Counting Defective Parking Functions

Thomas Prellberg

Queen Mary, University of London

Libra Visiting Professor of Diversity, University of Maine

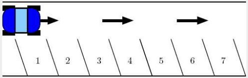
MIT Combinatorics Seminar

Massachusetts Institute of Technology
September 19, 2008

1. A parking problem (defective parking functions)
2. Enumeration (generating functions)
3. Asymptotics (limit distributions)
4. Conclusion and outlook

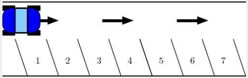
A parking problem

- Consider *n* parking spaces in a *one-way street*



A parking problem

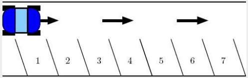
- Consider *n parking spaces* in a *one-way street*



- Drivers of *m cars* choose their preferred parking space

A parking problem

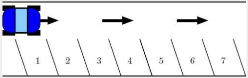
- Consider *n* parking spaces in a *one-way street*



- Drivers of *m* cars choose their preferred parking space
  - If the space is free, the driver parks the car there
  - If the space is occupied, the driver parks in the next free space
  - If no free space is available, the driver leaves

A parking problem

- Consider n parking spaces in a *one-way street*

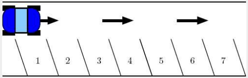


- Drivers of m cars choose their preferred parking space
    - If the space is free, the driver parks the car there
    - If the space is occupied, the driver parks in the next free space
    - If no free space is available, the driver leaves

- n^m sequences of choices

A parking problem

- Consider n parking spaces in a *one-way street*



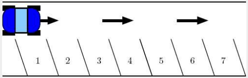
- Drivers of m cars choose their preferred parking space
  - If the space is free, the driver parks the car there
  - If the space is occupied, the driver parks in the next free space
  - If no free space is available, the driver leaves

- n^m sequences of choices
- If all drivers park successfully, the sequence is called a

parking function

A parking problem

- Consider n parking spaces in a *one-way street*



- Drivers of m cars choose their preferred parking space
  - If the space is free, the driver parks the car there
  - If the space is occupied, the driver parks in the next free space
  - If no free space is available, the driver leaves

- n^m sequences of choices
- If all drivers park successfully, the sequence is called a

  parking function

- If k drivers fail to park, the sequence is called a

  defective parking function of degree k

- $n = 2$ parking spaces, $m = 2$ cars:

- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: $11, 12, 21$
  - $k = 1$: 22

- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: $11, 12, 21$
  - $k = 1$: 22
- $n = 3$ parking spaces, $m = 3$ cars:

- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: $11, 12, 21$
  - $k = 1$: 22
- $n = 3$ parking spaces, $m = 3$ cars:
  - $k = 0$: $111, 112, 121, 211, 113, 131, 311, 122,$
    $212, 221, 123, 132, 213, 231, 312, 321$

- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: $11, 12, 21$
  - $k = 1$: 22
- $n = 3$ parking spaces, $m = 3$ cars:
  - $k = 0$: $111, 112, 121, 211, 113, 131, 311, 122,$ $212, 221, 123, 132, 213, 231, 312, 321$
  - $k = 1$: $133, 313, 331, 222, 223, 232, 322, 233, 323, 332$

- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: $11, 12, 21$
  - $k = 1$: 22
- $n = 3$ parking spaces, $m = 3$ cars:
  - $k = 0$: $111, 112, 121, 211, 113, 131, 311, 122,$
    $212, 221, 123, 132, 213, 231, 312, 321$
  - $k = 1$: $133, 313, 331, 222, 223, 232, 322, 233, 323, 332$
  - $k = 2$: 333

- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: $11, 12, 21$
  - $k = 1$: 22
- $n = 3$ parking spaces, $m = 3$ cars:
  - $k = 0$: $111, 112, 121, 211, 113, 131, 311, 122,$ $212, 221, 123, 132, 213, 231, 312, 321$
  - $k = 1$: $133, 313, 331, 222, 223, 232, 322, 233, 323, 332$
  - $k = 2$: 333

Theorem

Every permutation of a defective parking function of degree k is also a defective parking function of degree k.

The counting problem

Enumerate the number

$$\mathrm{cp}(n, m, k)$$

of assignments of m drivers to n spaces such that exactly k drivers leave

The counting problem

Enumerate the number

$$\mathrm{cp}(n, m, k)$$

of assignments of m drivers to n spaces such that exactly k drivers leave

The probabilistic question

What is the probability

$$p_{n,m}(k) = \frac{1}{n^m}\,\mathrm{cp}(n, m, k)$$

that for a randomly chosen assignment exactly k drivers leave? In particular, are there interesting limiting distributions?

Many equivalent or related formulations, for example

- Hashing with linear probing

  [Konheim and Weiss, 1966]

  [Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation

  [Majumdar and Dean, 2002]

Many equivalent or related formulations, for example

- Hashing with linear probing

  [Konheim and Weiss, 1966]

  [Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation

  [Majumdar and Dean, 2002]

Connections to other combinatorial objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, inversion in trees

$\mathrm{cp}(n,n,k)$	$k=0$	1	2	3	4	5	6
$n=1$	1						
2	3	1					
3	16	10	1				
4	125	107	23	1			
5	1296	1346	436	46	1		
6	16807	19917	8402	1442	87	1	
7	262144	341986	173860	41070	4320	162	1

$\mathrm{cp}(n,n,k)$: the number of car parking assignments of n cars to n spaces such that k cars are not parked.

Theorem

For $m \leq n$,

$$\mathrm{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

Theorem

For $m \leq n$,

$$\operatorname{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

Proof (adapted from Pollak ($m = n$), 1974).

- Consider a circular car park with m cars and spaces $(1, \ldots, n, n + 1)$.

Theorem

For $m \leq n$,

$$\mathrm{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

Proof (adapted from Pollak ($m = n$), 1974).

- Consider a circular car park with m cars and spaces $(1, \ldots, n, n + 1)$.
- There are $(n + 1)^m$ choices of sequences.

Theorem

For $m \leq n$,

$$\mathrm{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

Proof (adapted from Pollak ($m = n$), 1974).

- Consider a circular car park with m cars and spaces $(1, \ldots, n, n + 1)$.
- There are $(n + 1)^m$ choices of sequences.
- A sequence will be a parking function for the original problem on $(1, \ldots, n)$ *if and only if* space $n + 1$ remains empty.

Theorem

For $m \leq n$,

$$\mathrm{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

Proof (adapted from Pollak ($m = n$), 1974).

- Consider a circular car park with m cars and spaces $(1, \ldots, n, n+1)$.
- There are $(n+1)^m$ choices of sequences.
- A sequence will be a parking function for the original problem on $(1, \ldots, n)$ *if and only if* space $n+1$ remains empty.
- Space $n+1$ is empty with probability $(n+1-m)/(n+1)$.

□

Definition

For $r, s, k \in \mathbb{N}_0$ let $a(r, s, k)$ denote the number of choices for which r spaces remain empty, s spaces are occupied in the end and k people drive home.

Definition

For $r, s, k \in \mathbb{N}_0$ let $a(r, s, k)$ denote the number of choices for which r spaces remain empty, s spaces are occupied in the end and k people drive home.

- $n = s + r$ parking spaces, $m = s + k$ drivers
- $\mathrm{cp}(n, m, k) = a(n - m + k, m - k, k)$

Lemma (A recursion)

For $r, s, k \in \mathbb{N}_0$, the number of assignments of $s + k$ drivers to $s + r$ spaces such that r spaces remain empty, s spaces are occupied and k drivers leave is recursively defined by

$$a(r, s, k) =$$

Lemma (A recursion)

For $r, s, k \in \mathbb{N}_0$, the number of assignments of $s+k$ drivers to $s+r$ spaces such that r spaces remain empty, s spaces are occupied and k drivers leave is recursively defined by

$$a(r,s,k) = \begin{cases} 1 & \text{if } r = s = k = 0, \\ \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s-1, i) & \text{if } k > 0, \\ a(r-1, s, 0) + \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s-1, i) & \text{if } k = 0 \text{ and } (r > 0 \text{ or } s > 0). \end{cases}$$

Lemma (A functional equation)

Let A be the formal power series in the three variables u, v, and t defined by

$$A(u,v,t) := \sum_{r,s,k\geq 0} a(r,s,k)\cdot u^r \tfrac{v^s t^k}{(s+k)!}\,.$$

Then $A(u,v,t)$ is the unique solution of

$$0 = (\tfrac{v}{t}e^t - 1)\cdot A(u,v,t) + (u-\tfrac{v}{t})\cdot A(u,v,0) + 1$$

in the ring of formal power series in u, v and t.

Solving

$$0 = \left(\frac{v}{t}e^t - 1\right) \cdot A(u,v,t) + \left(u - \frac{v}{t}\right) \cdot A(u,v,0) + 1$$

with the Kernel Method:

Solving

$$0 = (\tfrac{v}{t}e^t - 1) \cdot A(u, v, t) + (u - \tfrac{v}{t}) \cdot A(u, v, 0) + 1$$

with the Kernel Method:

- Setting the *kernel* $K(v, t) = \frac{v}{t}e^t - 1$ equal to zero gives

$$v = te^{-t}$$

Solving

$$0 = (\tfrac{v}{t} e^t - 1) \cdot A(u,v,t) + (u - \tfrac{v}{t}) \cdot A(u,v,0) + 1$$

with the Kernel Method:

- Setting the *kernel* $K(v,t) = \frac{v}{t} e^t - 1$ equal to zero gives

$$v = te^{-t}$$

- This is solved by $t = T(v)$ where T is the *tree function*

$$T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$$

Solving

$$0 = (\tfrac{v}{t}e^t - 1) \cdot A(u,v,t) + (u - \tfrac{v}{t}) \cdot A(u,v,0) + 1$$

with the Kernel Method:

- Setting the *kernel* $K(v,t) = \frac{v}{t}e^t - 1$ equal to zero gives

$$v = te^{-t}$$

- This is solved by $t = T(v)$ where T is the *tree function*

$$T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$$

- Now we can solve

$$0 = (u - \tfrac{v}{T(v)}) \cdot A(u,v,0) + 1$$

for $A(u,v,0)$.

Lemma (The explicit generating function)

The generating function for the car parking problem is given by

$$A(u,v,t) = \frac{1}{1-\frac{v}{t}e^t} + \frac{u-\frac{v}{t}}{1-\frac{v}{t}e^t} \cdot \frac{e^{T(v)}}{1-ue^{T(v)}}\,,$$

where $T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$ *is the tree function* ($T(v) = ve^{T(v)}$).

Lemma (The explicit generating function)

The generating function for the car parking problem is given by

$$A(u,v,t) = \frac{1}{1-\frac{v}{t}e^t} + \frac{u-\frac{v}{t}}{1-\frac{v}{t}e^t}\cdot\frac{e^{T(v)}}{1-ue^{T(v)}}\,,$$

where $T(v)=\sum_{i=1}^{\infty}\frac{i^{i-1}}{i!}\cdot v^i$ *is the tree function* $(T(v)=ve^{T(v)})$.

Now extract the coefficients $a(r,s,k)$ ...

After some further work (Lagrange inversion, resummation of quadruple sums) ...

After some further work (Lagrange inversion, resummation of quadruple sums) . . .

Theorem

The number of car parking assignments of m cars on n spaces such that at least k cars do not find a parking space is given by

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i} \ .$$

After some further work (Lagrange inversion, resummation of quadruple sums) . . .

Theorem

The number of car parking assignments of m cars on n spaces such that at least k cars do not find a parking space is given by

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i} \, .$$

- $S(n,m,k) = \sum_{j=k}^{m} \mathrm{cp}(n,m,j)$

After some further work (Lagrange inversion, resummation of quadruple sums) ...

Theorem

The number of car parking assignments of m cars on n spaces such that at least k cars do not find a parking space is given by

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i} \,.$$

- $S(n,m,k) = \sum_{j=k}^{m} \mathrm{cp}(n,m,j)$
- The car parking numbers $\mathrm{cp}(n,m,k)$ are given by

$$\boxed{\mathrm{cp}(n,m,k) = S(n,m,k) - S(n,m,k+1)}$$

- Remembering that $cp(n,m,0) = (n+1-m)(n+1)^{m-1}$, we can rewrite

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \mathrm{cp}(n-m+k+i-1,i,0)\cdot(m-k-i)^{m-i} \ .$$

- Remembering that $cp(n,m,0) = (n+1-m)(n+1)^{m-1}$, we can rewrite

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \mathrm{cp}(n-m+k+i-1, i, 0) \cdot (m-k-i)^{m-i} \ .$$

- If at least k cars do not find a parking space, then there are at least $\ell = n+k-m$ empty parking spaces, and we rewrite

$$S(n,m,\ell+m-n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot \mathrm{cp}(\ell+i-1, i, 0) \cdot (n-\ell-i)^{m-i} \ .$$

- Remembering that $cp(n,m,0) = (n+1-m)(n+1)^{m-1}$, we can rewrite

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \mathrm{cp}(n-m+k+i-1, i, 0) \cdot (m-k-i)^{m-i} \,.$$

- If at least k cars do not find a parking space, then there are at least $\ell = n+k-m$ empty parking spaces, and we rewrite

$$S(n,m,\ell+m-n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot \mathrm{cp}(\ell+i-1, i, 0) \cdot (n-\ell-i)^{m-i} \,.$$

- This expression leads to a direct combinatorial proof.

Theorem

The number of car parking assignments of m cars on n spaces such that at least *l parking space remain empty is given by*

$$S(n, m, \ell + m - n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot \mathrm{cp}(\ell + i - 1, i, 0) \cdot (n - \ell - i)^{m-i} .$$

Theorem

The number of car parking assignments of m cars on n spaces such that at least l parking space remain empty is given by

$$S(n,m,\ell+m-n)=\sum_{i=0}^{n-\ell}\binom{m}{i}\cdot \mathrm{cp}(\ell+i-1,i,0)\cdot(n-\ell-i)^{m-i}\ .$$

Proof.

Theorem

The number of car parking assignments of m cars on n spaces such that at least *l parking space remain empty is given by*

$$S(n,m,\ell+m-n)=\sum_{i=0}^{n-\ell}\binom{m}{i}\cdot \mathrm{cp}(\ell+i-1,i,0)\cdot(n-\ell-i)^{m-i}\ .$$

Proof.

- Assuming the ℓ-th empty space occurs at position $\ell+i$, there are i cars successfully parked in the $\ell+i-1$ spaces to the left of it, which is counted by $\mathrm{cp}(\ell+i-1,i,0)$.

Theorem

The number of car parking assignments of m cars on n spaces such that at least *l parking space remain empty is given by*

$$S(n, m, \ell + m - n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot \mathrm{cp}(\ell + i - 1, i, 0) \cdot (n - \ell - i)^{m-i} \, .$$

Proof.

- Assuming the ℓ-th empty space occurs at position $\ell + i$, there are i cars successfully parked in the $\ell + i - 1$ spaces to the left of it, which is counted by $\mathrm{cp}(\ell + i - 1, i, 0)$.
- Selecting these i cars out of all m cars can be done in $\binom{m}{i}$ different ways.

Theorem

The number of car parking assignments of m cars on n spaces such that at least *l parking space remain empty is given by*

$$S(n, m, \ell + m - n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot \mathrm{cp}(\ell + i - 1, i, 0) \cdot (n - \ell - i)^{m-i} .$$

Proof.

- Assuming the ℓ-th empty space occurs at position $\ell + i$, there are i cars successfully parked in the $\ell + i - 1$ spaces to the left of it, which is counted by $\mathrm{cp}(\ell + i - 1, i, 0)$.
- Selecting these i cars out of all m cars can be done in $\binom{m}{i}$ different ways.
- The remaining $m - i$ cars are assigned to the $n - \ell - i$ rightmost spaces in $(n - \ell - i)^{m-i}$ different ways.

Theorem

The number of car parking assignments of m cars on n spaces such that at least *l parking space remain empty is given by*

$$S(n, m, \ell + m - n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot \mathrm{cp}(\ell + i - 1, i, 0) \cdot (n - \ell - i)^{m-i} \; .$$

Proof.

- Assuming the ℓ-th empty space occurs at position $\ell + i$, there are i cars successfully parked in the $\ell + i - 1$ spaces to the left of it, which is counted by $\mathrm{cp}(\ell + i - 1, i, 0)$.
- Selecting these i cars out of all m cars can be done in $\binom{m}{i}$ different ways.
- The remaining $m - i$ cars are assigned to the $n - \ell - i$ rightmost spaces in $(n - \ell - i)^{m-i}$ different ways.
- Sum over all possible values of i.

$\square$

Notice that

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}$$

are partial sums occurring in

Lemma (Abel's Binomial Identity, 1826)

For all $a, b \in \mathbb{R}$, $m \in \mathbb{N}_0$,

$$\sum_{i=0}^{m} \binom{m}{i} \cdot a \cdot (a+i)^{i-1} \cdot (b-i)^{m-i} = (a+b)^m$$

Notice that

$$S(n,m,k) = \sum_{i=0}^{m-k} \tbinom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}$$

are partial sums occurring in

Lemma (Abel's Binomial Identity, 1826)

For all $a, b \in \mathbb{R}, m \in \mathbb{N}_0$,

$$\sum_{i=0}^{m} \binom{m}{i} \cdot a \cdot (a+i)^{i-1} \cdot (b-i)^{m-i} = (a+b)^m$$

- In fact, $S(n,m,0) = n^m$ gives a combinatorial proof for Abel's identity for $a = n-m$ and $b = m$.

- Consider the distribution of the probability of a parking function with defect k,

$$p_{n,m}(k) = \frac{1}{n^m}\,\mathrm{cp}(n, m, k)\ ,$$

for n (parking spaces) or m (cars) large.

- Consider the distribution of the probability of a parking function with defect k,

$$p_{n,m}(k) = \frac{1}{n^m} \operatorname{cp}(n, m, k) \ ,$$

  for n (parking spaces) or m (cars) large.
- Different regimes:
  - $m \ll n$: cars park with probability 1
  - $m \sim \lambda n$, $\lambda < 1$: discrete limit law
  - ($\sqrt{m} \ll m - n \ll m$: exponential distribution)
  - $m - n \sim \lambda\sqrt{m}$: linear-exponential distribution
  - ($\sqrt{m} \ll n - m \ll m$: exponential distribution)
  - $m \sim \lambda n$, $\lambda > 1$: discrete limit law
  - $m \gg n$: cars leave with probability 1

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n,m,k) = \sum_{j=k}^{m} p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \to 0$

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n,m,k) = \sum_{j=k}^{m} p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \to 0$
- $m \sim \lambda n$, $\lambda < 1$: discrete limit law

$$P_{n,m}(k) \to (1-\lambda) \sum_{l=0}^{k} (-1)^{k-l} \frac{(l+1)^{k-l}}{(k-l)!} \lambda^{k-l} e^{\lambda(l+1)}$$

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n,m,k) = \sum_{j=k}^{m} p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \to 0$
- $m \sim \lambda n$, $\lambda < 1$: discrete limit law

$$P_{n,m}(k) \to (1-\lambda) \sum_{l=0}^{k} (-1)^{k-l} \frac{(l+1)^{k-l}}{(k-l)!} \lambda^{k-l} e^{\lambda(l+1)}$$

- $m \sim \lambda n$, $\lambda > 1$: discrete limit law

$$P_{n,m}(n-m+k) \to k \sum_{l=0}^{\infty} \frac{(l+k)^{l-1}}{l!} \lambda^{l} e^{-\lambda(l+k)}$$

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n,m,k) = \sum_{j=k}^{m} p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \to 0$
- $m \sim \lambda n$, $\lambda < 1$: discrete limit law

$$P_{n,m}(k) \to (1-\lambda) \sum_{l=0}^{k} (-1)^{k-l} \frac{(l+1)^{k-l}}{(k-l)!} \lambda^{k-l} e^{\lambda(l+1)}$$

- $m \sim \lambda n$, $\lambda > 1$: discrete limit law

$$P_{n,m}(n-m+k) \to k \sum_{l=0}^{\infty} \frac{(l+k)^{l-1}}{l!} \lambda^{l} e^{-\lambda(l+k)}$$

- $m \gg n$: $P_{n,m}(k) \to 1$

The interesting scaling regime $m - n \sim \lambda\sqrt{m}$

The interesting scaling regime $m - n \sim \lambda\sqrt{m}$

Theorem

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$

The interesting scaling regime $m - n \sim \lambda\sqrt{m}$

Theorem

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor) = \begin{cases} e^{-2x(x-y)} & \textit{if } x > y, \\ 1 & \textit{otherwise.} \end{cases}$$

Therefore we have approximately

$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor) = \int_0^1 \frac{x-y}{\sqrt{2\pi\alpha^3(1-\alpha)}} \cdot \exp\left(-\frac{(x-(1-\alpha)y)^2}{2\alpha(1-\alpha)}\right) d\alpha\,.$$

Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\begin{aligned}
&\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor) \\
&\quad = \int_0^1 \frac{x-y}{\sqrt{2\pi\alpha^3(1-\alpha)}} \cdot \exp\left(-\frac{(x-(1-\alpha)y)^2}{2\alpha(1-\alpha)}\right) d\alpha\,.
\end{aligned}$$

Under the substitution $\alpha = \frac{u(x-y)}{x+u(x-y)}$ this integral simplifies to

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \sqrt{\frac{x(x-y)}{u^3}} \cdot \exp\left(-x\cdot(x-y)\cdot\frac{(1+u)^2}{2u}\right) du \\
&= \exp(-2x(x-y))\,.
\end{aligned}$$

$\square$

Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor)$$
$$= \int_0^1 \frac{x-y}{\sqrt{2\pi\alpha^3(1-\alpha)}} \cdot \exp\left(-\frac{(x-(1-\alpha)y)^2}{2\alpha(1-\alpha)}\right) d\alpha\,.$$

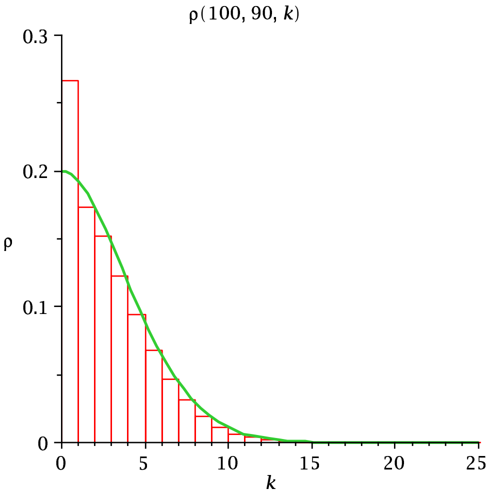
Under the substitution $\alpha = \frac{u(x-y)}{x+u(x-y)}$ this integral simplifies to

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \sqrt{\frac{x(x-y)}{u^3}} \cdot \exp\left(-x\cdot(x-y)\cdot\frac{(1+u)^2}{2u}\right) du$$
$$= \exp(-2x(x-y))\,.$$

$\square$

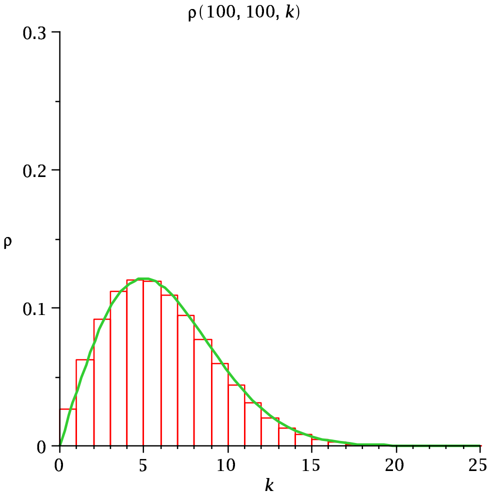
NB: neither Maple nor Mathematica can do the above integral!

Less cars than parking spaces ($n > m$):



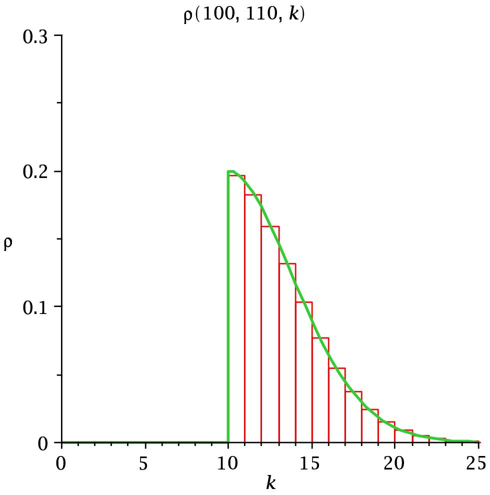
$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Equal number of cars and parking spaces ($n = m$):



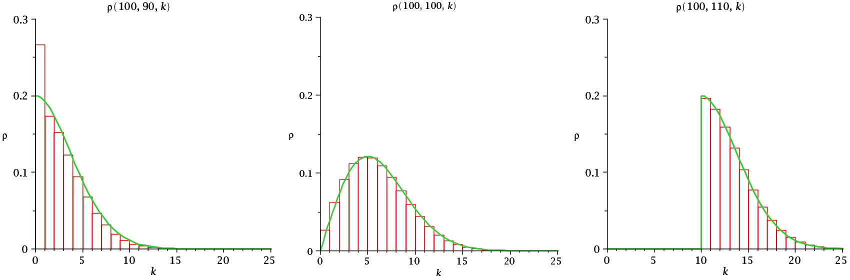
$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

More cars than parking spaces ($n < m$):



$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

The approximation is surprisingly accurate:



$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

What is the probability that the car park is full?

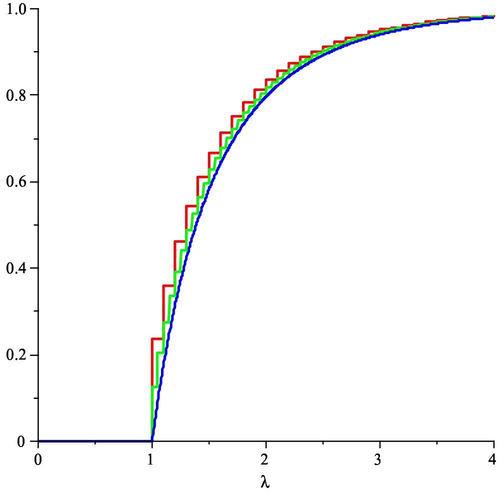
What is the probability that the car park is full?

Theorem

Let $\lambda \in \mathbb{R}^+$. Then the limiting probability that in a random assignment of $\lfloor \lambda n \rfloor$ drivers to n spaces all spaces are occupied is

$$\lim_{n\to\infty} \frac{\mathrm{cp}(n, \lfloor \lambda n \rfloor, \lfloor \lambda n \rfloor - n)}{n^{\lfloor \lambda n \rfloor}} = \begin{cases} 0 & \text{if } \lambda \leq 1, \\ 1 - \frac{1}{\lambda} \cdot T\left(\lambda e^{-\lambda}\right) & \text{if } \lambda > 1. \end{cases}$$

$n = 10, 20, \infty$:



$$\left.\frac{\mathrm{cp}(n,m,m-n)}{n^m}\right|_{m=\lfloor \lambda n \rfloor} \approx 1 - \frac{1}{\lambda} T(\lambda e^{-\lambda})$$

Reference for this work:

- P. J. Cameron, D. Johannsen, T. Prellberg, and P. Schweitzer, "Counting defective parking functions", Electron. J. Combinat. **15** (2008) R92 (arXiv:0803.0302v2) (submitted to EJC on March 3, 2008)

Reference for this work:

- P. J. Cameron, D. Johannsen, T. Prellberg, and P. Schweitzer, “Counting defective parking functions”, Electron. J. Combinat. **15** (2008) R92 (arXiv:0803.0302v2)
  (submitted to EJC on March 3, 2008)

Related work by Alois Panholzer, TU Wien:

- “Limiting distribution results for a discrete parking problem”, GOCPS 2008
  (talk presented on March 5, 2008)
- “On a discrete parking problem”, AofA 2008
  (talk presented on April 17, 2008)

Summary:

- Introduced interesting and apparently new parking problem
- Solved the counting problem (Kernel method)
- Discussed limiting probability distributions

Summary:

- Introduced interesting and apparently new parking problem
- Solved the counting problem (Kernel method)
- Discussed limiting probability distributions

Outlook:

- So far only "weak limit laws" - can refine analysis
- Extension to several generalised parking problems in the literature possible

The End