Long-time behaviour of Ljapunov stretching and entropy production in a piecewise-linear intermittent map

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1 Setup in Thomas's notation

Thomas suggests that for an interval map T, we study the decay of correlation between two functions g and f via

$$\sum_{k=0}^{\infty} z^k \left(\mu_\lambda(f \circ T^k g) - \mu_\lambda(g)\mu_\lambda(f)\right) = \mu_\lambda(g(1 - z\mathcal{L})^{-1} f) - (1 - z)^{-1} \tag{1}$$

If $\mu_\lambda(f, g) = 1$. Here in fact we are going to be interested in only one function: the average Ljapunov stretching is obtained by looking at

$$Q = \ln(T'(x)). \tag{2}$$

In particular,

$$\begin{aligned} \lambda_n &= \mu_\lambda(\ln((T^n)')) = \int_0^1 \ln((T^n)')\mathrm{d}x = \int_0^1 \ln\left(\frac{\mathrm{d}T^n(x)}{\mathrm{d}x}\right)\mathrm{d}x \\ &= \mu_\lambda\left(\ln \prod_{k=0}^{n-1} \frac{\mathrm{d}T^{k+1}(x)}{\mathrm{d}T^k(x)}\right) = \mu_\lambda\left(\sum_{k=0}^{n-1} \ln T' \circ T^k\right) = \mu_\lambda\left(\sum_{k=0}^{n-1} Q \circ T^k\right) \end{aligned} \tag{3}$$

Thus, what we shall calculate is

$$\sum_{k=0}^{\infty} z^k \mu_\lambda\left(Q \circ T^k\right) = \mu_\lambda(Q \cdot (1 - z\mathcal{L})^{-1}(1)), \tag{4}$$

and the individual terms of the sum required shall be extracted from this generating function via contour integration.

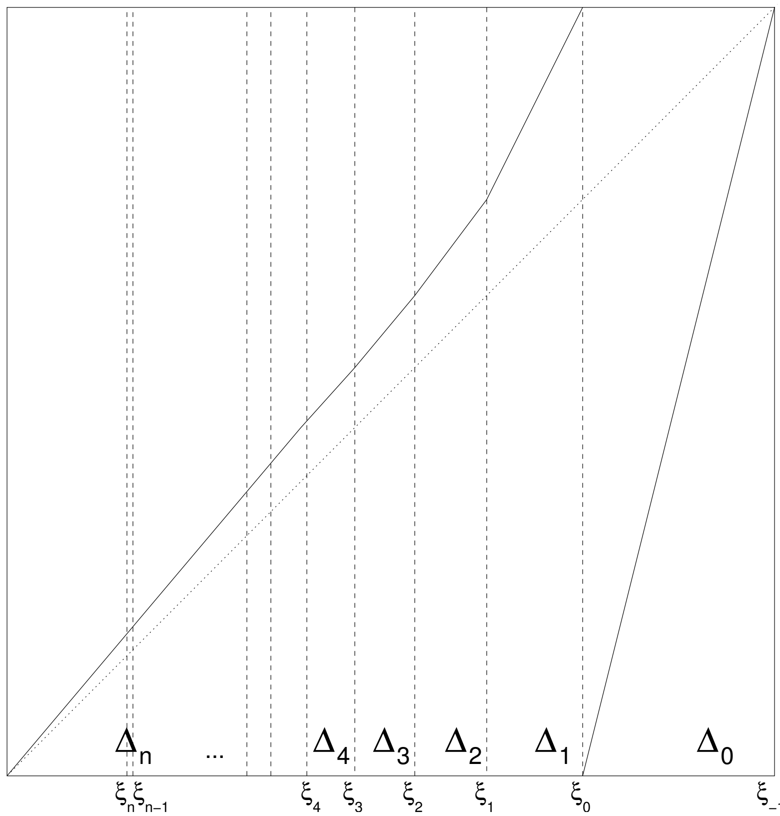

Figure 1: The mapping (bold lines) and partition (dashed lines) considered on the unit interval.

1.1 The Gaspard-Wang piecewise linearisation of the Pomeau-Manneville map

Figure 1 shows the system we are interested in. It is piecewise linear, with slopes such that, for $n > 0$,

$$T(\xi_n) = \xi_{n-1}. \tag{5}$$

The unit interval is divided into an infinite number of partition parts labeled by n, such that the length of part n is $\Delta_n = e^{-v_n}$ and is bounded by ξ_n and ξ_{n-1}. The slopes are thus given by

$$\eta_n = \frac{\xi_{n-2} - \xi_{n-1}}{\xi_{n-1} - \xi_n} = \frac{\Delta_{n-1}}{\Delta_n}, \tag{6}$$

which Thomas calls $1/c_n$. Later we shall specify to $\Delta_n = \frac{1-\Delta_0}{\zeta(\beta)} \frac{1}{n^\beta}$, so that near $x = 0$, i.e. for $n \gg 1$,

$$\xi_n = \sum_{i=n+1}^{\infty} \Delta_i = \frac{1-\Delta_0}{\zeta(\beta)} \sum_{i=n+1}^{\infty} \frac{1}{i^\beta} \simeq \frac{1-\Delta_0}{\zeta(\beta)} \int_n^\infty \frac{1}{i^\beta} \mathrm{d}i = \frac{1-\Delta_0}{\zeta(\beta)} \frac{1}{\beta - 1} \frac{1}{n^{\beta-1}} \tag{7}$$

or

$$n \simeq \left(\frac{1-\Delta_0}{\zeta(\beta)} \frac{1}{\beta - 1} \frac{1}{\xi_n} \right)^{1/(\beta-1)}. \tag{8}$$

Thus

$$T(\xi_n) = \xi_{n-1} = \xi_n + \Delta_n = \xi_n + \frac{1-\Delta_0}{\zeta(\beta)} \frac{1}{n^\beta}, \tag{9}$$

which approximates to

$$T(\xi_n) \simeq \xi_n + C\xi_n^{\beta/(\beta-1)}, \tag{10}$$

with

$$C = \frac{1-\Delta_0}{\zeta(\beta)} \left((\beta - 1) \frac{\zeta(\beta)}{1-\Delta_0} \right)^{\beta/(\beta-1)} = (\beta-1)^{\beta/(\beta-1)} \left(\frac{1-\Delta_0}{\zeta(\beta)} \right)^{1/(\beta-1)}. \tag{11}$$

We shall be interested in the parameter regimes $z = \beta/(\beta - 1) \geq 2$, $3/2 \leq z < 2$ and $1 < z < 3/2$. These translate to $1 < \beta \leq 2$, $2 < \beta \leq 3$ and $\beta > 3$ respectively. Fully anomalous behaviour will be found in the first of these regimes.

1.2 Frobenius-Perron operator

We now consider the action of the Frobenius-Perron or transfer operator, $\mathcal{L}$, on a density. As pointed out by Thomas, it is convenient to split it into two parts, $\mathcal{L}_L$ and $\mathcal{L}_R$, corresponding to left and right preimages under the mapping T of a point x respectively. In the partition part n, their action on a piecewise constant function ψ is

$$(\mathcal{L}_L\psi)_n = c_{n+1}\psi_{n+1} \tag{12}$$

and

$$(\mathcal{L}_R\psi)_n = c_0\psi_0 \tag{13}$$

according to the Frobenius-Perron equation. Also

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R. \tag{14}$$

The operator we are interested in is $(1 - z\mathcal{L})^{-1}$. Splitting $(1 - z\mathcal{L})$ as above and inverting we get

$$(1 - z\mathcal{L})^{-1} = [1 - (1 - z\mathcal{L}_L)^{-1} z\mathcal{L}_R]^{-1}(1 - z\mathcal{L}_L)^{-1}. \tag{15}$$

This expression is built up from the formulae (12,13) as follows:

$$[(1 - z\mathcal{L}_L)^{-1}\psi]_n = \sum_{k=0}^{\infty} z^k (\mathcal{L}_L^k\psi)_n = \sum_{k=0}^{\infty} z^k \prod_{l=1}^{k} c_{n+l}\psi_{n+k}, \tag{16}$$

where the product by definition equals

$$\prod_{l=1}^{k} c_{n+l} = \Delta_{n+k}/\Delta_n = e^{-v_{n+k}+v_n}. \tag{17}$$

Thus

$$[(1 - z\mathcal{L}_L)^{-1} z\mathcal{L}_R\psi]_n = e^{v_n - v_0} \sum_{k=0}^{\infty} z^{k+1} e^{-v_{n+k}} \psi_0, \tag{18}$$

only depending on ψ in the far-right partition part.

Inverting this operator we obtain

$$\left([1 - (1 - z\mathcal{L}_L)^{-1} z\mathcal{L}_R]^{-1}\psi\right)_n = \psi_n + \frac{e^{v_n - v_0}\sum_{k=0}^{\infty} z^{k+1}e^{-v_{n+k}}}{1 - \sum_{k=0}^{\infty} z^{k+1}e^{-v_k}}\psi_0, \tag{19}$$

which can be used in (15) along with (16) to give

$$\left((1 - z\mathcal{L})^{-1}\psi\right)_n = e^{v_n}\sum_{k=0}^{\infty} z^k e^{-v_{n+k}}\psi_{n+k} + \frac{\sum_{k=0}^{\infty} z^k e^{-v_k}\psi_k e^{v_n}\sum_{j=0}^{\infty} z^{j+1}e^{-v_{n+j}}}{1 - \sum_{k=0}^{\infty} z^{k+1}e^{-v_k}}\psi_0. \tag{20}$$

Now put

$$Q_n = \int_{\xi_n}^{\xi_{n-1}} Q(x)\mathrm{d}x \tag{21}$$

and $\psi = 1$, so we have an expression for (4)

$$\begin{aligned} \mu_\lambda(Q\cdot(1-z\mathcal{L})^{-1}(1)) &= \sum_{n=0}^{\infty} Q_n \left[e^{v_n} \sum_{k=0}^{\infty} z^k e^{-v_{n+k}} + \frac{\sum_{k=0}^{\infty} z^k e^{-v_k} e^{v_n} \sum_{j=0}^{\infty} z^{j+1} e^{-v_{n+j}}}{1-\sum_{k=0}^{\infty} z^{k+1} e^{-v_k}} \right] \\ &= \frac{\sum_{n=0}^{\infty} Q_n e^{v_n} \sum_{j=0}^{\infty} z^j e^{-v_{n+j}}}{1-\sum_{k=0}^{\infty} z^{k+1} e^{-v_k}}. \end{aligned} \tag{22}$$

2 Conversion to Tasaki-Gaspard notation

(22) is now close to the corresponding expression (22) derived in the Tasaki-Gaspard [1] paper. However, rather than considering the generating function above, they use

$$\mu_\lambda(Q\cdot(z-\mathcal{L})^{-1}(1)). \tag{23}$$

This is related to that suggested by Thomas using the following conversion:

$$\mu_\lambda(Q\cdot(z-\mathcal{L})^{-1}(1)) = \frac{1}{z}\mu_\lambda(Q\cdot(1-\frac{1}{z}\mathcal{L})^{-1}(1)). \tag{24}$$

Thus in Thomas's language, we obtain

$$\mu_\lambda(Q\cdot(z-\mathcal{L})^{-1}(1)) = \frac{\sum_{n=0}^{\infty} Q_n e^{v_n} \sum_{j=0}^{\infty} z^{-j} e^{-v_{n+j}}}{z-\sum_{k=0}^{\infty} z^{-k} e^{-v_k}}. \tag{25}$$

That this is indeed equivalent to their formula is apparent on setting $\tilde{\rho}_n = 1$, $B_n(0) = Q_n$ and using the following relations for the functions $\Phi(z), Z(z)$ and $\Psi(z)$, specifying now as at the end of p.1.

$$\Phi(z) = Q_0 + \frac{1}{z} e^{-v_0} \sum_{l=1}^{\infty}\sum_{k=0}^{\infty} z^{-k} e^{v_l - v_{k+l}} Q_l. \tag{26}$$

$$Z(z) = z - \sum_{k=0}^{\infty} z^{-k} e^{-v_k}. \tag{27}$$

$$\Psi(z) = e^{v_0} \sum_{k=0}^{\infty} z^{-k} e^{-v_k}. \tag{28}$$

To get to their final form still requires a fair amount of algebra. However, we have only considered unit density ($\tilde{\rho}_n = 1$) so it will be sufficient for us to consider the simpler form

$$\mu_\lambda(Q\cdot(z-\mathcal{L})^{-1}(1)) = e^{v_0}\left[\frac{z\Phi(z)}{Z(z)} - Q_0\right]. \tag{29}$$

In actuality, the form derived from Thomas's method is simplest to treat so we proceed with (25). This is convergent for $|z|>1$, but to obtain the desired coefficients of a power series expansion of the generating function we shall need

$$\mu_\lambda(Q\cdot\mathcal{L}^t(1)) = \oint\frac{\mathrm{d}z}{2\pi\imath}z^t\mu_\lambda(Q\cdot(z-\mathcal{L})^{-1}(1)). \tag{30}$$

The contour for the integral should be a circle around the origin with radius greater than 1, and to evaluate it will require knowledge of the analytic structure of the generating function (25) *inside* the unit circle.

2.1 Analytic continuation

From hereon, we specialise to the specific setting of Gaspard-Wang, with

$$e^{-v_n} = \Delta_n = \frac{1-\Delta_0}{\zeta(\beta)}\frac{1}{n^\beta}, \quad n>0 \tag{31}$$

and

$$\Delta_0 := 1-a. \tag{32}$$

Next we use the following identity, obtained from a simple change of variables in the definition of the Gamma function:

$$\frac{1}{k^\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty \mathrm{d}s s^{\beta-1}e^{-ks}. \tag{33}$$

Also useful is the corresponding relation for the (Hurwitz) zeta function,

$$\zeta_q(\beta) := \sum_{k=0}^\infty\frac{1}{(k+q)^\beta} = \sum_{k=0}^\infty\frac{1}{\Gamma(\beta)}\int_0^\infty\mathrm{d}s s^{\beta-1}e^{-(k+q)s} = \frac{1}{\Gamma(\beta)}\int_0^\infty\mathrm{d}s\frac{s^{\beta-1}e^{-qs}}{1-e^{-s}}. \tag{34}$$

Thus for the numerator of (25) we obtain

$$\begin{aligned}
\mathcal{Q}(z) &:= \sum_{n=0}^\infty Q_n e^{v_n}\sum_{j=0}^\infty z^{-j}e^{-v_{n+j}} \\
&= Q_0e^{v_0}\sum_{j=0}^\infty z^{-j}e^{-v_j} + \sum_{n=1}^\infty Q_n n^\beta\sum_{j=0}^\infty\frac{1}{z^j(n+j)^\beta} \\
&= Q_0e^{v_0}\left(z-Z(z)\right) + \sum_{n=1}^\infty Q_n n^\beta\frac{1}{\Gamma(\beta)}\int_0^\infty\mathrm{d}s\frac{s^{\beta-1}e^{-ns}}{1-z^{-1}e^{-s}} \\
&= Q_0e^{v_0}\left(z-Z(z)\right) + \frac{z}{\Gamma(\beta)}\sum_{n=1}^\infty Q_n n^\beta\int_0^\infty\mathrm{d}s\frac{s^{\beta-1}e^{-ns}}{z-e^{-s}},
\end{aligned} \tag{35}$$

and for the denominator,

$$
\begin{aligned}
Z(z) &= z - \sum_{k=0}^{\infty} z^{-k} e^{-v_k} = z - (1-a) - \frac{a}{\zeta(\beta)} \sum_{k=1}^{\infty} z^{-k} k^{-\beta} \\
&= (z-1) + a \left[1 - \frac{1}{\zeta(\beta)} \sum_{k=1}^{\infty} \frac{1}{\Gamma(\beta)} \int_0^{\infty} \mathrm{d}s s^{\beta-1} z^{-k} e^{-ks} \right] \\
&= (z-1) + a \left[1 - \frac{1}{\zeta(\beta)\Gamma(\beta)} \int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1} z^{-1} e^{-s}}{1 - z^{-1} e^{-s}} \right] \\
&= (z-1) + \frac{a}{\zeta(\beta)\Gamma(\beta)} \left[\int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1} e^{-s}}{1 - e^{-s}} - \int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1} e^{-s}}{z - e^{-s}} \right] \\
&= (z-1) \left[1 + \frac{a}{\zeta(\beta)\Gamma(\beta)} \int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1} e^{-s}}{(1-e^{-s})(z-e^{-s})} \right]. \qquad (36)
\end{aligned}
$$

For bounded Q_n these expressions are analytic everywhere in z except for the unit interval (where $z = e^{-s}$, for some s).

Note also that

$$
Z(z) = (z-1) \left[1 + \frac{a}{\zeta(\beta)\Gamma(\beta)} \sum_{k=1}^{\infty} \int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1} e^{-ks}}{z - e^{-s}} \right], \qquad (37)
$$

so that, if these expressions converge near $z = 1$, we have

$$
\lim_{z \to 1} (z-1) \mu_\lambda (Q \cdot (z - \mathcal{L})^{-1}(1)) = \lim_{z \to 1} (z-1) \frac{\mathcal{Q}}{Z} = \int_0^1 \mathrm{d}\lambda Q(x) \tilde{\rho}(x), \qquad (38)
$$

with

$$
\begin{aligned}
\tilde{\rho}_n &= \frac{\frac{1}{\Gamma(\beta)} n^\beta \int_0^\infty \mathrm{d}s \frac{s^{\beta-1} e^{-ns}}{1-e^{-s}}}{\left[1 + \frac{a}{\zeta(\beta)\Gamma(\beta)} \sum_{k=1}^\infty \int_0^\infty \mathrm{d}s \frac{s^{\beta-1} e^{-ks}}{1-e^{-s}} \right]} \\
&= n^\beta \zeta_n(\beta) \left[1 + \sum_{k=1}^{\infty} e^{-v_k} k^\beta \zeta_k(\beta) \right]^{-1}, \quad n > 0, \qquad (39)
\end{aligned}
$$

and

$$
\tilde{\rho}_0 = \frac{1}{1-a} \left[1 + \sum_{k=1}^{\infty} e^{-v_k} k^\beta \zeta_k(\beta) \right]^{-1}. \qquad (40)
$$

Thus $\tilde{\rho}(x)$ is the invariant density, where it exists. We point out that the normalization factor can be written as

$$\left[1+\sum_{k=1}^{\infty} e^{-v_k} k^{\beta} \zeta_k(\beta)\right] = 1 + a\frac{\zeta(\beta-1)}{\zeta(\beta)}, \tag{41}$$

which diverges for $\beta < 2$.

2.2 Extracting individual terms

We are interested in evaluating the contour integral

$$\mu_\lambda(Q \cdot \mathcal{L}^t(1)) = \oint \frac{\mathrm{d}z}{2\pi\imath} z^t \frac{\mathcal{Q}(z)}{Z(z)}, \tag{42}$$

where the contour can now be deformed, as suggested in [1], to

$$C \equiv \{z|,\ |z| = \epsilon, |z-1| = \epsilon \,\mathrm{or}\, z = q \pm \imath 0 \,(\epsilon < q < 1-\epsilon)\}. \tag{43}$$

We neglect the simple pole of magnitude less than 1 on the negative real axis, since this generates exponential decay which will not be seen in the long-time limit.

For the part $|z| = \epsilon$, we require the residue

$$\lim_{z\to 0} z^{t+1}\frac{\mathcal{Q}(z)}{Z(z)} = z^{t+1}\frac{Q_0 e^{v_0}\,(z - Z(z)) + \frac{z}{\Gamma(\beta)}\sum_{n=1}^{\infty} Q_n n^{\beta}\int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1}e^{-ns}}{z-e^{-s}}}{(z-1)\left[1+\frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}}{z-e^{-s}}\right]} = 0 \tag{44}$$

since $Q(x)$ is bounded and the additional factors of z in the numerator go to 0. As hinted at above, for $|z-1| = \epsilon$

$$\lim_{z\to 1}(z-1)z^t\frac{\mathcal{Q}(z)}{Z(z)} = \int_0^1 \mathrm{d}\lambda Q(x)\tilde{\rho}(x) = \sum_{n=0}^{\infty} Q_n \tilde{\rho}_n \tag{45}$$

where $\tilde{\rho}$ is a probability density. In the case that the bracketed sum in (39) diverges ($\beta \leq 2$), consider

$$\lim_{z\to 1}\frac{Q_0 e^{v_0} + \frac{1}{\Gamma(\beta)}\sum_{n=1}^{\infty} Q_n n^{\beta}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ns}}{z-e^{-s}}}{\left[1+\frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}}{z-e^{-s}}\right]}. \tag{46}$$

Since $\lim_{n\to\infty} Q_n n^{\beta} = Q(0)\frac{a}{\zeta(\beta)}$, and this lower bound on $Q(x)$ is approached monotonically, we have

$$\begin{aligned}\lim_{z\to 1}&\frac{Q_0 e^{v_0} + \frac{1}{\Gamma(\beta)}\sum_{n=1}^{\infty} Q_n n^{\beta}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ns}}{z-e^{-s}}}{\left[1+\frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}}{z-e^{-s}}\right]} = \\ Q(0) + \lim_{z\to 1}&\frac{(Q_0 e^{v_0} - Q(0)) + \frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{n=1}^{\infty}(Q_n e^{v_n} - Q(0))\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ns}}{z-e^{-s}}}{\left[1+\frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}}{z-e^{-s}}\right]} \\ &= Q(0) + 0.\end{aligned} \tag{47}$$

In fact here we also have $Q(0) = \log(1) = 0$. The denominator diverges faster than the numerator here since

$$Q_n e^{v_n} = \log\left(\frac{n}{n-1}\right)^{\beta} = \beta \log\left(\frac{n}{n-1}\right) = -\beta\log(1-\frac{1}{n}) \sim \beta(\frac{1}{n}+\frac{1}{2n^2}+\ldots). \tag{48}$$

What remains is the integral along the cut $[0,1]$, i.e.

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^1 \frac{\mathrm{d}q}{2\pi\imath} q^t\left[\frac{\mathcal{Q}(q-\imath\epsilon)}{Z(q-\imath\epsilon)} - \frac{\mathcal{Q}(q+\imath\epsilon)}{Z(q+\imath\epsilon)}\right] &= \lim_{\epsilon\to 0}\int_0^1 \frac{\mathrm{d}q}{2\pi\imath} q^t\left[\frac{2\imath\Im\left(\mathcal{Q}(q-\imath\epsilon)Z(q+\imath\epsilon)\right)}{|Z(q+\imath\epsilon)|^2}\right] \\ = \lim_{\epsilon\to 0}\int_0^1 \frac{\mathrm{d}q}{\pi} q^t\left[\frac{\Re\mathcal{Q}(q-\imath\epsilon)\Im Z(q+\imath\epsilon)+\Im\mathcal{Q}(q-\imath\epsilon)\Re Z(q+\imath\epsilon)}{|Z(q+\imath\epsilon)|^2}\right]&, \end{aligned}\tag{49}$$

where $\Re$ and $\Im$ denote taking real and imaginary parts respectively. Using

$$\begin{aligned}\lim_{\epsilon\to 0}\Im Z(q\pm\imath\epsilon) &= \lim_{\epsilon\to 0}\Im\left((q\pm\imath\epsilon-1)\left[1+\frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}}{q\pm\imath\epsilon-e^{-s}}\right]\right) \\ &= \lim_{\epsilon\to 0}\mp(q-1)\left[\frac{a}{\zeta(\beta)\Gamma(\beta)}\sum_{k=1}^{\infty}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}\epsilon}{(q-e^{-s})^2+\epsilon^2}\right] \\ &= \frac{\pm\pi a}{\zeta(\beta)\Gamma(\beta)}(-\log q)^{\beta-1} \end{aligned}\tag{50}$$

via the integral

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^{\infty}\mathrm{d}s\frac{s^{\beta-1}e^{-ks}\epsilon}{(q-e^{-s})^2+\epsilon^2} &= \pi\int_0^{\infty}\mathrm{d}s s^{\beta-1}e^{-ks}\delta(q-e^{-s}) \\ = \pi\int_1^0\frac{\mathrm{d}y}{-y}(-\log y)^{\beta-1}y^k\delta(q-y) &= \frac{\pi}{q}(-\log q)^{\beta-1}q^k, \end{aligned}\tag{51}$$

and similarly

$$\lim_{\epsilon\to 0}\Im\mathcal{Q}(q-\imath\epsilon) = \sum_{k=0}^{\infty}Q_k e^{v_k}q^k\lim_{\epsilon\to 0}\Im Z(q+\imath\epsilon), \tag{52}$$

we have, in the long time limit,

$$\begin{aligned}\mu_\lambda(Q\cdot\mathcal{L}^t(1)) &= \sum_{n=0}^{\infty}Q_n\tilde{\rho}_n + \lim_{\epsilon\to 0}\int_0^1\frac{\mathrm{d}q}{\pi}q^t\left[\frac{\Re\mathcal{Q}(q-\imath\epsilon)\Im Z(q+\imath\epsilon)+\Im\mathcal{Q}(q-\imath\epsilon)\Re Z(q+\imath\epsilon)}{|Z(q+\imath\epsilon)|^2}\right] \\ = \textstyle\sum_{n=0}^{\infty}Q_n\tilde{\rho}_n &+ \int_0^1\mathrm{d}q q^t\frac{a(-\log q)^{\beta-1}}{\zeta(\beta)\Gamma(\beta)}\left[\frac{\hat{\mathcal{Q}}(q)+(q-1)\hat{\Omega}(q)\sum_{k=0}^{\infty}Q_k e^{v_k}q^k}{(q-1)^2\hat{\Omega}(q)^2+\left(\frac{\pi a(-\log q)^{\beta-1}}{\zeta(\beta)\Gamma(\beta)}\right)^2}\right], \end{aligned}\tag{53}$$

having put

$$\hat{\Omega}(q) = \left[1 + \frac{a}{\zeta(\beta)\Gamma(\beta)} \sum_{k=1}^{\infty} \int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1}e^{-ks}}{q-e^{-s}}\right] := \left[1 + \frac{a}{\zeta(\beta)\Gamma(\beta)} \int_0^{\infty} \mathrm{d}s \mathcal{P} \frac{s^{\beta-1}e^{-s}}{(1-e^{-s})(q-e^{-s})}\right] \tag{54}$$

in $\lim_{\epsilon\to 0} \Re Z(q+\imath\epsilon) = (q-1)\hat{\Omega}(q)$ and

$$\hat{\mathcal{Q}}(q) = \lim_{\epsilon\to 0} \Re\mathcal{Q}(q-\imath\epsilon) := Q_0 e^{v_0}\left(q-(q-1)\hat{\Omega}(q)\right) + \frac{q}{\Gamma(\beta)} \sum_{n=1}^{\infty} Q_n n^{\beta} \int_0^{\infty} \mathrm{d}s \mathcal{P} \frac{s^{\beta-1}e^{-ns}}{q-e^{-s}}. \tag{55}$$

3 Long time behaviour

As β varies, the behaviour of (53) near $q=1$ is governed by that of

$$\lim_{\epsilon\to 0} \frac{\Im Z(q+\imath\epsilon)}{|Z(q+\imath\epsilon)|^2} = \frac{k_1(-\log q)^{\beta-1}}{\left[(q-1)^2\hat{\Omega}(q)^2 + k_2(-\log q)^{2(\beta-1)}\right]}, \tag{56}$$

which is in turn controlled by $\hat{\Omega}(q)$ near $q=1$.

3.1 Asymptotic behaviour of $\hat{\Omega}$

To evaluate the behaviour of $\hat{\Omega}(q)$ near $q=1$, we draw inspiration from the appendix of [3]. In particular, we are interested in the principal value of the integral $\int_0^{\infty} \mathrm{d}s \frac{s^{\beta-1}e^{-s}}{(1-e^{-s})(q-e^{-s})}$. The integral from 1 to ∞ is clearly finite, so we consider, setting $t=e^{-s}$

$$\int_0^1 \mathrm{d}s \frac{s^{\beta-1}e^{-s}}{(1-e^{-s})(q-e^{-s})} = \int_{1/e}^1 \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(q-t)}. \tag{57}$$

This is then split into three parts and the substitution $q=1-\epsilon$ is made:

$$\int_{1/e}^{1-3\epsilon/2} \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(1-\epsilon-t)} + \int_{1-3\epsilon/2}^{1-\epsilon/2} \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(1-\epsilon-t)} + \int_{1-\epsilon/2}^{1} \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(1-\epsilon-t)}. \tag{58}$$

The middle term of this sum is where we must take the principal value; putting $t=1-\epsilon-\epsilon y$ we obtain

$$\mathcal{P}\int_{1-3\epsilon/2}^{1-\epsilon/2} \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(1-\epsilon-t)} = -\frac{1}{\epsilon}\mathcal{P}\int_{1/2}^{-1/2} \mathrm{d}y \frac{(-\log(1-\epsilon(1+y)))^{\beta-1}}{(1+y)y} \tag{59}$$

$$= \frac{1}{\epsilon}\lim_{a\to 0}\int_a^{1/2} \mathrm{d}y \frac{1}{y}\left(\frac{(-\log(1-\epsilon(1+y)))^{\beta-1}}{(1+y)} - \frac{(-\log(1-\epsilon(1-y)))^{\beta-1}}{(1-y)}\right).$$

Taking the limit as $\epsilon \to 0$, this reduces to

$$\epsilon^{\beta-2} \lim_{a\to 0} \int_a^{1/2} \mathrm{d}y \frac{1}{y} \left((1+y)^{\beta-2} - (1-y)^{\beta-2}\right), \tag{60}$$

and the remaining integral is clearly finite upon expanding the bracket in a power series

$$\begin{aligned}
&\lim_{a\to 0} \int_a^{1/2} \mathrm{d}y \frac{1}{y} \left((1+y)^{\beta-2} - (1-y)^{\beta-2}\right) \\
&= \lim_{a\to 0} \int_a^{1/2} \mathrm{d}y \left(2(\beta-2) + \frac{1}{3}(\beta-2)(\beta-3)(\beta-4)y^2 + \mathcal{O}y^4\right) \\
&= \int_0^{1/2} \mathrm{d}y \left(2(\beta-2) + \frac{1}{3}(\beta-2)(\beta-3)(\beta-4)y^2 + \mathcal{O}y^4\right). \qquad (61)
\end{aligned}$$

The other two terms in (58) can be estimated to have similar limiting behaviour as $q \to 1$. Performing the same substitutions and limits as for the centre term, the third integral becomes

$$\lim_{\epsilon\to 0} \int_{1-\epsilon/2}^{1} \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(1-\epsilon-t)} \sim \epsilon^{\beta-2} \int_{-1}^{-1/2} \mathrm{d}y \frac{(1+y)^{\beta-2}}{y}, \tag{62}$$

and the first transforms to

$$\lim_{\epsilon\to 0} \int_{1/e}^{1-\epsilon/2} \mathrm{d}t \frac{(-\log t)^{\beta-1}}{(1-t)(1-\epsilon-t)} \sim \epsilon^{\beta-2} \int_{1/2}^{\frac{1}{\epsilon}(1-1/e)-1} \mathrm{d}y \frac{(1+y)^{\beta-2}}{y}. \tag{63}$$

Both of these terms thus grow as $\epsilon^{\beta-2}$, except for the latter in the special case of $\beta = 2$. Then the upper limit of the integral leads to a term $\log(\epsilon)$ in the limit of $\epsilon \to 0$.

These results on the asymptotic behaviour of $\hat{\Omega}$ as $q \to 1$ are summarised in the following set of equations:

$$\lim_{q\to 1} \hat{\Omega}(q) \sim \begin{cases} \hat{\Omega}(1), & \beta > 2 \\ \log(1-q), & \beta = 2 \\ (1-q)^{\beta-2}, & \beta < 2. \end{cases} \tag{64}$$

3.2 Long-time behaviour of Ljapunov stretching

The asymptotics for $\hat{\Omega}(q)$ quickly lead to the results of [1]

$$\lim_{q\to 1} \frac{k_1(-\log q)^{\beta-1}}{\left[(q-1)^2\hat{\Omega}(q)^2 + k_2(-\log q)^{2(\beta-1)}\right]} = \begin{cases} K_1(1-q)^{\beta-3}, & \beta > 2 \\ K_2\left((1-q)\log^2(1-q)\right)^{-1}, & \beta = 2 \\ K_3(1-q)^{1-\beta}, & \beta < 2, \end{cases} \tag{65}$$

and then, using the integral representation of the beta function

$$\int_0^1 \mathrm{d}q q^t (1-q)^\alpha = \frac{\Gamma(t+1)\Gamma(\alpha+1)}{\Gamma(t+\alpha+2)}, \tag{66}$$

we obtain the long-time behaviour via the generalised initial/final value theorem for Laplace transforms. The result for $z = 2$ follows after converting $\int_0^1 \mathrm{d}q \frac{q^t}{(1-q)\log^2(1-q)}$ to a Laplace transform and performing an integration by parts.

$$\lim_{t\to\infty} \mu_\lambda(Q \cdot \mathcal{L}^t(1)) = \begin{cases} \sum_{n=0}^\infty Q_n \tilde{\rho}_n + \frac{K_1'}{t^{\beta-2}} \hat{\mathcal{Q}}(1), & \beta > 2 \\ \frac{K_2'}{\log t} \hat{\mathcal{Q}}(1), & \beta = 2 \\ \frac{K_3'}{t^{2-\beta}} \hat{\mathcal{Q}}(1), & \beta < 2, \end{cases} \tag{67}$$

where

$$\hat{\mathcal{Q}}(1) = Q_0 e^{v_0} + \frac{1}{\Gamma(\beta)} \sum_{n=1}^\infty Q_n n^\beta \int_0^\infty \mathrm{d}s \mathcal{P} \frac{s^{\beta-1} e^{-ns}}{1-e^{-s}} = \hat{\Omega}(1) \sum_{n=0}^\infty Q_n \tilde{\rho}_n. \tag{68}$$

Finally, by integrating over time t, we come to an expression for the long-time behaviour of the Ljapunov stretching:

$$\lambda_n = \mu_\lambda \left(\sum_{k=0}^{n-1} Q \circ T^k \right) \sim \begin{cases} n, & \beta > 2 \\ \frac{n}{\log n}, & \beta = 2 \\ n^{\beta-1}, & \beta < 2, \end{cases} \tag{69}$$

which nicely matches the predictions of [2].

4 Entropy

The entropy of a system quantifies the information gained about initial conditions upon interation. That is to say, the rate at which 'nearby' points diverge under iteration tells us how quickly we can distinguish those two points, given measuring aparatus of limited precision. The following way of calculating such a quantity requires the notion of a partition and refinement of that partition under the mapping.

Figure (2) illustrates the first stages of such a process. A Pomeau-Manneville-like mapping

$$f(x) = \begin{cases} g(x) = x + 2^{z-1}x^z, & g^{-1}(0) \le x < g^{-1}(1) \\ h(x) = 2x - 1, & h^{-1}(0) \le x < h^{-1}(1), \end{cases} \tag{70}$$

and its first iteration

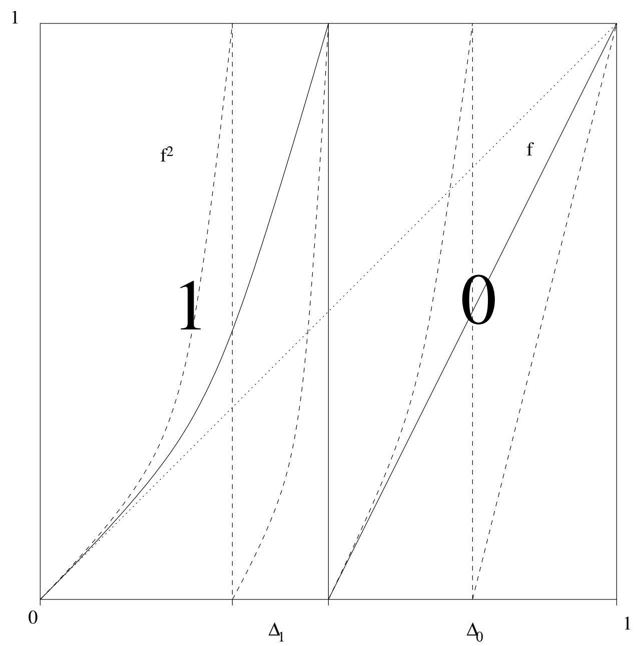

Figure 2: An iterated Pomeau-Manneville-like mapping and the natural partition of the interval. f is the mapping and f^2 the once-iterated mapping. The large 0 and 1 identify the two parts of the natural partition.

$$f^2(x) = \begin{cases} g\circ g = x + 2^{z-1}x^z + 2^{z-1}(x+2^{z-1}x^z)^z, & g^{-2}(0) \le x < g^{-2}(1) \\ h\circ g = 2(x+2^{z-1}x^z) - 1, & g^{-1}h^{-1}(0) \le x < g^{-1}h^{-1}(1) \\ g\circ h = 2x - 1 + 2^{z-1}(2x-1)^z, & h^{-1}g^{-1}(0) \le x < h^{-1}g^{-1}(1) \\ h\circ h = 2(2x-1) - 1, & h^{-2}(0) \le x < h^{-2}(1) \end{cases} \tag{71}$$

are shown, and one sees that if we consider only being able to distinguish points as either being in the interval $\mathbf{0} = \Delta_0 = [g^{-1}(1) = 1/2, 1)$ or its complement, $\mathbf{1}$, that after one iteration we can distinguish points as belonging to one of four initial sets, given by the domains of the four branches of f^2. After $n-1$ iterations, repeating this procedure distinguishes 2^n subintervals, given by the branches of f^n.

This is equivalent to the identification of the dynamics with a symbolic dynamics for a two-symbol alphabet $\{0, 1\}$. That is to say, we identify an symbol sequence such as $01\ldots110$, of length n, with the set of points of the interval which are in part $\mathbf{0}$ initially, then in $\mathbf{1}$ after one iteration, $\mathbf{1}$ again at the $(n-3)^{\text{th}}$ and $(n-2)^{\text{th}}$ iterates, and finally in $\mathbf{0}$ after the $(n-1)^{\text{th}}$ iteration, at time n.

This identification is important, since it allows us to then assign dynamically-associated probabilities to sequences. These should give, in some sense, the chance that a point chosen at random follows a trajectory with that symbol sequence. In the case where an invariant probability measure exists for the underlying mapping, that is a natural choice. Here this may not be true (for $z \ge 2$) so we just use the Lebesgue measure of the corresponding set. Once we have identified sequences with probabilities, then the average amount of information obtained by measuring an n-digit symbol sequence is

$$H_n := \sum_{i=1}^{2^n} -\mu_\lambda(P_i^n) \log \mu_\lambda(P_i^n), \tag{72}$$

where P_i^n is the i^{th} part of the n^{th} partition refinement, or in other words the i^{th} symbol sequence of length n (given an appropriate ordering).

4.1 Entropy generation in the Gaspard-Wang map

To allow for explicitness, from this point we revert to consideration only of the piecewise-linear mapping described in subsection 1.1. We now ask the question: If we know the first n digits of the symbol sequence for a trajectory, then how much information do we gain by observing the trajectory after one further iteration of the system? If the system has ended up in part $\mathbf{0}$, then the answer is simple, since our mapping is just linear there - if the n-digit sequence corresponded to an interval of length $\mu_\lambda(P_i^n) = L_i^n$, then the two possible sequences obtained at the n^{th} iteration have lengths $L_i^n \Delta_0$ and $L_i^n(1-\Delta_0)$, meaning that a fraction Δ_0 of L_i^n remains in part $\mathbf{0}$, and the rest goes to part $\mathbf{1}$. Thus the contribution of this interval to the average information increases by

$$
\begin{aligned}
&L_i^n \log L_i^n - (L_i^n \Delta_0 \log(L_i^n \Delta_0) + L_i^n (1-\Delta_0)\log(L_i^n(1-\Delta_0))) \\
&\quad = \quad L_i^n(\Delta_0 \log \Delta_0 + (1-\Delta_0)\log(1-\Delta_0)). \qquad (73)
\end{aligned}
$$

Trajectories that end up in part **1** involve a slight complication, in that how long the trajectory has spent in that part affects the ratio into which the interval containing that trajectory splits. Consider the set of points from the previous paragraph which went to **1** at the n^{th} iteration, that is a set of length $L_i^n(1-\Delta_0)$. At the next time step, a fraction $\frac{\Delta_1}{\sum_{j=1}^{\infty}\Delta_j}$ of that interval, i.e. $L_i^n\Delta_1$, will be mapped uniformly onto **0**, and $1-\frac{\Delta_1}{\sum_{j=1}^{\infty}\Delta_j}$ to **1**. However, what remains in **1** is not mapped uniformly; in fact by the construction of the mapping, only a fraction $\frac{\Delta_2}{\sum_{j=2}^{\infty}\Delta_j}$ of that interval will get mapped to **0** at the next step, i.e. $L_i^n\Delta_2$. Asymptotically, after m steps in part **1**, this probability decreases to

$$
\frac{\Delta_m}{\sum_{j=m}^{\infty}\Delta_j} = \frac{\beta-1}{m}, \quad m \to \infty. \qquad (74)
$$

This demonstrates the weakly repelling nature of the fixed point of the mapping, in that there is some tendency for trajectories that enter a region near the fixed point to stay there for a long time.

So, if a symbol sequence is n digits ending in 1 followed by m zeros, the $(n+m)^{\text{th}}$ iteration increases the average information by

$$
\begin{aligned}
&L_i^n \sum_{j=m}^{\infty} \Delta_j \log(L_i^n \sum_{j=m}^{\infty} \Delta_j) - \\
&\quad \left(L_i^n \Delta_m \log(L_i^n \Delta_m) + L_i^n \textstyle\sum_{j=m+1}^{\infty} \Delta_j \log(L_i^n \sum_{j=m+1}^{\infty}\Delta_j)\right) \\
&= -L_i^n \left(\Delta_m \log\left(\frac{\Delta_m}{\sum_{j=m}^{\infty}\Delta_j}\right) + \textstyle\sum_{j=m+1}^{\infty} \Delta_j \log\left(\frac{\sum_{j=m+1}^{\infty}\Delta_j}{\sum_{j=m}^{\infty}\Delta_j}\right)\right). \qquad (75)
\end{aligned}
$$

This formula generalises the previous for $m=0$, remembering that $\sum_{j=0}^{\infty}\Delta_j = 1$.

Summing up all the 2^{n-1} intervals that cover **0** at time n, we define

$$
L_n := \sum_{i=1}^{2^{n-1}} L_i^n = \mu_\lambda(\chi_0(T^{n-1})), \qquad (76)
$$

the total Lebesgue measure of points which are in part **0** at time n ($\chi_i(x)$ is the indicator function for the region Δ_i). Also defining

$$
\phi_m := \left(\Delta_m \log\left(\frac{\Delta_m}{\sum_{j=m}^{\infty}\Delta_j}\right) + \sum_{j=m+1}^{\infty} \Delta_j \log\left(\frac{\sum_{j=m+1}^{\infty}\Delta_j}{\sum_{j=m}^{\infty}\Delta_j}\right)\right), \qquad (77)
$$

we can finally write the total change in entropy from all 2^n partition parts (branches of T^n) generated by the n^{th} iteration as

$$h_n := \sum_{i=0}^{n} L_i \phi_{n-i}. \tag{78}$$

4.1.1 Calculation of entropy generation

To calculate the quantity h_n in the long-time limit, we shall use the same technique as was used for the Ljapunov stretching. We look at a generating function for h_n

$$h(z) := \sum_{j=0}^{\infty} \frac{h_j}{z^{j+1}} = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{L_i \phi_{j-i}}{z^{j+1}}. \tag{79}$$

This can be rewritten as

$$h(z) = \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \frac{L_i \phi_{j-i}}{z^{j+1}} = \sum_{i=0}^{\infty} \frac{L_i}{z^{i+1}} \sum_{q=0}^{\infty} \frac{\phi_q}{z^q}, \tag{80}$$

using the substitution $q = j - i$. This can be expressed using (25) with $Q(x) = \chi_0(x)$ and setting

$$\phi(z) := \sum_{q=0}^{\infty} \frac{\phi_q}{z^q} \tag{81}$$

as

$$\begin{aligned} h(z) = \frac{1}{z} \sum_{i=0}^{\infty} \frac{\mu_\lambda(\chi_0(T^{i-1}))}{z^i} \phi(z) &= \frac{\phi(z)}{z} \left(1 + \mu_\lambda \left(\chi_0 \sum_{i=0}^{\infty} \frac{\mathcal{L}^i(1)}{z^{i+1}} \right) \right) \\ &= \frac{\phi(z)}{z} \left(1 + \mu_\lambda \left(\chi_0 (z - \mathcal{L})^{-1}(1) \right) \right) \\ &= \frac{\phi(z)}{z} \left(1 + \frac{z - Z(z)}{Z(z)} \right) = \frac{\phi(z)}{Z(z)}. \end{aligned} \tag{82}$$

Since the long-time (large n) dynamics of λ_n were governed by $Z(z)$, it is immediately apparent that H_n also shows the same behaviour, or for individual terms that is

$$\lim_{t\to\infty} h_t = \begin{cases} \frac{\phi(1)}{\hat{\Omega}(1)} + \frac{K_1'}{t^{\beta-2}} \phi(1), & \beta > 2 \\ \frac{K_2'}{\log t} \phi(1), & \beta = 2 \\ \frac{K_3'}{t^{2-\beta}} \phi(1), & \beta < 2. \end{cases} \tag{83}$$

5 Pesin's formula for the Gaspard-Wang map

Comparing equations (67) and (83), we see that in fact these two quantities, the rate of stretching and of information production, will be equal in the limit $n \to \infty$, provided that

$$\hat{\mathcal{Q}}(1) = \phi(1). \tag{84}$$

From the defining equations (35) and (77), the two sides of this equation can be seen to reduce to the following:

$$\begin{aligned}
\mathcal{Q}(1) &= \sum_{n=0}^{\infty} Q_n 1/\Delta_n \sum_{j=0}^{\infty} \Delta_{n+j} = \sum_{n=0}^{\infty} -\log\left(\frac{\Delta_{n-1}}{\Delta_n}\right) \sum_{j=0}^{\infty} \Delta_{n+j} \\
&= \sum_{n=0}^{\infty} (\log \Delta_n - \log \Delta_{n-1}) \sum_{j=0}^{\infty} \Delta_{n+j} \\
&= \left(\sum_{n=0}^{\infty} \Delta_n \log \Delta_n - \sum_{n=0}^{\infty} \Delta_n \log \Delta_{n-1}\right) + \sum_{n=0}^{\infty} (\log \Delta_n - \log \Delta_{n-1}) \sum_{j=1}^{\infty} \Delta_{n+j} \\
&= \sum_{n=0}^{\infty} \Delta_n \log \Delta_n - \sum_{n=1}^{\infty} \Delta_n \log \Delta_{n-1} + \left(\sum_{n=0}^{\infty} \Delta_{n+1} \log \Delta_n - \sum_{n=0}^{\infty} \Delta_{n+1} \log \Delta_{n-1}\right) \\
&\qquad + \sum_{n=0}^{\infty} (\log \Delta_n - \log \Delta_{n-1}) \sum_{j=2}^{\infty} \Delta_{n+j} \\
&= \sum_{n=0}^{\infty} \Delta_n \log \Delta_n - \sum_{n=0}^{\infty} \Delta_{n+1} \log \Delta_{n-1} + \sum_{n=0}^{\infty} (\log \Delta_n - \log \Delta_{n-1}) \sum_{j=2}^{\infty} \Delta_{n+j} \\
&= \sum_{n=0}^{\infty} \Delta_n \log \Delta_n - \lim_{j\to\infty} \sum_{n=0}^{\infty} \Delta_{n+j} \log \Delta_{n-1} = \sum_{n=0}^{\infty} \Delta_n \log \Delta_n
\end{aligned} \tag{85}$$

and similarly

$$\begin{aligned}
\phi(1) &= \sum_{m=0}^{\infty} \left(\Delta_m \log \Delta_m + \sum_{j=m+1}^{\infty} \Delta_j \log\left(\sum_{j=m+1}^{\infty} \Delta_j\right) - \sum_{j=m}^{\infty} \Delta_j \log\left(\sum_{j=m}^{\infty} \Delta_j\right)\right) \\
&= \sum_{m=0}^{\infty} \Delta_m \log \Delta_m + \lim_{m\to\infty} \sum_{j=m+1}^{\infty} \Delta_j \log\left(\sum_{j=m+1}^{\infty} \Delta_j\right) = \sum_{m=0}^{\infty} \Delta_m \log \Delta_m.
\end{aligned} \tag{86}$$

Thus we have the equality

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \lim_{n\to\infty} \frac{H_n}{n}, \tag{87}$$

which is Pesin's equation relating the Ljapunov exponent of a one-dimensional system to its Kolmogorov-Sinai entropy. Moreover, when $\beta \leq 2$ this equality still holds in the generalised sense

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{\beta-1}} = \lim_{n\to\infty} \frac{H_n}{n^{\beta-1}}. \tag{88}$$

It remains to point out that the final expression for $\phi(1)$ and $\mathcal{Q}(1)$ is well-defined. This is easily seen since

$$\begin{aligned}
\sum_{n=0}^{\infty} \Delta_n \log \Delta_n &= \Delta_0 \log \Delta_0 + \sum_{n=1}^{\infty} \Delta_n \log \Delta_n \\
&= \Delta_0 \log \Delta_0 + \sum_{n=1}^{\infty} \frac{(1-\Delta_0)}{\zeta(\beta) n^\beta} \log \frac{(1-\Delta_0)}{\zeta(\beta) n^\beta} \\
&= \Delta_0 \log \Delta_0 + (1-\Delta_0) \log \frac{(1-\Delta_0)}{\zeta(\beta)} + \sum_{n=1}^{\infty} \frac{(1-\Delta_0)}{\zeta(\beta) n^\beta} \log \frac{1}{n^\beta} \\
&= \Delta_0 \log \Delta_0 + (1-\Delta_0) \log \frac{(1-\Delta_0)}{\zeta(\beta)} - \frac{(1-\Delta_0)}{\zeta(\beta)} \sum_{n=1}^{\infty} \frac{\beta \log n}{n^\beta} \\
&= \Delta_0 \log \Delta_0 + (1-\Delta_0) \log \frac{(1-\Delta_0)}{\zeta(\beta)} - \beta(1-\Delta_0) \frac{\zeta'(\beta)}{\zeta(\beta)} \\
&= \Delta_0 \log \Delta_0 + (1-\Delta_0) \left(\log(1-\Delta_0) - \log \zeta(\beta) - \beta \frac{\partial \log(\zeta(\beta))}{\partial \beta} \right),
\end{aligned} \tag{89}$$

which is finite for $\beta > 1$.

6 Pressure

We now consider a more general system, and in turn a more powerful method for the calculation of rates of stretching and entropy production. Specifically we allow for 'escape' from our system, in a controlled fashion, and investigate the decay of measure remaining in the unit interval upon iteration of the mapping. This can be done using a transfer operator, as in section 1, and in fact we shall use this technique to calculate the mean stretching rate. First off however, to get at the decay of measure, which will be characterised by a *generalised escape rate*, we perform a rather direct calculation. This requires taking a closer look at how the partition of figure 2 is refined upon backwards iteration.

Figure 3 shows how the system looks when escape is allowed in one particular way. It is the same system as that of figure 1, only the partition part 0 of figure 2, which corresponds to the interval Δ_0 in figure 1, is split into two parts. The right hand full branch of the mapping now has domain $(1 - \Delta_0(1 - \Delta_e), 1]$. There is thus an interval of length $\Delta_0 \Delta_e$ between the left and right branches for which the mapping is undefined and points in that region are considered 'lost'

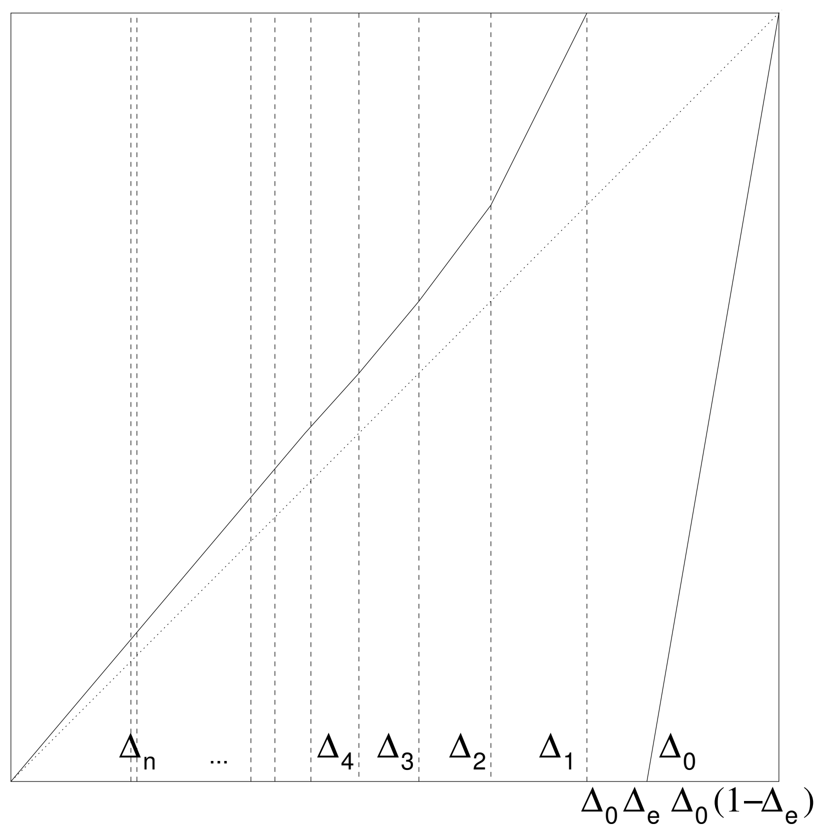

Figure 3: Piecewise linear map with escape region $\Delta_0\Delta_e$.

under iteration. What remains in the interval after n iterations is encoded in the *topological partition function*.

For this system, the partition is refined in essentially the same way as for the original closed one. After one step, the interval splits into the parts 0 and 1. After two steps, part 0, that is Δ_0, splits into two parts and so does part 1 - into Δ_1 and what remains. After n steps, the 2^n preimages of the initial interval each contain one of the 2^n branches of $f^n(x)$. 2^{n-1} of these will occur in Δ_0, 2^{n-2} in Δ_1 and so on, finishing at the left hand side of the interval with two parts Δ_{n-1} and $\sum_{i=n}^{\infty} \Delta_i$. That this correctly accounts for all parts is simple to see since

$$\left(\sum_{i=0}^{n-1} 2^{n-1-i} \right) + 1 = 2^n. \tag{90}$$

However, since we now have to account for escape, at each step 2^{n-1} new 'holes' will appear, which must also be accounted for. With the simple placement of the initial escape region in figure 3, this is easily done.

6.1 Topological Partition Function

With knowledge of how the canonical partition is refined under backwards iteration, we can calculate the total length of partition parts that remain in the interval for n steps. Moreover, we can write down their sizes explicity. There will be 2^n subintervals, whose lengths we denote by $l_i^{(n)}$. The topological partition function, $Z_n^{\mathrm{top}}(q)$ is defined as [1]

$$Z_n^{\mathrm{top}}(q) = \sum_{i=1}^{2^n} (l_i^{(n)})^q. \tag{91}$$

Thus in particular, the total Lebesgue measure of points remaining in the interval after n steps is given by $Z_n^{\mathrm{top}}(1)$. The parameter q allows us to assign varying levels of significance to small or large subintervals. For example, $Z_n^{\mathrm{top}}(0)$ merely counts the total number of intervals, regardless of their size.

Here we briefly point out a connection to the transfer operator method of the first section. It is clear that

$$Z_n^{\mathrm{top}}(1) = \mu_\lambda(\chi_I(f^n(x))) = \int_0^1 \mathrm{d}x \chi_I(f^n(x)) = \int_0^1 \mathrm{d}x \mathcal{L}_e^n(1)(x)\chi_I(x) = \int_0^1 \mathrm{d}x \mathcal{L}_e^n(1)(x), \tag{92}$$

[1] In fact the topological partition function as normally defined would be written as $\sum_{i=1}^{2^n} |f^{n\prime}(x_i)|^{-q}$, where the x_i are chosen from distinct members of the 2^n parts of the n^{th} preimage of the initial partition. This makes the connection to transfer operators more explicit and might make the connection with stretching rates rather obvious, since there also we are essentially looking to replace the slopes by the inverse width of intervals - this should be checked. That this reduces to what I consider above has only been proved for hyperbolic maps [4].

where $\mathcal{L}_e$ is now the Frobenius-Perron operator for the open system and χ_I is the indicator function for I, the unit interval. In fact a generalised transfer operator can be defined to generate $Z_n^{\mathrm{top}}(q)$ in the same way (see the footnote on the previous above).

Rather than seeing how the transfer operator of section 1 must be modified to account for escape, we procede directly. Initially we have $Z_0^{\mathrm{top}}(q) = 1$. One step splits this interval into two parts such that

$$Z_1^{\mathrm{top}}(q) = [\Delta_0(1-\Delta_e)]^q + \left(\sum_{i=1}^{\infty} \Delta_i\right)^q . \tag{93}$$

It is easy to see that whenever f^n is a simple linear branch in $l_i^{(n)}$ (there are 2^{n-1} of these at the n^{th} step), then at the next step, it splits by this same procedure, that is.

$$l_i^{(n)} \to \left\{ l_i^{(n)} \Delta_0(1-\Delta_e), l_i^{(n)} \sum_{i=1}^{\infty} \Delta_i \right\} . \tag{94}$$

When f^n is more complicated in $l_i^{(n)}$, then the right-most part of the piecewise-linearization in that subinterval (e.g. Δ_1 at step 1) becomes a full linear branch, separated from the rest by a gap, at the next step. The rest is left unchanged. Thus

$$Z_2^{\mathrm{top}}(q) = [\Delta_0(1-\Delta_e)]^q \left([\Delta_0(1-\Delta_e)]^q + \left(\sum_{i=1}^{\infty} \Delta_i\right)^q \right) + (\Delta_1(1-\Delta_e))^q + \left(\sum_{i=2}^{\infty} \Delta_i\right)^q . \tag{95}$$

Proceding iteratively, we quickly arrive at the general expression

$$Z_n^{\mathrm{top}}(q) = \sum_{i=0}^{n-1} (\Delta_i(1-\Delta_e))^q \, Z_{n-1-i}^{\mathrm{top}}(q) + \left(\sum_{j=n}^{\infty} \Delta_j\right)^q , \quad n > 0. \tag{96}$$

This is comparable to (90), and in fact reduces to it for $q = 0$. That is, $Z_n^{\mathrm{top}}(q)$ is made up of 2^n parts, 2^{n-1} of which occur in Δ_0, 2^{n-2} in Δ_1 and so on. At each step, $\Delta_0(1-\Delta_e)$ contains a uniformly compressed copy of the step before - precisely since the action of the mapping in $\Delta_0(1-\Delta_e)$ is to uniformly cover the whole interval. Likewise, $\Delta_1(1-\Delta_e)$ contains a uniformly compressed copy of the partition from 2 steps before, since the mapping takes $\Delta_1(1-\Delta_e)$ uniformly over the whole interval after two steps.

Equation (96) easily allows us to obtain a generating function for $Z_n^{\mathrm{top}}(q)$.

$$
\begin{aligned}
\sum_{n=0}^{\infty} z^n Z_n^{\mathrm{top}}(q) &= 1 + \sum_{n=1}^{\infty} z^n \left[\sum_{i=0}^{n-1} \left(\Delta_i (1-\Delta_e)\right)^q Z_{n-1-i}^{\mathrm{top}}(q) + \left(\sum_{j=n}^{\infty} \Delta_j \right)^q \right] \\
&= \sum_{n=1}^{\infty} z^n \sum_{i=0}^{n-1} \left(\Delta_i (1-\Delta_e)\right)^q Z_{n-1-i}^{\mathrm{top}}(q) + \sum_{n=0}^{\infty} z^n \left(\sum_{j=n}^{\infty} \Delta_j \right)^q \\
&= \sum_{i=0}^{\infty} \sum_{n=i+1}^{\infty} z^n \left(\Delta_i (1-\Delta_e)\right)^q Z_{n-1-i}^{\mathrm{top}}(q) + \sum_{n=0}^{\infty} z^n \left(\sum_{j=n}^{\infty} \Delta_j \right)^q \\
&= z(1-\Delta_e)^q \sum_{i=0}^{\infty} z^i \Delta_i^q \sum_{n=0}^{\infty} z^n Z_n^{\mathrm{top}}(q) + \sum_{n=0}^{\infty} z^n \left(\sum_{j=n}^{\infty} \Delta_j \right)^q \\
&= \frac{\sum_{n=0}^{\infty} z^n \left(\sum_{j=n}^{\infty} \Delta_j \right)^q}{1 - z(1-\Delta_e)^q \sum_{i=0}^{\infty} z^i \Delta_i^q}. \qquad (97)
\end{aligned}
$$

Individual terms can be extracted from this generating function via contour integration - a suitable contour allowing us to easily calculate the asymptotic behaviour as $n \to \infty$. Note that here the generating function will converge for $|z| < 1$, rather than for $|z| > 1$, which was the case in previous sections. This is purely a cosmetic change (due to my becoming more familiar with generatingfunctionology).

It is also worth noting at this point that a consistency check can be made by examining (97) as $\Delta_e \to 0$, that is, taking the limit of the closed system. The simplest case is when $q = 1$, so that

$$
\sum_{n=0}^{\infty} z^n Z_n^{\mathrm{top}}(1) = \frac{\sum_{n=0}^{\infty} z^n \sum_{j=n}^{\infty} \Delta_j}{1 - z \sum_{i=0}^{\infty} z^i \Delta_i} = \frac{\sum_{j=0}^{\infty} \Delta_j \sum_{n=0}^{j} z^n}{1 - z \sum_{i=0}^{\infty} z^i \Delta_i} = \frac{\sum_{j=0}^{\infty} \Delta_j \left(\frac{z^{j+1}-1}{z-1} \right)}{1 - z \sum_{i=0}^{\infty} z^i \Delta_i} = \frac{1}{1-z}. \qquad (98)
$$

This gives $Z_n^{\mathrm{top}}(1) = 1$, as required for a system with no escape.

A second consistency check is the case $q = 0$, for which $Z_n^{\mathrm{top}}(q)$ is the partition counting function. In that case

$$
\sum_{n=0}^{\infty} z^n Z_n^{\mathrm{top}}(0) = \frac{\sum_{n=0}^{\infty} z^n}{1 - z \sum_{i=0}^{\infty} z^i} = \frac{1/(1-z)}{1 - z/(1-z)} = \frac{1}{1-2z}, \qquad (99)
$$

which has Maclaurin series

$$
\frac{1}{1-2z} = \sum_{i=0}^{\infty} (2z)^i, \qquad (100)
$$

so that $Z_n^{\mathrm{top}}(0) = 2^n \square$ This is correct whether the system is open or closed.

6.1.1 What can we do with the Topological Partition Function?

The first thing we shall consider is $Z_n^{\text{top}}(1)$, which directly tells us the total length of points which remain in the interval for n iterations. The second is to differentiate it, which leads straight to the entropy. It's no surprise that the preceding calculation closely echoed that of section 4.1.1.

Starting then with the first point,

$$H_n := -\sum_{i=1}^{2^n} \frac{l_i^{(n)}}{Z_n^{\text{top}}(1)} \log\left(\frac{l_i^{(n)}}{Z_n^{\text{top}}(1)}\right) = \frac{1}{Z_n^{\text{top}}(1)} \sum_{i=1}^{2^n} l_i^{(n)} \log\left(\frac{1}{l_i^{(n)}}\right) + \log\left(Z_n^{\text{top}}(1)\right). \tag{101}$$

This definition is consistent with that in (72), if instead of the Lebesgue measure of the partition parts $\mu_\lambda(P_i^n)$, we now use the scaled lengths $l_i^{(n)}/Z_n^{\text{top}}(1)$, which sum to unity and thus can be considered a sensible measure for the partition parts. The splitting of this expression into the two parts of (101) reflects the *escape rate formula* of Kantz-Grassberger, which generalises Pesin's theorem to open systems.

6.2 Average stretching

The final piece required to put together a generalised Pesin identity is the average stretching of nearby orbits in the system. This is defined analogously to the calculation in section 1 by

$$\Lambda_n := \int \mathrm{d}x \log f^{n\prime}(x), \tag{102}$$

where the integral should extend over the domain of $f^n(x)$. Using the product rule for derivatives, the sum rule for logarithms, and defining a tranfer operator $\mathcal{L}_e$ for the system with escape, this can be expressed as

$$\Lambda_n = \int_0^1 \mathrm{d}x \chi_I(f^n(x)) \log f^{n\prime}(x) = \sum_{i=0}^{n-1} \int_0^1 \mathrm{d}x \mathcal{L}_e^i(1)(x) \chi_I(f^{n-i}(x)) \log f'(x), \quad n > 1, \tag{103}$$

with

$$\Lambda_0 = \int_0^1 \mathrm{d}x \log x' = 0. \tag{104}$$

The generating function for Λ_n, is given by

$$\Lambda(z) := \sum_{n=0}^{\infty} z^n \Lambda_n = \sum_{n=1}^{\infty} z^n \sum_{i=0}^{n-1} \int_0^1 \mathrm{d}x \mathcal{L}_e^i(1)(x) \chi_I(f^{n-i}(x)) \log f'(x). \tag{105}$$

This can be rewritten as

$$
\begin{aligned}
\Lambda(z) = \sum_{i=0}^{\infty}\sum_{n=i+1}^{\infty} z^n \int_0^1 \mathrm{d}x \mathcal{L}_e^i(1)(x)\chi_I(f^{n-i}(x)) \log f'(x) \\
= \quad \sum_{i=0}^{\infty}\sum_{q=1}^{\infty} z^{q+i} \int_0^1 \mathrm{d}x \mathcal{L}_e^i(1)(x)\chi_I(f^{q}(x)) \log f'(x) \\
= \quad \sum_{q=1}^{\infty} z^q \int_0^1 \mathrm{d}x \chi_I(f^q(x)) \log f'(x) \sum_{i=0}^{\infty} z^i \mathcal{L}_e^i(1)(x). \qquad (106)
\end{aligned}
$$

To procede further, we require both a way to calculate integrals of a function over the domain of $f^n(x)$ and an expression for the transfer operator $\mathcal{L}_e$ - or at least it's generating function.

The latter can be derived simply, following the calculation of section 1.2. The result is, for the operator acting on a space of functions constant in each Δ_k,

$$
\left[\sum_{i=0}^{\infty} z^i \mathcal{L}_e^i(1)\right]_k = \frac{\sum_{i=0}^{\infty} z^i \Delta_{i+k}/\Delta_k}{1 - z(1-\Delta_e)\sum_{j=0}^{\infty} z^j \Delta_j}. \qquad (107)
$$

This leads straight to the expression (97) for $\sum_{n=0}^{\infty} z^n Z_n^{\mathrm{top}}(1)$ by using (92).

It is also straightforward to obtain integrals such as $\int_0^1 \mathrm{d}x \chi_I(f^n(x))F(x)$, provided F is simple enough. In particular, $F(x) = \log f'(x)$ is a constant (denoted by F_k) over any Δ_k, as is the generating function for $\mathcal{L}_e$, so things are easy. Equation (96) expressed the domain of $f^n(x)$ as $Z_n^{\mathrm{top}}(1)$, broken down into the partition parts in each Δ_k. Thus

$$
\int_0^1 \mathrm{d}x \chi_I(f^n(x))F(x) = \sum_{k=0}^{n-1} \Delta_k (1-\Delta_e) Z_{n-k-1}^{\mathrm{top}}(1) F_k + \sum_{j=n}^{\infty} \Delta_j F_j. \qquad (108)
$$

Noting that $F_k = \log \frac{\Delta_{k-1}}{\Delta_k}, \quad k > 0$, with $F_0 = -\log \Delta_0(1-\Delta_e)$, the full expression for $\Lambda(z)$, (105), becomes

$$
\begin{aligned}
\Lambda(z) \quad = \quad \sum_{n=1}^{\infty} z^n \left(\sum_{k=0}^{n-1} \Delta_k (1-\Delta_e) Z_{n-k-1}^{\mathrm{top}}(1) F_k \left[\sum_{i=0}^{\infty} z^i \mathcal{L}_e^i(1)\right]_k \right. \\
\left. + \sum_{j=n}^{\infty} \Delta_j F_j \left[\sum_{i=0}^{\infty} z^i \mathcal{L}_e^i(1)\right]_j \right). \qquad (109)
\end{aligned}
$$

The first half of the bracketed part is

$$
\begin{aligned}
&\sum_{n=1}^{\infty} z^n \sum_{k=0}^{n-1} \Delta_k (1-\Delta_e) Z^{\text{top}}_{n-k-1}(1) F_k \left[\sum_{i=0}^{\infty} z^i \mathcal{L}_e^i(1) \right]_k \\
&\quad = \quad \frac{(1-\Delta_e)}{1 - z(1-\Delta_e)\sum_{i=0}^{\infty} z^i \Delta_i} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} z^{n+k+1} Z^{\text{top}}_n(1) F_k \sum_{j=0}^{\infty} z^j \Delta_{k+j} \\
&\quad = \quad \frac{(1-\Delta_e)\sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \Delta_{j+n}}{\left[1 - z(1-\Delta_e)\sum_{i=0}^{\infty} z^i \Delta_i\right]^2} \sum_{k=0}^{\infty} z^{k+1} F_k \sum_{j=0}^{\infty} z^j \Delta_{k+j}. \qquad (110)
\end{aligned}
$$

and the second can be written as

$$
\sum_{n=1}^{\infty} z^n \sum_{j=n}^{\infty} \Delta_j F_j \left[\sum_{i=0}^{\infty} z^i \mathcal{L}_e^i(1) \right]_j = \frac{1}{1 - z(1-\Delta_e)\sum_{i=0}^{\infty} z^i \Delta_i} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} z^n F_j \sum_{k=0}^{\infty} z^k \Delta_{k+j}, \qquad (111)
$$

where the sums reduce to

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z^{n+k} \Delta_{k+j+n} \left[\log \Delta_{n+j-1} - \log \Delta_{n+j}\right] \\
&\quad = \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} z^{n+k} \Delta_{k+j+n} \log \Delta_{n+j} - \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z^{n+k} \Delta_{k+j+n} \log \Delta_{n+j} \\
&\quad = \quad \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} z^n \Delta_{j+n} \log \frac{\Delta_j}{\Delta_{n+j}}. \qquad (112)
\end{aligned}
$$

In addition, the sum over k in (110) gives

$$
\begin{aligned}
&\sum_{k=0}^{\infty} z^k F_k \sum_{j=0}^{\infty} z^j \Delta_{k+j} = \\
&\quad = \quad -\log\left(\Delta_0(1-\Delta_e)\right) \sum_{j=0}^{\infty} z^j \Delta_j + \sum_{k=1}^{\infty} \log \frac{\Delta_{k-1}}{\Delta_k} \sum_{j=0}^{\infty} z^{k+j} \Delta_{k+j} \\
&\quad = \quad -\log\left(\Delta_0(1-\Delta_e)\right) \sum_{j=0}^{\infty} z^j \Delta_j + \sum_{j=1}^{\infty} z^j \Delta_j \log \frac{\Delta_0}{\Delta_j} \\
&\quad = \quad -\sum_{j=0}^{\infty} z^j \Delta_j \log(\Delta_j(1-\Delta_e)). \qquad (113)
\end{aligned}
$$

We are now in a position to state our main result: writing (109) in the suggestive form

$$\Lambda(z) = \frac{-z(1-\Delta_e)\sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \Delta_{j+n}}{\left[1-z(1-\Delta_e)\sum_{i=0}^{\infty} z^i \Delta_i\right]^2} \sum_{j=0}^{\infty} z^j \Delta_j \log(\Delta_j(1-\Delta_e))$$
$$+\frac{1}{1-z(1-\Delta_e)\sum_{i=0}^{\infty} z^i \Delta_i} \sum_{n=1}^{\infty} z^n \sum_{j=0}^{\infty} \Delta_{j+n} \log \frac{\Delta_j}{\Delta_{n+j}}, \qquad (114)$$

we see that this will be asymptotically equal to

$$\Lambda(z) = -\left[\frac{\partial}{\partial q} \sum_{n=0}^{\infty} z^n Z_n^{\mathrm{top}}(q)\right]_{q=1} = -\left[\frac{\partial}{\partial q} \frac{\sum_{n=0}^{\infty} z^n \left(\sum_{j=0}^{\infty} \Delta_{j+n}\right)^q}{1-z(1-\Delta_e)^q \sum_{i=0}^{\infty} z^i \Delta_i^q}\right]_{q=1} \qquad (115)$$

if

$$\sum_{n=1}^{\infty} z^n \sum_{j=0}^{\infty} \Delta_{j+n} \log \frac{\Delta_j}{\Delta_{n+j}} \sim -\sum_{n=0}^{\infty} z^n \sum_{j=0}^{\infty} \Delta_{j+n} \log \left(\sum_{j=0}^{\infty} \Delta_{j+n}\right). \qquad (116)$$

holds as $z \to 1$, i.e. if there is equality of coefficients for large n. A little thought reveals that this is merely the statement that, for large n, the leftmost branch of $f^n(x)$ can have its average slope estimate by the reciprocal of its width $\left(\sum_{j=0}^{\infty} \Delta_{j+n}\right)$, that is to say the graph is, on average, linear there. Although of course the mapping is nonlinear there - indeed it maintains the neutral fixed point at zero - the width goes to zero as $n^{1-\beta}$, and the statement is true. After removing preasymptotic terms, it reduces to

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} \frac{1}{j^\beta} \log \left(1-\frac{n}{j}\right)^\beta \sim \sum_{j=n+1}^{\infty} \frac{1}{j^\beta} \log n^{1-\beta} \qquad (117)$$

References

[1] S. Tasaki and P. Gaspard, *Spectral Properties of a Piecewise Linear Intermittent Map* J. Stat. Phys. **109** (2002), 803-820

[2] P. Gaspard and X.-J. Wang, *Sporadicity: Between Periodic and Chaotic Dynamical Behaviors* Proc. Natl. Acad. Sci. USA **85** (1988), 4591-4595

[3] T. Miyaguchi and Y. Aizawa *Spectral analysis and an area-preserving extension of a piecewise linear intermittent map* Phys. Rev. E **75** 066201 (2007)

[4] C. Beck and F. Schlögl *Thermodynamics of Chaotic Systems* Cambridge Nonlinear Science Series (1993)