MTH5105 Differential and Integral Analysis 2009-2010

Solutions 3

1 Exercise for Feedback/Assessment

1) The function $\tanh : \mathbb{R} \to \mathbb{R}$ is given by

$$x \mapsto \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)} \, .$$

(a) Prove that tanh is differentiable and that $\tanh' = 1 - \tanh^2$. [4 marks]

(b) Prove that tanh is strictly increasing, and hence invertible. [4 marks]

(c) Prove that $\lim_{x\to\pm\infty} \tanh(x) = \pm 1$ and hence that $\tanh(\mathbb{R}) = (-1, 1)$. [4 marks]

(d) Prove that $\mathrm{artanh} = \tanh^{-1} : (-1, 1) \to \mathbb{R}$ is differentiable, and that

$$\mathrm{artanh}'(x) = \frac{1}{1 - x^2} \, .$$

[4 marks]

(e) Prove the identity $\mathrm{artanh}(a) + \mathrm{artanh}(b) = \mathrm{artanh}\left(\frac{a+b}{1+ab}\right)$ for $a, b \in (-1, 1)$ by considering the derivative of the function

$$f(x) = \mathrm{artanh}(x) + \mathrm{artanh}(b) - \mathrm{artanh}\left(\frac{x+b}{1+xb}\right)$$

for fixed $b \in (-1, 1)$. [4 marks]

Solution:

(a) $\tanh = f/g$ where $f(x) = \exp(x) - \exp(-x)$ and $g(x) = \exp(x) + \exp(-x)$. f and g are sums of differentiable functions, hence differentiable. f/g is differentiable at all points x where $g(x) \neq 0$. As $\exp(\mathbb{R}) = \mathbb{R}^+$, $g(x) > 0$ for all $x \in \mathbb{R}$, so tanh is differentiable for all $x \in \mathbb{R}$. [2 marks]

We compute directly

$$\begin{aligned} \tanh'(x) = &\frac{(\exp(x) + \exp(-x))^2 - (\exp(x) - \exp(-x))^2}{(\exp(x) + \exp(-x))^2} \\ =&1 + \left(\frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}\right)^2 = 1 - \tanh(x)^2 \end{aligned}$$

[2 marks]

(b) Continuing the calculation in (a), we find

$$\begin{aligned}\tanh'(x) &= \frac{(\exp(x)+\exp(-x))^2 - (\exp(x)-\exp(-x))^2}{(\exp(x)+\exp(-x))^2)} \\ &= \frac{4}{(\exp(x)+\exp(-x))^2} > 0\ .\end{aligned}$$

[2 marks]

Applying Theorem 2.4, tanh is strictly increasing on (any closed and bounded subinterval of $\mathbb{R}$, therefore also on) $\mathbb{R}$. [1 mark]

By the corollary to Theorem 4.2, tanh is invertible. [1 mark]

(c) As $\lim\limits_{x\to\infty} \exp(-x) = 0$, we have $\lim\limits_{x\to\infty} \tanh(x) = \lim\limits_{x\to\infty} \dfrac{1-\exp(-2x)}{1+\exp(-2x)} = 1$ and from the symmetry $\tanh(-x) = -\tanh(x)$ it follows that $\lim\limits_{x\to-\infty} \tanh(x) = -1$. [3 marks]

As tanh is strictly increasing, $\tanh(\mathbb{R}) = (-1,1)$. [1 marks]

(d) By Theorem 4.6, artanh is differentiable at $b = \tanh(a)$ for all a such that $\tanh'(a) \neq 0$. As $\tanh' > 0$, artanh is differentiable on $\tanh(\mathbb{R}) = (-1,1)$. [2 marks]

With $b = \tanh(a)$ we have

$$\operatorname{artanh}'(b) = \frac{1}{\tanh'(a)} = \frac{1}{1-\tanh(a)^2} = \frac{1}{1-b^2}\ .$$

[2 marks]

(e) We compute

$$f'(x) = \frac{1}{1-x^2} - \frac{1}{1-\left(\frac{x+b}{1+xb}\right)^2}\frac{(1+xb)-(x+b)b}{(1+xb)^2} = 0$$

[2 marks]

By Theorem 2.5, f is constant on (any closed and bounded subinterval of $\mathbb{R}$, therefore also on) $\mathbb{R}$. [1 mark]

Finally, $f(0) = \operatorname{artanh}(0) = 0$, so $f(x) = 0$ for all $x \in (1,1)$. [1 mark]

2 Extra Exercises

2) (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continous inverse.

(b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove: If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} f'(x) = 0$.

Solution:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$.
f is differentiable with continuous derivative $f'(x) = 3x^2$. We have $f'(0) = 0$.
The inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{1/3}$.
As f is strictly increasing on $\mathbb{R}$, f is injective. $f(\mathbb{R}) = \mathbb{R}$ implies that f is surjective as well, so f is bijective.
As f is differentiable, it is continuous. Therefore f^{-1} is also continuous.

(b) This can be disproved by a counterexample.
Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be given by $f(x) = -x$.
f is differentiable and $f'(x) = -1$ for $x \geq 0$.
$\lim_{x\to 0} f(x) = 0$, but $\lim_{x\to 0} f'(x) = -1$.

3) Using the Intermediate Value Theorem, prove that a continuous function maps intervals to intervals.

Solution:

We use the following characterisation of an interval: $I \subseteq \mathbb{R}$ is an interval if and only if for all $x_1, x_2 \in I$ with $x_1 < x_2$,

$$x_1 < c < x_2 \Rightarrow c \in I \ .$$

Let $J = f(I)$. We need to show that J is an interval, i.e. for all $y_1, y_2 \in J$ with $y_1 < y_2$, $y_1 < c < y_2 \Rightarrow c \in J$:

Let $y_1, y_2 \in J$ with $y_1 < y_2$. Then there exist $x_1, x_2 \in I$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

As $y_1 \neq y_2$, necessarily $x_1 \neq x_2$ also, so either $x_1 < x_2$ or $x_2 < x_1$.

Consider, without loss of generality, the case $x_1 < x_2$. By assumption, f is a continuous function on I, so it is a continuous function on $[x_1, x_2]$ (or $[x_2, x_1]$, if $x_2 < x_1$).

Hence, by the intermediate value theorem, for all c with $y_1 < c < y_2$ there exists an $a \in [x_1, x_2]$ such that $f(a) = c$.

This implies that $c \in J$.

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