MTH5105 Differential and Integral Analysis 2008-2009

Exercises 1

Exercise 1: Investigate differentiability of

(a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x|x+1|$,

(a) $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto (x+1)|x+1|$,

and find the derivative, if it exists.

Solution: For $x \neq -1$, both functions are differentiable since they are products of differentiable functions.

[3 marks]

To discuss differentiability at $a = -1$, consider, for $x \neq -1$,

$$\frac{f(x) - f(-1)}{x - (-1)} = x\frac{|x+1|}{x+1} = \begin{cases} x & x > -1 \\ -x & x < -1 \end{cases}$$

As $x \to -1$, the left and right limits differ. Therefore f is not differentiable at $x = -1$.

[2 marks]

Similarly, for $x \neq -1$,

$$\frac{g(x) - g(-1)}{x - (-1)} = (x+1)\frac{|x+1|}{x+1} = |x+1|$$

Here the limit exists as $x \to -1$. Therefore g is differentiable at $x = -1$.

[2 marks]

The derivatives are

$$f'(x) = \begin{cases} -2x - 1 & x < -1 \\ \text{undefined} & x = -1 \\ 2x + 1 & x > -1 \end{cases}$$

and

$$g'(x) = \begin{cases} -2x - 2 & x < -1 \\ 0 & x = -1 \\ 2x + 2 & x > -1 \end{cases}$$

or, simply, $g'(x) = 2|x+1|$.

[3 marks]

Exercise 2: Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at zero and find $f'(0)$.

Find $f'(x)$ for $x \neq 0$ assuming that $\sin' = \cos$.

Give a rough sketch of the curve $f'(x)$ for small x and mark $f'(0)$ clearly on your sketch.

Solution: Consider the difference quotient

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin(1/x^2) - 0}{x} = x \sin(1/x^2) \ .$$

Since $|\sin(1/x^2)| \leq 1$,

$$\frac{f(x) - f(0)}{x - 0} \to 0$$

as $x \to 0$. Therefore f is differentiable at zero with $f'(0) = 0$.

[4 marks]

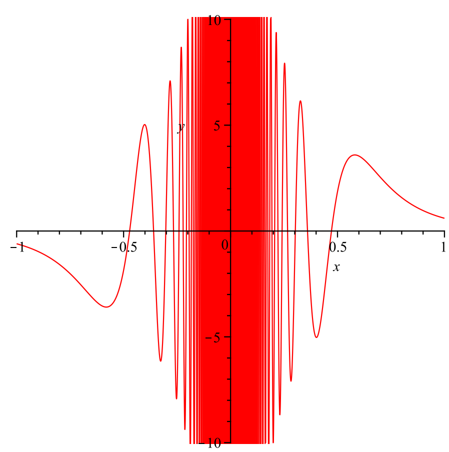
For $x \neq 0$ differentiation gives

$$f'(x) = 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2) \ .$$

[3 marks]

Graph of $f'(x)$:

For $x \to 0$, $2x \sin(1/x^2) \to 0$ and the second term dominates. The graph of f' oscillates rapidly with increasing amplitude as $x \to 0$. At zero, the derivative is zero.



[3 marks]

Exercise 3: Let $f : [-1, 1] \to \mathbb{R}$ be continuous on $[-1, 1]$, differentiable at zero and $f(0) = 0$. Show that the function

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

is continuous at zero. Is it continuous for $x \neq 0$?

Deduce that there is some number M such that

$$f(x)/x \leq M \quad \text{for all} \quad x \in [-1, 1] \setminus \{0\} \; .$$

Solution: A function g is continuous at a if $\lim_{x\to a} g(x) = g(a)$.

With $a = 0$, this gives

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0)$$

so g is continuous at 0.

[4 marks]

For $x \neq 0$, g is continuous since it is a quotient of continuous functions.

[3 marks]

By the boundedness principle, a continuous function on a closed interval attains it maximum and minimum.

Therefore there exists a number M such that $g(x) \leq M$ for all $x \in [-1, 1]$.

[3 marks]

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