

Counting Defective Parking Functions

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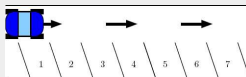
October 4-5, 2008

- 1 A parking problem (defective parking functions)
- 2 Enumeration (generating functions)
- 3 Asymptotics (limit distributions)
- 4 Conclusion and outlook

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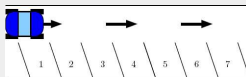
A parking problem

- Consider n parking spaces in a *one-way street*



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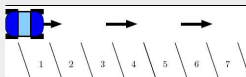
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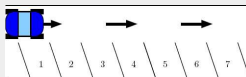
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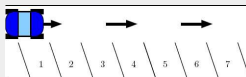
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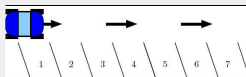
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- If k drivers fail to park, the sequence is called a *defective parking function of degree k*

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Theorem

Every permutation of a defective parking function of degree k is also a defective parking function of degree k .

The counting problem

Enumerate the number

$$cp(n, m, k)$$

of assignments of m drivers to n spaces such that exactly k drivers leave

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The probabilistic question

What is the probability

$$p_{n,m}(k) = \frac{1}{n^m} \text{cp}(n, m, k)$$

that for a randomly chosen assignment exactly k drivers leave? In particular, are there interesting **limiting distributions**?

Many equivalent or related formulations, for example

- Hashing with linear probing

[Konheim and Weiss, 1966]

[Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation

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Connections to other combinatorial objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, inversion in trees

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$cp(n, n, k)$	$k = 0$	1	2	3	4	5	6
$n = 1$	1						
2	3	1					
3	16	10	1				
4	125	107	23	1			
5	1296	1346	436	46	1		
6	16807	19917	8402	1442	87	1	
7	262144	341986	173860	41070	4320	162	1

$cp(n, n, k)$: the number of car parking assignments of n cars to n spaces such that k cars are not parked.

Theorem

For $m \leq n$,

$$\text{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

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- Space $n + 1$ is empty with probability $(n + 1 - m)/(n + 1)$.



Definition

For $r, s, k \in \mathbb{N}_0$ let $a(r, s, k)$ denote the number of choices for which r spaces remain empty, s spaces are occupied in the end and k people drive home.

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- $n = s + r$ parking spaces, $m = s + k$ drivers
- $\text{cp}(n, m, k) = a(n - m + k, m - k, k)$

Lemma (A functional equation)

Let A be the formal power series in the three variables u , v , and t defined by

$$A(u, v, t) := \sum_{r, s, k \geq 0} a(r, s, k) \cdot u^r \frac{v^s t^k}{(s+k)!}.$$

Then $A(u, v, t)$ is the unique solution of

$$0 = \left(\frac{v}{t} e^t - 1\right) \cdot A(u, v, t) + \left(u - \frac{v}{t}\right) \cdot A(u, v, 0) + 1$$

in the ring of formal power series in u , v and t .

Solving

$$0 = \left(\frac{v}{t}e^t - 1\right) \cdot A(u, v, t) + \left(u - \frac{v}{t}\right) \cdot A(u, v, 0) + 1$$

with the **Kernel Method**:

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- Setting the *kernel* $K(v, t) = \frac{v}{t}e^t - 1$ equal to zero gives

$$v = te^{-t}$$

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$$T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$$

- Now we can solve

$$0 = \left(u - \frac{v}{T(v)}\right) \cdot A(u, v, 0) + 1$$

for $A(u, v, 0)$.

Lemma (The explicit generating function)

The generating function for the car parking problem is given by

$$A(u, v, t) = \frac{1}{1 - \frac{v}{t}e^t} + \frac{u - \frac{v}{t}}{1 - \frac{v}{t}e^t} \cdot \frac{e^{T(v)}}{1 - ue^{T(v)}},$$

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Now extract the coefficients $a(r, s, k) \dots$

After some further work (Lagrange inversion, resummation of quadruple sums) ...

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Theorem

*The number of car parking assignments of m cars on n spaces such that **at least** k cars do not find a parking space is given by*

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}.$$

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- $S(n, m, k) = \sum_{j=k}^m \text{cp}(n, m, j)$
- The car parking numbers $\text{cp}(n, m, k)$ are given by

$$\text{cp}(n, m, k) = S(n, m, k) - S(n, m, k+1)$$

- Remembering that $cp(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$, we can rewrite

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} cp(n-m+k+i-1, i, 0) \cdot (m-k-i)^{m-i}.$$

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- If at least k cars do not find a parking space, then there are at least $\ell = n + k - m$ empty parking spaces, and we rewrite

$$S(n, m, \ell + m - n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot cp(\ell + i - 1, i, 0) \cdot (n - \ell - i)^{m-i}.$$

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- This expression leads to a direct combinatorial proof.

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Proof.

- Assuming the ℓ -th empty space occurs at position $\ell + i$, there are i cars successfully parked in the $\ell + i - 1$ spaces to the left of it, which is counted by $\text{cp}(\ell + i - 1, i, 0)$.

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- The remaining $m - i$ cars are assigned to the $n - \ell - i$ rightmost spaces in $(n - \ell - i)^{m-i}$ different ways.
- Sum over all possible values of i .



Notice that

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}$$

are partial sums occurring in

Lemma (Abel's Binomial Identity, 1826)

For all $a, b \in \mathbb{R}, m \in \mathbb{N}_0$,

$$\sum_{i=0}^m \binom{m}{i} \cdot a \cdot (a+i)^{i-1} \cdot (b-i)^{m-i} = (a+b)^m$$

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- In fact, $S(n, m, 0) = n^m$ gives a combinatorial proof for Abel's identity for $a = n - m$ and $b = m$.

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for n (parking spaces) or m (cars) large.

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- Different regimes:
 - $m \ll n$: cars park with probability 1
 - $m \sim \lambda n$, $\lambda < 1$: discrete limit law
 - $(\sqrt{m} \ll m - n \ll m$: exponential distribution)
 - $m - n \sim \lambda\sqrt{m}$: linear-exponential distribution
 - $(\sqrt{m} \ll n - m \ll m$: exponential distribution)
 - $m \sim \lambda n$, $\lambda > 1$: discrete limit law
 - $m \gg n$: cars leave with probability 1

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Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n \rightarrow \infty} P_{n, n + \lfloor y\sqrt{n} \rfloor}(\lfloor x\sqrt{n} \rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$

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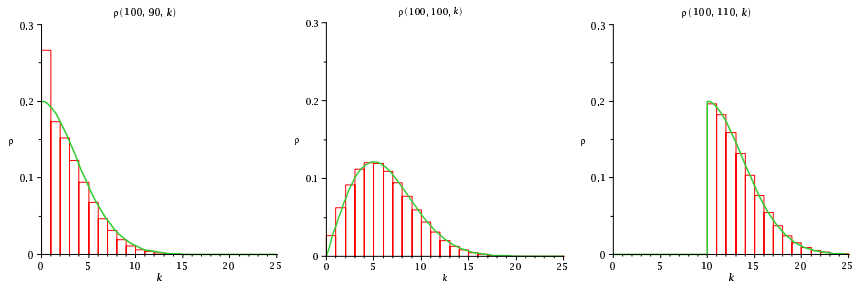
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Therefore we have approximately

$$\frac{\text{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

The approximation is surprisingly accurate:



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Outlook:

- So far only “weak limit laws” - can refine analysis
- Extension to several generalised parking problems in the literature possible

The End