MAS115 Calculus I
Week 3

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Revision: Composition, Scaling, Trigonometry

- Composition of functions
  Remember: $(f \circ g)(x)$ is different from $(f \cdot g)(x)$
- Scaling of functions: transform graph of

$$y = f(x)$$

  to graph of

$$y = cf(ax + b) + d$$

- Trigonometric functions
  - Reading Assignment: Chapter 1.6

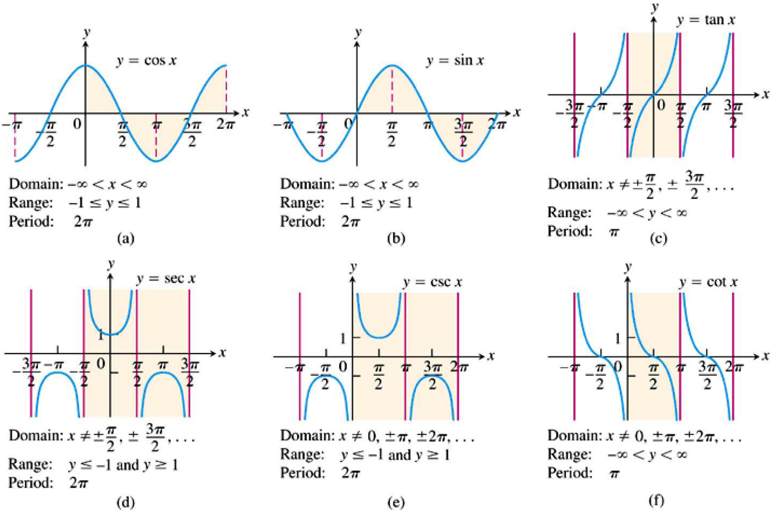
Periodic functions

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

$$\sin(\theta + 2\pi) = \sin\theta$$
$$\cos(\theta + 2\pi) = \cos\theta$$
$$\tan(\theta + \pi) = \tan\theta$$

Graphs of trigonometric functions



Trigonometric identities

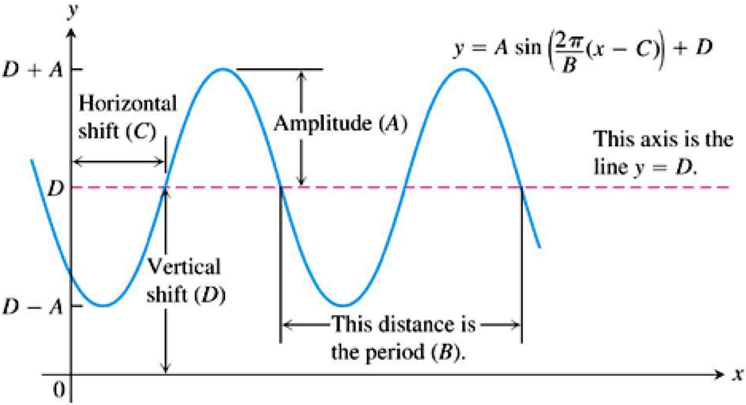
Read on your own:

- Symmetries
- Special values
- Addition formulae
- Double-angle and half-angle formulae
- Law of sines
- Law of cosines

Relevant for exercise class ...

Shifting and scaling of trigonometric functions

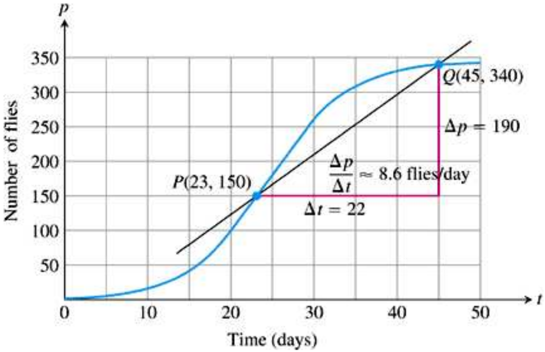
$$f(x) = A\sin\left[\frac{2\pi}{B}(x-C)\right] + D$$



Limits

Average rate of change

Growth of a fruit fly population

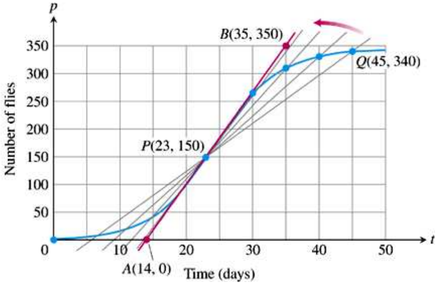


- Average rate of change over 22 days (day 23 to day 45)?
- Growth rate on day 23?

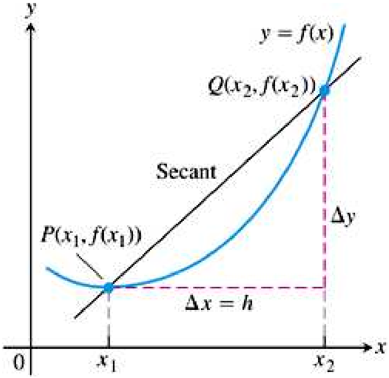
Average rate of change

Growth of a fruit fly population

Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$



Average rate of change



DEFINITION **Average Rate of Change over an Interval**

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Animation!

Limits

To move from

- average rates of change

to

- instantaneous rates of change

we first of all need to consider

limits

Informal definition of a limit

Definition (Limit (informal))

Let $f(x)$ be defined on an open interval about x_0 *except possibly at x_0 itself*. If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L \,,$$

which is read "the limit of $f(x)$ as x approaches x_0."

Why informal?
What does "arbitrarily close" and "sufficiently close" mean?
We'll deal with that later ...

Example

Consider

$$f(x) = \frac{x^2 - 1}{x - 1}$$

with $x_0 = 1$

- $f(x)$ is not defined for $x_0 = 1$
- we can simplify for $x \neq 1$

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$$

- this suggests that

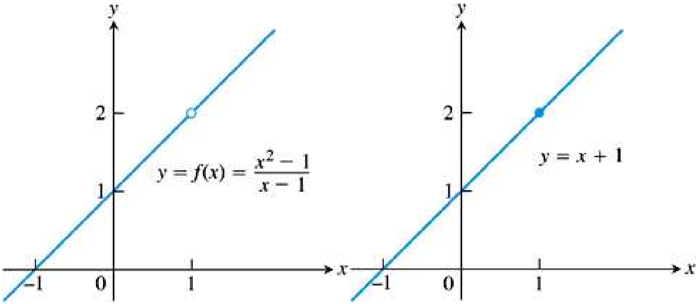
$$\lim_{x \to 1} f(x) = 1 + 1 = 2$$

Example

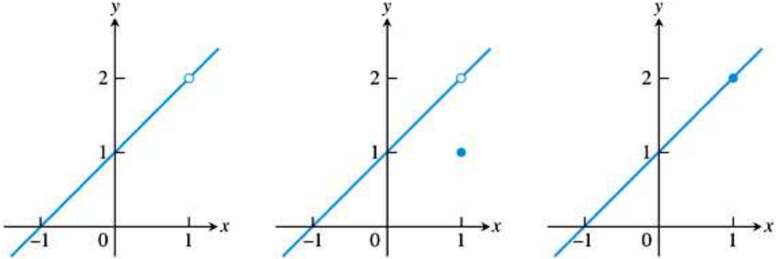
TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Example

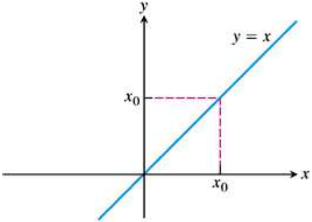


$f(x_0)$ is irrelevant!

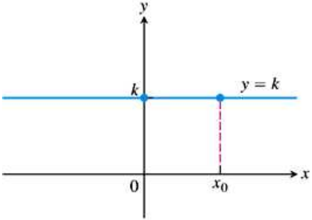


(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Limits at every point

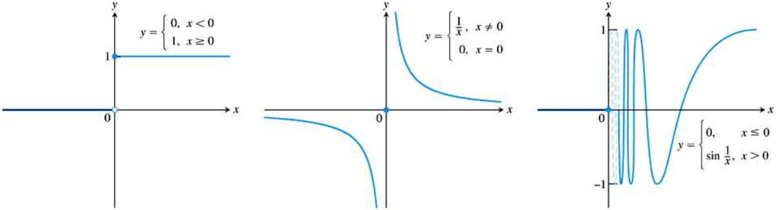


(a) Identity function



(b) Constant function

Limits can fail to exist



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

values that jump, become "larger", or oscillate rapidly

Revision

- Periodic functions
- Average rate of change
- Limits

Limit laws

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x\to c}(f(x) + g(x)) = L + M$$
The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x\to c}(f(x) - g(x)) = L - M$$
The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$$
The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x\to c}(k\cdot f(x)) = k\cdot L$$
The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$
The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then
$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$
provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)
The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Examples

- $\lim_{x\to c}(x^3-4x-2)=c^3-4c-2$
- $\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}=\frac{c^4+c^2-1}{c^2+5}$
- $\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{4(-2)^2-3}=\sqrt{13}$

So sometimes you can just "plug in the value of x"

Some consequences

THEOREM 2 **Limits of Polynomials Can Be Found by Substitution**

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 **Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero**

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Eliminating a common factor

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$$

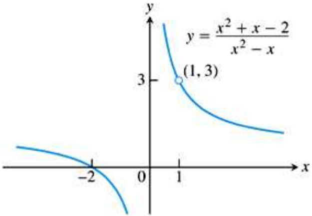
- substitution of $x = 1$?
- algebraic simplification:

$$\frac{x^2+x-2}{x^2-x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x}$$

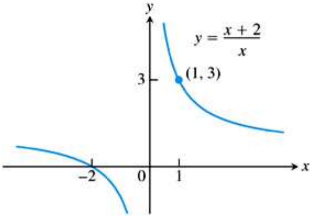
- therefore

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = 3$$

Eliminating a common factor



(a)



(b)

Creating and cancelling a common factor

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

- substitution of $x = 0$?
- algebraic simplification (trick):

$$\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{\sqrt{x^2+100}-10}{x^2}\,\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \\
&= \frac{(x^2+100)-100}{x^2(\sqrt{x^2+100}+10)} \\
&= \frac{1}{\sqrt{x^2+100}+10}
\end{aligned}$$

- therefore

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}$$

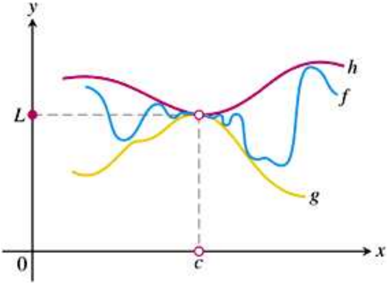
The Sandwich Theorem

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.



Application

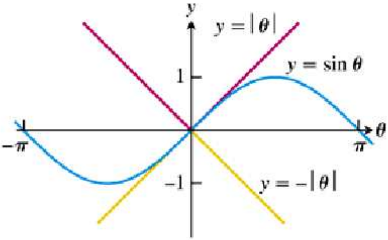
Show that $\lim_{\theta\to 0}\sin\theta = 0$

- From the definition of $\sin\theta$ it follows that

$$-|\theta| \le \sin\theta \le |\theta|$$

- We have $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$
- Using the sandwich theorem, we therefore conclude

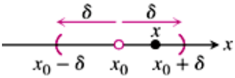
$$\lim_{\theta\to 0}\sin\theta = 0$$



(a)

Trying to be more precise

- We have used informal phrases such as "sufficiently close" — what do they mean?
- A picture might help ...



- Let's be precise: instead of

  "For x sufficiently close to x_0 ..."

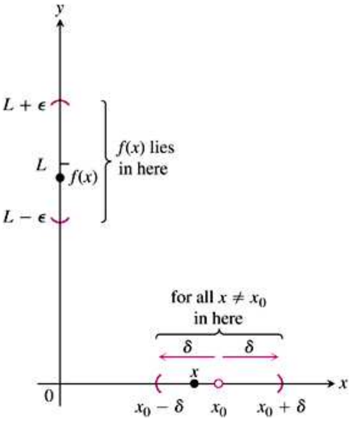
  write

  "There is a $\delta > 0$ such that for all $0 < |x - x_0| < \delta$ ..."

Revisiting the definition

Let $f(x)$ be defined on an open interval about x_0 *except possibly at* x_0 *itself*. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L\ .$$



The precise definition of a limit

DEFINITION **Limit of a Function**

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Animation!

Revision

- Limit laws
- Some useful "tricks"
- $\epsilon - \delta$ definition of limit

The precise definition of a limit

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

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Animation!

How to find δ for a given ϵ

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

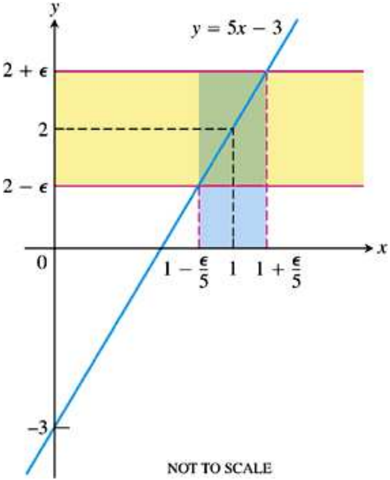
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

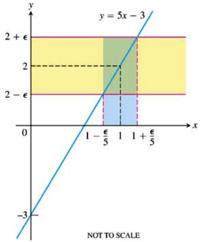
Example

Show that $\lim_{x\to1}(5x-3)=2$:



Example

Show that $\lim_{x\to 1}(5x-3) = 2$:



1. $|f(x) - L| < \epsilon$:

$$|(5x-3)-2| < \epsilon \quad \Leftrightarrow \quad |5x-5| < \epsilon$$
$$\Leftrightarrow \quad |x-1| < \frac{1}{5}\epsilon$$

Therefore

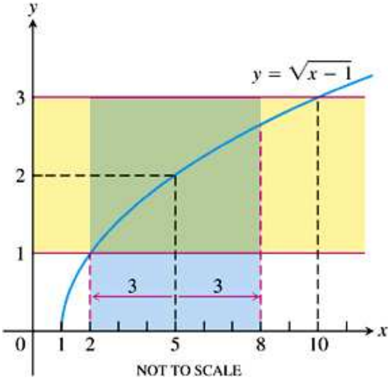
$$(a,b) = (1-\frac{\epsilon}{5}, 1+\frac{\epsilon}{5})$$

2. Find δ: Choose $\delta = \frac{1}{5}\epsilon$. Then

$$(1-\delta, 1+\delta) = (1-\frac{\epsilon}{5}, 1+\frac{\epsilon}{5})$$

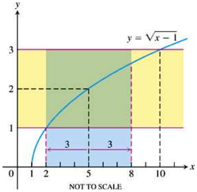
Example

Find a $\delta > 0$ such that $|\sqrt{x-1} - 2| < 1$ for all $0 < |x-5| < \delta$:



Example

Find a $\delta > 0$ such that $|\sqrt{x-1} - 2| < 1$ for all $0 < |x-5| < \delta$:



1. $|f(x) - L| < \epsilon$:

$$\begin{aligned} |\sqrt{x-1} - 2| < \epsilon \quad &\Leftrightarrow \quad 1 < \sqrt{x-1} < 3 \\ &\Leftrightarrow \quad 2 < x < 10 \end{aligned}$$

   Therefore

$$(a, b) = (2, 10)$$

2. Find δ: Choose $\delta = 3$. Then

$$(5 - \delta, 5 + \delta) = (2, 8) \subset (2, 10)$$

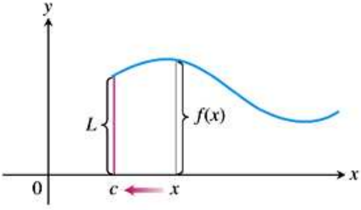
One-sided limits

- To have a *limit* L as $x \to c$, a function f must be defined on both sides of c (two-sided limit)
- If f fails to have a limit as $x \to c$, it may still have a one-sided limit if the approach is only from the right (*right-hand limit*) or from the left (*left-hand limit*)
- We write

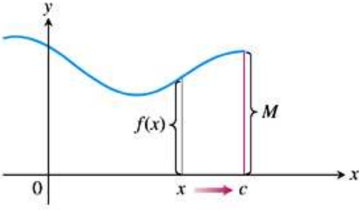
$$\lim_{x\to c^+} f(x) = L \quad \text{or} \quad \lim_{x\to c^-} f(x) = M$$

- The symbol $x \to c^+$ means that we only consider values of x greater than c. The symbol $x \to c^-$ means that we only consider values of x less than c.

One-sided limits and limits at infinity

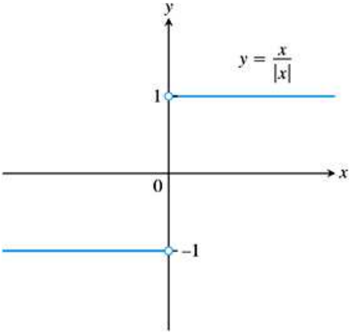


(a) $\lim_{x \to c^+} f(x) = L$



(b) $\lim_{x \to c^-} f(x) = M$

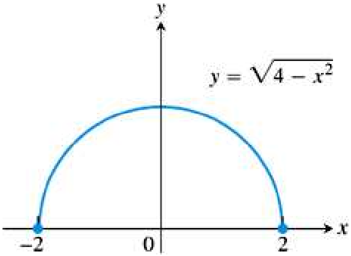
Examples



$$f(x) = \frac{x}{|x|}$$

- $\lim_{x\to 0^+} f(x) = 1$
- $\lim_{x\to 0^-} f(x) = -1$
- $\lim_{x\to 0} f(x)$
  does not exist

Examples



$$f(x) = \sqrt{4 - x^2}$$

- $\lim_{x \to 2^-} f(x) = 0$
- $\lim_{x \to 2^+} f(x)$
  does not exist
- $\lim_{x \to -2^-} f(x)$
  does not exist
- $\lim_{x \to -2^+} f(x) = 0$

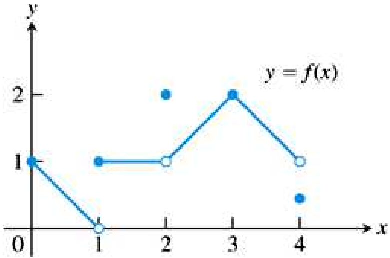
Connection between limits and one-sided limits

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example



c	$\lim_{x\to c} f(x)$	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$
0	n.a.	n.a.	1
1	n.a.	0	1
2	1	1	1
3	2	2	2
4	n.a.	1	n.a.

Precise definitions of one-sided limits

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

Limit laws, theorems for limits of polynomials and rational functions, and the sandwich theorem all hold for one-sided limits.