MTH5105 Differential and Integral Analysis 2008-2009

Exercises 3

Exercise 1: The functions sinh and cosh are given by

$$\sinh : \mathbb{R} \to \mathbb{R}\,, \qquad x \mapsto \frac{1}{2}(\exp(x) - \exp(-x))\,,$$
$$\cosh : \mathbb{R} \to \mathbb{R}\,, \qquad x \mapsto \frac{1}{2}(\exp(x) + \exp(-x))\,.$$

(a) Prove that sinh and cosh are differentiable and that $\sinh' = \cosh$ and $\cosh' = \sinh$. [2 marks]

(b) Prove that the function

$$f(x) = \cosh^2(x) - \sinh^2(x)$$

is constant by considering $f'(x)$.
What is the value of the constant? [4 marks]

(c) Prove the identity $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$ by considering the function

$$f(x) = \cosh(x+b) - \cosh(x)\cosh(b) - \sinh(x)\sinh(b)$$

for fixed $b \in \mathbb{R}$. [4 marks]
Hint: you may use that if $g : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $g'' = g$ then there exist $c, d \in \mathbb{R}$ such that $g(x) = c\exp(x) + d\exp(-x)$.

(d) Prove that sinh is invertible. [2 marks]

(e) Prove that $\sinh(\mathbb{R}) = \mathbb{R}$. [4 marks]
Hint: show that $\sinh(2x) > x$ for $x > 0$, and mimic the proof of the statement that $\exp(\mathbb{R}) = \mathbb{R}^+$.

(f) Prove that $\mathrm{arsinh} = \sinh^{-1}$ is differentiable, and that

$$\mathrm{arsinh}'(x) = \frac{1}{\sqrt{1+x^2}}\,.$$

[4 marks]

Exercise 2: (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continous inverse. [5 marks]

(b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove:
If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} f'(x) = 0$. [5 marks]

The deadline is 12.15 on Monday, 2nd February. Please hand in your coursework at the end of Monday's lecture or to my office MAS113 immediately afterwards.

Thomas Prellberg, January 2009