Queen Mary
University of London

B. Sc. Examination by course unit 2011

MTH5105 Differential and Integral Analysis

Duration: 2 hours

Date and time: 10 June 2011, 10.00–12.00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): T. Prellberg

Question 1 [25 marks] Let a, b be real numbers with $a < b$, and let the function $f : [a, b] \to \mathbb{R}$ be continuous.

(a) State the Boundedness Principle and the Intermediate Value Theorem.

(b) Explain why $\int_a^b f(x)dx$ exists, and why

$$m = \inf\{f(x) : x \in [a, b]\} \quad \text{and} \quad M = \sup\{f(x) : x \in [a, b]\}$$

are finite.

(c) Show that

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M \; .$$

(d) By using the Boundedness Principle, the Intermediate Value Theorem and the previous result, prove that there exists a real number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx \; .$$

Question 2 [25 marks]

(a) Let the real-valued function f be infinitely often differentiable at zero. Write down the Taylor series of f about zero. Define what is meant by the radius of convergence of this Taylor series.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1/(1-x^2)$. Using the geometric series, write down the Taylor series of f about 0. What is its radius of convergence?

(c) Given $0 < \rho < 1$, why do we know that the Taylor series of $f(x) = 1/(1 - x^2)$ about 0 is uniformly convergent for all $|x| \leq \rho$?

(d) Prove that for all $|x| < 1$

$$\operatorname{artanh}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \; .$$

[*Hint:* $\operatorname{artanh}(x) = \int\limits_0^x \frac{dt}{1-t^2}$ *for* $|x| < 1$]

Question 3 [25 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Give the definition of f being differentiable at $a \in \mathbb{R}$, and say what is meant by the derivative of f at a.

(b) Prove directly from the definition that $f(x) = 1/x^2$ is differentiable at a for any $a \neq 0$ and find $f'(a)$.

(c) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are functions with $f(x) \leq g(x)$, and suppose that $f(c) = g(c)$ for some $c \in \mathbb{R}$. By comparing $\dfrac{f(x) - f(c)}{x - c}$ and $\dfrac{g(x) - g(c)}{x - c}$ for $x > c$ and $x < c$, prove that if f and g are both differentiable at c then $f'(c) = g'(c)$.

Question 4 [25 marks]

(a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a, b]$.

(i) Define the lower sum $L(f, P)$ and upper sum $U(f, P)$ of f with respect to the partition P, and show that $L(f, P) \leq U(f, P)$.

(ii) Define the lower and the upper integrals of f on $[a, b]$, give the definition of f being Riemann integrable on $[a, b]$, and state what is meant by the Riemann integral of f on $[a, b]$.

(iii) State Riemann's condition for f to be Riemann integrable.

(b) Now consider $f(x) = 1/(1 + x)$ on the interval $[0, 1]$.

(i) For each $n \in \mathbb{N}$, define $P_n = \{0, 1/n, 2/n, \ldots, (n - 1)/n, 1\}$. Calculate $L(f, P_n)$ and $U(f, P_n)$ and deduce that f is Riemann integrable on $[0, 1]$.

(ii) Using the Fundamental Theorem of Calculus, find $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{n + k}$.

End of Paper