MAS115 Calculus I

Week 12

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Revision

- Improper Integrals
- Tests for Convergence/Divergence of Integrals

Polar Coordinates

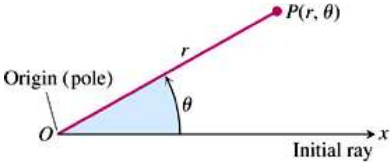
How can we describe a point P in the plane?

- give x and y coordinates:

$$(x, y) \qquad \text{Cartesian coordinates}$$

- Alternatively, we could decide to give

$$(r, \theta) \qquad \text{polar coordinates}$$

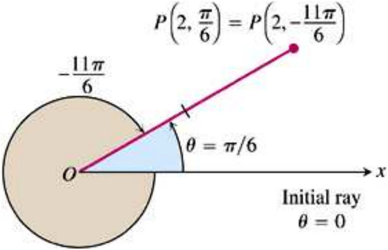


- r: the distance from the origin, O
- θ: the angle between OP and the positive x-direction

Polar Coordinates

A slight complication: while Cartesian coordinates are unique, polar coordinates are not!

- the angle θ can vary by multiples of 2π



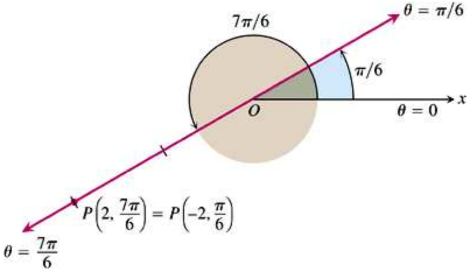
$$(r, \theta) = (r, \theta + 2\pi)$$

- if $r = 0$, the angle θ can assume any value

Polar Coordinates

A slight complication: while Cartesian coordinates are unique, polar coordinates are not!

- We allow negative values for r

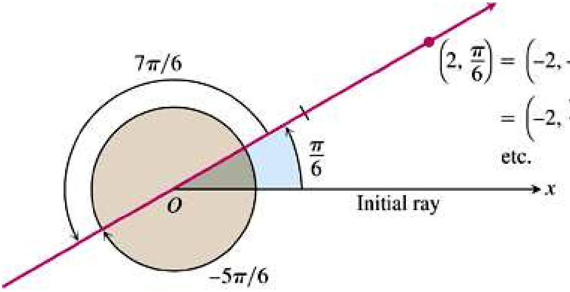


$$(r, \theta) = (-r, \theta + \pi)$$

Note: sometimes negative r is excluded (distances should not be negative), but we will find it useful for calculations.

Example

Find all polar coordinates of the point $(2, \pi/6)$:

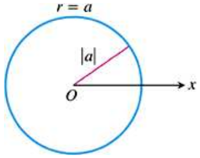


- $r = 2$: $\theta = \pi/6,\ \pi/6 \pm 2\pi,\ \pi/6 \pm 4\pi,\ \pi/6 \pm 6\pi,\ \ldots$
- $r = -2$: $\theta = 7\pi/6,\ 7\pi/6 \pm 2\pi,\ 7\pi/6 \pm 4\pi,\ 7\pi/6 \pm 6\pi,\ \ldots$

Graphing in Polar Coordinates

Some graphs have simple equations in polar coordinates

- a circle about the origin:



$$r = a$$

Note: $r = a$ and $r = -a$ both describe the *same* circle of radius $|a|$.

- a line through the origin:

$$\theta = \theta_0$$

Note: Here it becomes convenient to have allowed negative r. Otherwise the graph of $\theta = \theta_0$ would only be a ray ending at the origin.

Inequalities in Polar Coordinates

Example: find the graphs of

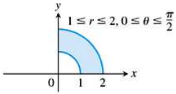
(a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/2$

(b) $-3 \leq r \leq 2$ and $\theta = \pi/4$

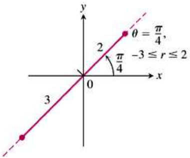
(c) $r \leq 0$ and $\theta = \pi/4$

(d) $2\pi/3 \leq \theta \leq 5\pi/6$

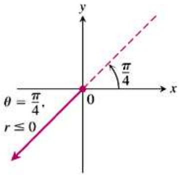
(a)



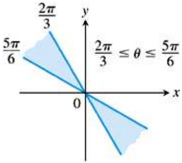
(b)



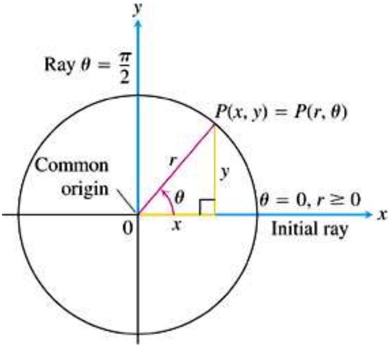
(c)



(d)



Relating Polar and Cartesian Coordinates

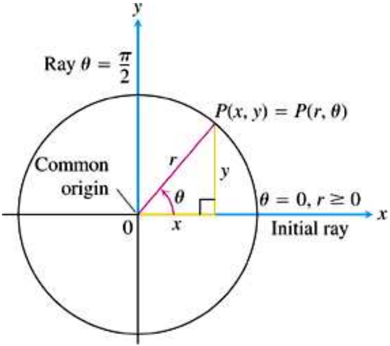


Converting polar coordinates to Cartesian coordinates:

$$x = r\cos\theta\,, \quad y = r\sin\theta$$

- given (r, θ), we can uniquely compute (x, y)

Relating Polar and Cartesian Coordinates



Converting Cartesian coordinates to polar coordinates:

$$r^2 = x^2 + y^2 \ , \quad \tan\theta = y/x$$

- given (x, y), we have to choose one of many polar coordinates.

Usual convention: $r \geq 0$ and $0 \leq \theta < 2\pi$
(if $r = 0$, choose also $\theta = 0$ for uniqueness)

Equivalent Polar and Cartesian Equations

Examples:

$$
\begin{array}{rlrl}
\text{polar:} & & \text{Cartesian:} & \\
r\cos\theta = & 2 & x = & 2 \\
r^2\cos\theta\sin\theta = & 4 & xy = & 4 \\
r^2\cos 2\theta = & 1 & y^2 = & x^2 - 1 \\
r(1 - 2\cos\theta) = & 1 & y^2 = & (x+1)(3x+1) \\
r + \cos\theta = & 1 & (x^2+y^2)^2 = & 2x(y^2 - x^2)
\end{array}
$$

Sometimes, polar coordinates are a lot simpler!

Converting Between Polar and Cartesian Equations

- Cartesian to polar

$$
\begin{aligned}
& x^2+(y-3)^2 = && 9 \\
\Leftrightarrow\quad & (x^2+y^2)-6y+9 = && 9 \\
\Leftrightarrow\quad & r^2-6r\sin\theta = && 0 \\
\Leftrightarrow\quad & r=0 \text{ or } && r=6\sin\theta
\end{aligned}
$$

  Therefore $r=6\sin\theta$ describes a circle centred at $(0,3)$ with radius 3.

- Polar to Cartesian

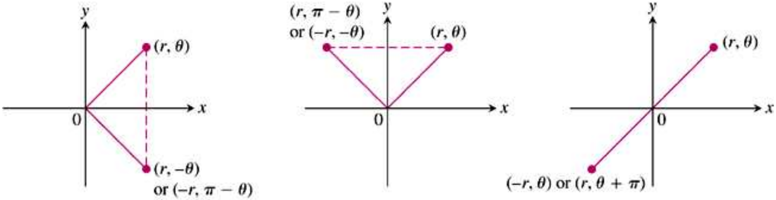
$$
r=\frac{4}{2\cos\theta-\sin\theta}
$$

  is equivalent to $2r\cos\theta-r\sin\theta=4$ or $2x-y=4$. We therefore have the equation of a line

$$
y=2x-4
$$

Symmetry in Polar Coordinates

Tests for Symmetry



(a) About the x-axis (b) About the y-axis (c) About the origin

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph

The Slope of a Polar Curve

Given $r = f(\theta)$, compute the slope of the curve:

- The slope is still dy/dx, so think of x and y as given by the *parameter* θ:

$$\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}$$

- Therefore

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

  with

$$\begin{aligned} dx/d\theta &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ dy/d\theta &= f'(\theta)\sin\theta + f(\theta)\cos\theta \end{aligned}$$

Graphing a Polar Curve

Graph $r = 1 - \cos\theta$:

- Symmetry: $\cos\theta = \cos(-\theta)$ so both (r, θ) and $(r, -\theta)$ are on the curve:

  The curve is symmetric about the x-axis

- Monotonicity: $\cos\theta$ is monotonically decreasing on $[0, \pi]$:

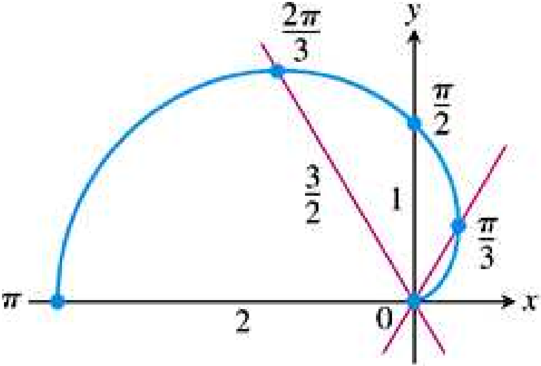
  As θ increases from 0 to π
  $r = 1 - \cos\theta$ increases from 0 to 2

- A small table of values:

θ :	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r = 1 - \cos\theta$:	0	$1/2$	1	$3/2$	2

Graphing a Polar Curve

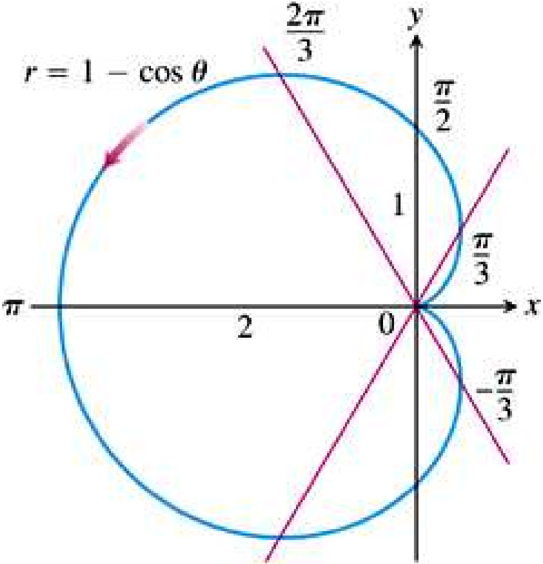
Use symmetry and monotonicity, start with table of values



θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

Graphing a Polar Curve

Use symmetry and monotonicity, start with table of values



θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

Graphing a Polar Curve

Find horizontal and vertical tangents to $r = f(\theta) = 1 - \cos\theta$:

- Recall $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ with

$$\begin{aligned} dx/d\theta &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ dy/d\theta &= f'(\theta)\sin\theta + f(\theta)\cos\theta \end{aligned}$$

- Compute

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta} \\ &= \frac{1-\cos\theta}{\sin\theta} \cdot \frac{1+2\cos\theta}{1-2\cos\theta} \end{aligned}$$

- Horizontal tangents at

$$\theta = 0\ , \quad \theta = \pm\frac{2}{3}\pi$$

- Vertical tangents at

$$\theta = \pi\ , \quad \theta = \pm\frac{1}{3}\pi$$

The End