MTH5105 Differential and Integral Analysis 2010-2011

Exercises 4

There are two sections. Questions in Section 1 will be used for feedback. Questions in Section 2 are voluntary but highly recommended. Starred questions are more difficult than unstarred ones.

1 Exercises for Feedback

1) Let the function $f\colon (0,\pi) \to \mathbb{R}$ be given by $x \mapsto \cos(x)$. Show that f is invertible and that the inverse $g(y) = f^{-1}(y)$ is differentiable. Find the derivative g'.

   Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g. What is the remainder term in the Lagrange form?

   Hence show that for $|y| \leq 1/2$

$$|g(y) - \pi/2 + y| \leq \sqrt{3}/18 \approx 0.096 .$$

2 Extra Exercises

2) Let $f : (-1,\infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0 .$$

   Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0 .$$

   Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$.
   Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h, $(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)} h^{(k)}$.

3) In the lecture we have shown that the number e can be expressed as

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!} .$$

   Show that remainder term r_n in

$$n!e = n! \sum_{k=0}^{n} \frac{1}{k!} + r_n .$$

   cannot be an integer. Hence deduce that e is irrational.

4*) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, and let $M_i = \sup_{x\in\mathbb{R}} |f^{(i)}(x)|$ for $i = 0, 1, 2$. Show that

$$M_1^2 \leq 4M_0M_2 .$$

   Hint: apply Taylor's theorem to $f(a+h)$.

The deadline is 5.00pm (strict) on Monday 14th February. Please hand in your coursework to the orange coursework box on the second floor. Coursework will be returned during the exercise class immediately following the deadline.

Thomas Prellberg, February 2011