MAS115 Calculus I

Week 10

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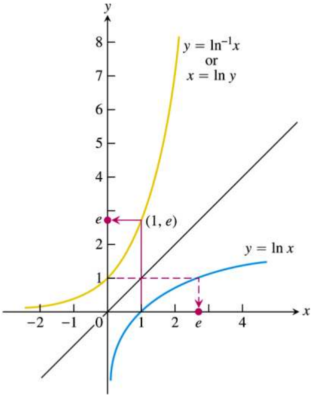
Revision

- One-to-One Functions
- Inverse Functions
- Derivatives of Inverse Functions
- Natural Logarithm: $\ln x = \int_1^x \frac{dt}{t}$
- Properties of $\ln x$
- Use of $\ln x$ for Integration

The Exponential Function

- $\ln x$ has domain $\mathbb{R}^+$ and range $\mathbb{R}$
- $\ln x$ is strictly increasing, therefore invertible



Definition (Exponential Function)

For every $x \in \mathbb{R}$, $\exp x = \ln^{-1} x$.

Exponential Function and Real Powers

- $1 = \ln e$, so that $\exp 1 = e$
- $r = \ln(e^r)$, so that $\exp r = e^r$ (for $r \in \mathbb{Q}$)

$\exp x$ is defined for real x, but so far we have only dealt with rational powers. For base e, it makes now sense to introduce real exponents:

Definition

For every $x \in \mathbb{R}$, $e^x = \exp x$ $(= \ln^{-1} x)$.

- $a = \ln(\exp a)$, and, for $a > 0$, $a = \exp(\ln a)$

We also define real powers of positive real numbers a:

Definition

For every $x \in \mathbb{R}$ and $a > 0$, $a^x = \exp(x \ln a)$.

Properties of the Exponential Function

$e^x = \exp(x)$ obeys the familiar laws of exponents:

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

Proof of 1.:

$$\begin{aligned}
\exp(x_1)\cdot\exp(x_2) &= \exp\ln(\exp(x_1)\cdot\exp(x_2)) \\
&= \exp(\ln\exp(x_1)+\ln\exp(x_2)) \\
&= \exp(x_1+x_2)\,.
\end{aligned}$$

□

Differentiating and Integrating $\exp x$

As $e^x = f^{-1}(x)$ with $f(x) = \ln x$ and $f'(x) = 1/x$, we find

$$\frac{d}{dx}e^x = \frac{1}{f'(f^{-1}(x))} = f^{-1}(x) = e^x \ .$$

Therefore

$$\frac{d}{dx}e^x = e^x$$
$$\int e^x dx = e^x + C$$

By the chain rule,

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

so that also

$$\int e^{f(x)}f'(x)dx = e^{f(x)} + C \, .$$

Example

Solve the initial value problem

$$y' = e^{-y}2x \text{ for } x > \sqrt{3} \text{ with } y(2) = 0:$$

- Rewrite $e^y y' = 2x$ and integrate both sides:

$$e^y = x^2 + C$$

- Determine C from $y(2) = 0$:

$$e^0 = 2^2 + C \quad \Rightarrow \quad C = -3.$$

- Take logarithms to get

$$y = \ln(x^2 - 3)$$

which is valid for $x > \sqrt{3}$.

What is e?

We defined e via $\ln e = 1$, and gave $e = 2.718281828459\ldots$.

Theorem (The Number e as a Limit)

$$e = \lim_{x \longrightarrow 0} (1+x)^{1/x}$$

Proof.

$$\begin{aligned}
\ln\left(\lim_{x\longrightarrow 0}(1+x)^{1/x}\right) &= \lim_{x\longrightarrow 0}\left(\ln(1+x)^{1/x}\right) \\
&= \lim_{x\longrightarrow 0}\left(\frac{1}{x}\ln(1+x)\right) \\
&= \lim_{x\longrightarrow 0}\frac{\ln(1+x)-\ln(1)}{x} \\
\text{let } f(t)=\ln t: \quad &= \lim_{h\longrightarrow 0}\frac{f(1+h)-f(1)}{h} \\
&= f'(1) = 1 = \ln(e)
\end{aligned}$$

$\square$

General Exponential Functions

Consider base $a > 0$.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a \,,$$

so that

$$\frac{d}{dx}a^x = a^x \ln a$$
$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Examples:

$$\begin{aligned}
\frac{d}{dx}2^x &= 2^x \ln 2 \quad \text{with } \ln 2 \approx 0.6931 \\
\int_0^1 3^x dx &= \left.\frac{3^x}{\ln 3}\right|_0^1 = \frac{2}{\ln 3} \quad \text{with } \ln 3 \approx 1.0986 \\
\frac{d}{dx}x^x &= \frac{d}{dx}e^{x \ln x} = e^{x \ln x}\frac{d}{dx}(x \ln x) = x^x(1 + \ln x)
\end{aligned}$$

General Exponential Functions

The inverse of $y = a^x$ is

$$\log_a x \text{ , the logarithm of } x \text{ with base } a,$$

provided $a > 0$ and $a \neq 1$.

- $x = \log_a(a^x)$ and, for $x > 0$, $x = a^{\log_a x}$
- $\ln x = \ln\left(a^{\log_a x}\right) = \ln\left(e^{\log_a x \cdot \ln a}\right) = \log_a x \cdot \ln a$

We therefore have

$$\log_a x = \frac{\ln x}{\ln a}$$

For calculations, always express $\log_a$ in terms of ln before differentiating or integrating.

Example:

$$\log_2 3 = \frac{\ln 3}{\ln 2} \approx \frac{1.0986}{0.6931} \approx 1.585$$

Indeterminate Powers

Just like "0/0", the forms "0^0", "1^∞", and "∞^0" are also indeterminate. We handle them using logarithms.

- Compute $\lim_{x\to 0^+} x^x$:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

  and therefore $\lim_{x\to 0^+} x^x = e^0 = 1$.

- Compute $\lim_{x\to 0+}(1+x)^{1/x}$:

$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{1/(1+x)}{1} = \lim_{x\to 0^+} \frac{1}{1+x} = 1$$

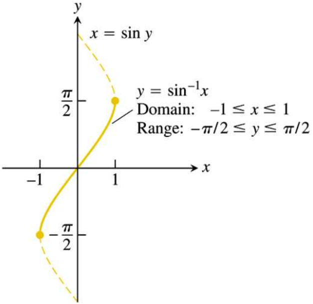
  and therefore $\lim_{x\to 0^+}(1+x)^{1/x} = e^1 = e$.

Revision

- Exponential Function $\exp x = \ln^{-1} x$
- $\exp x = e^x$, $e = 2.71828\ldots$
- Differentiating and Integrating $\exp x$
- General Exponential Functions and Logarithms
- Indeterminate Powers

Inverse Trigonometric Functions

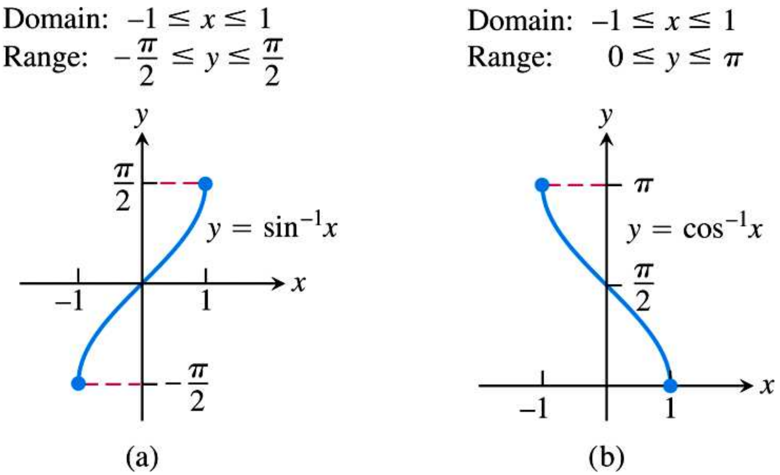
- sin, cos, sec, csc, tan, cot are not one-to-one unless the domain is restricted:



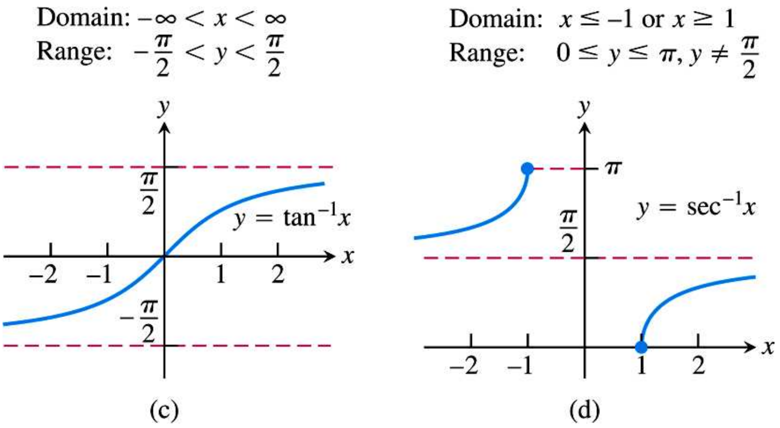
- once the domains are suitably restricted, we can define

$$\arcsin x = \sin^{-1} x \qquad \operatorname{arccsc} x = \csc^{-1} x$$
$$\arccos x = \cos^{-1} x \qquad \operatorname{arcsec} x = \sec^{-1} x$$
$$\arctan x = \tan^{-1} x \qquad \operatorname{arccot} x = \cot^{-1} x$$

Inverse Trigonometric Functions

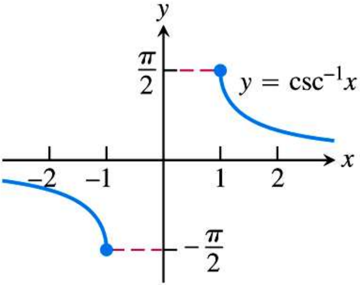


Inverse Trigonometric Functions



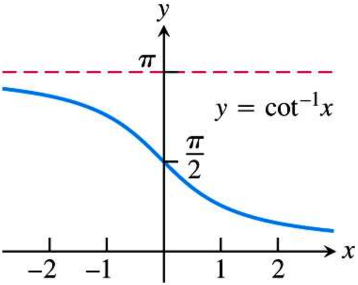
Inverse Trigonometric Functions

Domain: $x \leq -1$ or $x \geq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



(e)

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



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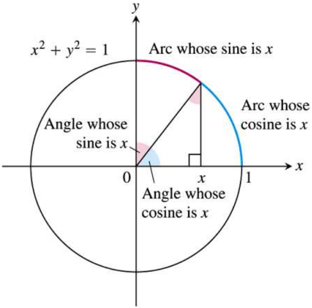
Inverse Trigonometric Functions

Caution: $\sin^{-1} x \neq (\sin x)^{-1}$ (unfortunately this is inconsistent: $\sin^2 x = (\sin x)^2$). Best to avoid $\sin^{-1} x$ and use $\arcsin x$ etc. instead.

The "arc" explained:

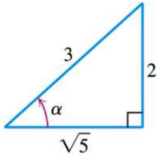


Observe that

$$\arcsin x + \arccos x = \pi/2$$

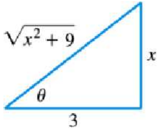
Inverse Trigonometric Functions

- If $\alpha = \arcsin(2/3)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, $\cot\alpha$: Construct right triangle with $\sin\alpha = 2/3$:



  Read off $\cos\alpha = \sqrt{5}/3$, $\tan\alpha = 2/\sqrt{5}$, $\sec\alpha = 3/\sqrt{5}$, $\csc\alpha = 3/2$, $\cot\alpha = \sqrt{5}/2$.

- Find $\sec\arctan(x/3)$: Construct right triangle with $\tan\theta = x/3$:



  Read off $\sec\theta = \sqrt{x^2+9}/3$.

Differentiating $\arcsin x$

- Differentiate $\sin y = x$:

$$\cos y \frac{dy}{dx} = 1 \ .$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

for $-\pi/2 < y < \pi/2$.

Therefore, for $|x| < 1$,

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

and, conversely,

$$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C \ .$$

Similiarly, $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ etc.

Derivatives of Inverse Trigonometric Functions

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1+u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1+u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

Integrals Leading to Inverse Trigonometric Functions

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1.	$\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$	(Valid for $u^2 < a^2$)
2.	$\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$	(Valid for all u)
3.	$\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{u}{a}\right\rvert + C$	(Valid for $\lvert u\rvert > a > 0$)

Example

Find the line tangent to $y = \operatorname{arccot} x$ at $x = -1$:

- $\operatorname{arccot}(-1) = \frac{1}{2}\pi - \arctan(-1) = \frac{1}{2}\pi + \arctan 1 = \frac{1}{2}\pi + \frac{1}{4}\pi = \frac{3}{4}\pi$.
- $\frac{d}{dx} \operatorname{arccot} x\Big|_{x=-1} = -\frac{1}{1+x^2}\Big|_{x=-1} = -\frac{1}{2}$.
- The equation of the line is

$$y = \frac{3\pi}{4} - \frac{1}{2}(x+1)$$

Evaluate $\int_0^1 \frac{dx}{1+x^2}$:

- We have

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^2} &= \arctan x\Big|_0^1 \\ &= \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4} \end{aligned}$$

Example

Evaluate

$$\int \frac{dx}{\sqrt{4x - x^2}} \ :$$

- Trick: complete the square!

$$4x - x^2 = 4 - (x-2)^2$$

- Now integrate

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x - x^2}} &= \int \frac{dx}{\sqrt{4 - (x-2)^2}} \\
\text{let } u = x - 2: \quad &= \int \frac{du}{\sqrt{4 - u^2}} \\
&= \arcsin \frac{u}{2} + C \\
&= \arcsin \left(\frac{x}{2} - 1 \right) + C
\end{aligned}$$

Example

Evaluate

$$\int \frac{dx}{4x^2+4x+2} \quad :$$

- Trick: complete the square!

$$4x^2+4x+2=(2x+1)^2+1$$

- Now integrate

$$\begin{aligned}
\int \frac{dx}{4x^2+4x+2} &= \int \frac{dx}{(2x+1)^2+1} \\
\text{let } u=2x+1: \quad &= \int \frac{\frac{1}{2}du}{u^2+1} \\
&= \frac{1}{2}\arctan u + C \\
&= \frac{1}{2}\arctan(2x+1) + C
\end{aligned}$$

Hyperbolic Functions

Split $\exp x$ into even and odd part:

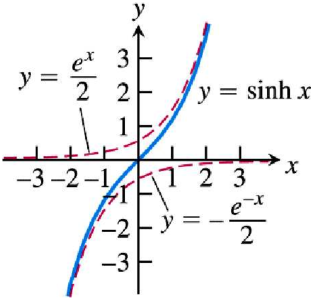
$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even function}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd function}}$$

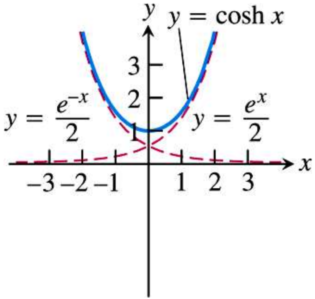
We define the "hyperbolic sine" and "hyperbolic cosine" as

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

and define tanh, coth, sech, and csch in analogy to trigonometric functions.

Hyperbolic Sine and Cosine

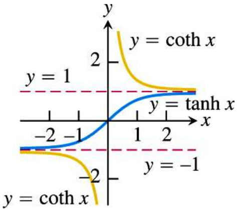


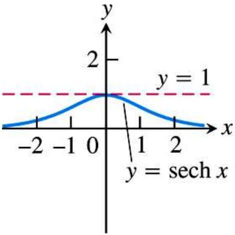


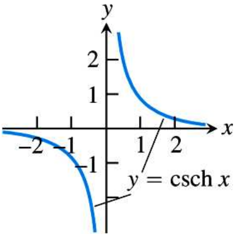
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

tanh, coth, sech, and csch







$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

Identities for Hyperbolic Functions

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Similarities between trigonometric and hyperbolic functions are no accident (but an explanation needs complex numbers and complex functions).

Revision

- Inverse Trigonometric Functions
- Differentiating and Integrating
- Trick: Completing the Square
- Hyperbolic Functions

Derivatives of Hyperbolic Functions

Formulas for derivatives follow directly from the definition:

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

Example

- Find $\int_0^1 \sinh^2 x\,dx$:

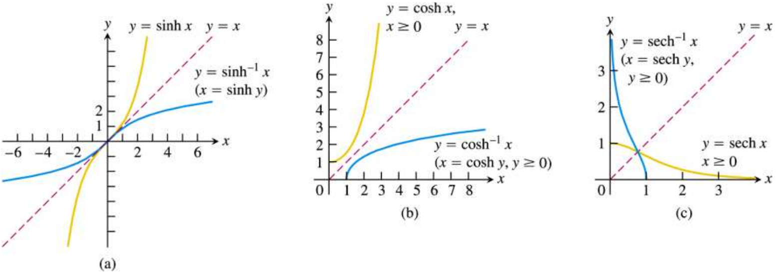
$$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} = \frac{1}{8}(e^2 - e^{-2}) - \frac{1}{2} \approx 0.40672$$

- Find $\int_0^{\ln 2} 4e^x \sinh x\,dx$:

$$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} dx$$
$$= \int_0^{\ln 2} (2e^{2x} - 2)dx = \left[e^{2x} - 2x\right]_0^{\ln 2}$$
$$= e^{2\ln 2} - 2\ln 2 - 1 = 2^2 - 2\ln 2 - 1 \approx 1.6137$$

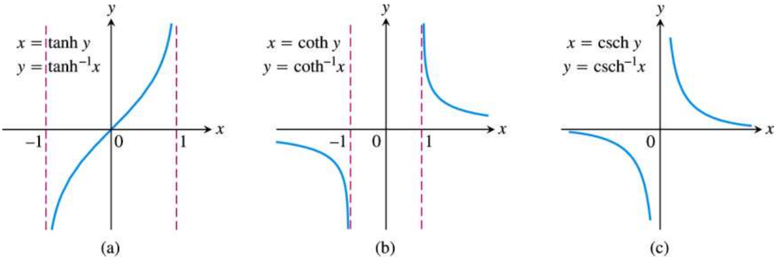
Inverse Hyperbolic Functions

As with trigonometric functions, restrict the domains and invert:



Inverse Hyperbolic Functions

As with trigonometric functions, restrict the domains and invert:



Differentiating Inverse Hyperbolic Functions

As with inverse trigonometric functions, compute:

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \qquad u \neq 0$$

Integrals Leading to Inverse Hyperbolic Functions

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Example

Find $\int_0^1 \frac{2dx}{\sqrt{3+4x^2}}$:

- Substitute $u = 2x$:

$$\int \frac{2dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{3+u^2}}$$

- Use $\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}(u/a) + C$:

$$\int \frac{2dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

- Now compute the definite integral:

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg|_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) \approx 0.98665$$

Techniques of Integration

- Basic properties (Thomas' Calculus, Chapter 5)
- Rules (substitution, integration by parts - today)
- Basic formulas, integration tables (Thomas' Calculus, pages T1-T6)
- Procedures to simplify integrals (bag of tricks, methods)

this needs practice, practice, practice, ...:
exerciseclass 9, coursework 10/11, and end-of-term test

Integration Tables

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

Integration Tricks

Procedures for Matching Integrals to Basic Formulas

Procedure	**Example**
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x+\tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x+\tan x}{\sec x+\tan x}$ $= \dfrac{\sec^2 x + \sec x\tan x}{\sec x+\tan x}$

Integration by Parts

Differentation $\longleftrightarrow$ Integration

- chain rule $\longleftrightarrow$ substitution

$$\int f(g(x))g'(x)dx = \int f(u)du \ , \quad u = g(x)$$

- product rule $\longleftrightarrow$?

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Integrate:

$$\int \frac{d}{dx}(f(x)g(x))\,dx = \int \left(f'(x)g(x) + f(x)g'(x)\right)dx$$

Therefore

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

Example

Evaluate

$$\int x \cos x \, dx \quad :$$

Choose

$$u = x \,, \qquad dv = \cos x \, dx$$

then

$$du = dx \,, \qquad v = \sin x$$

and

$$\int u \, dv = uv - \int v \, du$$

gives

$$\begin{aligned} \int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

Example

Are there other choices of u and dv for

$$\int x \cos x \, dx \ ?$$

Some choices are

1. $u = 1$ and $dv = x \cos x \, dx$
2. $u = x$ and $dv = \cos x \, dx$
3. $u = \cos x$ and $dv = x \, dx$
4. $u = x \cos x$ and $dv = dx$

Which one should we choose?

- $u = 1$ and $dv = x \cos x \, dx$:
  Computing v is the same as the original problem: no good!
- $u = x$ and $dv = \cos x \, dx$:
  Done above, works!

Example

Other choices of u and dv for

$$\int x \cos x \, dx \quad :$$

- $u = \cos x$ and $dv = x\,dx$:
  Now $du = -\sin x \, dx$ and $v = x^2/2$, so that

  $$\int x \cos x \, dx = \frac{1}{2}x^2 \cos x + \int \frac{1}{2}x^2 \sin x \, dx$$

  This makes the situation worse!
- $u = x \cos x$ and $dv = dx$:
  Now $du = (\cos x - x \sin x)dx$ and $v = x$, so that

  $$\int x \cos x \, dx = x^2 \cos x - \int x(\cos x - x \sin x)dx$$

  This again is worse!

Integration by Parts

General advice:

- Choose dv to have "as much of the integrand as possible", provided you can compute v.
- If it looks more complicated after doing integration by parts, it's most likely not right. Try something else.
- Remember: generally

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

Example

Evaluate

$$\int \ln x \, dx \ :$$

Choose $u = \ln x$ and $dv = dx$, so that

$$du = \frac{1}{x} dx \ , \quad v = x$$

Integrate by parts:

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int dx = x \ln x - x + C \end{aligned}$$

Could we have obtained this by guessing?

$$\frac{d}{dx}(x \ln x) = 1 \cdot \ln x + x \frac{1}{x} = \ln x + 1 = \ln x + \frac{d}{dx} x$$

Well, maybe ...

The End