MTH5105 Differential and Integral Analysis 2008-2009

Exercises 2

Exercise 1: Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g \qquad \text{and} \qquad g' = -f \ .$$

Show that between every two zeros of f there is a zero of g and between every two zeros of g there is a zero of f. [8 marks]

Solution: Choose $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b) = 0$.
As f is differentiable on $\mathbb{R}$, the assumptions of Rolle's Theorem are satisfied on $[a, b]$, i.e. f continuous on $[a, b]$ and differentiable on (a, b). [2 marks]

Therefore there exists a $c \in (a, b)$ such that $f'(c) = 0$. [2 marks]

As $f' = g$, $g(c) = f'(c) = 0$. [2 marks]

An analogous argument is valid with f and g exchanged. [2 marks]

Exercise 2: Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable ($f'' = (f')'$) with

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f(1) = 1 \ .$$

Show that there exists a $c \in (0, 1)$ such that $f''(c) > 1$. [10 marks]

Solution: As f is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, 1]$, i.e. f continuous on $[0, 1]$ and differentiable on $(0, 1)$. [2 marks]

Therefore there exists a $d \in (0, 1)$ such that

$$f'(d) = \frac{f(1) - f(0)}{1 - 0} = 1 \ .$$

[2 marks]

As f' is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, d]$, i.e. f' continuous on $[0, d]$ and differentiable on $(0, d)$. [2 marks]

Therefore there exists a $c \in (0, d)$ such that

$$f''(c) = \frac{f'(d) - f'(0)}{d - 0} = \frac{1}{d} \ .$$

[2 marks]

As $d \in (0, 1)$, $1/d > 1$. [2 marks]

Exercise 3: (a) Let $g : [a, b] \to \mathbb{R}$ be differentiable and $g'(a) < 0 < g'(b)$.
Show that g attains its minimal value for some $c \in (a, b)$. Deduce that $g'(c) = 0$. [8 marks]

(b) Let $f : [a, b] \to \mathbb{R}$ be differentiable and $f'(a) < s < f'(b)$.
Using $g(x) = f(x) - sx$, show that there exists a $c \in (a, b)$ such that $f'(c) = s$. [4 marks]

This shows that the derivative of differentiable functions satisfies the intermediate value property. Note that the derivative doesn't have to be continuous, so this is different from the intermediate value theorem for continuous functions.

Solution: (a) As g is differentiable on $[a, b]$, g is continuous on $[a, b]$ and therefore attains its minimum on $[a, b]$. [2 marks]

As $g'(a) < 0$, there exists an $a' > a$ with $g(a') < g(a)$. Similarly, as $g'(b) > 0$, there exists a $b' < b$ with $g(b') < g(b)$. [3 marks]

Note: a more detailed proof of this was given three extra marks. For example, one could write that $g'(a) < 0$ implies that there exists an $a' > a$ with $(g(a') - g(a))/(a' - a) < 0$.

Therefore the minimum is not attained at a or b, but at some point in (a, b). [1 mark]

As g is differentiable at $c \in (a, b)$, $g'(c) = 0$ (by Theorem 6). [2 marks]

(b) $g(x) = f(x) - sx$ is differentiable on $[a, b]$ and $g'(x) = f'(x) - s$ implies $g'(a) = f'(a) - s < 0$ and $g'(b) = f'(b) - s > 0$. [2 marks]

Therefore, by part (a), $g'(c) = 0$ for some $c \in (a, b)$.
This implies $f'(c) = s$. [2 marks]

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