 0$: two fixed points; a stable one at $x^* = 0$, an unstable one at $x^* = -r$

$r = 0$: one half-stable fixed point at $x^* = 0$

$r > 0$: two fixed points; an unstable one at $x^* = 0$, a stable one at $x^* = r$

The fixed point at $x^* = 0$ collides with the fixed point at $x^* = r$ and changes stability. In contrast to the saddle-node bifurcation, fixed points don't disappear in a transcritical bifurcation.

<u>Example:</u> Show that $\dot{x} = x(1 - x^2) - a(1 - e^{-bx})$ undergoes a transcritical bifurcation at $x = 0$:

$f(x) = x(1-x^2) - a(1-e^{-bx})$ satisfies $f(0) = 0$, so $x^* = 0$ is a fixed point. An expansion for small x gives

$$f(x) = x - x^3 - a(1 - 1 + bx - (bx)^2/2 + O(x^3)) = (1-ab)x + \frac{ab^2}{2}x^2 + O(x^3)$$

We find a transcritical bifurcation when $ab = 1$, where necessarily also $ab^2 \neq 0$. The second fixed point occurs at $x^* \approx 2(ab-1)/ab^2$.

Example: Show that $\dot{x} = r \log x + x - 1$ undergoes a transcritical bifurcation at $x = 1$:

$f(x) = r \log x + x - 1$ satisfies $f(1) = 0$, so $x^* = 0$ is a fixed point. An expansion for x close to 1 gives

$$f(x) = r\left((x-1) - (x-1)^2/2 + O((x-1)^3)\right) + x - 1 = (r+1)(x-1) - \frac{r}{2}(x-1)^2 + O((x-1)^3)$$

We find a transcritical bifurcation at $r_c = -1$.

Pitchfork Bifurcation: fixed point changes stability in "systems with lef-right symmetry"

Two kinds: supercritical and subcritical pitchfork bifurcation

Supercritical Pitchfork Bifurcation

Normal Form: $\dot{x} = rx - x^3$

Where is the symmetry? Invariant under change of variables $x \to -x$.

$r < 0$: one stable fixed point at $x^* = 0$

$r = 0$: one weakly stable fixed point $x^* = 0$

$r > 0$: three fixed points; an unstable one at $x^* = 0$, two stable ones at $x^* = \pm\sqrt{r}$

Bifurcation diagram has the shape of a pitchfork, hence it's name.

Example: $\dot{x} = -x + \beta \tanh x$ as β changes

Example: $\dot{x} = rx - x^3$ using potentials

Exercises: 3.1.5; 3.2.4,6; Sample Exam Question 1 Lecture

Subcritical Pitchfork Bifurcation 9/10:

Normal Form: $\dot{x} = rx + x^3$ 25/10/12

Here the cubic term destabilises.

$r < 0$: three fixed points; a stable fixed point at $x^* = 0$, two unstable ones at $x^* = \pm\sqrt{-r}$

$r = 0$: one weakly unstable fixed point $x^* = 0$

$r > 0$: one unstable fixed point at $x^* = 0$

Bifurcation diagram has the shape of an inverted pitchfork.

Note that for $r > 0$ the unstable fixed point is "enhanced" by the term $+x^3$, which leads to blowup. To stabilise, one needs to add another term, such as

$$\dot{x} = rx + x^3 - x^5$$

Now the unstable branches "turn around" and become stable. Near the origin, this is still a subcritical pitchfork bifurcation, but now we find "hysteretic" behaviour: changing r changes the state of the system, but this is not necessarily reversible!

Imperfect Bifurcations and Catastrophes

Perturbing a supercritical pitchfork bifurcation by adding an imperfection

$$\dot{x} = h + rx - x^3$$

The symmetry apparent at $h = 0$ is broken if $h \neq 0$.

To analyse this, keep r fixed and change h. A graphical analysis shows that for $r < 0$ nothing much changes, but for $r > 0$ the fixed point structure changes depending on the value of h. Indeed, we find saddle-node bifurcations when

$$0 = \frac{d}{dx}(h + rx - x^3) = r - 3x^2$$

i.e. at $x = \pm\sqrt{r/3}$. Inserting this into $0 = h + rx - x^3$ gives bifurcation curves for the critical value $h = h_c(r)$ as a function of r as

$$h_c(r) = \pm\frac{2r\sqrt{r}}{3\sqrt{3}}$$

These curves delineate regions of "similar" behaviour. Note that the two branches of $h_c(r)$ meet in a "cusp point".

It is instructive to plot this in a conventional bifurcation diagram, now with h fixed but r changing. For $h \neq 0$, the pitchfork splits into two pieces. One piece is now a line of stable fixed points, whereas the other piece has two lines of stable and unstable fixed points.

The full picture shows only in three-dimensional plots in which we show the fixed points as a function of r and h. We see an *catastrophe* develop as the surface of fixed points folds over on itself.

Example: bead on a tilted wire

This scenario may seem complicated, but a simple mechanical model allows to understand what's happening. Tilting a mechanical model with two locally stable equilibria leads to one of these becoming unstable, causing the system to jump to the remaining stable fixed point.

1.3 Flows on the Circle

We consider

$$\dot{\theta} = f(\theta)$$

where θ is now an angle and $f(\theta)$ is a vector field on a circle. (Equivalently, we could have considered $\dot{x} = f(x)$ with periodic x and analysing the dynamics modulo the period of f.) The main change is that now periodic motion is possible in the form of rotation around the circle.

Example: $\dot{\theta} = \sin\theta$

non-Example: $\dot{\theta} = \theta$

Uniform Oscillation:

$\dot{\theta} = \omega$ with ω a constant gives

$$\theta = \theta_0 + \omega t$$

θ changes by 2π after a *period* $T = 2\pi/\omega$.

Example: How often will two oscillators with $\omega_1 \neq \omega_2$ meet?

Relevance of phase difference, beat phenomenon.

Non-uniform Oscillators:

$$\dot{\theta} = \omega - a\sin\theta$$

$a = 0$: uniform oscillation

$a > 0$ but small: $\dot{\theta}$ assumes maximum/minimum at $\pm\pi/2$

$a \to \omega$: $\dot{\theta}$ approaches zero, trajectory slows down near bottleneck

$a \geq \omega$: system stops oscillating through saddle-node bifurcation

Plot $f(\theta)$ versus θ in Cartesian coordinates, but also vector fields on the circle.

Exercises: 3.4.2,16, 3.5.8, 3.6.2 Lecture 11/12: 1/11/12

Example: Linear stability analysis for $\dot{\theta} = \omega - a \sin\theta$

Example: Oscillation period for $\dot{\theta} = \omega - a \sin\theta$

We find

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

which for $a = 0$ reduces to $T = 2\pi/\omega$ and for $a \to \omega$ diverges as

$$T \sim \frac{\pi\sqrt{2}}{\sqrt{\omega}\sqrt{\omega - a}}$$

This square root divergence is generic for saddle-node bifurcations. Locally, near the bottle neck, we can write the dynamical system in the normal form

$$\dot{x} = r + x^2$$

whence we get $t_1 - t_0 = \int_{x_0}^{x_1} \frac{dx}{r+x^2} = \frac{1}{\sqrt{r}} \arctan(x/\sqrt{r})|_{x_0}^{x_1}$ and thus for $r \to 0$ we find

$$t_1 - t_0 \sim \pi/\sqrt{r}$$

One could say that there is a saddle-node ghost moving through the bottle neck.

Example: Estimate the period of $\dot{\theta} = \omega - a \sin\theta$ using the normal form method.

Overdamped pendulum with constant torque: Approximate

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

by neglecting the inertial term $mL^2\ddot{\theta}$ to get

$$b\dot{\theta} + mgL\sin\theta = \Gamma$$

To simplify this, divide through by mgL,

$$\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \sin\theta$$

Now let $\gamma = \Gamma/mgL$ and $\tau = (mgL/b)t$ (i.e. measure time in units of b/mgL) to get

$$\theta' = \gamma - \sin\theta$$

where $\theta' = d\theta/d\tau$. But this is just what the equation we have been discussing all along.

Here, γ is the ratio of the applied torque Γ to the maximal gravitational torque mgL. If $\gamma > 1$ then the pendulum rotates continually. As $\gamma = 1$ a fixed point appears at $\gamma = \pi/2$ and splits into a pair of stable and unstable fixed points. Finally, as $\gamma = 0$ the stable fixed point is at the bottom and the unstable one at the top.

2 Two-dimensional Flows

2.1 Linear Systems

A two-dimensional linear dynamical system is given by

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where $a, b, c, d \in \mathbb{R}$, or, equivalently

$$\dot{\mathbf{x}} = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Linear Superposition Principle: if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ for $c_1, c_2 \in \mathbb{R}$.

$\mathbf{x} = \mathbf{0}$ is a fixed point, and solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualised as trajectories in the (x, y)-plane, the *phase plane* of the system.

Example: simple harmonic oscillator

$$m\ddot{x} + kx = 0$$

Let $v = \dot{x}$ to get

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

where $\omega^2 = k/m$.

At each point (x, v) in the phase plane, there is a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$, representing an *vector field* in the plane.

Trajectories in the phase plane are closed ellipses $\omega^2 x^2 + v^2 = 1$, representing oscillations. The origin is a fixed point.

Example: Show the qualitatively different phase plots occurring for $\dot{x} = A\mathbf{x}$ with $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$ for $a \in \mathbb{R}$.

The motion is uncoupled ($\dot{x}$ does not depend on y and vice versa), and we find $x = x_0 e^{at}$ and $y = y_0 e^{-t}$. We distinguish (a) $a < -1$, (b) $a = -1$, (c) $-1 < a < 0$, (d) $a = 0$, and (e) $a > 0$.

A graphical inspection shows stable nodes (a),(b),(c), a star or symmetric node (b), a line of fixed points (d), and a saddle point (e) with a stable and an unstable manifold.

Exercises: 4.2.7; 4.4.4; 5.1.2,9

READING WEEK: read about Insect Outbreak and Fireflies (sections 3.7 and 4.5)

Lecture 13/14: 15/11/12

The stable manifold of a saddle point $\mathbf{x}^*$ is defined as the set of all points reaching $\mathbf{x}^*$ as $t \to \infty$, and the unstable manifold of a saddle point $\mathbf{x}^*$ is defined as the set of all points having been at $\mathbf{x}^*$ as $t \to -\infty$,

A typical trajectory approaches the unstable manifold as $t \to \infty$ and approaches the stable manifold as $t \to -\infty$.

Some Terminology: In (a),(b),(c) $\mathbf{x}^* = \mathbf{0}$ is an *attractive fixed point*, i.e. all trajectories in a neighbourhood of $\mathbf{x}^*$ approach x^* as $t \to \infty$ (if this is true for the whole plane then $\mathbf{x}^*$ is called *globally attractive*).

In (a), (b), (c), (d) $\mathbf{x}^* = \mathbf{0}$ is *Liapunov stable*, i.e. all trajectories starting sufficiently close to $\mathbf{x}^*$ remain close to it for all time. (d) is an example of a fixed point that is Liapunov stable, but not attracting. This is called *neutrally stable.*

The simple harmonic oscillator has a neutrally stable fixed point at the origin. On the other hand, $\dot{\theta} = 1 - \cos\theta$ has an attracting fixed point at $\theta = 0$, which is however not Liapunov stable.

If a fixed point is both attracting and Liapunov stable then it is called *stable*.

(e) has a fixed point that is *unstable*, i.e. neither attracting nor Liapunov stable.

Classification of Linear Systems:

Using some linear algebra, we find that solutions of $\dot{x} = A\mathbf{x}$ are linear superpositions of $\mathbf{x} = e^{\lambda t}\mathbf{v}$ where λ satisfies the *characteristic equation*

$$\det(\lambda - A) = 0$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ this yields

$$0 = (\lambda - a)(\lambda - d) - bc = \lambda^2 - \tau\lambda + \Delta$$

where $\tau = a + d$ is the trace of A and $\Delta = ad - bc$ is the determinant of A. We find

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \quad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Generally, $\lambda_1 \neq \lambda_2$, and the complete solution of $\dot{x} = A\mathbf{x}$ is given by

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of A associated to the corresponding eigenvalues λ_1 and λ_2.

Example: Solve $\dot{x} = x + y$, $\dot{y} = 4x - 2y$ for $(x_0, y_0) = (2, -3)$.

Example: Plot the phase portrait for the preceding dynamical system.

Example: Sketch a typical phase portrait for the case $\lambda_2 < \lambda_1 < 0$.

Example: What happens if the eigenvalues are complex numbers?

Example: What happens if the eigenvalues are equal?

Putting it together: General classification of Fixed Point Structure in the (Δ, τ)-plane.

- $\Delta < 0$: saddle point (real eigenvalues with opposite sign)

- $\Delta > 0$ and $\tau^2 > 4\delta$: nodes (real eigenvalues with same sign)

- $\Delta > 0$ and $\tau^2 < 4\delta$: spirals and centres (complex conjugate eigenvalues)

On the borderline $\Delta = 0$ we find non-isolated fixed points (at least one eigenvalue is zero), and on the borderline $\tau^2 = 4\delta$ we find stars and degenerate nodes (both eigenvalues are equal). If $\tau = 0$ and $\Delta > 0$ we find centres (the eigenvalues are purely imaginary).

Example: Classify the fixed point for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

Exercises: any of 5.2.3-5.2.10; 5.2.11 Lecture 15/16: 29/11/11

2.2 Phase Plane

We now turn to two-dimensional *nonlinear* systems. We consider general smooth

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

or $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in vector notation. At each point $\mathbf{x}$ in the phase plane there is a velocity vector $\dot{\mathbf{x}}$. Trajectories $\mathbf{x}(t)$ flow along this vector field, and the entire phase plane is filled with trajectories.

Generally nonlinear systems will be analytically intractable, so we will want to determine the quantitative behaviour by considering

1. fixed points $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$
2. closed orbits $\mathbf{x}(t + T) = \mathbf{x}(t)$
3. trajectories near fixed points and closed orbits
4. stability of fixed points and closed orbits

Example: $\dot{x} = x + e^{-y}$, $\dot{y} = -y$.

There is only one fixed point $(x^*, y^*) = (-1, 0)$. To consider its stability, notice that $\dot{y} = -y$ implies that $y(t) \to 0$ as $t \to \infty$). Therefore $e^{-y(t)} \to 1$, and $\dot{x} \approx x + 1$ for large t. On the x-axis $\dot{x} = x + 1$ precisely, so the fixed point is unstable.

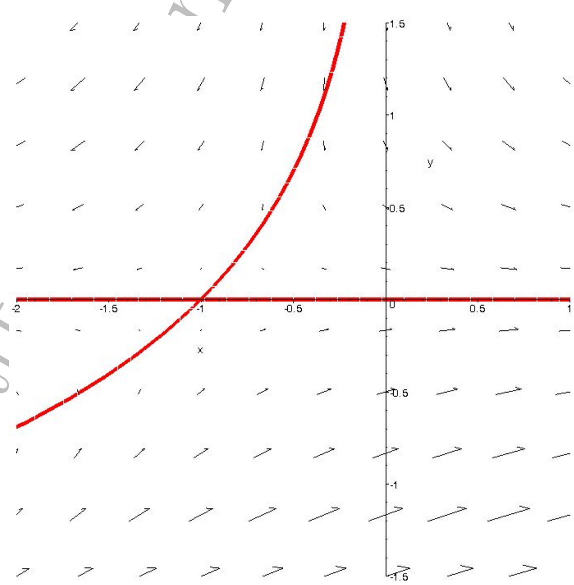
To sketch the phase portrait, it is helpful to consider the so-called *nullclines*, which is the name for curves where $\dot{x} = 0$ or $\dot{y} = 0$, i.e. where the flow is purely vertical or horizontal.

We find $\dot{x} = 0$ for $x = -e^{-y}$ and $\dot{y} = 0$ for $y = 0$. These two lines separate the phase plane into four regions in which the flow points north-east, north-west, south-west, and south-east, respectively. This is sufficient to give a general picture of the phase portrait.

Using Maple, for example, we can confirm the sketch.

```
with(plots):
plot1:=fieldplot([x+exp(-y),-y],x=-2..1,y=-1.5..1.5,grid=[10,10]):
plot2:=implicitplot({y=0,x+exp(-y)=0},x=-2..1,y=-1.5..1.5,thickness=5):
display(plot1,plot2,view=[-2..1,-1.5..1.5]);
```

produces



Existence, Uniqueness, and Topological Consequences

The existence and uniqueness theorem encountered earlier can be generalised to higher dimensions.

Existence and Uniqueness Theorem: Let $\mathbf{f}(x)$ and $\partial f_i/\partial x_j$ be continuous in a neighbourhood $D \in \mathbb{R}^n$ with $x_0 \in D$. Then there exists a unique solution of the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0$$

on some open interval $(-\tau, \tau)$.

As a result, trajectories through a point are well-defined. In other words, trajectories cannot intersect. This implies that closed orbits separate trajectories in the phase plane. We will see later that this has important consequences (Poincare-Bendixson Theorem).

Fixed points and linearisation:

Consider the effect of a small perturbation near a fixed point (x^*, y^*) of a non-linear system

$$\dot{x} = f(x, y) \;, \quad \dot{y} = g(x, y) \;.$$

Denote $u = x - x^*$ and $v = y - y^*$, so u and v are small.

We find

$$\begin{aligned} \dot{u} = \dot{x} = f(x^* + u, y^* + v) &\approx u\frac{\partial f}{\partial x}(x^*, y^*) + v\frac{\partial f}{\partial y}(x^*, y^*) \\ \dot{v} = \dot{y} = g(x^* + u, y^* + v) &\approx u\frac{\partial g}{\partial x}(x^*, y^*) + v\frac{\partial g}{\partial y}(x^*, y^*) \end{aligned}$$

or

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx A \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$A = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(x^*, y^*)}$$

is the Jacobian matrix at the fixed point (x^*, y^*). This means we can expect to use the results obtained for linear systems to discuss non-linear systems. This turns out to be correct for saddles, nodes, and spirals, i.e. as long as we stay away from the borders of the regions. Centres, degenerate nodes, stars, or non-isolated fixed points require some extra care.

Example: Classify all fixed points of $\dot{x} = -x + x^3$, $\dot{y} = -2y$.

Example: Linearisation gives the wrong result for $\dot{x} = -y + ax(x^2+y^2)$, $\dot{y} = x + ay(x^2+y^2)$.

In the linearisation, the origin is always a centre. However, a change to polar coordinates (r, θ) gives $\dot{r} = ar^3$ and $\dot{\theta} = 1$, which implies that for $a > 0$ there is an unstable spiral and for $a < 0$ there is a stable spiral.

As borderline cases are not stable under perturbation, it is convenient to consider another coarser classification of fixed points. The robust cases are

- Repelling fixed points: both eigenvalues have positive real part
- Attracting fixed points: both eigenvalues have negative real part
- Saddle points: eigenvalues have opposite signs

The marginal cases are then

- Centres: both eigenvalues are imaginary
- Higher-order or non-isolated fixed points: at least one eigenvalue is zero

or in other words, marginal cases are present when the real part of at least one eigenvalue is zero.

If both eigenvalues have non-zero real part, the fixed point is called *hyperbolic*. Hyperbolic fixed points are structurally stable: a small perturbation cannot change the structure of the phase portrait near the fixed point.

The Lotka-Volterra model:

Two species, e.g. rabbits and sheep, are individually modelled by logistic growth, and rabbits (x) grow faster than sheep (y). In the absence of sheep ($y = 0$), the rabbit population might grow as $\dot{x} = x(3-x)$ and in the absence of rabbits ($x = 0$), the sheep population might grow as $\dot{y} = y(2-y)$. Now assume that rabbits and sheep compete for the same resource, with rabbits being affected more than sheep. This can be described by a dynamical system such as

$$\dot{x} = x(3-x) - 2xy$$

$$\dot{y} = y(2-y) - xy$$

Fixed points are $(0,0)$, $(0,2)$, $(3,0)$ and $(1,1)$. The Jacobi matrix is

$$\begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$$

We discuss the four fixed points individually:

- $(0,0)$: here the matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, and the eigenvalues are 3 and 2, we have an unstable node. The slow eigenvector is $(0,1)$.
- $(0,2)$: here the matrix is $\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$, and the eigenvalues are -1 and -2, we have stable node. The slow eigenvector is $(1,-2)$.
- $(3,0)$: here the matrix is $\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$, and the eigenvalues are -3 and -1, we have a stable node. The slow eigenvector is $(3,-1)$.
- $(1,1)$: here the matrix is $\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, and the eigenvalues are $-1 \pm \sqrt{2}$, we have a saddle point.

This suffices to produce a sketch of the phase portrait. The stable manifold of the saddle point separates two basins of attraction for the fixed points $(0,2)$ and $(3,0)$, respectively. The stable manifold is a basin boundary, consisting of two trajectories called separatrices.

Returning to the model at hand, we conclude that species competing for the same limited resource cannot coexist.

Exercises: 6.1.3,8; 6.2.2; 6.3.4,16 Lecture

<u>Conservative Systems:</u> 17/18:

In a mechanical system described by $m\ddot{x} = F(x)$ with $F(x) = -dV/dx$ the total 6/12/11
energy

$$E = m\dot{x}^2/2 + V(x)$$

is conserved:

$$\frac{d}{dt}E = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}(m\ddot{x} - F(x)) = 0$$

Generally, a real-valued function that is constant on trajectories (and is non-constant on open sets) is called a conserved quantity. A system that has a conserved quantity is called conservative.

A conservative system cannot have any attracting fixed points.

Assume there is an attracting fixed point. Then the conserved quantity would assume the same value in the whole basin of attraction, i.e. it would be constant in a neighbourhood of the fixed point. This is a contradiction.

A conservative system can have saddles and centres.

Example: $\ddot{x} = -dV/dx$ for $V(x) = -x^2/2 + x^4/4$

We rewrite this as $\dot{x} = y$ and $\dot{y} = x - x^3$. Fixed points are $(0,0)$, $(-1,0)$, and $(1,0)$. The first one is a saddle, the linearisation of the latter two would indicate centres. Considering the conserved quantity

$$E = y^2/2 - x^2/2 + y^4/4$$

confirms this. Trajectories are closed curves defined by the contours of constant energy. There are two neutrally stable centres at $(-1,0)$ and $(1,0)$. Note that there are two special trajectories starting and ending at the same saddle point, so-called homoclinic orbits. They separate trajectories encircling each centre and trajectories encircling both centres.

Note that the presence of a conserved quantity implies that centres are made more robust:

Theorem on nonlinear centres for conservative systems: Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{f}$ is continuously differentiable, and a conserved quantity $E(\mathbf{x})$. If $\mathbf{x}^*$ is an isolated fixed point which is a local minimum of E then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

Sketch of proof: Near the fixed point, a trajectory lies on a closed contour of E. The trajectory goes all the way around the fixed point, unless it stops at another fixed point. But $\mathbf{x}^*$ is isolated, so this cannot happen.

Reversible Systems:

The presence of a symmetry of the dynamical system can have a profound effect on its behaviour. Here we consider systems with time-reversal symmetry, i.e.

systems have a symmetry involving the change of variables $t \to -t$.

An example for such a system is a mechanical system of the form $m\ddot{x} = F(x)$, as $\ddot{x}$ remains unchanged under $t \to -t$. The equivalent system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= F(x)/m\end{aligned}$$

is invariant if we change $t \to -t$ **and** $y \to -y$ (both time and velocity change sign). Associated trajectories in the phase plane get reflected at the x-axis, and the direction of traversal changes, i.e. every trajectory has a "twin".

We use this as a motivation to define an *reversible system* as a second-order system invariant under $t \to -t$ and $y \to -y$.

$$\begin{aligned}\dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}$$

is reversible if $f(x, y)$ is odd in y and $g(x, y)$ is even in y.

Theorem on nonlinear centres for reversible systems: Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{f}$ is continuously differentiable. If this system is reversible, and if the origin is a linear centre for this system, then sufficiently close to the origin all trajectories are closed curves.

Sketch of proof: A trajectory in the upper half plane staring on the positive x-axis and ending on the negative x-axis gets completed to a closed curve by the associated time-reversed trajectory in the lower half plane.

Example: $\dot{x} = y - y^3$, $\dot{y} = -x - y^2$ has a linear center at the origin ($A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$) and is reversible, therefore has a linear centre. The other two fixed points at $(-1, \pm 1)$ are saddles ($A = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix}$). The saddle points are joined by a pair of trajectories called *heteroclinic orbits*.

2.3 Limit Cycles

A *limit cycle* is an isolated closed trajectory: neigbouring trajectories are not closed but spiral away or toward the limit cycle.

A limit cycle can be stable (all neighbouring trajectories spiral toward it), unstable (all neighbouring trajectories spiral away), or half-stable.

Stable limit cycles occur in models of systems with robust oscillations. They cannot occur in linear systems.

Example: $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$ for $r \geq 0$ has a stable limit cycle on the unit circle $r = 1$. The origin is an unstable spiral.

Example: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ (van der Pol-oscillator) has a stable limit cycle.

Exercises: 6.5.2; 6.6.3, 7.1.3 Lecture

How do we decide whether a dynamical system allows oscillations or not? 19/20:

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1. Gradient systems are systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}) = -\nabla V(\mathbf{x})$.

   Theorem: Closed orbits are impossible in gradient systems.

   Proof: Along a closed orbit with period T,

   $$0 = V(\mathbf{x}(T)) - V(\mathbf{x}(0)) = \int_0^T \frac{d}{dt} V(\mathbf{x}(t))dt = \int_0^T (\nabla V \cdot \dot{\mathbf{x}})dt = -\int_0^T (\dot{\mathbf{x}} \cdot \dot{\mathbf{x}})dt < 0$$

   unless $\dot{\mathbf{x}} = \mathbf{0}$, which only occurs for a fixed point. Therefore there are no closed orbits.

   Example: There are no closed orbits for $\dot{x} = \sin y$, $\dot{y} = x \cos y$, as we have a gradient function $V(x, y) = -x \sin y$.

2. Given a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point $\mathbf{x}^*$, a *Liapunov function* is a continuously differentiable function $V(\mathbf{x})$ with (i) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}^*) = 0$, and (ii) $\dot{V} < 0$ along trajectories, provided $\mathbf{x} \neq \mathbf{x}^*$.

   Theorem: If a dynamical system has a Liapunov function with respect to a fixed point $\mathbf{x}^*$, then this fixed point is globally asymptotically stable. In particular, the dynamical system has no closed orbits.

Sketch of proof: All trajectories have $\dot{V} < 0$, so V keeps decreasing until the fixed point is reached.

Example: $\dot{x} = -x + 4y$, $\dot{y} = -x - y^3$ has no closed orbits: $V(x, y) = x^2 + 4y^2$ satisfies $\dot{V} = 2x\dot{x} + 8y\dot{y} = -2x^2 - 8y^4$ and hence is a Liapunov function with respect to $(x^*, y^*) = (0, 0)$.

Existence of closed orbits

Poincaré-Bendixson Theorem: suppose that

- (1) $R \subset \mathbb{R}^2$ is closed and bounded,
- (2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R,
- (3) R does not contain any fixed points, and
- (4) there is a trajectory $C \subset R$ starting in R and remaining in R for all future times).

Then either C is a closed orbit, or it spirals towards a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

To check assumptions (1),(2),(3) is easy. In order to check assumption (4), it suffices to find an *trapping region* R, a closed and connected set such that the vector field points "inward" everywhere on the boundary.

Example: $\dot{r} = r(1 - r^2) + \mu r \cos\theta$, $\dot{\theta} = 1$ has a closed orbit for $0 \leq \mu < 1$:

We can find $0 < r_{min} < r_{max}$ such that $\dot{r} > 0$ for r_{min} and $\dot{r} < 0$ for r_{max}. Then the annulus $r_{min} \leq r \leq r_{max}$ is a trapping region that does not contain any fixed points (as $\dot{\theta} > 0$).

Example: $\dot{x} = -x + ay + x^2y$, $\dot{y} = b - ay - x^2y$ for $a, b > 0$:

To construct a trapping region, it is instructive to start with the nullclines $y = x/(a + x^2)$ and $y = b/(a + x^2)$. These separate the first quadrant in regions with flows into the NE, NW, SE, and SW directions.

Observing that $\dot{x} + \dot{y} < 0$ for $x > b$ then leads to the construction of the trapping region (figure 7.3.5).

There is a fixed point in the trapping region, however, if this fixed point is repelling then by considering a modified trapping region obtained by removing a small disk around the fixed point we can conclude that there exists a closed orbit.
No Chaos in the Phase Plane: the Poincaré-Bendixson Theorem basically states that the most complicated dynamical behaviour one can observe in a confined region of the phase plane is a limit cycle. Nothing more complicated is possible, which is essentially due to the fact that trajectories cannot cross each other.

To see more complicated behaviour, we need to move to three-dimensional systems.

Exercises: 7.2.6,10; 7.3.1

3 Chaos

Outlook

3.1 Lorenz Equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

with parameters $\sigma, r, b > 0$ are the *Lorenz equations.* They result from an extremely simplified model for turbulence in the atmosphere. They also describe exactly the motion of a particular water wheel (it's in the book, but the derivation could already form the basis for an MSc project): in a nutshell, on a wheel that rotates with slight damping, leaky buckets are fixed and get filled when they are on top. And that's it already.

Fixed Points:

The fixed points are $(0, 0, 0)$ (corresponding to a non-rotating water wheel) and $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ (corresponding to uniform rotation to the left or right) if $r > 1$. As $r \to 1$, we find a pitchfork bifurcation.

Symmetry:

$(x, y, z) \leftrightarrow (-x, -y, z)$: solutions are either symmetric or have a "twin"

Volume contraction:

The volume in phase space shrinks exponentially fast: we have $\dot{V} = \int_V \nabla \cdot \dot{\mathbf{x}} dV$ and compute

$$\nabla \cdot \dot{\mathbf{x}} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\sigma - 1 - b < 0$$

This implies that there are no quasiperiodic solutions (solutions that densely fill a torus) or repellers. Therefore all fixed points must be sinks or saddles, and similarly for closed orbits.

Linear Stability:

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x \\ y & x & -b \end{pmatrix}$$

Origin: stable in z-direction, in (x, y)-plane attractor for $r < 1$, saddle for $r > 1$

The other two fixed points are stable for $1 < r < r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ and are surrounded by "saddle cycles".

Global Stability for $r < 1$:

For $r < 1$, $V(x, y, t) = x^2/\sigma + y^2 + z^2$ is a Lyapunov function wrt $(0, 0, 0)$, i.e. the origin is globally asymptotically stable.

What happens if $r > 1$?

Trajectories are attracted to a bounded set of measure zero. This set does not contain fixed points or limit cycles (all fixed points or closed orbits are repellers or saddles). Numerically, we find a "strange attractor" with a "fractal structure".

Exponential separation of nearby trajectories:

Numerically, we observe that "on" the attractor trajectories with small initial distance δ_0 separate exponentially fast: $\delta_t \approx \delta_0 e^{\lambda t}$ until the size of the attractor is reached.

This basically makes long-term prediction of dynamics impossible: we speak about sensitive dependance on initial conditions.

3.2 Defining Chaos

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions:

1. no attractive fixed points, limit cycles or such
2. no noise, the irregular behaviour is based on non-linearities
3. exponential separation of nearby trajectories

Example: $\dot{x} = x$ satisfies 2 and 3 but not 1 (fixed point at infinity).

An *attractor* is a closed set A with the following properties:

1. A is invariant under the dynamics
2. A attracts an open set of initial conditions
3. A is minimal

Example: $\dot{x} = x - x^3$, $\dot{y} = -y$: the interval $[-1, 1]$ satisfies 1 and 2 but not 3.

Finally, a *strange attractor* is an attractor that exhibits sensitive dependence on initial conditions.