# MTH5105 Differential and Integral Analysis 2009-2010

Solutions 2

## 1 Exercise for Feedback/Assessment

1) Suppose that $f : [0,1] \to \mathbb{R}$ is continuously differentiable.

   (a) Show that there is some number $M$ such that $|f'(x)| \leq M$ for all $x$. [8 marks]

   (b) Using the Mean Value Theorem, or otherwise, prove that

$$|f(x) - f(y)| \leq M|x - y|$$

   for all $x, y \in [0,1]$. [12 marks]

<u>Solution:</u>

(a) $f$ is continuously differentiable on $[0,1]$, which means that the derivative $f'$ is continous on $[0,1]$. [2 marks]
As $f'$ is continuous on a closed and bounded interval, it attains minimum and maximum. Thus there exist $L, U \in \mathbb{R}$ such that $L \leq f'(x) \leq U$ for all $x \in [0,1]$. [3 marks]
Therefore $|f'(x)| \leq M = \max(|L|, |U|)$ for all $x \in [0,1]$. [3 marks]
(Alternatively, use that $|f'|$ is continuous on $[0,1]$ and therefore has an upper bound.)

(b) For any $x, y \in [0,1]$ with $x < y$, $f$ is continuous on $[x, y]$ and differentiable on $(x, y)$. Therefore we can apply the Mean Value Theorem to $f$ on the interval $[x, y]$. [2 marks]
The MVT implies that there exists a $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \ .$$

[4 marks]

By (a), $|f'(c)| \leq M$. [2 marks]
Therefore

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq M \ ,$$

which implies $|f(y) - f(x)| \leq M|y - x|$. [3 marks]
This inequality is symmetric in $x$ and $y$ and trivially true if $x = y$, so that we can drop the restriction $x < y$. (This could have been argued earlier: without loss of generality, let $x < y$...) [1 mark]

## 2 Extra Exercises

2) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g \qquad \text{and} \qquad g' = -f \ .$$

Show that between every two zeros of $f$ there is a zero of $g$ and between every two zeros of $g$ there is a zero of $f$.

Solution:

Choose $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b) = 0$.
As $f$ is differentiable on $\mathbb{R}$, the assumptions of Rolle's Theorem are satisfied on $[a, b]$, i.e. $f$ continuous on $[a, b]$ and differentiable on $(a, b)$.

Therefore there exists a $c \in (a, b)$ such that $f'(c) = 0$.

As $f' = g$, $g(c) = f'(c) = 0$.

An analogous argument is valid with $f$ and $g$ exchanged.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable ($f'' = (f')'$) with

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f(1) = 1 \; .$$

Show that there exists a $c \in (0, 1)$ such that $f''(c) > 1$.

Solution:

As $f$ is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, 1]$, i.e. $f$ continuous on $[0, 1]$ and differentiable on $(0, 1)$.

Therefore there exists a $d \in (0, 1)$ such that

$$f'(d) = \frac{f(1) - f(0)}{1 - 0} = 1 \; .$$

As $f'$ is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, d]$, i.e. $f'$ continuous on $[0, d]$ and differentiable on $(0, d)$.

Therefore there exists a $c \in (0, d)$ such that

$$f''(c) = \frac{f'(d) - f'(0)}{d - 0} = \frac{1}{d} \; .$$

As $d \in (0, 1)$, $1/d > 1$.

Thomas Prellberg, January 2010