# The Mathematics of the Casimir Effect

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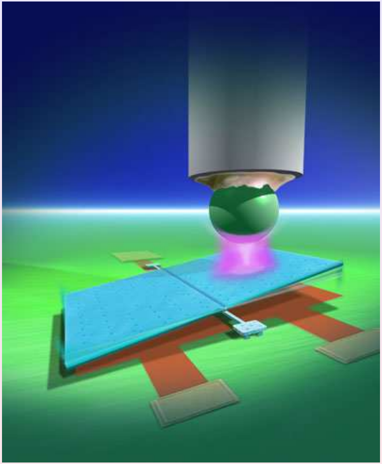
LSE Centre for Philosophy of Natural and Social Science
June 28, 2010



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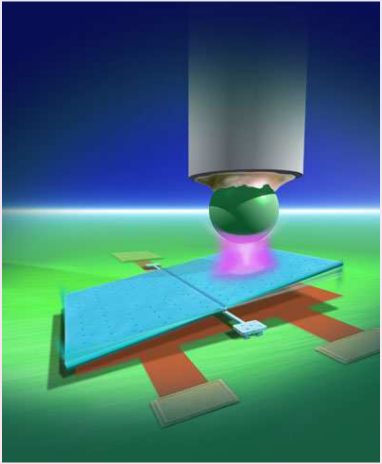

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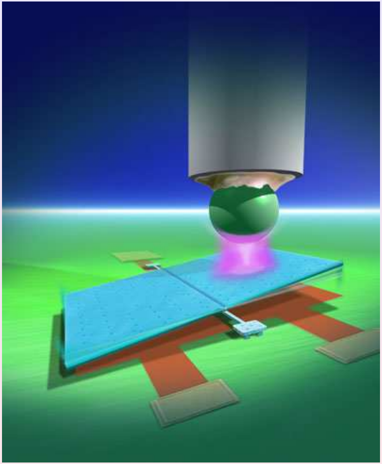

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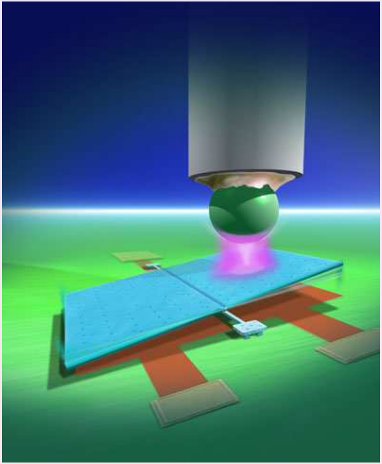

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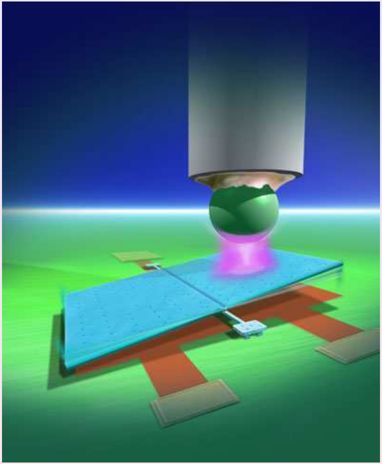

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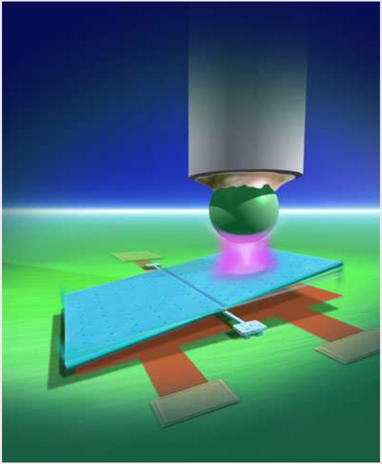

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# Topic Outline

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- student of Ehrenfest, worked with Pauli and Bohr
- *retarded Van-der-Waals-forces*

$$E = \frac{23}{4\pi}\hbar c \frac{\alpha}{R^7}$$

Casimir and Polder, 1948

- *Force between cavity walls*

$$F = -\frac{\pi^2 \hbar c}{240}\frac{A}{d^4}$$

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# The Electromagnetic Field

- The electromagnetic field, described by the *Maxwell Equations*, satisfies the *wave equation*

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{x},t) = 0$$

- Fourier-transformation ($\vec{x} \leftrightarrow \vec{k}$) gives

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\vec{A}(\vec{k},t) = 0 \quad \text{with } \omega = c|\vec{k}|$$

  which, for each $\vec{k}$, describes a **harmonic oscillator**

- Quantising harmonic oscillators is easy...

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- Quantising harmonic oscillators is easy...

# Quantisation of the Field

- Each harmonic oscillator can be in a discrete state of energy

$$E_m(\vec{k}) = \left(m + \frac{1}{2}\right)\hbar\omega \quad \text{with } \omega = c|\vec{k}|$$

- Interpretation: $m$ photons with energy $\hbar\omega$ and momentum $\hbar\vec{k}$
- In particular, the ground state energy $\frac{1}{2}\hbar\omega$ is non-zero!
- This leads to a **zero-point energy** density of the field

$$\frac{E}{V} = 2\int E_0(\vec{k})\,\frac{d^3k}{(2\pi)^3}$$

(factor 2 due to polarisation of the field)

- **Caveat:** this quantity is infinite...

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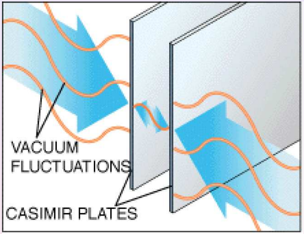
The zero-point energy shifts due to a restricted geometry




- In the presence of the boundary

$$E_{discrete} = \sum_n E_{0,n}$$

is a sum over discrete energies $E_{0,n} = \frac{1}{2}\hbar\omega_n$

- In the absence of a boundary

$$E = 2V \int E_0(\vec{k}) \frac{d^3k}{(2\pi)^3}$$

**The difference of the infinite zero-point energies is finite!**

$$\Delta E = E_{discrete} - E = -\frac{\pi^2 \hbar c}{720} \frac{L^2}{d^3}$$

for a box of size $L \times L \times d$ with $d \ll L$

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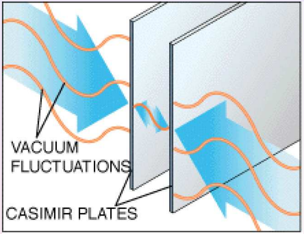


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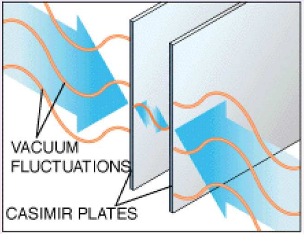


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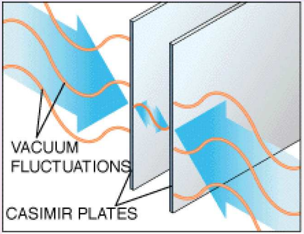


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- Experimental verification
  - Sparnaay (1958): 'not inconsistent with'
  - van Blokland and Overbeek (1978): experimental error of 50%
  - Lamoreaux (1997): experimental error of 5%
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  - Geometry dependence
  - Dynamical Casimir effect
  - Real media: non-zero temperature, finite conductivity, roughness, ...

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  infinite vacuum zero-point energy due to fluctuations of QED field
- Lifshitz (1956): Macroscopic Theory
  limiting case of fluctuations in dielectrica, no QED field
- Milloni and Shih (1992): QED with Normal Ordering
  formal subtraction of vacuum zero-point energy
- Schwinger (1967): Source Theory
  polarisation sources in plates, vacuum is void

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# Outline

# Spectral Theory

- Consider $-\Delta$ for a compact manifold $\Omega$ with a smooth boundary $\partial\Omega$
- On a suitable function space, this operator is self-adjoint and positive with pure point spectrum
- One finds formally

$$E_{discrete} = \frac{1}{2}\hbar c \,\mathrm{Trace}(-\Delta)^{1/2}$$

*This would be a different talk — let's keep it simple for today*

- Choose

$$\Omega = [0, L] \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2}$$

with Dirichlet boundary conditions $f(0) = f(L) = 0$.

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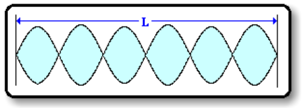

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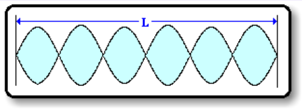


- The solutions are standing waves with wavelength $\lambda$ satisfying

$$n\frac{\lambda}{2} = L$$

- We therefore find

$$E_{0,n} = \frac{1}{2}\hbar c \frac{n\pi}{L}$$

- The zero-point energies are given by

$$E_{discrete} = \frac{\pi}{2L}\hbar c \sum_{n=0}^{\infty} n \qquad \text{and} \qquad E = \frac{\pi}{2L}\hbar c \int_0^{\infty} t\, dt$$

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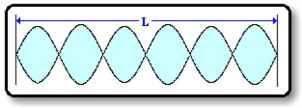


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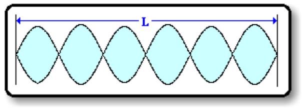


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# The Mathematical Problem

- We need to make sense of

$$\Delta E = E_{discrete} - E = \frac{\pi}{2L}\hbar c \left( \sum_{n=0}^{\infty} n - \int_0^{\infty} t \, dt \right)$$

- More generally, consider

$$\Delta(f) = \sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t) \, dt$$

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# Divergent Series

*On the Whole, Divergent Series are the Works of the Devil and it's a Shame that one dares base any Demonstration upon them. You can get whatever result you want when you use them, and they have given rise to so many Disasters and so many Paradoxes. Can anything more horrible be conceived than to have the following oozing out at you:*

$$0 = 1 - 2^n + 3^n - 4^n + etc.$$

*where n is an integer number?*

Niels Henrik Abel

# Summing Divergent Series

- Some divergent series can be summed in a sensible way ...

$$S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

- Cesaro summation: let $S_N = \sum_{n=0}^{N} (-1)^n$ and compute

$$S = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N = \frac{1}{2}$$

- Abel summation:

$$S = \lim_{x\to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}$$

- Borel summation, Euler summation, ...: again $S = \frac{1}{2}$

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# An Aside: Madelung's Constant M

- The energy of a single ion in a large NaCl crystal

$$E = -\frac{z^2 e^2}{4\pi\epsilon_0 r_0} M$$

where

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- Evaluate $M$ by summing over spheres with increasing radius

$$M = -6 + 12/\sqrt{2} - 8/\sqrt{3} + 6/2 - 24/\sqrt{5} + \dots$$

Unfortunately, the resulting series *diverges*!

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$$M = -1.74756\dots$$

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$$M = -12\pi \sum_{m,n=1,3,5\dots} \operatorname{sech}^2\left(\frac{1}{2}\pi\sqrt{m^2+n^2}\right)$$

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- The energy of a single ion in a large NaCl crystal

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  - Heat kernel regularisation $\tilde{f}(s) = \sum_{n=0}^{\infty} f(n)\, e^{-sn}$

    in particular, $\sum_{n=0}^{\infty} n\, e^{-sn} = \frac{e^s}{(e^s-1)^2} = \frac{1}{s^2} - \frac{1}{12} + O(s^2)$

    Compare with $\int_0^\infty t\, e^{-st} dt = \frac{1}{s^2}$: divergent terms cancel
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- Regularisation result should be independent of the method used
- In particular, for a reasonable class of *cutoff functions*

$$g(t;s) \quad \text{with} \quad \lim_{t\to\infty} g(t;s) = 0 \quad \text{and} \quad \lim_{s\to 0^+} g(t;s) = 1$$

  replacing $f(t)$ by $f(t)g(t;s)$ should give the same result for $s \to 0^+$
- We need to study

$$\lim_{s\to 0^+} \Delta(fg) = \lim_{s\to 0^+} \left( \sum_{n=0}^{\infty} f(n)g(n;s) - \int_0^{\infty} f(t)g(t;s)\,dt \right)$$

*Two mathematically sound approaches are*

- *Euler-Maclaurin Formula*
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# The Euler-Maclaurin Formula

Leonhard Euler, 1707 - 1783

Colin Maclaurin, 1698 - 1746

# The Euler-Maclaurin Formula

A formal derivation (Hardy, *Divergent Series*, 1949)

- Denoting $Df(x) = f'(x)$, the Taylor series can be written as

$$f(x+n) = e^{nD} f(x)$$

- It follows that

$$\sum_{n=0}^{N-1} f(x+n) = \frac{e^{ND}-1}{e^{D}-1} f(x) = \frac{1}{e^{D}-1}\left(f(x+N)-f(x)\right)$$

$$= \left(D^{-1} + \sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1}\right)\left(f(x+N)-f(x)\right)$$

$$\sum_{n=0}^{\infty} f(x+n) - \int_0^{\infty} f(x+t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} D^{k-1} f(x)$$

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## Theorem (Euler-Maclaurin Formula)

*If $f \in C^{2m}[0, N]$ then*

$$\sum_{n=0}^{N} f(n) - \int_0^N f(t)\, dt = \frac{1}{2}(f(0) + f(N)) + \\ + \sum_{k=1}^{m-1} \frac{B_{2k}}{(2k)!} \left( f^{(2k-1)}(N) - f^{(2k-1)}(0) \right) + R_m$$

*where*

$$R_m = \int_0^N \frac{B_{2m} - B_{2m}(t - \lfloor t \rfloor)}{(2m)!} f^{(2m)}(t)\, dt$$

*Here, $B_n(x)$ are Bernoulli polynomials and $B_n = B_n(0)$ are Bernoulli numbers*

# Applying the Euler-Maclaurin Formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\,dt = -\sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0)$$

- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ we find $RHS = -\frac{B_1}{1!} f(0) - \frac{B_2}{2!} f'(0)$

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# The Abel-Plana Formula

Niels Henrik Abel, 1802 - 1829

Giovanni Antonio Amedeo Plana, 1781 - 1864

# The Abel-Plana Formula

> *. . . a remarkable summation formula of Plana . . .*

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> *The only two places I have ever seen this formula are in Hardy's book and in the writings of the "massive photon" people — who also got it from Hardy.*

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The only other applications I am aware of, albeit for convergent series, are

- $q$-Gamma function asymptotics (Adri B Olde Daalhuis, 1994)
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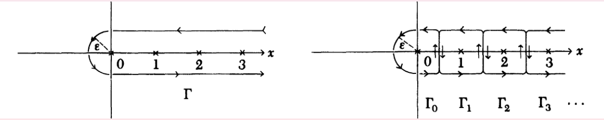
# The Abel-Plana Formula

- Use Cauchy's integral formula $f(\zeta) = \frac{1}{2\pi i} \oint_{\Gamma_\zeta} \frac{f(z)}{z-\zeta} \, dz$ together with

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z-n}$$

to get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{\Gamma_n} \frac{f(z)}{z-n} \, dz = \frac{1}{2i} \int_\Gamma \cot(\pi z) f(z) \, dz$$



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- Rotate the upper and lower arm of $\Gamma$ by $\pm\pi/2$ to get

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy)) \coth(\pi y)\, dy$$

- A similar trick gives

$$\int_0^{\infty} f(t)\, dt = \frac{i}{2} \int_0^{\infty} (f(iy) - f(-iy))\, dy$$

Taking the difference gives the elegant result

Skip precise statement

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\, dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy$$

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## Theorem (Abel-Plana Formula)

*Let $f : \mathbb{C} \to \mathbb{C}$ satisfy the following conditions*

(a) $f(z)$ *is analytic for* $\Re(z) \geq 0$ *(though not necessarily at infinity)*

(b) $\lim_{|y|\to\infty} |f(x+iy)|e^{-2\pi|y|} = 0$ *uniformly in $x$ in every finite interval*

(c) $\int_{-\infty}^{\infty} |f(x+iy) - f(x-iy)|e^{-2\pi|y|}\, dy$ *exists for every $x \geq 0$ and tends to zero for $x \to \infty$*

(d) $\int_0^{\infty} f(t)\, dt$ *is convergent, and* $\lim_{n\to\infty} f(n) = 0$

*Then*

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t)\, dt = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\, dy$$

# Applying the Abel-Plana Formula

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- Introducing suitable cutoff functions $g(t;s)$ one can justify applying this to divergent series
- For $f(t) = t$ one finds $RHS = -2 \int_0^{\infty} \frac{y\,dy}{e^{2\pi y} - 1}$

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# Outline

# Conclusion

- Mathematical question posed in theoretical physics
- Some really nice, old formulæ from classical analysis
- The result has been verified in the laboratory
- The motivation for this talk:
  - Jonathan P Dowling "The Mathematics of the Casimir Effect" Math Mag **62** (1989) 324

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*I had a feeling once about Mathematics - that I saw it all. Depth beyond depth was revealed to me - the Byss and Abyss. I saw - as one might see the transit of Venus or even the Lord Mayor's Show - a quantity passing through infinity and changing its sign from plus to minus. I saw exactly why it happened and why the tergiversation was inevitable but it was after dinner and I let it go.*

Sir Winston Spencer Churchill, 1874 - 1965

# The End

- Define for an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ the zeta function

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

- If the zeta function has an analytic extension up to 0 then define the regularised infinite sum by

$$\sum_{n=1}^{\infty} \log \lambda_n = -\zeta_\lambda'(0)$$

- Alternatively, the regularised infinite product is given by

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- Let $\lambda_n = p_n$ be the $n$-th prime so that

$$\prod_p p = e^{-\zeta_p'(0)} \qquad where \qquad \zeta_p(s) = \sum_p p^{-s}$$

- A result from Landau and Walfisz (1920) gives

$$e^{\zeta_p(s)} = \prod_{n=1}^{\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$$

- From this it takes a few steps to calculate

$$\zeta_p'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2\log(2\pi)$$

- This can be made rigorous, so that

$$\prod_p p = 4\pi^2$$

- Corollary: there are infinitely many primes

Back to main talk

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