MTH5105 Differential and Integral Analysis

Lecture Notes 2009-2010, Week 1 to Week 5

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Contents

0 Revision

Lecture 1: 11/01/10

Let $\mathcal{D} \subseteq \mathbb{R}$ be a domain (e.g. interval or all of $\mathbb{R}$).

Definition 0.1. *Let $f : \mathcal{D} \to \mathbb{R}$.*

(a) *$\underline{f \text{ is continuous at } a \in \mathcal{D}}$ if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\ .$$

(b) *$\underline{f \text{ is continuous}}$ if f is continuous at all $a \in \mathcal{D}$.*

(c) *$\underline{f(x) \text{ tends to the limit } L \in \mathbb{R} \text{ as } x \text{ tends to } a \in \mathcal{D}}$, $\lim_{x \to a} f(x) = L$, if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\ .$$

Remark. We use the short-hand notation $\lim_{x \to a} f(x) = f(a)$ to indicate that both (a) $\lim_{x \to a} f(x) = L$ exists and (b) $f(a) = L$.

Theorem 0.2. *Let $f : \mathcal{D} \to \mathbb{R}$. f is continuous at $a \in \mathcal{D}$ if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof. Let $f : \mathcal{D} \to \mathbb{R}$.

"$\Rightarrow$" Let f be continuous at $a \in \mathcal{D}$. Then

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\ .$$

If we set $L = f(a)$, then it follows that we can write

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\ .$$

But this implies $\lim_{x \to a} f(x) = L$, so $\lim_{x \to a} f(x) = f(a)$ as needed.

"$\Leftarrow$" Let $\lim_{x \to a} f(x) = f(a)$. Then

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\ .$$

Additionally, for $x = a$, we have $|f(a) - f(x)| = 0 < \varepsilon$, so that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\ .$$

This implies that f is continuous at $a \in \mathcal{D}$.

$\square$

Theorem 0.3. *If* $f : \mathcal{D} \to \mathbb{R}$ *is continuous at* $a \in D$ *and* $b = f(a) \neq 0$ *then* $f(x) \neq 0$ *nearby, i.e.*

$$\exists \delta > 0\ \forall x \in \mathcal{D} : \ |x - a| < \delta \Rightarrow f(x) \neq 0 \ .$$

Proof. f is continuous at a, and $b = f(a)$, so that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} : \ 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon \ .$$

Now pick $\varepsilon = |b|$ so that $|f(x) - b| < |b|$. Then

$$|b| > |f(x) - b| \geq ||f(x)| - |b|| \geq |b| - |f(x)|$$

or, equivalently, $|f(x)| > 0$.

Therefore, by choosing ε as we did, we have shown

$$\exists \delta > 0\ \forall x \in \mathcal{D} : \ |x - a| < \delta \Rightarrow f(x) \neq 0 \ .$$

$\square$

Reminder. Use the triangle inequality $|x + y| \leq |x| + |y|$ (Δ) to show

$$|x - y| \geq ||x| - |y|| \ .$$

Proof. We need to show both (a) $|x - y| \geq |x| - |y|$ and (b) $|x - y| \geq |y| - |x|$.

(a) is equivalent to $|x| \leq |x - y| + |y|$, but

$$|x| = |(x - y) + y| \leq |x - y| + |y| \quad \text{by } (\Delta).$$

(b) is equivalent to $|y| \leq |x - y| + |x|$, but

$$|y| = |(y - x) + x| \leq |y - x| + |x| \quad \text{by } (\Delta).$$

$\square$

Lecture 2:
14/01/10

1 Differentiation

Let $\mathcal{D} \subseteq \mathbb{R}$ be a domain without isolated points (to allow limits at all points of $\mathcal{D}$).

Definition 1.1. *Let $f : \mathcal{D} \to \mathbb{R}$.*

(a) f is differentiable at $a \in \mathcal{D}$ if the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

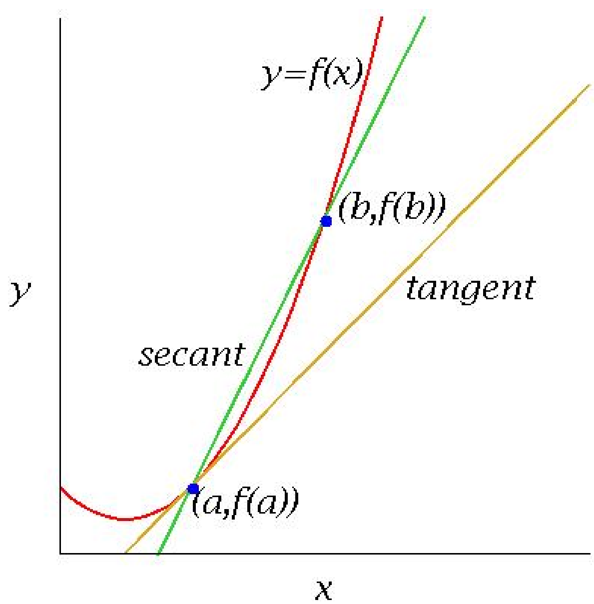
exists. $f'(a)$ is the derivative of f at a.

(b) f is differentiable if f is differentiable at all $a \in \mathcal{D}$. The function $f' : \mathcal{D} \to \mathbb{R}$ given by $x \mapsto f'(x)$ is the derivative of f.

Remark. Geometric interpretation: the difference quotient

$$\frac{f(b) - f(a)}{b - a}$$

is the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$, and the limit $f'(a)$ is the slope of the tangent line at $(a, f(a))$ of the graph of f.



Examples.

1) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is differentiable at every $a \in \mathbb{R}$:

We have

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a} = x + a$$

and

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a}(x + a) = 2a \ .$$

The derivative is $f' : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$.

2) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is not differentiable at $a = 0$:

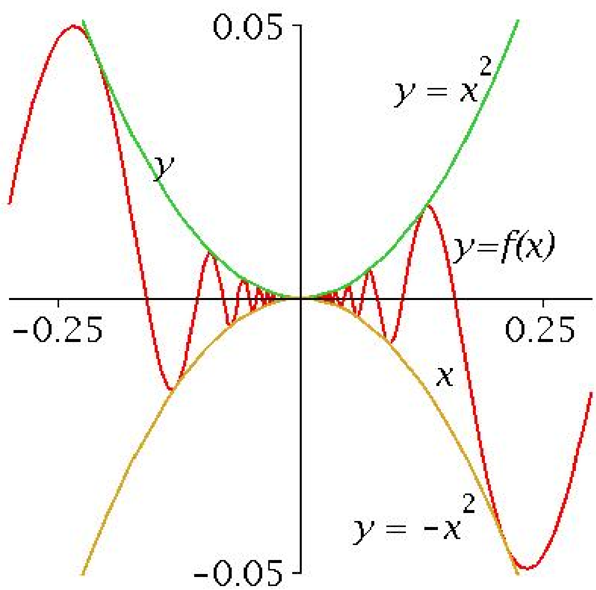
We have

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

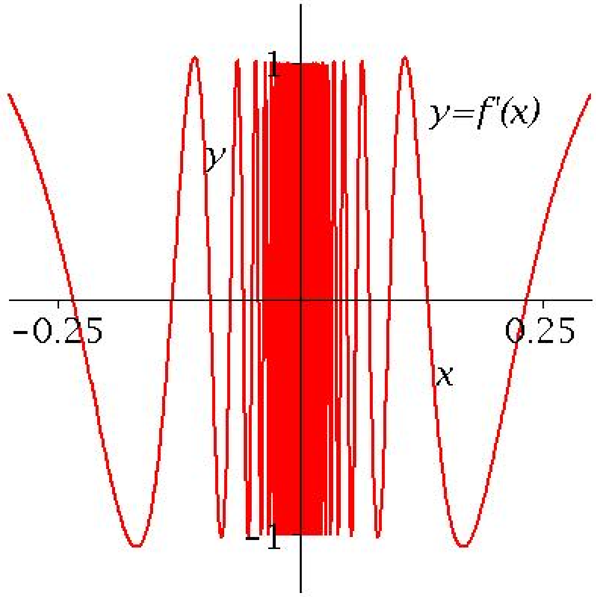
and $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist.

3) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at $a = 0$:

This is unclear from the graph of f, as f "wobbles" near zero.



Plotting the derivative doesn't help much, either:



We have

$$\frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x}$$

and noting that $\left|\sin \frac{1}{x}\right| \leq 1$, we have

$$\left|\lim_{x\to 0} x \sin \frac{1}{x}\right| = \lim_{x\to 0} \left|x \sin \frac{1}{x}\right| \leq \lim_{x\to 0} |x| = 0$$

and therefore $f'(0) = \lim\limits_{x\to 0} x \sin \frac{1}{x} = 0$.

Lemma 1.2. *$f : \mathcal{D} \to \mathbb{R}$ is differentiable at a if and only if there exist $s, t \in \mathbb{R}$ and $r : \mathcal{D} \to \mathbb{R}$ such that*

(1) $f(x) = s + t(x - a) + r(x)(x - a)$ for all $x \in \mathcal{D}$, and

(2) $\lim\limits_{x\to a} r(x) = 0$.

Remark. These properties say that $f(x)$ can be approximated by a linear function $y = s + t(x - a)$ for x close to a. Lecture 3: 15/01/10

Proof. "$\Rightarrow$" Let f be differentiable at a. We define $r : \mathcal{D} \to \mathbb{R}$ by

$$r(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} - f'(a) & x \neq a \\ 0 & x = a \end{cases}.$$

For $x \neq a$ it follows that

$$f(x) = f(a) + f'(a)(x-a) + r(x)(x-a) \ .$$

For $x = a$, this identity holds as well, as it reduces to $f(a) = f(a)$. Therefore (1) holds with $s = f(a)$ and $t = f'(a)$. To show (2) we compute

$$\lim_{x \to a} r(x) = f'(a) - f'(a) = 0 \ .$$

"$\Leftarrow$" Inserting $x = a$ into (1) gives $f(a) = s$, so that (1) gives

$$f(x) = f(a) + t(x-a) + r(x)(x-a) \ ,$$

and therefore

$$\frac{f(x) - f(a)}{x - a} = t + r(x) \ .$$

Now (2) implies that the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = t + \lim_{x \to a} r(x) = t$$

exists.

$\square$

Remark. If $f(x) = s + t(x-a) + r(x)(x-a)$ with $\lim_{x \to a} r(x) = 0$, then f is differentiable at a with $s = f(a)$ and $t = f'(a)$. The equation of the tangent at a of the graph of f is therefore

$$y = f(a) + f'(a)(x-a) \ .$$

Theorem 1.3. *If $f : \mathcal{D} \to \mathbb{R}$ is differentiable at $a \in \mathcal{D}$ then f is continuous at a.*

Proof. By Lemma 1.2,

$$f(x) = s + t(x-a) + r(x)(x-a)$$

with $\lim_{x \to a} r(x) = 0$. Therefore $\lim_{x \to a} f(x) = s = f(a)$. $\square$

Remark. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is continuous at 0 but not differentiable. The converse of Theorem 1.3 is therefore not true.

Theorem 1.4. *Let $f, g : \mathcal{D} \to \mathbb{R}$ be differentiable at $a \in \mathcal{D}$ and let $c \in \mathbb{R}$. Then $f + g$, cf, fg, and f/g (if $g(a) \neq 0$) are differentiable at a. We have*

(a) $(f + g)' = f' + g'$,

(b) $(cf)' = cf'$,

(c) $(fg)' = f'g + fg'$ *(product rule), and*

(d) $(f/g)' = (f'g - fg')/g^2$ *(quotient rule).*

Proof. (a) This is easy.

(b) This is a special case of (c) with the constant function $g(x) = c$.

(c) Write

$$\frac{f(x)g(x) - f(a)g(a)}{x - a} = \frac{f(x) - f(a)}{x - a} g(x) + f(a) \frac{g(x) - g(a)}{x - a} \ .$$

As f and g are differentiable at a and g is continuous at a by Theorem 1.3,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a) \ .$$

(d) By Theorem 1.3, g is continuous at a. $g(a) \neq 0$, therefore $g(x) \neq 0$ nearby, i.e.

$$\exists \delta > 0 \ \forall x \in \mathcal{D} : \ |x - a| < \delta \Rightarrow g(x) \neq 0 \ .$$

Therefore $f(x)/g(x)$ is defined near a, and

$$\frac{\dfrac{f(x)}{g(x)} - \dfrac{f(a)}{g(a)}}{x - a} = \frac{1}{g(x)g(a)} \left(\frac{f(x) - f(a)}{x - a} g(x) - f(a) \frac{g(x) - g(a)}{x - a} \right) \ .$$

The limit as $x \to a$ exists on the right-hand-side, and therefore

$$\left(\frac{f}{g} \right)' (x) = \frac{1}{g(x)^2} \left(f'(a)g(a) - f(a)g'(a) \right) \ .$$

$\square$

Example. Show that

$$\left(\frac{1}{f'} \right) = -\frac{f'}{f^2} \ :$$

(a) Use the quotient rule with constant function 1 in numerator:

$$\left(\frac{1}{f'}\right) = \frac{0 \cdot f - 1 \cdot f'}{f^2} = -\frac{f'}{f^2} \ .$$

(b) Use the product rule with $g = 1/f$, so that $fg = 1$, and differentiate this:

$$0 = (fg)' = f'g + fg' \qquad \text{and therefore} \qquad g' = -\frac{f'g}{f} = -\frac{f'}{f^2} \ .$$

Remark. All the derivatives from Calculus we shall assume as known. This is not cheating, as we can prove every single one in principle.

Lecture 4: 18/01/10

Theorem 1.5. *let $f : \mathcal{D} \to \mathbb{R}$ be differentiable at $a \in D$, and let $g : f(\mathcal{D}) \to \mathbb{R}$ be differentiable at $b = f(a)$. Then $g \circ f : \mathcal{D} \to \mathbb{R}$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a))f'(a) \ .$$

Remark. To get an idea for the formula, let us write

$$\frac{g \circ f(x) - g \circ f(a)}{x - a} = \frac{g \circ f(x) - g \circ f(a)}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \ .$$

It looks like we can easily take the limit of $x \to a$ on the right-hand side. However, the problem is that $f(x) - f(a)$ might be zero for $x \neq a$, and we need to be more careful because of this.

Proof. By Lemma 1.2 we have

(1) $f(x) = f(a) + f'(a)(x - a) + r(x)(x - a)$, and

(2) $g(y) = g(b) + g'(b)(y - b) + s(x)(y - b)$

with $\lim_{x \to a} r(x) = 0$ and $\lim_{y \to b} s(y) = 0$. Define $s(b) = 0$ so that s is continuous at b. Let $y = f(x)$ to get

$$\begin{aligned} g \circ f(x) - g(b) &= (g'(b) + s(f(x)))\,(f(x) - b) \\ &= (g'(b) + s(f(x)))\,(f'(a) + r(x))\,(x - a) \\ &= g'(b)f'(a)(x - a) + t(x)(x - a) \ , \end{aligned}$$

where $t(x) = s(f(x))f'(a) + g'(b)r(x) + s(f(x))r(x)$. Then

$$\begin{aligned}\lim_{x\to a} t(x) &= \lim_{x\to a} \left(s(f(x))f'(a) + g'(b)r(x) + s(f(x))r(x)\right) \\ &= \lim_{x\to a} s(f(x))f'(a) + g'(b)\lim_{x\to a} r(x) + \lim_{x\to a} s(f(x)) \lim_{x\to a} r(x) \ .\end{aligned}$$

Now $\lim_{x\to a} r(x) = 0$, and also $\lim_{x\to a} s(f(x)) = 0$ (for the latter we crucially need that s is continuous at b), so that

$$\lim_{x\to a} t(x) = 0 \ .$$

Thus $g \circ f$ is differentiable at a with $(g \circ f)'(a) = g'(b)f'(a) = g'(f(a))f'(a)$. $\square$

2 The Mean Value Theorem

Theorem 2.1. *If a function $f : [a, b] \to \mathbb{R}$ has a maximum (or minimum) at $c \in (a, b)$ and is differentiable at c, then $f'(c) = 0$.*

Proof. If f has a minumum at c then $-f$ has a maximum at c, so it suffices to consider the case of f having a maximum at c. By assumption f is differentiable at c, so

$$d = f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. Restricting to the one-sided limits, we have furthermore

$$d = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

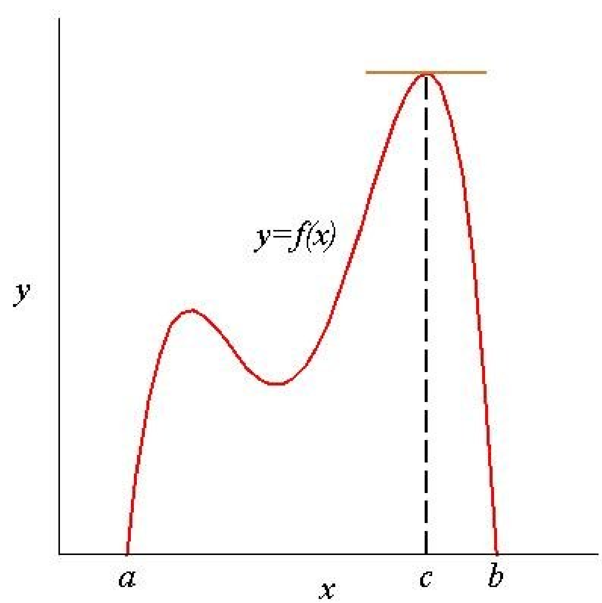
and

$$d = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \ .$$

Therefore $d = 0$. □

Lecture 5: 21/01/10

Theorem 2.2 (Rolle)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$ then there exists a $c \in (a, b)$ such that $f'(c) = 0$.*



Proof. We consider three cases:

(1) $f(x) = 0$ for all $x \in (a, b)$. Then $f'(x) = 0$ for all $x \in (a, b)$.

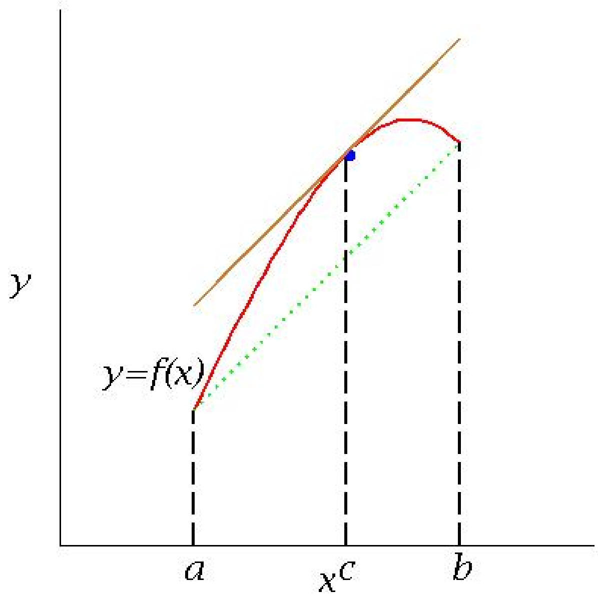
(2) $f(x) > 0$ for some $x \in (a, b)$. Then f is maximal at some $c \in [a, b]$ and $f(c) \geq f(x) > 0$. As $f(a) = f(b) = 0$, c must be different from a or b, so f is maximal at some $c \in (a, b)$. By Theorem 2.1 it follows that $f'(c) = 0$.

(2) $f(x) < 0$ for some $x \in (a, b)$. Then f is minimal at some $c \in [a, b]$ and $f(c) \leq f(x) < 0$. As $f(a) = f(b) = 0$, c must be different from a or b, so f is minimal at some $c \in (a, b)$. By Theorem 2.1 it follows that $f'(c) = 0$.

□

Theorem 2.3 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$



Proof. The equation of the straight line through the points $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) .$$

Taking the difference between $y = f(x)$ and this equation, we define the auxiliary function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) .$$

By construction, h is continuous on $[a,b]$ and differentiable on (a,b). Moreover

$$h(a) = 0 \qquad \text{and} \qquad h(b) = 0 \ ,$$

so that Rolle's Theorem applies: there exists a $c \in (a,b)$ such that $h'(x) = 0$. Now

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so that $h'(x) = 0$ implies $f'(x) = \dfrac{f(b) - f(a)}{b - a}$ as claimed. □

Remark. Geometric interpretation: there exists a tangent to the graph of f which is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$.

We continue with some applications of the Mean Value Theorem.

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).*

(a) If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on $[a,b]$, i.e. $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

(b) If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on $[a,b]$, i.e. $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Proof. (a) Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. Applying the Mean Value Theorem to f on $[x_1, x_2]$, we have that there exists a $c \in (x_1, x_2)$ with

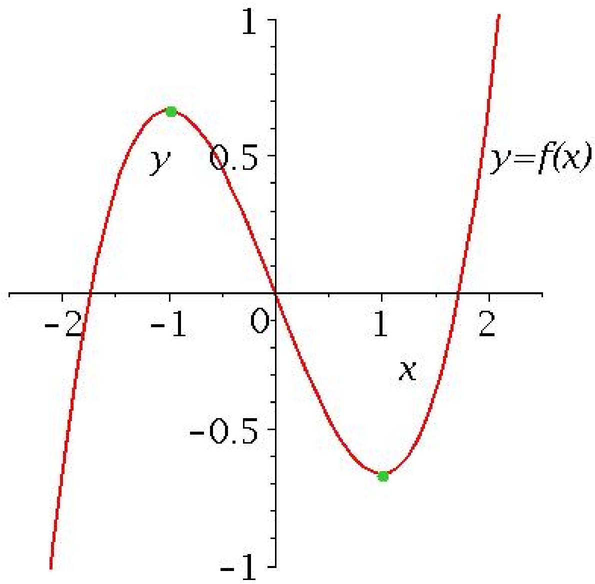
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 \ .$$

Therefore $f(x_2) - f(x_1) > 0$.

(b) Replace f by $-f$ and repeat.

□

Example. Find intervals on which $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \dfrac{x^3}{3} - x$ is strictly increasing or strictly decreasing.



As $f'(x) = x^2 - 1$, $f'(x) < 0$ on $(-1, 1)$ and $f'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$. Therefore f is strictly decreasing on $[-1, 1]$ and strictly increasing on $(-\infty, -1]$ and $[1, \infty)$.

Lecture 6: 22/01/10

Theorem 2.5. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$, i.e. $f(x) = f(a)$ for all $x \in [a, b]$.*

Proof. Let $x \in (a, b]$ and apply the Mean Value Theorem to f on $[a, x]$: there exists a $c \in (a, x)$ such that $\dfrac{f(x) - f(a)}{x - a} = f'(c) = 0$. Therefore $f(x) = f(a)$. $\square$

We conclude this section with presenting an Intermediate Value Theorem for differentiable functions. First recall the Intermediate Value Theorem for continuous functions.

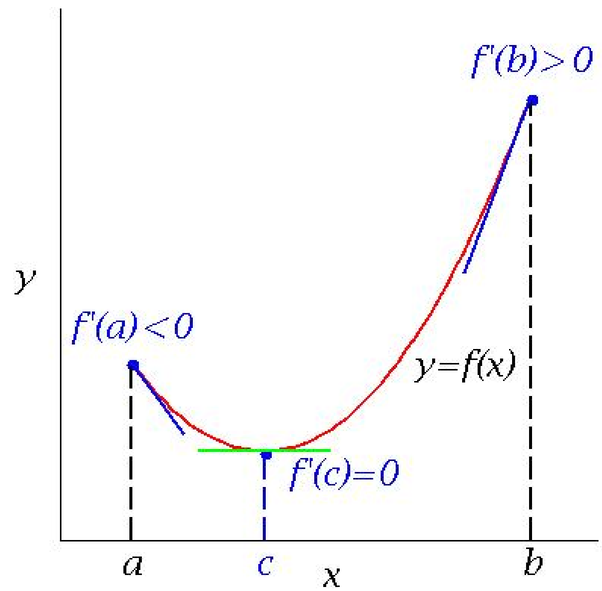
Theorem (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(a) < s < f(b)$. Then there exists a $c \in (a, b)$ such that $f(c) = s$.*

The following theorem looks very similar.

Theorem 2.6. *Let $f : [a, b] \to \mathbb{R}$ be differentiable and $f'(a) < s < f'(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = s$.*

Remark. This shows that the derivative of differentiable functions satisfies the intermediate value property. Note that the derivative doesn't have to be continuous, so this is different from the Intermediate Value Theorem for continuous functions.

Proof. Consider the case $s = 0$ first. We need to show that there exists a $c \in (a, b)$ such that $f'(c) = 0$:



As f is differentiable on $[a, b]$, f is continuous on $[a, b]$ and therefore attains its minimum on $[a, b]$. $f'(a) < 0$ implies that there exists an $a' > a$ with $(f(a') - f(a))/(a' - a) < 0$, thus there exists an $a' > a$ with $f(a') < f(a)$. Similarly, as $f'(b) > 0$, there exists a $b' < b$ with $f(b') < f(b)$. Therefore the minimum is not attained at the endpoints a or b, but at some point in (a, b). As f is differentiable at $c \in (a, b)$, $f'(c) = 0$ by Theorem 2.1. This concludes the proof for $s = 0$.

Now consider the general case of $s \neq 0$. We can reduce this to the case $s = 0$ by considering the function

$$g(x) = f(x) - sx \ .$$

g is differentiable on $[a, b]$ and $g'(x) = f'(x) - s$ implies $g'(a) = f'(a) - s < 0$ and $g'(b) = f'(b) - s > 0$. Therefore, $g'(c) = 0$ for some $c \in (a, b)$. This implies $f'(c) = s$.

$\square$

3 The Exponential Function

Definition 3.1. *A differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *with (a)* $f'(x) = f(x)$ *for all* $x \in \mathbb{R}$*, and (b)* $f(0) = 1$ *is called* *exponential function.*

Remark. We will show later that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies this definition. For now, we shall assume existence of such a function.

In items (A) to (J) we shall collect properties of an exponential function.

(A) $f(x)f(-x) = 1$.

Proof. Differentiate $h(x) = f(x)f(-x)$:

$$h'(x) = f'(x)f(x) + f(x)f'(x)(-1) = 0 \ ,$$

and by Theorem 2.5, h is constant and $h(0) = f(0)f(0) = 1$, so $h(x) = 1$. □

(B) $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Proof. If $f(x) = 0$ for some $x \in \mathbb{R}$ then $0 = f(x)f(-x) = 1$, a contradiction. □

(C) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and $g' = g$. Then there exists a $c \in \mathbb{R}$ such that $g = cf$.

Proof. Consider $h(x) = g(x)/f(x)$. By (B), h is defined on $\mathbb{R}$ and differentiable with

$$h'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2} = 0 \ .$$

Therefore h is constant, $h(x) = c$, and $g(x) = cf(x)$. □

(D) Definition 3.1 determines f uniquely.

Proof. Assume g satisfies Definition 3.1. Then (C) implies $g = cf$. As $g(0) = 1 = f(0)$, we have $c = 1$, so $g = cf$. □

Now that we have shown uniqueness, we will write $f(x) = \exp(x)$ for the function f defined by Definition 3.1.

Lecture 7: 25/01/10

Theorem 3.2. *For all $a, b \in \mathbb{R}$, $\exp(a+b) = \exp(a)\exp(b)$.*

Proof. Consider $g(x) = \exp(a+x)$. Then $g'(x) = \exp(a+x) = g(x)$, so $\exp(a+x) = c\exp(x)$ by (C). Letting $x = 0$, we find $\exp(a) = c$, so that $\exp(a+b) = c\exp(b) = \exp(a)\exp(b)$. □

Corollary. *For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, $\exp(na) = (\exp(a))^n$.*

Proof. We use mathematical induction in n: For $n = 1$, we have

$$\exp(1a) = \exp(a) = (\exp(a))^1 .$$

Next, assuming that we have shown that $\exp(na) = (\exp(a))^n$ for some $n \in \mathbb{N}$, we deduce that

$$\exp((n+1)a) = \exp(na)\exp(a) = (\exp(a))^n \exp(a) = (\exp(a))^{n+1} .$$

□

(E) $\exp(x) > 0$ for all $x \in \mathbb{R}$.

Proof. The function exp is differentiable, therefore continuous. By (B), $\exp(x) \neq 0$ for all $x \in \mathbb{R}$, and $\exp(0) = 1 > 0$. Assume now that (E) is false, i.e. there exists an $x \in \mathbb{R}$ for which $\exp(x) < 0$. By the Intermediate Value Theorem it follows that there exists a $c \in \mathbb{R}$ such that $\exp(c) = 0$, a contradiction. □

(F) exp is strictly increasing.

Proof. $\exp'(x) = \exp(x) > 0$, and the claim follows from Theorem 2.4. □

Theorem 3.3. *For all $x \in \mathbb{R}$, $\exp(x) > x$.*

Proof. $x < 0$: $\exp(x) > 0 > x$.
$x = 0$: $\exp(x) = 1 > 0$.
$x > 0$: By the Mean Value Theorem applied to $[0, x]$, there exists a $c \in (0, x)$ such that

$$\frac{\exp(x) - \exp(0)}{x - 0} = \exp(c) .$$

Moreover, $\exp(c) > \exp(0) = 1$ by (F). Therefore $\exp(x) - 1 = x\exp(c) > x$, and thus $\exp(x) > x + 1 > x$. □

(G) $\exp(\mathbb{R}) = \mathbb{R}^+$ $(= \{x \in \mathbb{R} : \ x > 0\})$.

Proof. (E) implies that $\exp(\mathbb{R}) \subseteq \mathbb{R}^+$. We need to show that also $\mathbb{R}^+ \subseteq \exp(\mathbb{R})$, i.e.

$$\forall c > 0 \ \exists x \in \mathbb{R} : \ \exp(x) = c \, .$$

Case 1: $c \geq 1$.
We have $\exp(0) = 1 \leq c < \exp(c)$. By the Intermediate Value Theorem applied to $[0, c]$, there exists an $x \in (0, c)$ such that $\exp(x) = c$.
Case 2: $0 < c < 1$.
Now $1/c > 1$ and as in Case 1 we can deduce that there exists an $x \in (0, 1/c)$ such that $\exp(x) = 1/c$. As $\exp(x)\exp(-x) = 1$, we have $\exp(-x) = c$. $\square$

Lecture 8: 28/01/10

(H) $\exp(1) = e$, where $e = \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$.

Proof. Recall the Bernoulli inequality: $(1 + x)^n \geq 1 + nx$ for all $x \geq -1$ and for all $n \in \mathbb{N}_0$.

1) Show that $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$ exists:

(a) $a_n = \left(1 + \dfrac{1}{n}\right)^n$ is increasing:
Using

$$\left(1 - \frac{1}{n^2}\right)\left(1 + \frac{1}{n-1}\right) = \frac{n^2-1}{n}\,\frac{n}{n-1} = 1 + \frac{1}{n} \, ,$$

it follows that

$$\begin{aligned} a_n = \left(1 + \frac{1}{n}\right)^n &= \left(1 - \frac{1}{n^2}\right)^n \left(1 + \frac{1}{n-1}\right)^n \\ &\geq \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n-1}\right)\left(1 + \frac{1}{n-1}\right)^{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} = a_{n-1} \, , \end{aligned}$$

where we have used the estimate $\left(1 - \dfrac{1}{n^2}\right)^n \geq 1 - \dfrac{1}{n}$ which follows from the Bernoulli inequality.

(b) $b_n = \left(1 + \dfrac{1}{n}\right)^{n+1}$ is decreasing:
From the Bernoulli inequality it follows that

$$\left(1 + \frac{1}{n^2-1}\right)^n \geq 1 + \frac{n}{n^2-1} \geq 1 + \frac{1}{n} \, .$$

Therefore

$$\begin{aligned} b_n &= \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) \\ &\le \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n^2-1}\right)^n = \left(1+\frac{1}{n-1}\right)^n = b_{n-1} \,. \end{aligned}$$

(c) Each b_m is an upper bound for the sequence (a_n) and each a_m is a lower bound for the sequence (b_n). Therefore the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. We find

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(a_n \left(1+\frac{1}{n}\right)\right) = \lim_{n\to\infty} a_n \,.$$

2) Show that

$$a_n = \left(1+\frac{1}{n}\right)^n \le \exp(1) \le \left(1+\frac{1}{n}\right)^{n+1} = b_n \ :$$

The Mean Value Theorem for exp on $[0, 1/n]$ implies that there exists a $c \in (0, 1/n)$ such that

$$\frac{\exp(1/n) - \exp(0)}{1/n - 0} = \exp(c)$$

so that $\exp(1/n) = 1 + \exp(c)/n$. As $1 \le \exp(c) \le \exp(1/n)$, we deduce that

$$1 + \frac{1}{n} \le \exp\left(\frac{1}{n}\right) \le 1 + \frac{1}{n}\exp\left(\frac{1}{n}\right) \,.$$

This implies firstly that

$$\left(1+\frac{1}{n}\right)^n \le \left(\exp\left(\frac{1}{n}\right)\right)^n = \exp(1) \,.$$

Secondly, $(1 - 1/n)\exp(1/n) \le 1$, so that $\exp(1/n) \le n/(n-1)$ for $n \ge 2$. Shifting n by one, we deduce that $\exp(1/(n+1)) \le (n+1)/n = 1 + 1/n$, so that

$$\left(1+\frac{1}{n}\right)^{n+1} \ge \left(\exp\left(\frac{1}{n+1}\right)\right)^{n+1} = \exp(1) \,.$$

$\square$

Corollary. $\exp(n) = e^n$ *for* $n \in \mathbb{Z}$.

Proof. $n \in \mathbb{N}$: $\exp(n) = (\exp(1))^n = e^n$.

$n = 0$: $\exp(0) = 1 = e^0$.

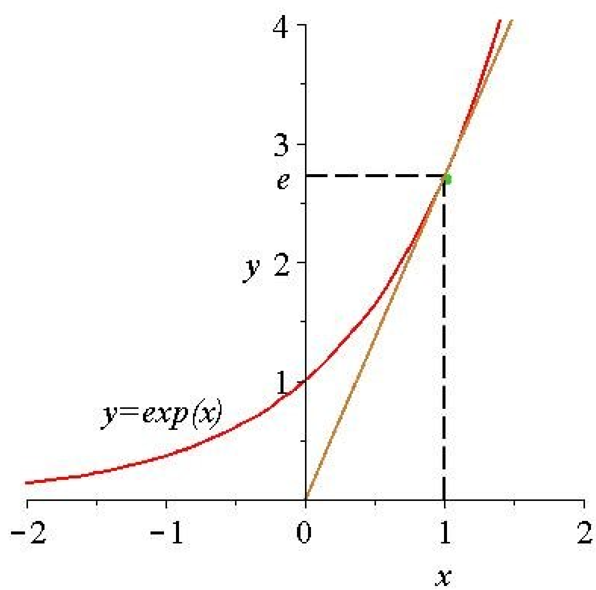
$-n \in \mathbb{N}$: $\exp(-n) = (\exp(n))^{-1} = e^{-n}$. $\square$

We also have $(\exp(n/m))^m = \exp(n) = e^n$, so that $\exp(n/m) = e^{n/m}$. Summarising we have proved the following result.

Theorem 3.4. *(1)* $\exp$ *is strictly increasing,*

(2) $\exp(\mathbb{R}) = \mathbb{R}^+$, *and*

(3) $\exp(x) = e^x$ *for all* $x \in \mathbb{Q}$.



4 Inverse Functions

Lecture 9: 29/01/10

Definition 4.1. *Let $f : \mathcal{D} \to \mathbb{R}$, and let $\mathcal{E} = f(\mathcal{D})$ be the image of f. Then f is invertible if there exists $g : \mathcal{E} \to \mathbb{R}$ such that*

$$g \circ f(x) = x \text{ for all } x \in \mathcal{D} \text{ and } f \circ g(x) = x \text{ for all } x \in \mathcal{E}.$$

g is an inverse of f.

Properties of the inverse:

1) The inverse is uniquely defined.

Proof. Let $\mathcal{E} = f(\mathcal{D})$ and $g_1, g_2 : \mathcal{E} \to]R$ be inverses of f. Let $y \in \mathcal{E}$. There exists an $x \in \mathcal{D}$ with $y = f(x)$ and

$$g_1(y) = g_1 \circ f(x) = x = g_2 \circ f(x) = g_2(y)\ ,$$

so $g_1 = g_2$. □

As the inverse is uniquely defined, we can write $g = f^{-1}$.

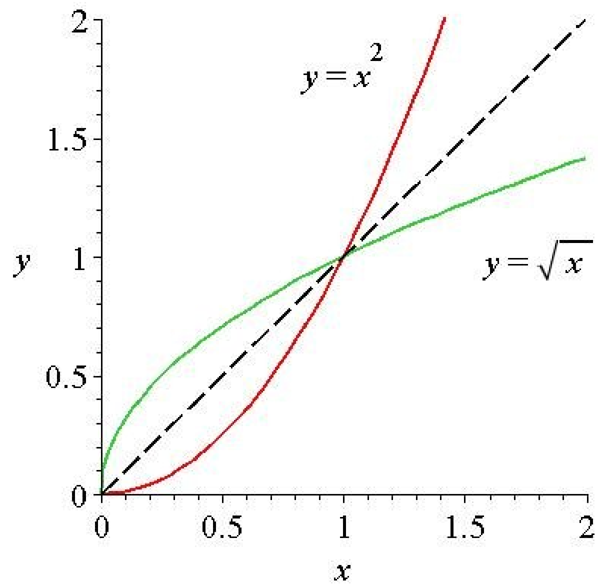
2) If f is invertible, then f^{-1} is invertible as well, and $(f^{-1})^{-1} = f$.

3) The graphs of f and f^{-1} are mirror images with respect to the straight line $y = x$.

Proof. $\text{Graph}(f) = \{(x, f(x)) : x \in \mathcal{D}\}$ and $\text{Graph}(f^{-1}) = \{(y, f^{-1}(y)) : y \in \mathcal{E}\} = \{(f(x), f^{-1} \circ f(x)) : x \in \mathcal{D}\} = \{(f(x), x) : x \in \mathcal{D}\}$ is its mirror image. □

Example.

$$\begin{aligned} f :&\mathbb{R}_0^+ \to \mathbb{R} & f(x) =&x^2 & f(\mathbb{R}_0^+) =&\mathbb{R}_0^+ \\ f^{-1} :&\mathbb{R}_0^+ \to \mathbb{R} & f^{-1}(x) =&\sqrt{x} & f^{-1}(\mathbb{R}_0^+) =&\mathbb{R}_0^+ \end{aligned}$$



Theorem 4.2. *$f : \mathcal{D} \to \mathbb{R}$ is invertible if and only if it is injective (one-to-one).*

Proof. "$\Rightarrow$" Let f be invertible and $f(x_1) = f(x_2)$. Then $x_1 = f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2) = x_2$.

"$\Leftarrow$" Let f be injective and let $\mathcal{E} = f(\mathcal{D})$. Then for each $y \in \mathcal{E}$ there is a unique $x \in \mathcal{D}$ such that $y = f(x)$. Define $g : \mathcal{E} \to \mathbb{R}$ via $g(y) = x$. Then

$$g \circ f(x) = g(y) = x \quad \forall x \in \mathcal{D} \text{ and}$$
$$f \circ g(y) = f(x) = y \quad \forall y \in \mathcal{E}\ .$$

$\square$

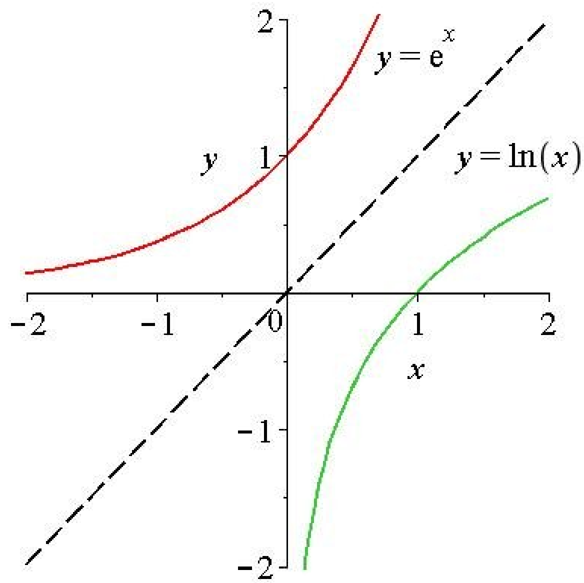
Corollary. *If $f : \mathcal{D} \to \mathbb{R}$ is strictly increasing (or decreasing) then f is invertible.*

Proof.

$$\left.\begin{array}{l} x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \\ x_1 > x_2 \Rightarrow f(x_1) > f(x_1) \end{array}\right\} \qquad \text{implies } x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

$\square$

Example. $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing, therefore invertible.

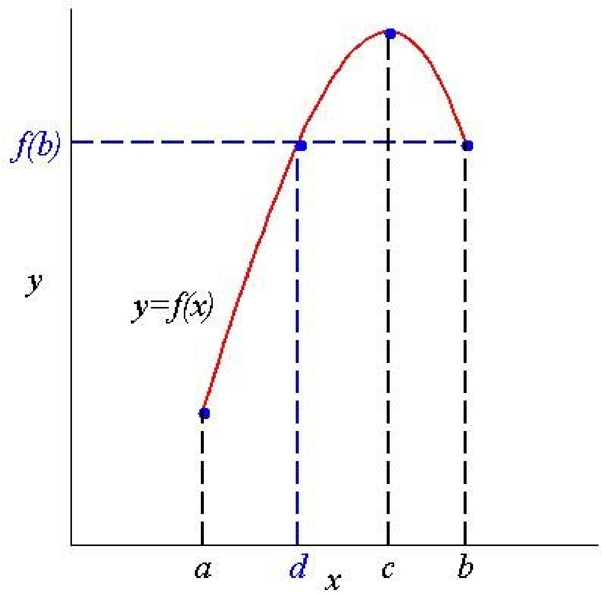


$$\exp(\mathbb{R}) = \mathbb{R}^+ \qquad\qquad \exp^{-1} = \log : \ \mathbb{R}^+ \to \mathbb{R} \ .$$

Let I be an interval ($a, b \in I$, $a \le c \le b \Rightarrow c \in I$). If $f : I \to R$ is continuous then $f(I)$ is an interval (by the Intermediate Value Theorem). Lecture 10: 01/02/10

Theorem 4.3. *Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. Then f attains its minimum and maximum at a or b.*

Proof. Without loss of generality, let $f(a) \le f(b)$. f is continuous, therefore f attains it's maximum in $c \in [a, b]$.



If $c < b$ then $f(a) \leq f(b) \leq f(c)$ and by the Intermediate Value Theorem there exists a $d \in [a, c]$ such that $f(d) = f(b)$. Now $d \leq c < b$ implies $d \neq b$, a contradiction to injectivity. Thus $c = b$ and f is maximal at b. An analogous argument shows that f is minimal at a. $\square$

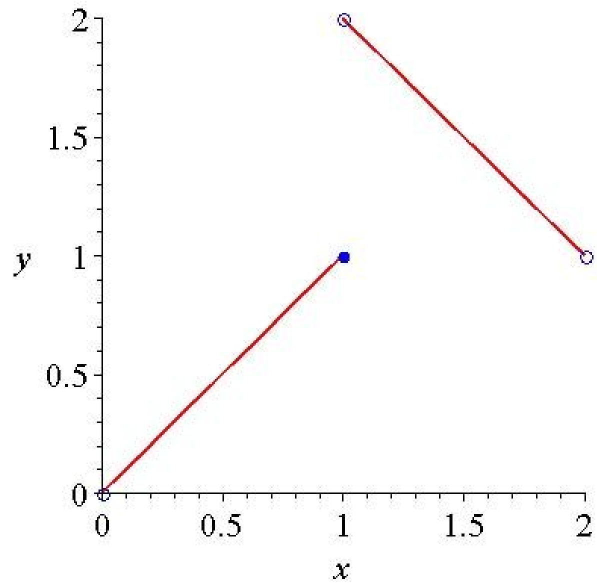
Theorem 4.4. *Let I be an interval and $f : I \to \mathbb{R}$ be continuous and injective. Then f is strictly increasing or decreasing.*

Proof. (1) Consider $I = [a, b]$ and assume without loss of generality that $f(a) < f(b)$. Let $x, y \in I$ with $x < y$. Then, by Theorem 4.3, f attains its maximum in b and therefore $f(x) \leq f(b)$. Restricted to the interval $[x, b]$, the minimum of f is attained at x, and thus $f(x) \leq f(y)$. As f is injective, in fact $f(x) < f(y)$.
(2) Consider now an arbitrary interval I. f is continuous and injective when restricted to any closed and bounded interval $[a, b] \subseteq I$, therefore by (1) it is strictly increasing or decreasing on $[a, b]$.

Now pick $u, v \in I$ with $u < v$ and assume without loss of generality that $f(u) < f(v)$. Let $x, y \in I$ with $x < y$, and choose a closed interval $[a, b] \subseteq I$ containing x, y, u, v. f is strictly increasing or decreasing on $[a, b]$, so $f(x) < f(y)$. $\square$

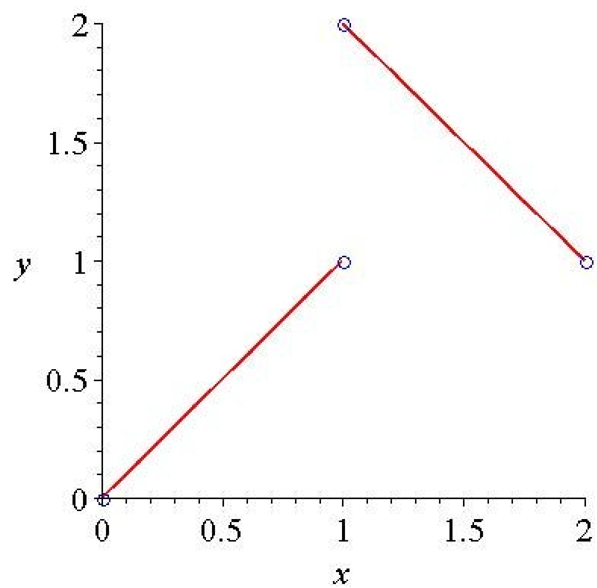
Examples.

1) $f:(0,2)\to\mathbb{R},\ f(x)=\begin{cases} x & x\in(0,1] \\ 3-x & x\in(1,2) \end{cases}.$



f is injective, but not strictly increasing or decreasing (it is not continuous).

2) $f:(0,1)\cup(1,2)\to\mathbb{R},\ f(x)=\begin{cases} x & x\in(0,1) \\ 3-x & x\in(1,2) \end{cases}.$



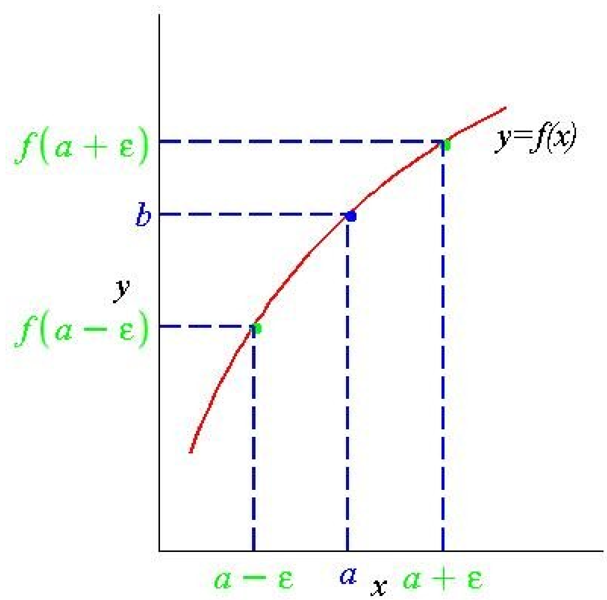
f is injective, continuous, but not strictly increasing or decreasing ($(0,1)\cup(1,2)$ is not an interval).

Lecture 11: 04/02/10

Theorem 4.5. *Let I be an interval and $f : I \to \mathbb{R}$ be continuous and injective. Then $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. Theorem 4.4 inplies that f is strictly increasing or decreasing. Consider the case of strictly increasing f. Let $a \in I$. Then $b = f(a) \in f(I)$, and we need to show that f^{-1} is continuous in b:

Fix $\varepsilon > 0$. If $y = f(x) \in f(I)$ satisfies $f(a-\varepsilon) < y < f(a+\varepsilon)$ then $a - \varepsilon < x < a + \varepsilon$.



Choose now

$$\delta := \min\{f(a+\varepsilon) - b, b - f(a-\varepsilon)\} \ .$$

Then $|y - b| < \delta$ implies $|x - a| < \varepsilon$, so f^{-1} is continuous in b. $\square$

Theorem 4.6. *Let I be an interval and $f : I \to \mathbb{R}$ be continuous and injective. Let f be differentiable at $a \in I$ and $b = f(a)$.*

(a) If $f'(a) = 0$ then f^{-1} is not differentiable at b.

(b) If $f'(a) \neq 0$ then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))} \ .$$

Proof. (a) Let $f'(a) = 0$ and assume f^{-1} is differentiable at $b = f(a)$. Then differentiating $x = f^{-1}(f(x))$ gives a contradiction:

$$1 = (f^{-1})'(f(a))f'(a) = 0 \ .$$

(b) Let $f'(a) \neq 0$. Define the difference quotient

$$A(y) = \frac{f^{-1}(y) - f^{-1}(b)}{y - b} \quad \text{for } y \neq b \ .$$

We need to show that $(f^{-1})'(b) = \lim_{y \to b} A(y)$ exists. Define now

$$B(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & x \neq a \ , \\ f'(a) & x = a \ . \end{cases}$$

Note that $\lim_{x \to a} B(x) = f'(a) = B(a)$, so B is continuous at a, and therefore continuous on I.

f^{-1} is continuous on $f(I)$, and so $B \circ f^{-1}$ is also continuous on $f(I)$. We compute

$$B \circ f^{-1}(y) = \begin{cases} \dfrac{y - b}{f^{-1}(y) - f^{-1}(b)} & y \neq b \ . \\ f'(a) & y = b \ . \end{cases}$$

Therefore $B \circ f^{-1}(y) = 1/A(y)$ for $y \neq b$ and

$$\lim_{y \to b} \frac{1}{A(y)} = B \circ f^{-1}(y) = f'(a) \ ,$$

so $(f^{-1})'(b)$ exists and equals $1/f'(a)$.

□

Examples.

1) Consider $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$. f is differentiable, and $f'(x) = 3x^2$. Moreover, $f(\mathbb{R}) = \mathbb{R}$ (and f is continuous by Theorem 1.3).

   $f' > 0$ on $(-\infty, 0)$ and on $(0, \infty)$, so by Theorem 2.4 f is strictly increasing on both $(-\infty, 0]$ and $[0, \infty)$, hence on all of $\mathbb{R}$.

   By the corollary to Theorem 4.2, f is invertible. (The inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ it is given by $x \mapsto x^{1/3}$).

   From Theorem 4.5 it follows that f^{-1} is continuous.

   From Theorem 4.6 it follows that f^{-1} is not differentiable at $x = 0$, and differentiable for all $x \neq 0$ with derivative

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3(x^{1/3})^2} = \frac{1}{3x^{2/3}} \ .$$

Lecture 12: 05/02/10

2) Consider $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x)$. $f(\mathbb{R}) = \mathbb{R}^+$, f is differentiable, and $f'(x) = \exp(x) > 0$.

   Therefore $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto \log(x)$ is differentiable, and

$$(f^{-1})'(x) = \frac{1}{\exp(\log(x))} = \frac{1}{x} \ .$$

General powers, exponentials, and logarithms

For $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$, we define

$$b^a = \exp(a \log(b)) \ .$$

We have $x^a = \exp(a \log(x))$ for $a \in \mathbb{R}$ and $x \in \mathbb{R}^+$, and differentiating using the chain rule gives

$$(x^a)' = \exp(a \log(x)) \frac{a}{x} = a x^{a-1} \ .$$

We have $b^x = \exp(x \log(b))$ for $b \in \mathbb{R}^+$ and $x \in \mathbb{R}$, and differentiating using the chain rule gives

$$(b^x)' = \exp(x \log(b)) \log(b) = \log(b) b^x \ .$$

For $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^+ \setminus \{1\}$ we define

$$\log_b(a) = \frac{\log(a)}{\log(b)} \ .$$

Considering the function $\log_b : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto \dfrac{\log x}{\log b}$, we find that for $x \in \mathbb{R}^+$

$$b^{\log_b(x)} = \exp\left(\log(b)\frac{\log(x)}{\log(b)}\right) = \exp(\log(x)) = x$$

and that for $x \in \mathbb{R}$

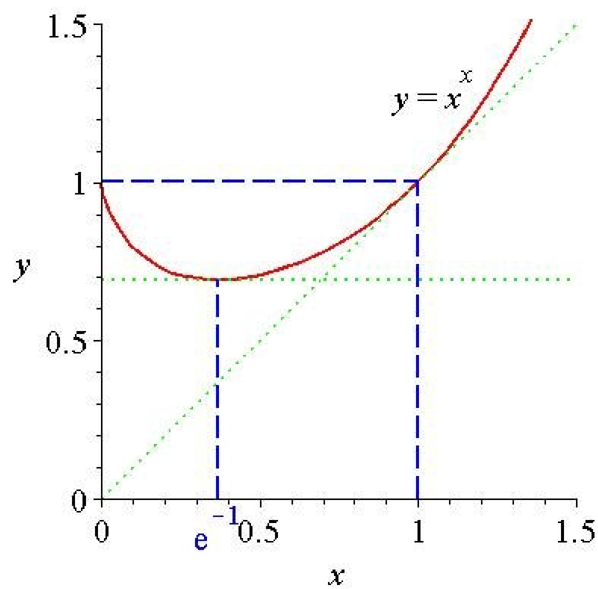
$$\log_b(b^x) = \frac{1}{\log(b)}\log(\exp(\log(b)x)) = \frac{1}{\log(b)}\log(b)x = x \ ,$$

so that $\log_b$ is the inverse of the function $x \mapsto b^x$.

Example.

The function $f : \mathbb{R}^+ \to \mathbb{R}$, $x \mapsto x^x$ is differentiable, and

$$f'(x) = (x^x)' = (\exp(x\log(x)))' = \exp(x\log x)\left(\log(x) + \frac{x}{x}\right) = (1 + \log x)x^x \ .$$



5 Higher Order Derivatives

Theorem 5.1 (Second Mean Value Theorem)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)) \ .$$

Proof. Consider the auxiliary function $h : [a,b] \to \mathbb{R}$ given by

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)) \ .$$

h is continuous on $[a,b]$ and differentiable on (a,b). By the Mean Value Theorem there exists a $c \in (a,b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} \ ,$$

and inserting the definition of h, we find

$$\begin{aligned} & f'(a)(g(b) - g(a)) - g'(c)(f(b) - f(a)) \\ = & \frac{1}{b-a}\Big(f(b)(g(b){-}g(a)){-}g(b)(f(b){-}f(a)){-}f(a)(g(b){-}g(a)){+}g(a)(f(b){-}f(a))\Big) = 0 \ . \end{aligned}$$

□

Remark. For $g(x) = x$, this reduces to the Mean Value Theorem.

If the derivative of a function $f : \mathcal{D} \to \mathbb{R}$ is again differentiable, we can consider the second derivative $f'' = (f')'$. We generalise this to higher order derivatives.

Definition 5.2. *Let $f : \mathcal{D} \to \mathbb{R}$ be n times differentiable at $a \in D$ for some $n \in \mathbb{N}_0$. We call $f^{(n)}$ the n-th derivative of f. It is given by*

$$f^{(0)}(a) = f(a) \quad \textit{and} \quad f^{k+1}(a) = (f^{(k)})'(a) \quad \textit{for } 0 \le k < n.$$

Remark. Conventionally, n-th derivatives are denoted by repeating dashes, i.e.

$$f = f^{(0)} \ , \quad f' = f^{(1)} \ , \quad f'' = f^{(2)} \ , \quad f''' = f^{(3)} \ , \quad f'''' = f^{(4)} \ ,$$

but this becomes cumbersome for large n. Lecture 13:

Example. For $n \in \mathbb{N}$, let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|x^n$. 08/02/10

(a) If $n \geq 1$ then $f'(x) = (n+1)|x|x^{n-1}$:

Consider three cases:

$x > 0$: $f(x) = x^{n+1}$, $f'(x) = (n+1)x^n$

$x < 0$: $f(x) = -x^{n+1}$, $f'(x) = -(n+1)x^n$

$x = 0$: $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} |x|x^{n-1} = 0$

(b) For $0 \leq k < n$, $f^{(k)}(x) = \left(\prod_{i=0}^{k-1}(n+1-i)\right)|x|x^{n-k}$:

Use mathematical induction in k:

$k = 0$:

$$f^{(0)}(x) = \left(\prod_{i=0}^{-1}(n+1-i)\right)|x|x^n = |x|x^n \ .$$

$k \to k+1$: For $k < n$,

$$\begin{aligned}
f^{(k+1)}(x) = (f^{(k)})'(x) &= \left(\prod_{i=0}^{k-1}(n+1-i)\right)(|x|x^{n-k})' \\
&= \left(\prod_{i=0}^{k-1}(n+1-i)\right)(n+1-k)|x|x^{n-k-1} \\
&= \left(\prod_{i=0}^{k}(n+1-i)\right)|x|x^{n-k}
\end{aligned}$$

So f is precisely n times differentiable.

Theorem 5.3 (Taylor's Theorem)**.** *Let $f : [a,x] \to \mathbb{R}$ be n times continuously differentiable (i.e. $f^{(n)}$ exists and is continuous) on $[a,x]$ and $(n+1)$ times differentiable on (a,x). Then there exists a $c \in (a,x)$ such that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \ .$$

Remark. A similar statement holds for $x < a$ (replace $[a,x]$ by $[x,a]$ and (a,x) by (x,a)).

Proof. Let

$$\begin{aligned}
F(t) =& f(t) + \frac{f'(t)}{1!}(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \ldots + \frac{f^{(n)}(t)}{n!}(x-t)^n \\
=& \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \ .
\end{aligned}$$

Then F is continuous on $[a,x]$ and differentiable on (a,x), and

$$\begin{aligned} F'(t) &= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\ &= \frac{f^{(n+1)}(t)}{n!}(x-t)^n \ . \end{aligned}$$

Applying Theorem 5.1 to $F(t)$ and $g(t) = (x-t)^{n+1}$ on $[a,x]$ shows that there exists a $c \in (a,x)$ such that $F'(c)(g(x)-g(a)) = g'(c)(F(x)-F(a))$. As $F(x) = f(x)$ and $g(x) = 0$, we find that

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n \left(0-(x-a)^{n+1}\right) = -(n+1)(x-c)^n \left(f(x)-F(a)\right) \ ,$$

so that

$$f(x) = F(a) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \ .$$

□

Remark. We call

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the <u>n-th degree Taylor polynomial of f at a</u> and

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

the <u>Lagrange form of the remainder term</u>. The

$$f(x) = T_{n,a}(x) + R_n$$

is also called <u>Taylor's formula</u>, and

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the <u>Taylor series of f at a</u>. Lecture 14:

Examples. 11/02/10

1) Estimate $e = \exp(1)$ using Taylor's formula:

For $f(x) = \exp(x)$, we have $f^{(k)}(x) = \exp(x)$, and thus

$$T_{n,0}(x) = \sum_{k=0}^{n} \frac{\exp(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{x^k}{k!}$$

and

$$R_n = \frac{\exp(c)}{(n+1)!}x^{n+1} \ .$$

Taylor's Theorem applied to $f = \exp$ on $[0,1]$ says that there exists a $c \in (0,1)$ such that

$$e = \exp(1) = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\exp(c)}{(n+1)!} \ .$$

Using that $\exp(c) < \exp(1) < (1+1/1)^2 = 4$, we find

$$\sum_{k=0}^{n} \frac{1}{k!} < e < \sum_{k=0}^{n+1} \frac{1}{k!} + \frac{3}{(n+1)!} \ .$$

Evaluating this chain of inequalities for $n = 10$ gives the bounds

$$2.718281826 < e < 2.718281901 \ .$$

Moreover, as

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| < \frac{4}{(n+1)!} \ ,$$

we find

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \ .$$

2) Show that $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in \mathbb{R}$:

Taylor's Theorem applied to $f = \exp$ on $[0,x]$ for $x > 0$, or on $[x,0]$ for $x < 0$, says that there exists a c with $|c| < |x|$ such that

$$|\exp(x) - T_{n,0}(x)| = |R_n| = \left| \frac{\exp(c)}{(n+1)!}x^{n+1} \right| \ .$$

Now $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, so that $R_n \to 0$ as $n \to \infty$.

3) Show that $\log(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$ for $1 < x \leq 2$:

For $f(x) = \log(x)$, we have $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f''' = 2/x^3$, ... From this we conjecture that for $k \geq 1$

$$f^{(k)}(x) = \frac{(-1)^k (k-1)!}{x^k} \,.$$

holds and prove this via mathematical induction (this is a standard argument which we omit here). We choose $a = 1$ and get

$$T_{n,1}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$

and

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = \frac{(-1)^n}{n+1}\left(\frac{x-1}{c}\right)^{n+1} \,.$$

Taylor's Theorem applied to $f = \log$ on $[1, x]$ for $1 < x \leq 2$ says that there exists a c with $c \in (1, x) \subseteq (1, 2)$ such that

$$|\log(x) - T_{n,1}(x)| = |R_n| \leq \frac{1}{n+1}\left|\frac{x-1}{c}\right|^{n+1} \,.$$

Now $0 < x - 1 \leq 1$ and $1 < c < x \leq 2$, so that $\left|\frac{x-1}{c}\right| < 1$. Therefore $R_n \to 0$ as $n \to \infty$.

(It can be shown that this result holds not only for $1 < x \leq 2$ but for $0 < x < 2$, or, equivalently, for $|x - 1| < 1$.)

We return now to our discussion of the exponential function.

(I) $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Proof. From Example 2) above. □

(J) $\lim_{x\to\infty} x^n \exp(-x) = 0$ for all $n \in \mathbb{N}_0$.

Proof. From (I) it follows that $\exp(x) > \frac{x^{n+1}}{(n+1)!}$ for $x > 0$ and $n \in \mathbb{N}_0$. Therefore

$$0 < x^n \exp(-x) < \frac{(n+1)!}{x} \,,$$

and, taking the limit of $x \to \infty$,

$$0 \leq \lim_{x\to\infty} x^n \exp(-x) \leq \lim_{x\to\infty} \frac{(n+1)!}{x} = 0 \,.$$

□

Lecture 15: 12/02/10

Theorem 5.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} \exp(-1/x) & x > 0 \, , \\ 0 & x \le 0 \, . \end{cases}$$

Then

$$f^{(k)}(x) = \begin{cases} P_k(1/x) \exp(-1/x) & x > 0 \, , \\ 0 & x \le 0 \, , \end{cases}$$

where P_k is a polynomial of at most degree $2k$.

Corollary. *The n-th degree Taylor polynomial of f at zero is $T_{n,0}(x) = 0$.*

Remark. While the Taylor polynomial is a good approximation to a function, it need not be. In this case all Taylor polynomials are zero, so $f(x) = R_n$ and the remainder does not get small.

When looking for the cause of this, one finds that close to zero the derivatives of f become arbitrarily large. From the Lagrange form of the remainder we know that for each $n \in \mathbb{N}_0$ there exists a $c_n \in (0, x)$ such that

$$\exp(-1/x) = R_{n-1} = \frac{f^{(n)}(c_n)}{n!} x^n \, .$$

This implies that for x fixed,

$$f^{(n)}(c_n) = \frac{n!}{x^n} \exp(-1/x) \to \infty \quad \text{as } n \to \infty.$$

In other words, no matter how close x is to zero, there exists a sequence (c_n) with $c_n \in (0, x)$ such that $\lim_{n \to \infty} f^{(n)}(c_n) = \infty$.

Proof (Theorem 5.4). We use mathematical induction in k. In the case $k = 0$ we only need to choose $P_0(1/x) = 1$. For the inductive step from k to $k + 1$, we need to compute the derivative of

$$f^{(k)}(x) = \begin{cases} P_k(1/x) \exp(-1/x) & x > 0 \, , \\ 0 & x \le 0 \, . \end{cases}$$

For $x < 0$ we find $f^{(k+1)}(x) = 0$, and for $x > 0$ we compute

$$\begin{aligned} f^{(k+1)}(x) =& P_k'(1/x)(-1/x^2)\exp(-1/x) + P_k(1/x)\exp(-1/x)(1/x^2) \\ =& (1/x^2)\left(P_k(1/x) - P_k'(1/x)\right)\exp(-1/x) \\ =& P_{k+1}(1/x)\exp(-1/x)\ , \end{aligned}$$

where $P_{k+1}(t) = t^2(P_k(t) - P_k'(t))$ is a polynomial of degree at most $2k+2$. For $x = 0$ we compute the left and right limits of the difference quotient separately. We have $\lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = 0$ and find

$$\begin{aligned} \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-} (1/x)P_k(1/x)\exp(-1/x) \\ &= \lim_{t\to\infty} tP_k(t)\exp(-t) = 0 \end{aligned}$$

by (J). This concludes the inductive step. □

Theorem 5.5 (L'Hospital's Rule)**.** *Let $f, g : \mathcal{D} \to \mathbb{R}$ be differentiable for $|x-a| < \varepsilon$ and let $g'(x) \neq 0$ for $0 < |x-a| < \varepsilon$. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ and if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ exists and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}\ .$$

Proof. We first show that $g(x) \neq 0$ for $0 < |x-a| < \varepsilon$. By assumption $g(a) = 0$. If $g(b) = 0$ for some b with $0 < |x-b| < \varepsilon$, then we apply Rolle's Theorem to g and find that there exists a c between a and b such that $g'(c) = 0$, but this contradicts the assumption that $g'(x) \neq 0$ for $0 < |x-a| < \varepsilon$.

Next, by the Second Mean Value Theorem applied to f and g, there exists a c between a and x such that

$$g'(c)(f(x) - f(a)) = f'(c)(g(x) - g(a))\ .$$

By assumption $f(a) = g(a) = 0$, and as $g(x) \neq 0$ as well as $g'(c) \neq 0$, we can write

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\ .$$

Finally, when $x \to a$ then necessarily $c \to a$, so that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} \ .$$

$\square$

Examples.

1) Apply l'Hospital's rule:

$$\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1+x}}{x} = \lim_{x \to 0} \frac{1/\sqrt{1+2x} - 1/\sqrt{1+x}}{1} = 1 - \frac{1}{2} = \frac{1}{2} \ .$$

2) Apply l'Hospital's rule twice:

$$\lim_{x \to 0} \frac{\exp(x) - 1 - x}{x^2} = \lim_{x \to 0} \frac{\exp(x) - 1}{2x} = \lim_{x \to 0} \frac{\exp(x)}{2} = \frac{1}{2} \ .$$

The rule also holds if $f(x), g(x) \to \infty$:

3)

$$\lim_{x \to 0} x \log(|x|) = \lim_{x \to 0} \frac{\log(|x|)}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = 0 \ .$$

4)

$$\lim_{x \to 0} |x|^x = \lim_{x \to 0} \exp(x \log(|x|)) = \exp(\lim_{x \to 0} x \log(|x|)) = \exp(0) = 1 \ .$$