MTH5105 Differential and Integral Analysis 2009-2010

Exercises 7

There are two sections. Questions in Section 1 will be marked and will form your coursework mark. Questions in Section 2 are voluntary but highly recommended.

1 Exercise for Feedback/Assessment

1) (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt \,.$$

(i) Why is f bounded? [2 marks]

(ii) Prove that F is bounded. [3 marks]

(iii) Prove that there exists a $c \in [a, b]$ such that

$$F(c) = \sup\{F(x) : x \in [a, b]\} \,.$$

[3 marks]

(iv) Now suppose that f is continuous, and that the point c from (iii) satisfies $c \in (a, b)$ What can you conclude about $f(c)$? [6 marks]

(b) Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove or disprove: if f^2 is Riemann integrable on $[a, b]$ then f is Riemann integrable on $[a, b]$. [6 marks]

2 Extra Exercises

2) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that if

$$\int_a^b f(x)\, dx = 0$$

then there exists a $c \in (a, b)$ such that $f(c) = 0$.

[Hint: use an antiderivative of f.]

3) Compute $\lim_{n\to\infty} f_n(x)$ and $\lim_{n\to\infty} f'_n(x)$ for the following functions:

(a) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{\sin(nx)}{\sqrt{n}} \,.$$

(b) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{n}(\sqrt{1 + n^2x^2} - 1) \,,$$

(c) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{1 + nx^2} \,.$$

If the limit doesn't exist, please indicate clearly for which values of x this is the case and give a brief indication why (no complete proof necessary).

4) For a bounded set $\Omega \subset \mathbb{R}$, show that

$$\sup_{y\in\Omega} |y| - \inf_{y\in\Omega} |y| \leq \sup_{y\in\Omega} y - \inf_{y\in\Omega} y \ .$$

[This is needed in the proof of Theorem 7.7.]

The deadline is 5.00pm (strict) on Monday 22nd March. Please hand in your coursework to the red coursework box on the ground floor.

Thomas Prellberg, March 2010