MTH5105 Differential and Integral Analysis 2009-2010

Exercises 9

These exercises do *not* constitute coursework, but their content is definitely examinable. Model solutions will be made available on the course webpage by the last day of term.

Exercises

1) (a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$ converges.
[You may use that $|\sin(t)| \leq |t|$ for all $t \in \mathbb{R}$.]

   (b) Show that the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$ converges uniformly for all $x \in \mathbb{R}$.

   (c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$$

   is differentiable.

2) Is the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sum_{k=1}^{\infty} \sin^2(x/k)$$

differentiable?

3) Let $f_n : [0, 1] \mapsto \mathbb{R}$ be a sequence of differentiable functions, and let $f : [0, 1] \mapsto \mathbb{R}$. Consider the statements

   (a) $f_n \to f$ pointwise,

   (b) $f_n \to f$ uniformly,

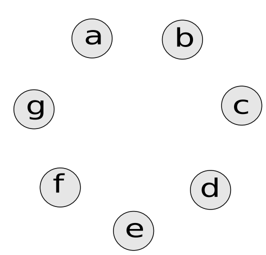
   (c) f_n' converges pointwise,

   (d) $f_n' \to f'$ pointwise,

   (e) f continuous,

   (f) f differentiable,

   (g) $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$,



and cleary indicate in the enclosed figure all implications by the appropriate arrows (“$\Longrightarrow$”).

Thomas Prellberg, March 2010