MAS205 Complex Variables 2004-2005

Exercises 9

Exercise 37: Use the residue theorem to calculate

$$\int_{\mathcal{C}} \frac{1}{z^2(z^2-4)} dz$$

where $\mathcal{C}$ is the positively oriented circle of radius 2 centred at 1.

Exercise 38: Use the residue theorem to calculate

$$\int_{\mathcal{C}} \frac{\sin z}{z(z+1)^2} dz$$

where $\mathcal{C}$ is the positively oriented circle of radius 3 centred at 0.

Exercise 39: Let $f(z) = z^8 - 7z^3 - 4$. Use Rouche's theorem to determine how many zeros of f (counted with multiplicity) have modulus strictly less than one. How many zeros of f (counted with multiplicity) lie in the annulus $\{z : 1 < |z|\}$?

Exercise 40: Let $f(z) = e^{-iz} - 20z + z^6$. Use Rouche's theorem to determine how many zeros of f (counted with multiplicity) lie in the annulus $\{z : 1 < |z| < 2\}$.

And for 50 extra marks (to top up your course work):

Exercise 41: Calculate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx \,.$$

Please hand in your solutions (to the yellow Complex Variables box on the ground floor) by 11am Tuesday 14th December

Thomas Prellberg, November 2004

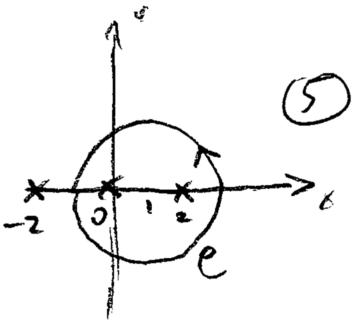
Sample solutions

37–40 : 25 marks each

41 : 50 bonus marks

37) $f(z) = \dfrac{1}{z^2(z^2-4)}$ singularities at $0, \pm 2$ 5

Residues : $\operatorname{Res}_0 f = 0$ as $f(z)$ is a function in z^2, 5
we have in Laurent series only <u>even</u> powers



5

$$\operatorname{Res}_2 f = \phi(2) \quad \left(\text{with } \phi(z) = \frac{1}{z^2(z+2)}\right)$$

$$= \frac{1}{2^2(2+2)} = \frac{1}{16} \quad 5$$

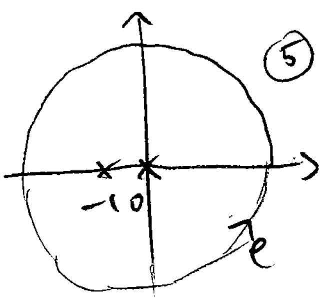
$$\left(\operatorname{Res}_{-2} f = \frac{1}{(-2)^2(-2-2)} = -\frac{1}{16} \quad \text{not needed}\right)$$

$$\int_C f(z)\,dz = 2\pi i\,(\operatorname{Res}_0 f + \operatorname{Res}_2 f) = 2\pi i\left(0 + \frac{1}{16}\right) = \frac{i\pi}{8} \quad 5$$

/25

38) $f(z) = \dfrac{\sin z}{z(z+1)^2}$ singularities at $0, -1$ 5

Residues : $\operatorname{Res}_0 f = 0$ (removable singularity) 5



5

$$\operatorname{Res}_{-1} f = \phi'(-1) \quad \left(\text{with } \phi(z) = \frac{\sin z}{z}\right)$$

$$= -\frac{\cos(-1)}{-1} - \frac{\sin(-1)}{(-1)^2} = \sin 1 - \cos 1 \quad 5$$

$$\int_C f(z)\,dz = 2\pi i\left(\operatorname{Res}_0 f + \operatorname{Res}_{-1} f\right) = 2\pi i(\sin 1 - \cos 1) \quad 5$$

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39) fund. thm of algebra: 8 zeros total (5)

now choose $F(z) = -7z^3$, $G(z) = z^8 - 4$ (4)

so that, for $|z| = 1$, $|G(z)| \leq 5 < 7 = |F(z)|$ (4)

F, G holom. on $\mathbb{C}$, Rouché applies (4)

Rouché $\rightsquigarrow$ $N_{F+G} = N_F = 3$ inside $|z| = 1$ (4)

Thus, $8 - 3 = 5$ zeros outside $|z| = 1$ (4)

(No zeros on $|z| = 1$, as $|f(z)| > 0$ here)

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40) i) choose $F(z) = -20z$, $G(z) = e^{-iz} + z^6$ (3)

so that, for $|z| = 1$, $|G(z)| \leq e + 1 < 20 = |F(z)|$ (3)

F, G holom on $\mathbb{C}$, Rouché applies (2)

Rouché $\rightsquigarrow$ $N_{F+G} = N_F = 1$ inside $|z| = 1$ (3)

ii) choose $F(z) = z^6$, $G(z) = e^{iz} - 20z$ (3)

so that, for $|z| = 2$, $|G(z)| \leq e^2 + 40 < 64 = |F(z)|$ (3)

F, G holom on $\mathbb{C}$, Rouché applies (2)

Rouché $\rightsquigarrow$ $N_{F+G} = N_F = 6$ inside $|z| = 2$ (3)

Thus, $6 - 1 = 5$ zeros in annulus $1 < |z| < 2$ (3)

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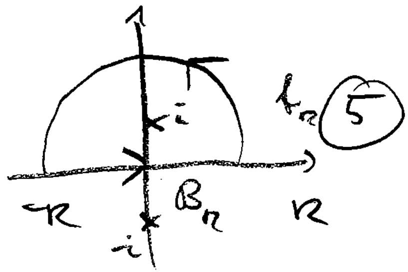
41) $$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}$$

second order poles at $\pm i$ 5

$$\text{Res}_i f = \left[\frac{1}{(z+i)^2}\right]'_{z=i} = -\frac{2}{(i+i)^3} = -\frac{i}{4}$$ 5



$C_R = A_R + B_R$ 5

choice of contour

$R > 1$: $$\int_{C_R} f(z)\,dz = 2\pi i \,\text{Res}_i f = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$$ 5

$$\int_{-R}^{R} f(x)\,dx + \int_{A_R} f(z)\,dz = \int_{C_R} f(z)\,dz = \frac{\pi}{2}$$ 10

$$\left|\int_{A_R} f(z)\,dz\right| \leq \frac{1}{(R^2-1)^2}\,\pi R \to 0 \text{ as } R \to \infty$$ 10

thus,

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \frac{\pi}{2}$$ 5

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