Queen Mary
University of London
BSc Examination 2001

MAS205 Complex Variables

`11th May 2001   2.30pm`

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
(i) Find all solutions $z \in \mathbb{C}$ of the equation

$$z^3 = 8.$$

(ii) Find all solutions $z \in \mathbb{C}$ of the equation

$$e^{3z} = 1.$$

(iii) How many roots of the polynomial $100z^{99} + z$ are enclosed by the unit circle $\{z \in \mathbb{C} : |z| = 1\}$?

A2.
(i) Evaluate

$$(a) \quad \lim_{z \to -i} \frac{z^2 + 4iz - 3}{z^2 + 1} \qquad\qquad (b) \quad \lim_{z \to \infty} \frac{(1-z)(2-3z)}{iz^2 + 4 + 2i}$$

(ii) Prove that $\lim_{z \to 0} (\bar{z} - z)/z$ does not exist.

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A3.
(i) Suppose that $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$, with $ad - bc \neq 0$. Prove that the inverse function f^{-1} is given by the formula $f^{-1}(z) = \frac{dz-b}{-cz+a}$.

(ii) Suppose $w = g(z) = \frac{z-1}{z-3}$. Show that the image under g of the line $Re(z) = 2$ is the unit circle $\{w \in \mathbb{C} : |w| = 1\}$.

A4. Let C be a *contour* parametrised by a piecewise smooth function $\gamma : [a, b] \to \mathbb{C}$. Define what is meant by the *contour integral*

$$\int_C f(z)\, dz$$

of the complex function f along the contour C.
Evaluate this integral when $f(z) = \bar{z}$ (the complex conjugate of z) and
(i) C is the straight line segment from $+2$ to -2.
(ii) C is the curve from $+2$ to -2 along the upper half of the radius-2 circle centred at 0.

Is it possible that the function $f(z) = \bar{z}$ has an *antiderivative* on $\mathbb{C}$? Find one or else give a reason why such an antiderivative cannot exist.

A5.
State the Residue Theorem.
Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic apart from a simple pole at $z_1 \in \mathbb{C}$, and an essential singularity at $z_2 \in \mathbb{C}$.
Let $r_1 \neq 0$ be the residue of f at z_1, and let $r_2 \neq 0$ be the residue of f at z_2.
Let $\mathcal{A}$ denote the collection of simple closed curves C in $\mathbb{C} \setminus \{z_1, z_2\}$.
Find how many elements there are in the set $\{\int_C f(z)\, dz : C \in \mathcal{A}\}$ in the case where

$$(a) \quad |r_1| \neq |r_2|, \qquad\qquad (b) \quad r_1 = r_2.$$

SECTION B *Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.
What does it mean to say that a function $f : \mathbb{C} \to \mathbb{C}$ is *differentiable* at a point $z_0 \in \mathbb{C}$?

If $f : \mathbb{C} \to \mathbb{C}$ has real and imaginary parts u and v respectively, what does it mean to say the functions u and v satisfy the *Cauchy-Riemann equations* at the point $z_0 \in \mathbb{C}$?

Prove that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 \in \mathbb{C}$ then its real and imaginary parts satisfy the Cauchy-Riemann equations at z_0.

For each of the following functions $f : \mathbb{C} \to \mathbb{C}$, determine the set of points at which f is differentiable. Justify your answer in each case.

(a) $f(x+iy) = 2x - 3iy$, $\quad (b)$ $f(x+iy) = 3x^2y - y^3 + i(3xy^2 - x^3)$, $\quad (c)$ $f(z) = |z|^2$.

B7.
(i) What is meant by an *isolated singularity* of a complex function f ?

What does it mean to say that such a singularity is a *pole of order m* ?

What is meant by the *residue* of f at an isolated singularity ?

(ii) Let m be a positive integer, and let $z_0 \in \mathbb{C}$. Let φ be a complex function which is holomorphic on a domain U containing z_0, and such that $\varphi(z_0) \neq 0$.
If $f(z) = \varphi(z)/(z - z_0)^m$ for all $z \in U$, prove that f has a pole of order m at z_0, and that its residue at z_0 is $\varphi^{(m-1)}(z_0)/(m-1)!$.
(Here $\varphi^{(m-1)}$ denotes the $(m-1)^{st}$ derivative of φ).

(iii) Let

$$f(z) = \frac{\cos z}{z^3}.$$

What is the order of its pole at $z_0 = 0$?

Compute the residue of f at $z_0 = 0$.

(iv) Let

$$g(z) = \frac{e^{iaz} - e^{ibz}}{z^3}, \quad \text{where } a, b \in \mathbb{R}.$$

If $a \neq b$, what is the order of its pole at $z_0 = 0$?

Compute the residue of g at $z_0 = 0$.

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B8.

(i) Write down the definition of the *radius of convergence* of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Find the Taylor series expansion $\sum_{n=0}^{\infty} a_n(z+1)^n$ of the function $f(z) = 1/(1-3z)$ about the point $z = -1$.

What is the radius of convergence of this Taylor series?

(ii) Suppose the power series $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence $R < \infty$, and the power series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R' < \infty$.

Prove or disprove that the power series $\sum_{n=0}^{\infty}(b_n + c_n)z^n$ has radius of convergence $R + R'$.

(iii) What does it mean to say a complex function f is *entire*?

Suppose the entire function f has Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. What is the radius of convergence of this Taylor series?

Suppose the complex function g is not a constant function, yet satisfies

$$\sup\{|g(z)| : |z| = r\} \leq 2 - 1/(1+r)^2 \quad \text{for all } r \geq 0.$$

Quoting a result from the course, or otherwise, prove that g is *not entire*.

B9.

(i) Use the Residue Theorem to evaluate

$$\int_C \frac{1}{z^2(z^2-4)} dz$$

where C is the positively oriented circle having centre $z = 1$ and radius 2.

(ii) Use the Residue Theorem to prove that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+16}\, dx = \frac{\sqrt{2}\pi}{16}.$$

Deduce the value of

$$\int_0^{\infty} \frac{1}{x^4+16}\, dx.$$

End of examination paper