MAS115 Calculus I

Week 8

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2007/08

Revision

- Extreme values
- Critical points
- Rolle's theorem
- Mean value theorem

Sections 4.3 and 4.4
(needed for coursework 7)

Indeterminate Forms and L'Hôpital's Rule

If $f(a) = 0$ and $g(a) = 0$, $f(a)/g(a)$ is a meaningless *indeterminate form*. How can we compute

$$\lim_{x \to a} \frac{f(x)}{g(x)} \ ?$$

Idea: consider linearisation

$$\begin{aligned} f(a + \Delta x) &\approx f(a) + f'(a)\Delta x = f'(a)\Delta x \\ g(a + \Delta x) &\approx g(a) + g'(a)\Delta x = g'(a)\Delta x \end{aligned}$$

and therefore

$$\frac{f(a + \Delta x)}{g(a + \Delta x)} \approx \frac{f'(a)\Delta x}{g'(a)\Delta x} = \frac{f'(a)}{g'(a)}$$

Can we prove this?

L'Hôpital's Rule (First Form)

Theorem

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} .$$

Proof.

$$\begin{aligned} \frac{f'(a)}{g'(a)} &= \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)} \end{aligned}$$

$\square$

Caution

- When using l'Hôpital's rule, always check for "0/0", i.e., check that $f(a) = 0$ and $g(a) = 0$.
- Do not compute $(\frac{f}{g})'(x)$, this is not the same as $\frac{f'(x)}{g'(x)}$ (I've seen second year students do just that ...).
- This sort of mistake is less likely to happen if you *understand* l'Hôpital's rule, rather than memorise the formula.

Examples

- $\lim_{x\to 0} \frac{3x-\sin x}{x} = \left.\frac{3-\cos x}{1}\right|_{x=0} = 2.$
- $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \left.\frac{\frac{1}{2}(1+x)^{-1/2}}{1}\right|_{x=0} = \frac{1}{2}.$
- $\lim_{x\to 0} \frac{1+\sin x}{1-x} = \left.\frac{\cos x}{-1}\right|_{x=0} = -1.$
  This is wrong! This was not of the form "0/0".
  The correct result is 1 (by substitution).
- $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \left.\frac{1-\cos x}{3x^2}\right|_{x=0} = \frac{0}{0}.$
  What's wrong here?
  Can we fix it?

L'Hôpital's Rule (Stronger Form)

Theorem

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Back to the last example:

- $\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$.

Using L'Hôpital's Rule

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Other Indeterminate Forms

So far, we have considered limits involving the indeterminate form "0/0". What about "∞/∞", "$\infty \cdot 0$", or "$\infty - \infty$"?

- "∞/∞": use

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{1/g(x)}{1/f(x)}$$

- "$\infty \cdot 0$": use

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a}\frac{g(x)}{1/f(x)}$$

- "$\infty - \infty$": use

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$$

The last case is slightly cumbersome to remember ...

Examples and Tricks

- $\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{h\to 0^+} \frac{\sin h}{h} = \lim_{h\to 0^+} \frac{\cos h}{1} = 1$

- $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$

- $\lim_{x\to\infty}(x - \sqrt{x^2+x}) = \lim_{x\to\infty} \frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-x}{x+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+1/x}} = -\frac{1}{2}$

Conclusion: use l'Hôpital's rule, but don't use it blindly!

Revision

- Indeterminate Forms
- L'Hôpital's Rule

Antiderivatives

Aim: given $f(x)$ and $f(x) = F'(x)$, find $F(x)$

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Reminder: a consequence of the Mean Value Theorem was

Corollary

If $f'(x) = g'(x)$ on (a, b) then $f(x) = g(x) + C$ for all $x \in (a, b)$.

Consequently

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Finding Antiderivatives

- Tables of formulas:

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Finding Antiderivatives

- List of rules:

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

- More advanced techniques will come later

Example

Given

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x \ ,$$

find all $F(x)$ with $F'(x) = f(x)$:

- $f(x) = 3g(x) + h(x)$ with $g(x) = x^{-1/2}$ and $h(x) = \sin 2x$.
- $G(x) = 2\sqrt{x} + C_1$ satisfies $G'(x) = g(x)$.
- $H(x) = -\frac{1}{2}\cos 2x + C_2$ satisfies $H'(x) = h(x)$.
- Therefore

$$F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C \ .$$

$(C = C_1 + C_2)$

Initial Value Problems and Differential Equations

Differential equation for unknown $y(x)$:

$$\frac{dy}{dx} = f(x)$$

Initial condition:

$$y(x_0) = y_0$$

Example: find the curve whose slope at (x, y) is $3x^2$ if $(1, -1)$ lies on the curve.

- Solve the differential equation $y' = 3x^2$:

$$y(x) = x^3 + C$$

- Evaluate C from $y(1) = -1$: $C = -2$.

Therefore

$$y(x) = x^3 - 2$$

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of f.

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

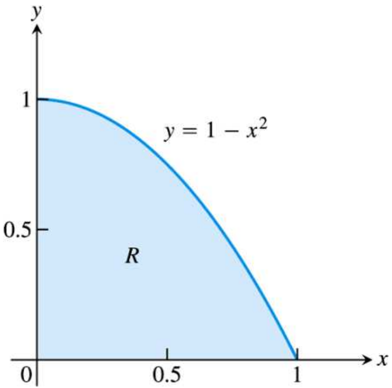
Examples:

- $\int 2x\,dx = x^2 + C$
- $\int \cos x\,dx = \sin x + C$
- $\int 2x + \cos x\,dx = x^2 + \sin x + C$

Integration

Estimating with Finite Sums

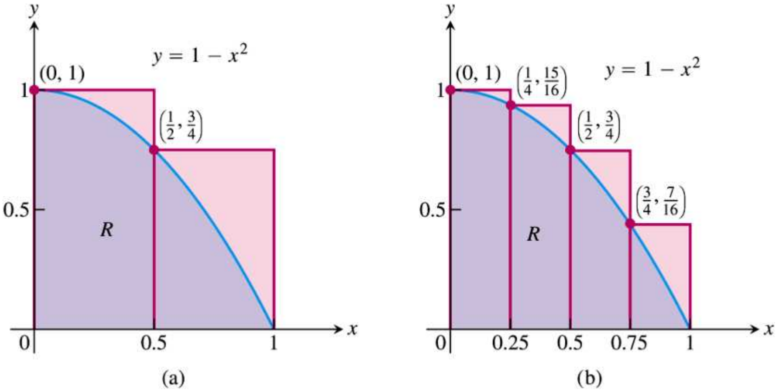
How can we compute the are between the x-axis and the curve $y = f(x)$?



Idea: approximate the area by “lots of small rectangles”

Estimating with Finite Sums

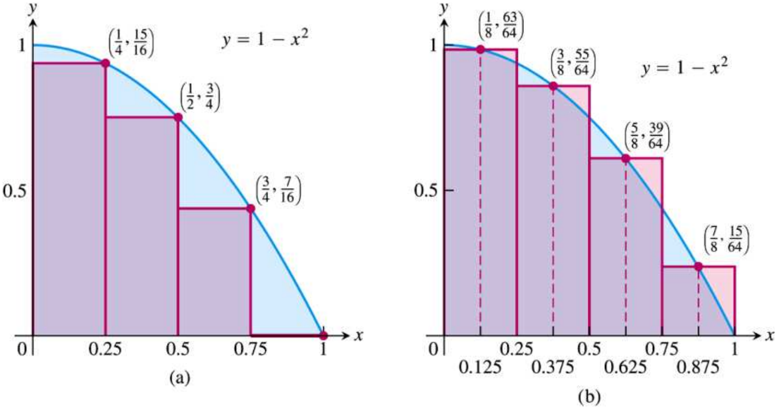
"more rectangles" $\Longrightarrow$ "better approximation"



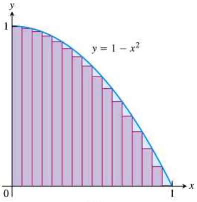
How should we pick the rectangles?
The choice of rectangles in the figure overestimate the area.

Estimating with Finite Sums

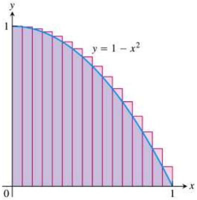
Alternatives: underestimate the area (left) or use "midpoint rule" (right)



Estimating with Finite Sums



(a)



(b)

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

Animation!

Estimating with Finite Sums

Summary:

- subdivide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$
- choose points c_k in the $k - th$ subinterval
- form the sum

$$f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$$

- different estimates:
  - upper sum: choose c_k such that $f(c_k)$is maximal
  - lower sum: choose c_k such that $f(c_k)$is minimal
  - midpoint rule: choose c_k in the middle of the interval

Sigma Notation

To handle sums with many terms, we need a better notation:

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) $$\sum_{k=1}^{n} a_k$$ a_k is a formula for the kth term.

The index k starts at $k = 1$.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

So, instead of writing

$$f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$$

we can write

$$\sum_{k=1}^{n} f(c_k)\Delta x$$

Sigma Notation

This needs practice

The sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

and rules

Algebra Rules for Finite Sums

1. *Sum Rule:* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule:* $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule:* $\sum_{k=1}^{n} c a_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

Example

Express the sum $1 + 3 + 5 + 7 + 9$ in sigma-notation:

- $\sum_{k=1}^{5}(2k - 1)$
- $\sum_{u=0}^{4}(2u + 1)$
- $\sum_{x=-3}^{1}(2x + 7)$

These sums are all equal to 25.

Example

The sum of the first n integers

$$\begin{aligned} S &= 1 \quad + 2 \quad + 3 \quad + \dots \quad + (n-1) \quad + n \\ &= n \quad + (n-1) \quad + (n-2) \quad + \dots \quad + 2 \quad + 1 \end{aligned}$$

so that

$$2S = (n+1)n$$

[Carl-Friedrich Gauß, ≈ 1784, seven years old!]
This shows that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Further Simple Sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The first n squares: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

The first n cubes: $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Such formulas can be proved by *mathematical induction*.

Revision

- Antiderivatives
- Initial Value Problems
- Indefinite Integrals
- Estimating Area with Sums

Limits of Finite Sums

Example: compute the area below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$.

- subdivide the interval into n subintervals of width $\Delta x = \frac{1}{n}$:

$$\left[0, \frac{1}{n}\right], \ \left[\frac{1}{n}, \frac{2}{n}\right], \ \left[\frac{2}{n}, \frac{3}{n}\right], \ \ldots, \ \left[\frac{n-1}{n}, \frac{n}{n}\right]$$

- choose lower sum: $c_k = \frac{k}{n}$ is rightmost point
- do the summation ...

Limits of Finite Sums

- do the summation:

$$
\begin{aligned}
\sum_{k=1}^{n} f(c_k)\Delta x &= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} \\
&= \sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\frac{1}{n} \\
&= \sum_{k=1}^{n}\left(\frac{1}{n}-\frac{k^2}{n^3}\right) \\
&= \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{1}{n}n - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= \ldots = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}
\end{aligned}
$$

Limits of Finite Sums

- lower sum:
$$R \geq \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

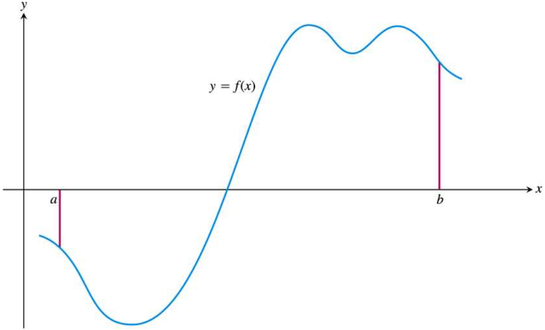
- upper sum:
$$R \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$$
(not done here)
- as $n \to \infty$, both sums tend to $\frac{2}{3}$
- any other choice of c_k would give the same result (why?)

Thereforem the area is equal to

$$R = \frac{2}{3}$$

Riemann Sums

- Allow f to be positive or negative



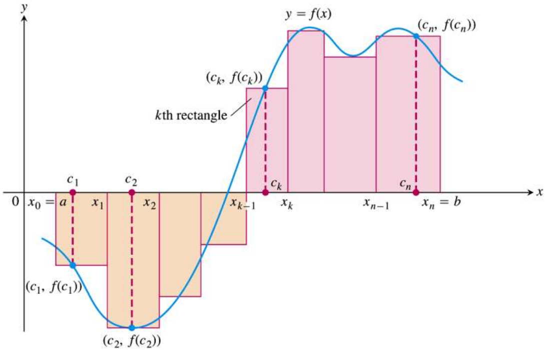
- Partition the interval $[a, b]$ by choosing $n - 1$ points between a and b:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

(i.e. Δx may vary!)

Riemann Sums

- Choose n points c_k between x_{k-1} and x_k



- The resulting sums are called Riemann sums for f on $[a, b]$
- Choose finer and finer partitions: take the limit such that the width of the largest subinterval tends to zero

Notation: for a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ we write $||P||$ for the width of the largest subinterval.

The Definite Integral

DEFINITION **The Definite Integral as a Limit of Riemann Sums**

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon .$$

Notation:

Upper limit of integration

The function is the integrand.

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

The Definite Integral

Shorthand notation:

$$\lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)dx$$

Note: as with the sigma-notation, the symbol for the "dummy variable" can be chosen as you wish

$$\int_a^b f(t)dt = \int_a^b f(x)dx$$

however, it is a bad idea to use a symbol already used for the upper/lower limit

The Definite Integral

When does the integral exist? When f is continuous:

THEOREM 1 The Existence of Definite Integrals
A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

When does the integral fail to exist? When f is "sufficiently discontinuous":
Example:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Here, $\int_0^1 f(x)dx$ does not exist as

- the upper sum is always 1
- the lower sum is always 0

Properties of Definite Integrals

TABLE 5.3 Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

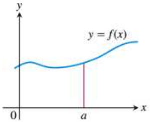
   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b]$ $\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b]$ $\Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

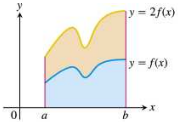
Properties of Definite Integrals



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

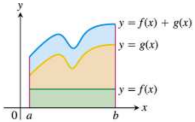
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

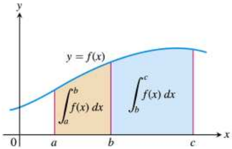
(Shown for $k = 2$.)



(c) *Sum:*

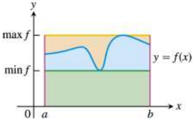
$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)



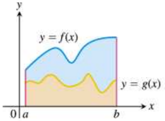
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Area under the Graph

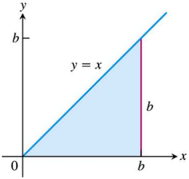
We now define the area as follows:

DEFINITION **Area Under a Curve as a Definite Integral**
If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example: $f(x) = x$, $a = 0$, $b > 0$

- graphically, $A = \frac{1}{2}b^2$
- using the definition



$$A = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{kb}{n}\cdot\frac{b}{n}$$

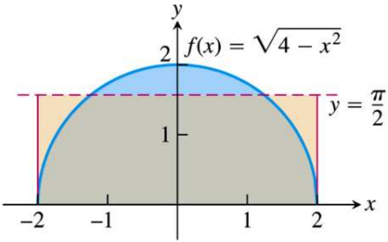
$$= \lim_{n\to\infty}\frac{b^2}{n^2}\sum_{k=1}^{n} k = \lim_{n\to\infty}\frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{b^2}{2}$$

Average Value of a Function

DEFINITION The Average or Mean Value of a Function
If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example:



$$\operatorname{av}(f) = \frac{1}{4}\int_{-2}^{2}\sqrt{4-x^2}dx = \frac{\pi}{2}$$

The End