ROOK PLACEMENTS IN YOUNG DIAGRAMS AND PERMUTATION ENUMERATION

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Abstract. Given two operators M and N subject to the relation $MN - qNM = p$, and a word w in M and N, the rewriting of w in normal form is combinatorially described by rook placements in a Young diagram. We give enumerative results about these rook placements, particularly in the case where $p = (1-q)/q^2$. This case naturally arises in the context of the PASEP, a random process whose partition function and stationary distribution are expressed using two operators D and E subject to the relation $DE - qED = D + E$ (matrix Ansatz). Using the link obtained by Corteel and Williams between the PASEP, permutation tableaux and permutations, we prove a conjecture of Corteel-Rubey about permutation enumeration. This result gives the generating function for permutations of given size with respect to the number of ascents and occurrences of pattern 13-2, this is also the moments of the q-Laguerre orthogonal polynomials.

1. Introduction

In the recent work of Postnikov [11], permutations were showed with a quite new description, as pattern-avoiding fillings of Young diagrams. More precisely, he made a correspondance between positive grassman cells, these pattern-avoiding fillings called Ꞁ-diagrams, and decorated permutations (which are permutations with a weight 2 on each fixed point). In particular, the usual permutations are in bijection with permutation tableaux, a subclass of Ꞁ-diagrams. Permutation tableaux have then been studied by Steingrìmsson, Williams, Burstein, Corteel, Nadeau [1, 5, 6, 13], and revealed themselves very useful for working on permutations.

Rather surprisingly, Corteel and Williams observed, and explained, a link between these permutation tableaux and the steady distribution of a classical process of statistical physics, the Partially Asymmetric Self-Exclusion Process (PASEP). This model is described in [7, 6]. A statement doing this link is that the steady probability of a given state in the process is proportional to the sum of weights of permutation tableaux of a given shape. The factor behind this proportionality is the partition function, which is the sum of weights of permutation tableaux of a given half-perimeter.

Another way of finding the steady distribution of the PASEP is the matrix ansatz [7]. Suppose that we have operators D and E, a row vector $\langle W|$ and a column vector $|V\rangle$ such that:

$$DE - qED = D + E, \qquad \langle W|E = \langle W|, \qquad D|V\rangle = |V\rangle, \qquad \text{and} \quad \langle W||V\rangle = 1.$$

Then, coding any state of the process by a word w of length n in D and E, the probability of the state w is given by $\langle W|w|V\rangle$, divided by the factor $\langle W|(D+E)^n|V\rangle$ which is the partition function.

We briefly describe how the matrix ansatz is related to permutation tableaux [6]. First, notice that there are unique polynomials $n_{i,j} \in \mathbb{Z}[q]$ such that:

$$(D+E)^n = \sum_{i,j\geq 0} n_{i,j} E^i D^j$$

This sum is called the normal form of $(D+E)^n$. It is particularly useful, since for example the sum of coefficients $n_{i,j}$ give an evaluation of $\langle W|(D+E)^n|V\rangle$. If D and E would commute, the expansion of $(D+E)^n$ would be described with binomial coefficients; but in this non-commutative context, the process of expanding and rewriting $(D+E)^n$ in normal form is combinatorially described by permutation tableaux, and each coefficient $n_{i,j}$ is a generating function for permutation tableaux satisfying certain conditions. Equivalently this can be done with the *alternative tableaux* defined by Viennot [20].

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One of the ideas at the origin of this article is the following. From D and E of the matrix ansatz, we define new operators:

$$\hat{D} = \frac{q-1}{q}D + \frac{1}{q}, \qquad \hat{E} = \frac{q-1}{q}E + \frac{1}{q}.$$

An immediate consequence is that:

$$\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1-q}{q^2}, \qquad \langle W|\hat{E} = \langle W|, \qquad \text{and} \quad \hat{D}|V\rangle = |V\rangle. \tag{1}$$

This new commutation relation is in a way much more simple than the one satisfied by D and E. It is close to the relation between creation and annihilation operators studied by physicists. Moreover from these definitions we have $q(y\hat{D}+\hat{E}) + (1-q)(yD+E) = 1+y$, for any parameter y. By isolating one term of the left-hand side and rising to the n with the binomial rule, we have the following inversion formulas between $(yD+E)^n$ and $(y\hat{D}+\hat{E})^n$:

$$(1-q)^n(yD+E)^n = \sum_{k=0}^{n}\binom{n}{k}(1+y)^{n-k}(-1)^k q^k (y\hat{D}+\hat{E})^k, \quad \text{and} \tag{2}$$

$$q^n(y\hat{D}+\hat{E})^n = \sum_{k=0}^{n}\binom{n}{k}(1+y)^{n-k}(-1)^k(1-q)^k(D+E)^k. \tag{3}$$

In particular, the first formula means that if we want to compute the coefficients of the normal form of $(yD+E)^n$, it is enough to compute the ones of $(y\hat{D}+\hat{E})^n$ for all n (taking the normal form is a linear operation).

Up to a factor which only depends on q, these operators $\hat{D}$ and $\hat{E}$ are also defined in [18] and [2]. In the first reference, Uchiyama, Sasamoto and Wadati used the new relation between $\hat{D}$ and $\hat{E}$ to find explicit matrices for these operators. They derive the eigenvalues and eigenvectors of $\hat{D}+\hat{E}$, and consequently the ones of $D+E$, in terms of orthogonal polynomials. In the second reference, Blythe, Evans, Colaiori and Essler also use these eigenvalues and obtain an integral form for $\langle W|(D+E)^n|V\rangle$. They also provide an exact integral-free formula of this quantity, although quite complicated since it contains three sum signs and several q-binomial coefficients.

In this article, instead of working on representations of $\hat{D}$ and $\hat{E}$ and their eigenvalues, we study the combinatorics of the rewriting in normal form of $(\hat{D}+\hat{E})^n$, and more generally $(y\hat{D}+\hat{E})^n$ for some parameter y. In the case of $\hat{D}$ and $\hat{E}$, the objects that appear are the *rook placements in Young diagrams*, long-known by combinatorists since the results of Kaplansky, Riordan, Goldman, Foata and Schützenberger (see [12] and references therein). This method is described in [19], and is the same that the one leading to permutation tableaux or alternative tableaux in the case of D and E.

Definition 1. *Let λ be a Young diagram. A rook placement of shape λ is a partial filling of the cells of λ with rooks (denoted by a circle $\circ$), such that there is at most one rook per row (resp. per column).*

For convenience, we distinguish with a cross ($\times$) each cell of the Young diagram that is not below (in the same column) or to the left (in the same row) of a rook (see Figures 3,4 and 5 further in next sections). We will see that the number of crosses is an important statistic on rook placements. This statistic was introduced in [8], as a generalisation of the inversion number for permutations. Indeed, if λ is a square of side length n, a rook placements R with n rooks may be seen as the graph of a permutation $\sigma \in \mathfrak{S}_n$, and in this view the number of crosses in R is the inversion number of σ.

Definition 2. *The weight of a rook placement R with r rooks and s crosses is $w(R) = p^r q^s$.*

Enumeration of rook placements leads to an evaluation of $\langle W|(y\hat{D}+\hat{E})^{n-1}|V\rangle$, hence of $\langle W|(yD+E)^{n-1}|V\rangle$ via the inversion formula (2). This is the main result of this article:

Theorem 1. *For any $n>0$, we have:*

$$\langle W|(yD+E)^{n-1}|V\rangle = \tfrac{1}{y(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{n-k} y^j\left(\tbinom{n}{j}\tbinom{n}{j+k} - \tbinom{n}{j-1}\tbinom{n}{j+k+1}\right)\right)\left(\sum_{i=0}^{k} y^i q^{i(k+1-i)}\right).$$

The combinatorial interpretation of this polynomial, in terms of permutations, is given in Proposition 15. When $y = 1$, this can be specialised in:

Theorem 2. *For any $n > 0$, we have:*

$$\langle W|(D+E)^{n-1}|V\rangle = \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\tbinom{2n}{n-k} - \tbinom{2n}{n-k-2}\right)\left(\sum_{i=0}^{k} q^{i(k+1-i)}\right).$$

These two results were conjectured by Corteel and Rubey. The earliest conjecture, when $y = 1$ and here stated as Theorem 2, was first proved by Prellberg [16] in May 2008 via semi-automatic methods (entirely non-combinatorial).

We can view this formula as a variation of the Touchard-Riordan formula [17]. It gives the q-enumeration of involutions of size $2n$ without fixed points with respect to the number of crossings, and is also the $2n$th moment of the q-Hermite polynomials. This formula is:

$$\sum_{I\in \mathrm{Inv}(2n,0)} q^{\mathrm{cr}(I)} = \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{\frac{k(k+1)}{2}}.$$

There is a combinatorial proof given by Penaud in [15]. It was in the goal of generalising this method that Rubey conjectured our Theorem 2, and gave an almost complete combinatorial proof. At the time of writing the last step of this combinatorial proof is still unsolved.

Besides references earlier mentioned, we have to point out the work of Williams [22], where we find a formula giving the coefficients of y^m in $\langle W|(yD+E)^n|V\rangle$. It was obtained by a more direct approach, via the enumeration of Ⅎ-diagrams, and was the only known polynomial formula for the distribution of a permutation pattern of length greater than 2 (See Proposition 15). Whereas Williams's work is rather focused on Ⅎ-diagrams, our results give more simple formulas in the case of permutations tableaux and permutations. Moreover Williams's formulas have also been obtained by Kasraoui, Stanton and Zeng in their work on orthogonal polynomials [9].

This article is organized as follows. In Section 2, we describe the link between rook placements and the rewriting of $(\hat{D}+\hat{E})^n$ in normal form. In Sections 3, 4, 5, we obtain enumerative results about rook placements; in particular Section 4 contains the bijective step of this enumeration. In Section 6, we use these results to prove Theorem 1, give the combinatorial interpretation of $\langle W|(yD+E)^n|V\rangle$, and some applications of the main theorem.

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Notations and conventions

We denote by $\mathrm{Par}(n-k,k)$ the set of Young diagram with exactly k rows and $n-k$ columns, allowing empty rows and columns. The integer n is the half-perimeter of the diagram $\lambda \in \mathrm{Par}(n-k,k)$, and we can see λ as an integer partition $(\lambda_1,\dots,\lambda_k)$ with $n-k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$. We use the French convention. We denote by $|\lambda|$ the number of cells in λ.

The North-East boundary of $\lambda \in \mathrm{Par}(n-k,k)$ is a path of n steps, k of them being vertical and $n-k$ horizontal. Reciprocally, for any word w of length n in $\hat{D}$ and $\hat{E}$, with k occurrences of $\hat{E}$, we define $\lambda(w) \in \mathrm{Par}(n-k,k)$ by the following rule: we read w from left to right, and draw one step East for each factor $\hat{D}$, and one step South for each factor $\hat{E}$.

We denote by $\mathrm{Inv}(n,k)$ the set of involutions on $\{1,\dots,n\}$ with k fixed points.

We use the classical q-analogs of integers, factorials, and binomials:

$$[n]_q = \frac{1-q^n}{1-q}, \qquad [n]_q! = \prod_{i=1}^{n} [i]_q, \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Proposition 1. [12] *The q-binomial coefficient is a polynomial with the following combinatorial interpretation:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \in Par(n-k,k)} q^{|\lambda|}, \qquad \textit{and} \qquad q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\lambda \in Par(n,k) \\ \lambda \textit{ has distinct} \\ \textit{non-zero parts}}} q^{|\lambda|}.$$

Definition 3. *The Delannoy numbers are defined for any $k, n \geq 0$ by:*

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \binom{n}{k} - \binom{n}{k-1}.$$

Proposition 2. *When $2k \leq n$, the number $\left\{ {n \atop k} \right\}$ counts the left factors of Dyck paths of n steps ending at height $n-2k$. In particular, $\left\{ {2n \atop n} \right\}$ is the nth Catalan number. They satisfy the relations:*

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}, \qquad \begin{Bmatrix} n \\ n+1-k \end{Bmatrix} = - \begin{Bmatrix} n \\ k \end{Bmatrix},$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 1, \qquad \textit{and} \begin{Bmatrix} n \\ k \end{Bmatrix} = 0 \textit{ if } k \notin \{0, \dots, n+1\}.$$

Proof. The number of left factors of Dyck paths of n steps ending at height $n-2k$ is easily seen to satisfy the same relations as $\left\{ {n \atop k} \right\}$: we just have to distinguish two cases whether the last step is going up or down. ☐

2. From operator relations to rook placements

In this section, we make the link between the coefficients of the normal form of $(\hat{D}+\hat{E})^n$, and rook placements in Young diagrams. This is done via a combinatorial description of the rewriting in normal form. When $q=1$, we can view it as a combinatorial statement of a classical result in statistical physics, Wick's theorem. The principle of this method is the same as the one described in this introduction, making the link between D and E and permutation tableaux, moreover the results of these section are present in [19] in a slightly different form.

From now on we assume that $\hat{D}$ and $\hat{E}$ are such that $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$ for some parameter p, which is a slight generalisation of the relation (1). As in the case of D and E, any word w in $\hat{D}$ and $\hat{E}$ can be uniquely written in normal form:

$$w = \sum_{i,j \geq 0} c_{i,j}(w) \hat{E}^i \hat{D}^j,$$

where $c_{i,j} \in \mathbb{Z}[p,q]$. We have:

$$\langle W | w | V \rangle = \sum_{i,j} c_{i,j}(w).$$

The combinatorial interpretation of this polynomial is given by the following proposition.

Proposition 3. *Let w be a word of length n in $\hat{E}$ and $\hat{D}$. Then $\langle W|w|V\rangle$ is the sum of weights of rook placements of shape $\lambda(w)$.*

Proof. Let us denote by T_w the sum of weights of rook placements of shape $\lambda(w)$. We prove with a recurrence on $|\lambda(w)|$, that $T_w = \langle W|w|V\rangle$.

The base case, $|\lambda(w)| = 0$, is the situation where the word w is already in normal form: $w = \hat{E}^i \hat{D}^j$ for some i and j. So we directly have $\langle W|w|V\rangle = 1$ from the properties of $\langle W|$ and $|V\rangle$ given in (1). This 1 corresponds to the unique rook placements of shape $\lambda(w)$, which contains no rook and no cross since there is no cell in this diagram.

Now we assume $|\lambda(w)| > 0$. It is possible to factorize w in $w = w_1\hat{D}\hat{E}w_2$. Indeed, this factor $\hat{D}\hat{E}$ correspond to a corner of $\lambda(w)$, and there is at least one corner since $|\lambda(w)| > 0$. The commutation relation of $\hat{D}$ and $\hat{E}$ gives:

$$w = q(w_1\hat{E}\hat{D}w_2) + p(w_1w_2), \qquad \text{hence } \langle W|w|V\rangle = q(\langle W|w_1\hat{E}\hat{D}w_2|V\rangle) + p(\langle W|w_1w_2|V\rangle).$$

So it remains to show that the same relation holds for T_w:

$$T_w = qT_{w_1\hat{E}\hat{D}w_2} + pT_{w_1w_2}.$$

To do this, we distinguish the rook placements of shape $\lambda(w)$ in two types, whether the corner corresponding to the factor $\hat{D}\hat{E}$ contains a cross or a rook (it cannot be empty: being a corner there is no cell above it or to its right that may contain a rook).

The sum of weights of rook placements of the first type is $qT_{w_1\hat{E}\hat{D}w_2}$. Indeed, when removing the corner the weight is divided by q and we can get any rook placement of shape $\lambda(w_1\hat{D}\hat{E}w_2)$.

Similarly, the sum of weights of rook placements of the second type is $pT_{w_1w_2}$. Indeed, when removing the corner, its row and its column, the weight is divided by p and we can get any rook placement of shape $\lambda(w_1w_2)$. $\square$

Since $(\hat{D}+\hat{E})^n$ expands into the sum of all words of length n in $\hat{D}$ and $\hat{E}$, we also obtain:

Proposition 4. *For any n, $\langle W|(\hat{D}+\hat{E})^n|V\rangle$ is equal to the sum of weights of all rook placements of half-perimeter n.*

If we add a parameter y, we can still expand $(y\hat{D}+\hat{E})^n$ and get the sum all words of length n in $\hat{D}$ and $\hat{E}$, but this time each word w has a coefficient y^m, where m is the number of occurrences of $\hat{D}$ in w. Via the correspondance between words and Young diagrams, the number of occurrences of $\hat{D}$ in w, is the number of columns in $\lambda(w)$. This leads to a refined version of the previous proposition.

Proposition 5. *For any n, $\langle W|(y\hat{D}+\hat{E})^n|V\rangle$ is the generating function for rook placements of half-perimeter n, the parameter y counting the number of columns.*

3. Basic results about rook placements

In this section we introduce the recurrence relation which will be used in the enumeration of rook placements, and we present two simple examples of enumeration. These two examples involves the q-binomial coefficients and the Delannoy numbers defined at the end of the introduction, and they are the beginning of a bijective explanation of the more general formulas we will show later.

Definition 4. *Let $T_{j,k,n}(p,q)$ be the sum of weights of rook placements of half-perimeter n, with k rows, and with j rows containing no rook (or equivalently, with $k-j$ rooks). We also define:*

$$T_{k,n}(p,q) = \sum_{j=0}^{k} T_{j,k,n}(p,q), \qquad T_n(p,q,y) = \sum_{k=0}^{n} y^k T_{k,n}(p,q),$$

so that $T_{k,n}(p,q)$ is the sum of weights of rook placements of half-perimeter n with k rows, and $T_n(p,q,y)$ is the generating function of rook placements of half-perimeter n, the parameter y counting the number of rows. Since there is an obvious transposition-symmetry, we can also view the parameter y as counting the number of columns.

These are polynomials in the variables p, q and y, so we will sometimes omit the arguments. From the previous section we know that $T_n(p,q,y)$ is equal to $\langle W|(y\hat{D}+\hat{E})^n|V\rangle$. In Figure 1 we give some examples of these polynomials.

$$T_{0,1,3} = pq + p + p, \qquad T_{1,1,3} = 1 + q + q^2, \qquad T_2 = 1 + (1+q+p)y + y^2.$$

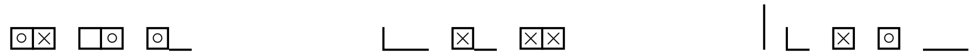

Figure 1. Some small values of $T_{j,k,n}$ and T_n, together with the rook placements corresponding to each term.

Proposition 6. *We have the following recurrence relation:*

$$T_{j,k,n} = T_{j-1,k-1,n-1} + q^j T_{j,k,n-1} + p[j+1]_q T_{j+1,k,n-1}. \tag{4}$$

Proof. The rook placements enumerated by $T_{j,k,n}$ may be distinguished into three types:

- the first column is of size strictly less than k,
- the first column is of size k and contains no rook,
- or the first column is of size k and contains exactly one rook.

We show that these three types lead respectively to the three terms of the recurrence relation.

The first case is the situation where the first step of the north-east boundary is a step down, or equivalently the first row is of size 0. Removing this step (or row) is a bijection between these first-type rook placements, and the ones enumerated by $T_{j-1,k-1,n-1}$, the first term of (4).

In the second case, the first column contains exactly j crosses, one per row without rook. So removing the first column is a bijection between the second-type rook placements, and the ones enumerated by $T_{j,k,n-1}$, and this bijection changes the weight by a factor q^j. This explains the second term of (4).

In the third case, removing the first column is not a bijection since there are several possibilities for the position of the rook in this column. But this map has the property that for any R enumerated by $T_{j+1,k,n-1}$, the preimage set of R contains $j+1$ elements, and their weights are $pw(R)$, $pqw(R)$, $\dots$, $pq^j w(R)$. This shows that the sum of weights of the third-type rook placements is the third term of (4), and completes the proof. □

Proposition 7. *For any k, n we have:*

$$T_{k,k,n} = \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{5}$$

Proof. We are counting rooks placements without any rook, *i.e.* such that all cells contain a cross. So this is a direct application of Proposition 1. □

This proposition is illustrated for example in Figure 1 where we see that $T_{1,1,3} = 1 + q + q^2 = [3]_q$. The second example of this section is more subtle and we begin with the following lemma.

Lemma 1. *Given a Young diagram λ, the number of rook placements of shape λ having no cross and exactly one rook per row is either 0 or 1. It is 1 in the case where the north-east boundary is a Dyck path (which means that the ith row of λ starting from the top contains at least i cells, for any i between 1 and the number of rows).*

Proof. Suppose R is a rook placement with no cross and exactly one rook per row. Then the i first rows contain i rooks, which are necessarly in i different columns. So the ith row contain at least i cells, *i.e.* the north-east boundary is a Dyck path.

It remains to prove there is a unique such rook placement in the case where the north-east boundary of a Young diagram λ is a Dyck path. We show there is only one way to build the rook placement starting from a empty diagram λ. First, notice that each corner of the diagram must contain a rook (as we saw in previous section, the general statement is that each corner contains either a rook or a cross). Then, if we consider the subdiagram of cells that are not in the same row or column of these rooks (see Figure 3), again all corners of this subdiagram must contain a rook by the same argument. We can even say that his north-east boundary is also a Dyck path: indeed, the boundary of the subdiagram is obtained from the boundary of the diagram by removing each occurrence of a step right followed by a step down. So we can conclude by recurrence.

FIGURE 2. Example of Young diagram whose north-east boundary is a Dyck path, *i.e.* doesn't go below the dotted line. The number of rows is $k = 6$, the half-perimeter is $n = 14$, and the path ends at height $n - 2k = 2$.

□

Proposition 8. *If $2k < n$, we have:*

$$T_{0,k,n}(p,0) = p^k \left\{ {n \atop k} \right\}. \tag{6}$$

Proof. We are counting rook placements with no cross (since q=0) and exactly k rooks, and each of these rook placement has weight p^k. We just have to prove that there are $\left\{ {n \atop k} \right\}$ such rook placements. Knowing that $\left\{ {n \atop k} \right\}$ is the number of Dyck paths of n steps ending at height $n-2k$, this is a consequence of the previous lemma. □

4. Rook placements and involutions

In this section we present the bijective step of the enumeration of rook placements. Indeed, the recurrence (4) is rather complicated to be solved directly, but thanks to this bijective step, we show that there is a simple relation between $T_{j,k,n}$ and $T_{0,k-j,n}$, and also that there is a simple recurrence relation satisfied by $T_{0,k,n}$.

Given a rook placement R of half-perimeter n, we define an involution $\alpha(R)$ by the following construction: label the north-east boundary of R with integers from 1 to n as shown in the left part of Figure 3. This way, each column or row has a label between 1 and n. If a column, or row, is labelled by i and does not contain a rook, it is a fixed point of $\alpha(R)$. And if there is a rook at the intersection of column i and row j, then $\alpha(R)$ sends i to j (and j to i).

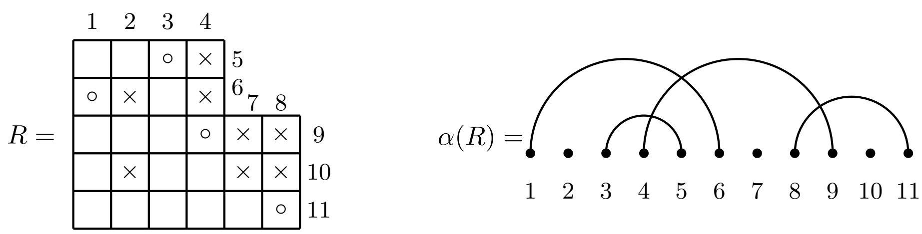

FIGURE 3. Example of a rook placement and its image by the map α.

Given a rook placement R of half-perimeter n, we also define a Young diagram $\beta(R)$ by the following construction (see Figure 4): if we remove all rows and columns of R containing a rook, the remaining cells form a Young diagram, which we denote by $\beta(R)$. We also define $\phi(R) = (\alpha(R), \beta(R))$.

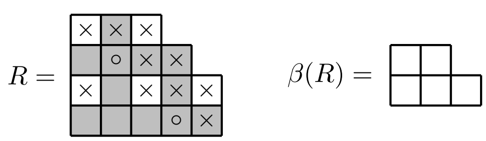

FIGURE 4. Example of a rook placement and its image by the map β. The grey cells are the ones that are in the same row (or column) as a rook. The Young diagram $\beta(R)$ is given by remaining cells (notice that all these remaining cells contain a cross).

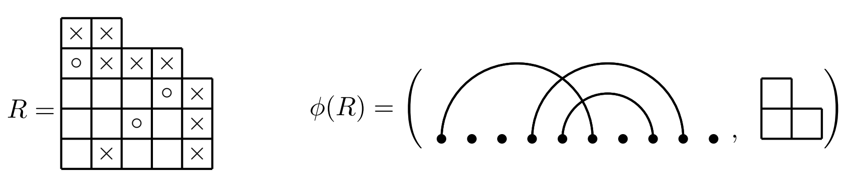

FIGURE 5. Example of a rook placement and its image by the map ϕ.

Proposition 9. *The map ϕ is a bijection between rooks placements in Young diagrams of half-perimeter n, and couples (I, λ) where I is an involution on $\{1, \dots, n\}$ and λ a Young diagram of half-perimeter $|Fix(\lambda)|$. If $\phi(R) = (I, \lambda)$, the number of rows (resp. columns) of λ is equal to the number of rows (resp. columns) without a rook in R.*

This is a classical argument so we don't give a complete proof. This bijection was already defined in [10], in terms of partial involutions, *i.e.* involutions on subsets of $\{1, \dots, n\}$. These partials involutions are equivalent to the couples (α, λ) in the sense that they are involutions with a weight 2 for each fixed point.

Indeed, remember the correspondance between Young diagram and words. A couple (I, λ) may be seen as an involution with two kinds of fixed points: those corresponding to vertical steps and those corresponding to horizontal steps. On the other hand, a partial involution I on $\{1, \dots, n\}$ may also be seen as an involution on $\{1, \dots, n\}$ with two kinds of fixed points: the ones that are not in the domain of I, and the ones that are in the domain and fixed by I.

Now that we have a bijection, we have to describe how the weight of a rook placements reads in the couple (I, λ). We need the following definitions on involutions.

Definition 5. *For any involution I, we define:*

- *an arch of the involution I as a couple (i,j) such that $i < j$ and $I(i) = j$,*
- *a crossing of the involution I as a pair of arches $((i, j), (k, l))$ such that $i < k < j < l$,*
- *the height of a fixed point $k \in Fix(I)$ as the number of arches (i, j) such that $i < k < j$.*

We denote by $cr(I)$ the number of crossings of I, and by $ht(k)$ the height of the fixed point k.

For example, consider the involution in Figure 5. There are two crossings, $((1, 6), (4, 9))$ and $((1, 6), (5, 8))$. The fixed points are $\mathrm{Fix}(I) = \{2, 3, 7, 9\}$ and their respective heights are 1, 1, 2 and 0.

Proposition 10. *Let $(I, \lambda) = \phi(R)$. Then:*

- *Each crossing of I corresponds to a cell of R containing a cross, having a rook to its left (in the same row) and a rook below (in the same column).*
- *Each triple (i, k, j) such that $i < k < j$, $k \in Fix(I)$ and (i, j) is an arch of I corresponds to a cell of R containing a cross, having either a rook to its left (in the same row) or a rook below (in the same column).*

Proof. The two statements are illustrated respectively in left part and right part of Figure 6.

- Let $((i, j), (k, l)$ be a crossing of I. Since $k < j$, the column k intersects the row j in some cell c. Then, (i, j) is an arch of I, which means that there is a rook at the intersection of column i and row j, to the left of the cell c. Similarly, (k, l) is an arch so there is a rook at the intersection of column k and row l, below the cell c.
- Let (i, k, j) be such that $i < k < j$, $k \in \mathrm{Fix}(I)$ and (i, j) is an arch of I. We suppose for example that j is the index of a column. Since $j < k$ the column j intersects the row k in some cell c. There is no rook below the cell c because j is a fixed point of I. But there is a rook in row k, at the left of c.

□

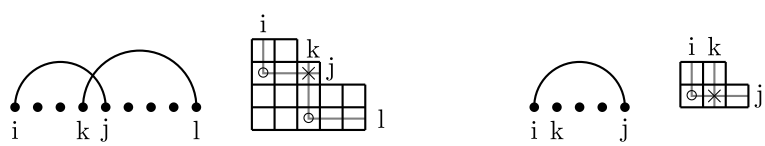

FIGURE 6. Interpretation of crossings, and sum of height of fixed points, in terms of rook placements.

Proposition 11. *The bijection ϕ has the property that if $\phi(R) = (I, \lambda)$ then the number of crosses in R is $|\lambda| + \mu(I)$, where μ is the statistic on involutions defined by:*

$$\mu(I) = cr(I) + \sum_{x \in Fix(I)} ht(x).$$

Proof. From the definition of the map β, we directly see that $|\lambda| = |\beta(R)|$ is the number of crosses in R with no rook in the same row and no rook in the same column. From Proposition 10, we know that the number of crossings counts the crosses of R with one rook to its left and one rook below, and the sum of heights of fixed points counts all remaining crosses. □

The previous proposition means that the number of crosses of rook placements is an additive parameter with respect to the decomposition $R \mapsto (I, \lambda)$. An additive parameter naturally leads to a factorisation of the corresponding generating functions so we get the following corollary:

Corollary 1. *For any j,k,n, we have:*

$$T_{j,k,n} = \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q T_{0,k-j,n}. \tag{7}$$

Proof. We assume that $n - 2k + 2j \geq 0$, since otherwise both sides are 0 (indeed, a rook placement enumerated by $T_{j,k,n}$ contains $k - j$ rooks, so it has at least $k - j$ different rows and $k - j$ different columns, so its half-perimeter is at least $2k - 2j$). Using the bijection ϕ, we can compute $T_{j,k,n}$ by summing the weights of couples (I, λ) where $I \in \mathrm{Inv}(n, n - 2k + 2j)$ and $\lambda \in \mathrm{Par}(n - 2k + 2j, j)$. Hence:

$$T_{j,k,n} = p^{k-j} \sum_{(\lambda,I)} q^{|\lambda|+\mu(I)} = \left(\sum_{\lambda} q^{|\lambda|} \right) \left(p^{k-j} \sum_{I} q^{\mu(I)} \right).$$

The first factor of the right-hand side is $\left[\begin{smallmatrix} n-2k+2j \\ j \end{smallmatrix}\right]_q$ by a direct application of Proposition 1. The second factor can be seen as a sum over couples (I, λ) where λ has 0 rows and $n - 2k + 2j$ columns (since there only one such Young diagram and it has weight 1). So using again the bijection ϕ, this second factor is $T_{0,k-j,n}$. □

Thanks to this factorisation property of $T_{j,k,n}$, we reduce the problem to knowing $T_{0,k,n}$. But this factorisation property also gives a recurrence relation satisfied by $T_{0,k,n}$.

Corollary 2. *We have the following recurrence relation:*

$$T_{0,k,n} = T_{0,k,n-1} + p[n + 1 - 2k]_q T_{0,k-1,n-1}. \tag{8}$$

Proof. When $j = 0$, the relation (4) gives:

$$T_{0,k,n} = T_{0,k,n-1} + pT_{1,k,n-1}.$$

Applying the previous corollary to the second term of this sum gives the desired expression. □

5. Enumeration of rook placements

In this section we solve the recurrence (8), and we obtain an expression for $T_{0,k,n}$ involving both q-binomials and Delannoy numbers, generalizing the two examples of Section 3. Using the factorisation property of $T_{j,k,n}$ and summing over j's, we obtain an expression for

$$T_{k,n} = \sum_{j=0}^{k} T_{j,k,n},$$

i.e. for the sum of weights of rook placements of half-perimeter n with k rows. This expression is rather lengthy, with a sum over three indices, but for certain values of p we can simplify it with the q-binomials identities of Lemma 2, so that in these particular specialisations we get expressions for $T_{k,n}$ and T_n without q-binomials.

Proposition 12. *When $p = 1 - q$, we have :*

$$T_{0,k,n}(1-q,q) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left\{ \begin{matrix} n \\ k-i \end{matrix} \right\}. \tag{9}$$

Proof. Let us denote by $f(k,n)$ the rhs of (9). The initial condition is $f(k,0) = T_{0,k,0} = \delta_{0k}$ so it remains to check the relation (8) when $p = 1-q$. Let us define:

$$A = \begin{bmatrix} n-1-2k+i \\ i \end{bmatrix}_q, \qquad B = q^{n-2k} \begin{bmatrix} n-1-2k+i \\ i-1 \end{bmatrix}_q,$$

$$C = \left\{ \begin{matrix} n-1 \\ k-i \end{matrix} \right\}, \qquad D = \left\{ \begin{matrix} n-1 \\ k-i-1 \end{matrix} \right\},$$

so that we have:

$$f(k,n) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (A+B)(C+D) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \Big(AC + BC + (A+B)D \Big).$$

After expanding this sum, the second term gives:

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} BC = -\sum_{i=0}^{k-1} (-1)^i q^{\frac{(i+1)(i+2)}{2}} q^{n-2k} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left\{ \begin{matrix} n-1 \\ k-i-1 \end{matrix} \right\},$$

(the sum is reindexed such that i becomes $i+1$), and the third term gives:

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (A+B)D = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left\{ \begin{matrix} n-1 \\ k-i-1 \end{matrix} \right\},$$

(noticing that the term where $i = k$ is 0). Adding the previous two identities gives:

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \Big(BC + AD + BD \Big) = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left\{ \begin{matrix} n-1 \\ k-i-1 \end{matrix} \right\} \big(1 - q^{n-2k+i+1}\big).$$

But we have $[n-2k+i+1]_q \left[\begin{smallmatrix} n-2k+i \\ i \end{smallmatrix}\right]_q = [n-2k+1]_q \left[\begin{smallmatrix} n-2k+i+1 \\ i \end{smallmatrix}\right]_q$, hence:

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \Big(BC + AD + BD \Big) = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i+1 \\ i \end{bmatrix}_q \left\{ \begin{matrix} n-1 \\ k-i-1 \end{matrix} \right\} \big(1 - q^{n-2k+1}\big)$$

$$= \big(1 - q^{n-2k+1}\big) f(k-1, n-1).$$

Since $\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} AC$ readily gives $f(k, n-1)$, we get the relation:

$$f(k,n) = f(k,n-1) + \big(1 - q^{n-2k+1}\big) f(k-1,n-1),$$

which is precisely (8) when $p = 1-q$. $\square$

Remark: The rook placements enumerated by $T_{0,k,n}$ contain exactly k rooks, so $T_{0,k,n}(p,q) = p^k T_{0,k,n}(1,q)$. It shows that there is no loss of generality in the assumption $p = 1-q$ of the previous proposition.

Now using (7) and (9), we have the following expression:

$$T_{j,k,n}(1-q,q) = \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q \sum_{i=0}^{k-j} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+2j+i \\ i \end{bmatrix}_q \left\{ \begin{matrix} n \\ k-j-i \end{matrix} \right\}. \tag{10}$$

And as in the previous remark, $T_{j,k,n}(p,q) = p^{k-j} T_{j,k,n}(1,q)$ so that we have an expression for any value of p. Summing it over j will give an expression for $T_{k,n}(p,q)$. For certain values of p, it is possible to simplify greatly this sum. To perform this simplification we need the following lemma.

Lemma 2. *For any $k, n \geq 0$ we have the following q-binomial identities:*

$$\sum_{j=0}^{k}(-1)^j q^{\frac{j(j+1)}{2}} \begin{bmatrix} n-j \\ n-k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q = 1, \tag{11}$$

$$\sum_{j=0}^{k}(-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n-j \\ n-k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q = q^{k(n-k)}, \tag{12}$$

$$\sum_{j=0}^{k}(-1)^j q^{\frac{(j-1)(j-2)}{2}} \begin{bmatrix} n-j \\ n-k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q = \frac{q^{(k+1)(n-k)} - q^{k(n-k)} + q^{k(n+1-k)} - q^{(k+1)(n+1-k)}}{q^{n-1}(1-q)}. \tag{13}$$

Proof. The first two are proved combinatorially using Proposition 1; actually we first prove the second one, which is slightly more simple. It seems there is no simple combinatorial proof of the third one so we prove it with a recurrence, which is quite similar to the one of Proposition 12.

- The left-hand side of (12) counts the pairs $(\lambda, \mu) \in \mathrm{Par}(n-k, k-j) \times \mathrm{Par}(n-k, j)$ such that μ has distinct parts, signed by $(-1)^j$, for any j between 0 and k. More precisely, λ is such that $n-k \geq \lambda_1 \geq \ldots \geq \lambda_{k-j} \geq 0$ and μ is such that $n-k > \mu_1 > \ldots > \mu_j \geq 0$. When $k-j > 0$, such a couple (λ, μ) satisfying $\lambda_{k-j} < \mu_j$ or $\mu = (\emptyset)$ may be paired with the following couple (λ', μ'):
$$\lambda' = (\lambda_1, \ldots, \lambda_{k-j-1}), \qquad \mu' = (\mu_1, \ldots, \mu_j, \lambda_{k-j}),$$
which satisfies $|\lambda| + |\mu| = |\lambda'| + |\mu'|$ but has opposite sign. The only couple which is not paired with any other is such that $\lambda_1 = \ldots = \lambda_k = n-k$ and $\mu = (\emptyset)$, it contributes to the sum by a $q^{k(n-k)}$.
- The proof of (11) is quite similar, this time the factor $q^{j(j+1)/2}$ means we count pairs (λ, μ) as before but such that $n-k \geq \mu_1 > \ldots > \mu_j > 0$, (because $j(j+1)/2 = 1 + \ldots + j$). This time the pairing is done by comparing the smallest non-zero part of λ with the smallest part of μ. Depending on the situation, one of these part is moved from λ to μ, or from μ to λ. The only couple (λ, μ) which is not paired with any other is such that $\lambda_1 = \ldots = \lambda_k = 0$ and $\mu = (\emptyset)$, and it contributes to the sum by a 1.
- When $k = 0$, both sides are equal to q. Let us denote by $g(n,k)$ the left-hand side of (13). We define:
$$A = q^{n-k}\begin{bmatrix} n-j \\ n-k \end{bmatrix}_q, \qquad B = \begin{bmatrix} n-j \\ n-k-1 \end{bmatrix}_q \qquad C = \begin{bmatrix} n-k-1 \\ j \end{bmatrix}_q, \qquad D = q^{n-k-j}\begin{bmatrix} n-k-1 \\ j-1 \end{bmatrix}_q,$$
so that $g(n+1,k+1) = \sum_{j=0}^{k}(-1)^k q^{(j-1)(j-2)/2}(A+B)(C+D)$. After expanding this product, we get the recurrence relation:
$$g(n+1,k+1) = q^{n-k} g(n,k) + g(n,k+1) - q^{n-k} g(n-1,k).$$
In view of the simple expression of the right-hand side of (13), it is straightforward to check that it satisfies the same relation.

□

Proposition 13.

$$T_{k,n}(1-q,q) = \binom{n}{k}, \qquad T_{k,n}\left(\tfrac{1-q}{q}, q\right) = \sum_{j=0}^{k} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} q^{(k-j)(n-k-j)}, \tag{14}$$

$$T_{k,n}\left(\tfrac{1-q}{q^2}, q\right) = \sum_{j=0}^{k} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \left(\tfrac{q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}}{(1-q)q^n} \right). \tag{15}$$

Proof. The three identities of this proposition comes respectively from (11), (12) and (13). We prove the last one, because it is the most important case, the two others are proved similarly but more simply. Multiplying (10) by q^{2j-2k} and summing over j gives:

$$T_{k,n}\left(\tfrac{1-q}{q^2}, q\right) = \sum_{j=0}^{k} q^{2j-2k} \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q \sum_{i=0}^{k-j}(-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+2j+i \\ i \end{bmatrix}_q \left\{ \begin{matrix} n \\ k-j-i \end{matrix} \right\}$$

$$= \sum_{\substack{0\le i,j\\ i+j\le k}} \left\{ {n \atop k-j-i} \right\} q^{2j-2k} \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+2j+i \\ i \end{bmatrix}_q .$$

Introducing $l = k-j-i$, we get:

$$T_{k,n}\left(\tfrac{1-q}{q^2}, q\right) = \sum_{l=0}^{k} \left\{ {n \atop l} \right\} \sum_{j=0}^{k-l} q^{2j-2k} \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q (-1)^{k-j-l} q^{\frac{(k-j-l)(k-j-l+1)}{2}} \begin{bmatrix} n-k+j-l \\ k-j-l \end{bmatrix}_q ,$$

and by replacing j with $k-l-j$ we have also:

$$\begin{aligned} T_{k,n}\left(\tfrac{1-q}{q^2}, q\right) &= \sum_{l=0}^{k} \left\{ {n \atop l} \right\} \sum_{j=0}^{k-l} q^{-2j-2l} \begin{bmatrix} n-2l-2j \\ k-l-j \end{bmatrix}_q (-1)^j q^{\frac{j(j+1)}{2}} \begin{bmatrix} n-2l-j \\ j \end{bmatrix}_q \\ &= \sum_{l=0}^{k} \left\{ {n \atop l} \right\} \sum_{j=0}^{k-l} (-1)^j q^{\frac{(j-1)(j-2)}{2}-1-2l} \frac{[n-2l-j]_q!}{[j]_q![k-l-j]_q![n-k-l-j]_q!} \\ &= \sum_{l=0}^{k} \left\{ {n \atop l} \right\} q^{-1-2l} \sum_{j=0}^{k-l} (-1)^j q^{\frac{(j-1)(j-2)}{2}} \begin{bmatrix} n-2l-j \\ n-l-k \end{bmatrix}_q \begin{bmatrix} n-l-k \\ j \end{bmatrix}_q . \end{aligned}$$

At this point we can apply the second identity of Lemma 2 with $n' = n-2l$ and $k' = k-l$, and get (15). □

Remark: By an obvious argument of symmetry by transposition, we have $T_{k,n} = T_{n-k,n}$, and this can be directly seen in (15). The summand $q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}$ is unchanged when k is replaced with $n-k$. A consequence is that in (15) instead of summing over js between 0 and k, we could sum over js between 0 and $\min(k, n-k)$. This is also true for the second identity of (14).

The last step of this section is the summing over k to get an expression for $T_n(\frac{1-q}{q^2}, q, y)$.

Proposition 14.

$$(1-q)q^n T_n\left(\tfrac{1-q}{q^2}, q, y\right) = (1+y)G(n) - G(n+1), \tag{16}$$

$$\textit{where} \quad G(n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ {n \atop j} \right\} \sum_{i=0}^{n-2j} y^{i+j-1} q^{i(n+1-2j-i)}.$$

Proof. First we define:

$$P_k = \sum_{i=0}^{k} y^i q^{i(k+1-i)}$$

We have to sum (15) times y^k for $k = 0 \ldots n$. This gives:

$$(1-q)q^n T_n\left(\tfrac{1-q}{q^2}, q, y\right) = \sum_{0\le j\le k\le n} y^k \left\{ {n \atop j} \right\} \left(q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)} \right)$$

$$= \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \left(\sum_{k=j}^{n} y^k q^{(k+1-j)(n-k-j)} - \sum_{k=j}^{n} y^k q^{(k-j)(n-k-j)} + \sum_{k=j}^{n} y^k q^{(k-j)(n+1-k-j)} - \sum_{k=j}^{n} y^k q^{(k+1-j)(n+1-k-j)} \right)$$

$$= \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \left(\sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+1-2j-i)} - \sum_{i=0}^{n-j} y^{i+j} q^{i(n-2j-i)} + \sum_{i=0}^{n-j} y^{i+j} q^{i(n+1-2j-i)} - \sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+2-2j-i)} \right),$$

by a reindexing of the second and third sums with $i = k-j$, and of the first and fourth sums with $i = k+1-j$. Since $(1-q)q^n T_n$ is a polynomial, we can discard all negative powers of q appearing in these sums. Modulo non-positive powers of q, these four sums are respectively equal to $y^{j-1}P_{n-2j}$, $y^j P_{n-1-2j}$, $y^j P_{n-2j}$, $y^{j-1}P_{n+1-2j}$. But we have to be careful when it comes to the constant terms in q. These constant terms are respectively:

$$[q^0]\sum_{i=1}^{n+1-j} y^{i+j-1}q^{i(n+1-2j-i)} = y^{n-j}\delta_{1\le n+1-2j\le n-j+1},$$

$$[q^0]\sum_{i=0}^{n-j} y^{i+j}q^{i(n-2j-i)} = 1+y^{n-j}\delta_{0\le n-2j\le n-j},$$

$$[q^0]\sum_{i=0}^{n-j} y^{i+j}q^{i(n+1-2j-i)} = 1+y^{n+1-j}\delta_{0\le n+1-2j\le n-j},$$

$$[q^0]\sum_{i=1}^{n+1-j} y^{i+j-1}q^{i(n+2-2j-i)} = y^{n+1-j}\delta_{1\le n+2-2j\le n+1-j}.$$

We see that these constant terms in q actually cancel two-by-two, so that it remains:

$$\begin{aligned}(1-q)q^nT_n\left(\tfrac{1-q}{q^2},q,y\right) &= \sum_{j=0}^{n}\left\{ {n \atop j} \right\}\Big((y^j+y^{j-1})P_{n-2j} - y^jP_{n-1-2j} - y^{j-1}P_{n+1-2j}\Big)\\ &= (1+y)\sum_{j=0}^{n}\left\{ {n \atop j} \right\}y^{j-1}P_{n-2j} - \sum_{j=0}^{n+1}\left(\left\{ {n \atop j-1} \right\}+\left\{ {n \atop j} \right\}\right)y^{j-1}P_{n+1-2j}\\ &= (1+y)G(n)-G(n+1), \qquad \text{where} \quad G(n)=\sum_{j=0}^{n}\left\{ {n \atop j} \right\}y^{j-1}P_{n-2j}.\end{aligned}$$

Since the polynomial P_{n-2j} is zero when $n-2j<0$, we can sum over js between 0 and $\lfloor n/2\rfloor$ in the definition of $G(n)$, so that we get (16). □

6. Application to permutation enumeration

In the previous section we have computed T_n, which is also equal to $\langle W|(y\hat{D}+\hat{E})^n|V\rangle$ thanks to the results of Section 2. Now, using the inversion formulas (2), we can compute $\langle W|(yD+E)^n|V\rangle$ and prove Theorem 1. At beginning of this section we describe the combinatorial interpretation of this polynomial with permutations and permutations tableaux. Then we prove Theorem 1 and Theorem 2, and give some applications.

Proposition 15. [4, 5, 6, 9, 13, 21] *For any $n\ge 1$ the following polynomials are equal:*

- $\langle W|y(yD+E)^{n-1}|V\rangle$,
- *the generating function for permutation tableau x of size n, the number of lines counted by y and the number of superfluous 1's counted by q,*
- *the generating function for permutations of size n, the number of ascents plus 1 counted by y and the number of 13-2 patterns counted by q,*
- *the generating function for permutations of size n, the number of weak excedences counted by y and the number of crossings counted by q,*
- *the nth momentum of the q-Laguerre polynomials.*

Proof. All this material is present in the references. See also the references for definitions, in particular there are several possible definitions for the q-Laguerre polynomials, the one we use is defined as a rescaled version of the Al-Salam-Chihara polynomials as in [9].

As said in the introduction, the link between the operators D and E of the matrix ansatz and the permutation tableaux was first exposed by Corteel and Williams in [6], showing the equality of the first two points. See also [21].

The classical bijections between permutations and Laguerre histories, namely the Françon-Viennot and Foata-Zeilberger bijections, give the equality of the three last points in the list. We recall that the nth moment of the q-Laguerre polynomials is the sum of weights of Laguerre histories of n steps. This is also present in [4].

To conclude on these equalities we can use the bijection between permutation tableaux and permutations found in [5]: the number of columns in permutations tableaux corresponds to the number of ascents in permutations, and the number of superfluous 1's corresponds to the number of 13-2 patterns. We also have to mention the previous results of Postnikov, who made the link between Ꞁ-diagrams, which generalize the permutations tableaux, and alignments in decorated permutations [11, 22]. ☐

We now give the formula for the polynomials of Proposition 15. This is the Theorem 1 stated in the introduction.

Theorem 1. *For any $n \geq 1$, we have:*

$$\langle W|(yD+E)^{n-1}|V\rangle = \tfrac{1}{y(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{n-k} y^j\left(\tbinom{n}{j}\tbinom{n}{j+k}-\tbinom{n}{j-1}\tbinom{n}{j+k+1}\right)\right)\left(\sum_{i=0}^{k} y^i q^{i(k+1-i)}\right).$$

Proof. Using the main result of the previous section (16) and the inversion formula (2), we obtain:

$$\langle W|(1-q)^n(yD+E)^{n-1}|V\rangle = (1-q)\sum_{k=0}^{n-1}\tbinom{n-1}{k}(1+y)^{n-1-k}(-q)^k\langle W|(y\hat{D}+\hat{E})^k|V\rangle$$

$$=\sum_{k=0}^{n-1}\tbinom{n-1}{k}(1+y)^{n-1-k}(-1)^k\Big((1+y)G(k)-G(k+1)\Big)=\sum_{k=0}^{n}\tbinom{n}{k}(1+y)^{n-k}(-1)^kG(k)$$

$$=\sum_{\substack{0\leq i\leq k\leq n\\ i\equiv k \bmod 2}}\tbinom{n}{k}(1+y)^{n-k}(-1)^k\left\{\begin{smallmatrix}k\\ \frac{k-i}{2}\end{smallmatrix}\right\}y^{(k-i)/2}P_i=\sum_{i=0}^{n}(-1)^i\left(\sum_{k=0}^{\lfloor\frac{n-i}{2}\rfloor}\tbinom{n}{2k+i}(1+y)^{n-2k-i}\left\{\begin{smallmatrix}2k+i\\ k\end{smallmatrix}\right\}y^k\right)P_i,$$

after a reindexing such that k becomes $2k+i$. It remains to simplify the sum between parentheses. After expanding the power of $1+y$, this sum is:

$$\sum_{k=0}^{\lfloor\frac{n-i}{2}\rfloor}\sum_{j=0}^{n-2k-i}\binom{n}{2k+i}\binom{n-2k-i}{j}\left\{\begin{matrix}2k+i\\ k\end{matrix}\right\}y^{k+j}$$

$$=\sum_{0\leq k,j}\frac{n!}{j!(n-2k-i-j)!}\left(\frac{1}{k!(k+i)!}-\frac{1}{(k-1)!(k+i+1)!}\right)y^{k+j}$$

$$=\sum_{0\leq k\leq m}\frac{n!}{(m-k)!(n-m-k-i)!}\left(\frac{1}{k!(k+i)!}-\frac{1}{(k-1)!(k+i+1)!}\right)y^{m}$$

$$=\sum_{m=0}^{n-i}y^m\left(\binom{n}{m}\sum_{k=0}^{m}\tbinom{m}{k}\tbinom{n-m}{k+i}-\binom{n}{m-1}\sum_{k=0}^{m}\tbinom{m-1}{k-1}\tbinom{n-m+1}{k+i+1}\right).$$

But thanks to the Vandermonde identity, the two sums over k may be simplified:

$$\sum_{k=0}^{m}\tbinom{m}{k}\tbinom{n-m}{k+i}=\binom{n}{m+i},\qquad \sum_{k=0}^{m}\tbinom{m-1}{k-1}\tbinom{n-m+1}{k+i+1}=\binom{n}{m+i+1},$$

and this completes the proof. ☐

Remark: The number $(\binom{n}{j}\binom{n}{j+k}-\binom{n}{j-1}\binom{n}{j+k+1})$ may be seen as the determinant of a 2×2-matrix of binomials. The Lindstöm-Gessel-Viennot lemma gives a combinatorial interpretation of this quantity in terms of lattice paths: it is the number of pairs of non-intersecting paths with start points $(1,0)$ and $(0,1)$, with end points $(j,n-j+1)$ and $(j+k+1,n-k-j)$, and only with unit steps up or right, as in Figure 7. In particular when $k=0$, this is the Narayana number $N(n+1,j+1)$.

Proposition 16. *The coefficient of y^m in $\langle W|(yD+E)^{n-1}|V\rangle$ is given by:*

$$[y^m]\langle W|(yD+E)^{n-1}|V\rangle=\frac{1}{(1-q)^n}\sum_{k=0}^{n}\sum_{j=m-k}^{m}(-1)^kq^{(m-j)(k+j+1-m)}\Big(\tbinom{n}{j}\tbinom{n}{j+k}-\tbinom{n}{j-1}\tbinom{n}{j+k+1}\Big).$$

Proof. We just have to expand the products in the equality of Theorem 1, since each of the factor between parentheses is a polynomial in y and their coefficients are explicit. ☐

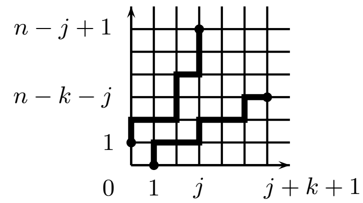

FIGURE 7. Interpretation of a determinant of binomials in terms of lattice paths. In this example, we have $n=8$, $j=3$, $k=2$.

In [22], Williams provides a different formula for the same polynomial, indeed $[y^m]\langle W|(yD+E)^{n-1}|V\rangle$ is also equal to:

$$q^{-k^2}\sum_{i=0}^{k-1}(-1)^i[k-i]^n q^{ki}\left(\binom{n}{i}q^{k-i}+\binom{n}{i-1}\right).$$

She shows that this polynomial is a q-analog of eulerian numbers that interpolates between Narayana number (when $q=0$), binomial coefficents (when $q=-1$), and of course eulerian numbers (when $q=1$).

We can also obtain these results from the expression of Proposition 16. For example, if we put $q=0$ in the previous equality, it tells that the number of permutations avoiding pattern 13-2 and with m ascents is:

$$\sum_{k=0}^{n}(-1)^k\left(\tbinom{n}{m}\tbinom{n}{m+k}-\tbinom{n}{m-1}\tbinom{n}{m+k+1}\right)=\binom{n}{m}^2+\sum_{k=1}^{n}(-1)^k\tbinom{n}{m}\tbinom{n}{m+k}+\sum_{k=1}^{n+1}(-1)^k\tbinom{n}{m-1}\tbinom{n}{m+k}$$
$$=\binom{n}{m}^2+\sum_{k=1}^{n}(-1)^k\binom{n+1}{m}\binom{n}{m+k}=\binom{n}{m}^2-\binom{n+1}{m}\sum_{k=0}^{m}(-1)^{k+m}\binom{n}{k}.$$

This alternate sum of binomials is itself the binomial $\binom{n-1}{m}$. So the number we get is $\binom{n}{m}^2-\binom{n+1}{m}\binom{n-1}{m}$. Although it is not the most common way to define it, this number is the Narayana number $N(n,m)$, as can be seen combinatorially using again the Lindström-Gessel-Viennot lemma.

We now give the specialization when $y=1$. This is the Theorem 2 stated in the introduction.

Theorem 2. *For any $n\geq 1$, we have:*

$$\langle W|(D+E)^{n-1}|V\rangle=\frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\tbinom{2n}{n-k}-\tbinom{2n}{n-k-2}\right)\left(\sum_{i=0}^{k}q^{i(k+1-i)}\right). \tag{17}$$

Proof. We just have to put $y=1$ in the equality of Theorem 1. We can simplify the resulting expression by using again the Vandermonde identity, indeed we have:

$$\sum_{j=0}^{n-k}\tbinom{n}{j}\tbinom{n}{j+k}=\sum_{j=0}^{n-k}\tbinom{n}{j}\tbinom{n}{n-j-k}=\binom{2n}{n-k},$$
$$\sum_{j=0}^{n-k}\tbinom{n}{j-1}\tbinom{n}{j+k+1}=\sum_{j=0}^{n-k}\tbinom{n}{j-1}\tbinom{n}{n-j-k-1}=\binom{2n}{n-k-2},$$

and the result follows. □

Among the several objects of the list in Proposition 15, the most studied are probably permutations and the pattern 13-2, see for example [3, 5, 13, 14]. In particular in [3, 14] we can find methods for obtaining, as a function of n for a given k, the number of permutations of size n with exactly k occurrences of pattern 13-2. By taking the Taylor series of (2), we obtain direct and quick proofs for these previous results. As an illustration we give the formulas for $k\leq 3$ in the following proposition.

Proposition 17. *The order 3 Taylor series of $\langle W|(D+E)^{n-1}|V\rangle$ is:*

$$\langle W|(D+E)^{n-1}|V\rangle=C_n+\binom{2n}{n-3}q+\frac{n}{2}\binom{2n}{n-4}q^2+\frac{(n+1)(n+2)}{6}\binom{2n}{n-5}q^3+O(q^4),$$

where C_n is the nth Catalan number.

Proof. On one side, we have $(1-q)^{-n} = 1+nq+\binom{n+1}{2}q^2+\binom{n+2}{3}q^3+O(q^4)$. On the other side, we have $\sum_{i=0}^{k} q^{i(k+1-i)} = 1+q\delta_{k=1}+2q^2\delta_{k=2}+2q^3\delta_{k=3}+O(q^4)$. The constant term is:

$$\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-2}\right)=\binom{2n}{n}-\binom{2n}{n-1}=C_n.$$

So this Taylor series is:

$$\left(1+nq+\tfrac{n(n+1)}{2}q^2+\tfrac{n(n+1)(n+2)}{6}q^3\right)\left(C_n-\left(\tbinom{2n}{n-1}-\tbinom{2n}{n-3}\right)q+\left(\tbinom{2n}{n-2}-\tbinom{2n}{n-4}\right)q^2-\left(\tbinom{2n}{n-3}-\tbinom{2n}{n-5}\right)q^3\right)+O(q^4).$$

After expanding the product, all coefficients may be seen as the product of $\binom{2n}{n}$ and a rational fraction of n. So the simplification is just a matter of simplifying rational fractions of n, which is straightforward. $\square$

More generally, a computer algebra system can provide higher order terms, for example it takes no more than a few seconds to obtain the following closed formula for $[q^{10}]\langle W|(D+E)^{n-1}|V\rangle$:

$$\tfrac{(2n)!}{10!(n+12)!(n-8)!}\Big(n^{13}+70\,n^{12}+2093\,n^{11}+32354\,n^{10}+228543\,n^9-318990\,n^8-17493961\,n^7-104051458\,n^6$$

$$-6828164\,n^5+2022876520\,n^4+6310831968\,n^3+5832578304\,n^2+14397419520\,n+5748019200\Big),$$

which is quite an improvement compared to the methods of [14]. Besides these exact formulas, the following proposition gives the asymptotic for permutations with a given fixed number of occurrences of pattern 13-2.

Theorem 3. *for any $m \geq 0$ we have the following asymptotic when n goes to infinity:*

$$[q^m]\langle W|(D+E)^{n-1}|V\rangle \sim \frac{4^n n^{m-\frac{3}{2}}}{\sqrt{\pi}m!}.$$

Proof. When n goes to infinity, the numbers $\binom{2n}{n-k}-\binom{2n}{n-k-2}$ are dominated by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It implies that in $(1-q)^n\langle W|(D+E)^{n-1}|V\rangle$, each higher order term grows at most as fast as the constant term C_n. On the other side, the coefficient of q^m in $(1-q)^{-n}$ is equivalent to $n^m/m!$. So we have the asymptotic:

$$[q^m]\langle W|(D+E)^{n-1}|V\rangle \sim \frac{C_n n^m}{m!}.$$

Knowing the asymptotic of the Catalan numbers, we can conclude the proof. $\square$

Since any occurrence of the pattern 13-2 in a permutation is also an occurrence of the pattern 1-3-2, a permutation with k occurrences of the pattern 1-3-2 has at most k occurrences of the pattern 13-2. So we get the following corollary.

Corollary 3. *Let $\psi_k(n)$ be the number of permutations in $\mathfrak{S}_n$ with at most k occurrences of the pattern 1-3-2. For any constant $C>1$ and $k \geq 0$, we have:*

$$\psi_k(n) \leq C\frac{4^n n^{k-\frac{3}{2}}}{\sqrt{\pi}k!}$$

when n is sufficiently large.

Proof. By the previous remark we have:

$$\psi_k(n) \leq \sum_{i=0}^{k}[q^i]\langle W|(D+E)^{n-1}|V\rangle,$$

so this is a consequence of Theorem 3, which gives the asymptotics of each of these terms. $\square$

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