The combinatorics of the leading root of Ramanujan's (and related) functions

Thomas Prellberg

School of Mathematical Sciences
Queen Mary, University of London

Mathematics Colloquium, University of Kent
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In Memoriam

Donald Preece († Jan 8, 2014)

Emeritus Professor of Mathematical Sciences, QMUL
Emeritus Professor of Combinatorial Mathematics, Kent

Topic Outline

Outline

Ramanujan's Function

$$A_q(x) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

Ramanujan's Function

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Rogers-Ramanujan Identities

$$A_q(-1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

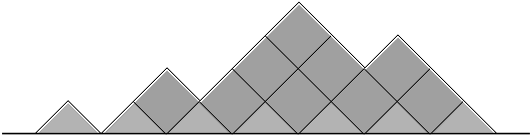
and

$$A_q(-q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

Related to partitions of integers into parts mod 5

Area-weighed Dyck paths

Count Dyck paths with respect to steps and enclosed area



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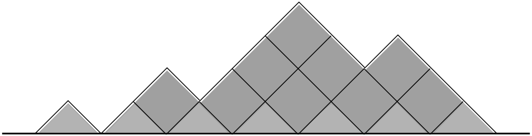

Generating function

$$G(x,q)=\sum_{m,n} c_{m,n}x^n q^m=\frac{A_q(x)}{A_q(x/q)}$$

x counts pairs of up/down steps, q counts enclosed area

Ramanujan's Lost Notebook, page 57

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 - \frac{xq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \dots}\right)$$

where

$$(t;q)_n = (1-t)(1-tq)(1-tq^2)\dots(1-tq^{n-1})$$

with y_1, y_2, y_3, y_4 explicitly given

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with y_1, y_2, y_3, y_4 explicitly given

More precisely,

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \quad y_2 = 0$$

$$y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)}, \quad y_4 = y_1 y_3$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

The roots of $A_q(x)$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=1}^{\infty}\left(1 - \frac{x q^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \ldots}\right)$$
$$= \prod_{n=0}^{\infty}\left(1 - \frac{x}{x_n(q)}\right)$$

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- Roots are positive, real, and simple (Al-Salam and Ismail, 1983)
- Ramanujan's expansion is an asymptotic series (Andrews, 2005)
- Relation to Stieltjes-Wigert polynomials (Andrews, 2005)
- Integral equation for roots (Ismail and Zhang, 2007)
- Combinatorial interpretation of y_k (Huber, 2008, and Huber and Yee, 2010)

The aim of this talk

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-x)^n}{(q;q)_n} = \prod_{n=0}^{\infty}\left(1 - \frac{x}{x_n(q)}\right)$$

$$qx_0(-q) = 1 + q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + 33q^7 + 70q^8 + 151q^9 + \ldots$$

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Goal

A combinatorial interpretation of the coefficients of the leading root

Painlevé Airy Function

$$\mathrm{Ai}_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(1-q^2)(1-q^4)\dots(1-q^{2n})}$$

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$$x_0(q) = 1 + q + q^2 + 2q^3 + 3q^4 + 6q^5 + 12q^6 + 25q^7 + 54q^8 + 120q^9 + \ldots$$

The coefficients of the leading root also seem to be positive integers

Painlevé Airy Function

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Connection Formula (Morita, 2011)

$$A_{q^2}\left(-\frac{q^3}{x^2}\right) = \frac{1}{(q;q)_\infty(-1;q)_\infty}\left\{\Theta\left(-\frac{x}{q},q\right)\mathrm{Ai}_q(-x) + \Theta\left(\frac{x}{q},q\right)\mathrm{Ai}_q(x)\right\}$$

with Theta Function

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n$$

Partial Theta Function

The Theta Function has roots $x_k(q) = q^k \quad k \in \mathbb{Z}$

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} (-x)^n = (q;q)_\infty (x;q)_\infty (q/x;q)_\infty$$

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The Partial Theta Function

$$\Theta_0(x,q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}(-x)^n$$

does not admit a "nice" product formula, but

$$x_0(q) = 1+q+2q^2+4q^3+9q^4+21q^5+52q^6+133q^7+351q^8+948q^9+\ldots$$

The coefficients of the leading root are positive integers (Sokal, 2012)

Outline

Jacobi Theta Function

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$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$$

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$$\vartheta(z + a + b\tau;\tau) = \exp(-\pi i b^2 \tau - 2\pi i b z)\vartheta(z;\tau)$$

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- Relation to modular group

$$\vartheta(z/\tau;-1/\tau)=(-i\tau)^{1/2}\exp(\pi i z^2/\tau)\vartheta(z;\tau)$$

Roots of the Jacobi Theta Function

- Jacobi Theta function

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} x^{2n}$$

where $q = \exp(\pi i \tau)$ and $x = \exp(\pi i z)$

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- Jacobi triple product

$$\sum_{n=-\infty}^{\infty} q^{n^2}x^{2n}=\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1}x^2)(1+q^{2m-1}/x^2)$$

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- Roots

$$x_k(q) = -q^{k+1/2} \qquad k \in \mathbb{Z}$$

q-Theta Function

- Combinatorialists prefer

$$\Theta(x,q)=\sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n=(q;q)_\infty(x;q)_\infty(q/x;q)_\infty$$

with q-product notation

$$(t;q)_n=\prod_{m=0}^{n-1}(1-tq^m)$$

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- Special case $R(x,q,0)$ of Rogers-Ramanujan Function

$$R(x,y,q) = \sum_{n=0}^{\infty} \frac{y^{\binom{n}{2}} x^n}{(q;q)_n}$$

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$$R(x,y,q) = \sum_{n=0}^{\infty} \frac{y^{\binom{n}{2}} x^n}{(q;q)_n}$$

- Euler identities

$$R(x,1,q) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}$$

and

$$R(x,q,q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty$$

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- $R(x,1,q)$ has no roots, whereas $R(x,q,q)$ has roots

$$x_k(1,q)=-q^{-k}\qquad k\in\mathbb{N}_0$$

Outline

The Leading Roots

Partial Theta Function $\Theta_0(x,q)=\sum_{n=0}^{\infty} q^{\binom{n}{2}}(-x)^n$

$$\Theta_0(x,q)=0$$

$$x=1+q+2q^2+4q^3+9q^4+21q^5+52q^6+133q^7+351q^8+948q^9+\ldots$$

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Ramanujan Function $A_q(x) = \sum_{n=0}^{\infty} q^{n^2}(-x)^n/(q;q)_n$

$A_{-q}(x/q) = 0$

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Key Identities

$\Theta_0(x, q)$ satisfies (Andrews and Warnaar, 2007)

$$\Theta_0(x, q) = (x; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (x; q)_n}$$

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$\mathrm{Ai}_q(x)$ satisfies (Gessel and Stanton, 1983)

$$\mathrm{Ai}_q(x) = (x; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+\binom{n}{2}} x^n}{(q^2; q^2)_n (x; q)_n}$$

Identities for the Roots

Partial Theta Function (Sokal, 2012)

$\Theta_0(x,q)=0$ if

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q;q)_n (qx;q)_{n-1}}$$

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Two identities for $\Theta_0(x, q)$

$\Theta_0(x, q)$ satisfies

$$\begin{aligned}\Theta_0(x, q) &= (q; q)_\infty (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (-x; q)_n} \\ &= (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} (-x)^n}{(q; q)_n (-x; q)_n}\end{aligned}$$

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- The first identity follows from Euler's identities
- The first and second identity follow from Heine's transformations for q-deformed hypergeometric functions

Proof of the first identity for $\Theta_0(x, q)$

$$
\begin{aligned}
\sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n \frac{(q;q)_\infty}{(q;q)_n (q^{n+1};q)_\infty} \\
&= (q;q)_\infty \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(q^{n+1})^m}{(q;q)_m} \\
&= (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(xq^m)^n}{(q;q)_n} \\
&= (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m} (-xq^m;q)_\infty \\
&= (q;q)_\infty (-x;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m (-x;q)_m}
\end{aligned}
$$

Two functional equations for $\xi_0(q)$

Lemma [Sokal]

$\xi_0(q)$ satisfies

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q;q)_n (\xi_0(q)q;q)_{n-1}}$$

and

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} \xi_0(q)^n}{(q;q)_n (\xi_0(q)q;q)_{n-1}}$$

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- This follows directly from the preceding identities for $\Theta_0(x,q)$

Proof of the first equation for $\xi_0(q)$

From

$$\Theta_0(x, q) = (q; q)_\infty (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (-x; q)_n}$$

it follows that

$$\Theta_0(x, q) = (q; q)_\infty (-xq; q)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (-xq; q)_{n-1}} \right]$$

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Hence $\Theta_0(-\xi_0(q),q) = 0$ implies that

$$0 = 1 - \xi_0(q) + \sum_{n=1}^{\infty} \frac{q^n}{(q;q)_n(\xi_0(q)q;q)_{n-1}}$$

Positivity

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- Letting $x^{(0)} = 1$ and iterating

$$x^{(N+1)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n (qx^{(N)};q)_{n-1}}$$

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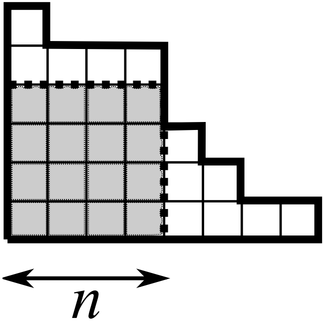
Is there an underlying combinatorial structure?

Outline

Ferrers Diagrams



a Ferrers diagram with Durfee square of size n

Why Ferrers Diagrams?

The generating function $G(x, y, q)$ of Ferrers diagrams with n-th largest row having length n for some positive integer n, enumerated with respect to width (x), height (y), and total area (q), is given by

$$G(x, y, q) = \sum_{n=1}^{\infty} \frac{(xy)^n q^{n^2}}{(qy; q)_n (qx; q)_{n-1}}$$

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Compare with

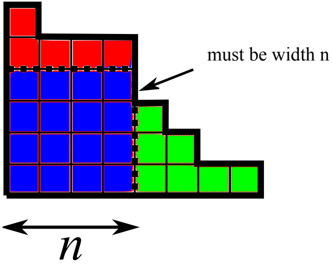
$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q; q)_n (qx; q)_{n-1}}$$

to get

$$x = 1 + G(x, 1, q)$$

Enumerating Ferrers Diagrams

$$G(x,y,q)=\sum_{n=1}^{\infty}(xy)^n q^{n^2}\frac{1}{(yq;q)_n}\frac{1}{(xq;q)_{n-1}}$$



The sum is over all sizes n of Durfee squares

Trees Decorated with Ferrers Diagrams

The functional equation

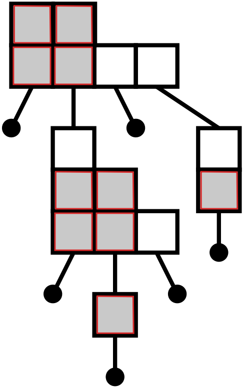
$$x = 1 + G(x, 1, q)$$

admits a combinatorial interpretation using the "theory of species"

Theorem (TP, 2012)

Let F_q be the species of Ferrers diagrams with n-th largest row having length n for some integer n, weighted by area (q), with size given by the width of the Ferrers diagram, augmented by the 'empty polyomino'. Then x enumerates F_q-enriched rooted trees (trees decorated such that the out-degree of the vertex matches the width of the Ferrers diagram) with respect to the total area of the Ferrers diagrams at the vertices of the tree.

Trees Decorated with Ferrers Diagrams

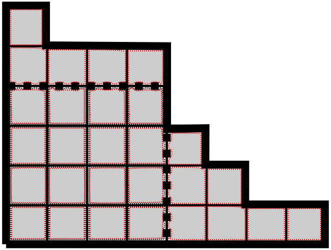


Tree with area 15, contributing q^{15} to the Partial Theta Function root

The Same Trees, But Different Area Weights

Partial Theta Function (TP, 2012)

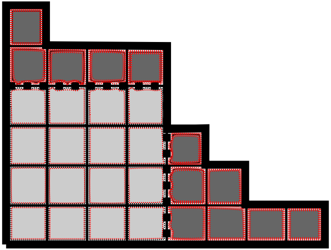
$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q;q)_n (qx;q)_{n-1}}$$



The Same Trees, But Different Area Weights

Ramanujan Function (TP, 2013)

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} x^n}{(q^2;q^2)_n (q^2 x;q^2)_{n-1}}$$

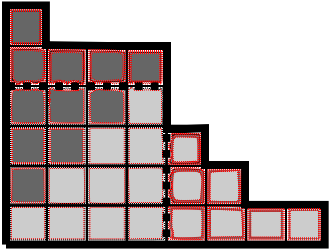


Count dark area twice

The Same Trees, But Different Area Weights

Painlevé Airy Function (TP, 2013)

$$x = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+\binom{n}{2}} x^n}{(q^2;q^2)_n (qx;q)_{n-1}}$$



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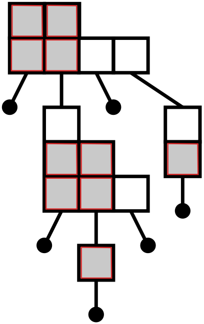
Trees Decorated with Ferrers Diagrams



- contributing q^{15} to the Partial Theta Function root
- contributing q^{20} to the Ramanujan Function root
- contributing q^{19} to the Painlevé Airy Function root

Outline

Why are Θ, A_q, Ai_q special?

Can we understand and generalise?

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These look similar, but their proofs have nothing in common!

Higher Roots for $A_q(x)$

Let $x = y/q^{2m}$ in

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Numerically, the m-th root seems to satisfy

$$\frac{1}{q^{2m+1}} (1 + (-1)^m q^{m+1} (\text{positive terms}))$$

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