MTH5105 Differential and Integral Analysis 2008-2009

Exercises 8

Exercise 1: For $x \in \mathbb{R}$, compute

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{(1+x^2)^n} \ .$$

Show that the convergence is not uniform.

[7 marks]

Solution: We have a geometric series with terms of the form aq^n where $a = x$ and $q = 1/(1+x^2)$. For $|q| < 1$ the sum is therefore $aq/(1-q)$.

$|q| < 1$ is equivalent to $x \neq 0$, in which case we find

$$f(x) = \frac{x}{(1+x^2)\left(1 - \frac{1}{1+x^2}\right)} = \frac{1}{x} \ .$$

[3 marks]

For $x = 0$, $f(x) = \sum_{n=1}^{\infty} 0 = 0$. Thus,

$$f(x) = \begin{cases} 0 & x = 0 \ , \\ 1/x & x \neq 0 \ . \end{cases}$$

[2 marks]

The convergence cannot be uniform, as the limiting function is discontinuous.

[2 marks]

[Alternatively, to directly show lack of uniform convergence you would need to consider the partial sums

$$f_N(x) = \sum_{n=1}^{N} \frac{x}{(1+x^2)^n} = \frac{1}{x} - \frac{1}{x(1+x^2)^N} \ .$$

Clearly something goes wrong with uniform convergence near zero. For instance, if you check what happens for $x = 1/N$ you will find that $f(1/N) - f_N(1/N)$ actually diverges as $N \to \infty$ (in fact, $f_N(1/N) \to 1$).]

Exercise 2: Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to a function f. Show that if $\lim_{n\to\infty} x_n = x$ then

$$\lim_{n\to\infty} f_n(x_n) = f(x) \ .$$

[8 marks]

Solution: We need to show that for all $\epsilon > 0$ there exists an n_0 such that $|f_n(x_n) - f(x)| < \epsilon$ for all $n \geq n_0$.

The key step is to use the triangle inequality

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \ .$$

[2 marks]

f_n converges uniformly to f, so for given $\epsilon_1 > 0$ there is an n_1 such that

$$|f_n(x) - f(x)| < \epsilon_1$$

for all $n \geq n_1$ *independently* of the value of x, so in particular

$$|f_n(x_n) - f(x_n)| < \epsilon_1$$

for all $n \geq n_1$.

[2 marks]

As f is a uniform limit of continuous functions f_n, f is continuous. Therefore, for given $\epsilon_2 > 0$ there is an n_2 such that

$$|f(x_n) - f(x)| < \epsilon_2$$

for all $n \geq n_2$.

[2 marks]

Now, for given ϵ choose $\epsilon_1 = \epsilon_2 = \epsilon/2$. Then for $n_0 = \max(n_1, n_2)$ we find that

$$|f_n(x_n) - f(x)| \leq \epsilon/2 + \epsilon/2 = \epsilon \ .$$

[2 marks]

Exercise 3: (a) Show that the following sequences of functions converge uniformly on the given intervals.

$$\text{(i)} \quad u_n(x) = (1-x)x^n \ , \qquad [0,1] \ ;$$

$$\text{(ii)} \quad v_n(x) = \frac{x^2}{1+nx^2} \ , \qquad \mathbb{R} \ .$$

[6 marks]

(b) Which of the following sequences of functions converge uniformly to $s(x) = 1$ on the interval $[0,1]$?

$$\text{(i)} \quad f_n(x) = (1+x/n)^2 \ ,$$

$$\text{(ii)} \quad g_n(x) = 1 + x^n(1-x)^n \ ,$$

$$\text{(iii)} \quad h_n(x) = 1 - x^n(1-x^n) \ .$$

[9 marks]

Solution: (a) On $[0,1]$, $u_n(x) = (1-x)x^n$ is non-negative and maximal at $x = n/(1+n)$ (compute u_n' to find this value), so that

$$0 \le u_n(x) \le u_n(n/(1+n)) = \frac{1}{n}\left(1 - \frac{1}{n+1}\right)^{n+1} < \frac{1}{n} \ .$$

Therefore $|u_n(x)| < e/n$ which tends to zero independent of x.

[3 marks]

On $\mathbb{R}$, $v_n(x) = x^2/(1+nx^2)$ is non-negative and bounded above by $1/n$, as

$$0 \le v_n(x) = \frac{1}{n} - \frac{1}{n(1+nx^2)} < \frac{1}{n} \ .$$

Therefore $|v_n(x)| < 1/n$ which tends to zero independent of x.

[3 marks]

(b) On $[0,1]$, $0 \le f_n(x) - s(x) = x^2/n^2 + 2x/n \le 3/n$. Therefore $|f_n(x) - s(x)| < 3/n$ which tends to zero independent of x.

Hence f_n converges uniformly to s. [3 marks]

On $[0,1]$, $0 \le g_n(x) - s(x) = (x(1-x))^n$. This is maximal at $x = 1/2$, and therefore $|g_n(x) - s(x)| \le 1/4^n$ which tends to zero independent of x.

Hence g_n converges uniformly to s. [3 marks]

On $[0,1]$, $0 \le s(x) - h_n(x) = x^n(1-x^n)$. However, this is maximal at $x_n = 2^{-1/n}$, and therefore $s(x_n) - h_n(x_n) = 1/4$ which does *not* tend to zero as n becomes large.

Hence h_n does not converge uniformly to s. [3 marks]