Queen Mary and Westfield College
University of London
BSc Examination by Course Units 1997

MAS 205 Complex Variables

Wednesday 28th May 1997, 2.30 p.m.

The duration of this examination is 2 hours.

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

SECTION A *Each question carries 12 marks. You should attempt ALL questions.*

A1.
For each of the examples below, either find the limit or prove that it does not exist:

(i) $$lim_{z\to\infty}\frac{3z^2+2}{z^2+5}$$

(ii) $$lim_{z\to 0}\frac{\sin z}{z}$$

(iii) $$lim_{z\to 0}\frac{(\bar{z})^2}{z}$$

(iv) $$lim_{z\to 0}\frac{(\bar{z})^2}{z^2}$$

A2.

Let $f = u+iv$ be a complex function of a complex variable $z = x+iy$. Write down the definition of the *derivative* of f at z_0. What are meant by the *Cauchy-Riemann equations* and when do they hold ?

(i) Prove (from the definition of derivative) that the function $f(z) = z^2$ is differentiable everywhere, and has derivative $2z$.

(ii) Let $f(z) = u(x,y) + iv(x,y)$ be the function defined by:

$$u = \frac{x}{x^2+y^2} \quad v = \frac{y}{x^2+y^2}$$

Show that f is not differentiable at any point $z \neq 0$.

A3.

(i) Let

$$f(z) = \frac{1}{z^2-4}$$

Find the Taylor series for f about the point $z = 0$. For what values of z does this series converge ?

(ii) Let

$$g(z) = \frac{1}{z^2-5z+6}$$

Find the Laurent series for g valid for $0 < |z-2| < 1$ and compute the residue of g at $z = 2$.

A4.

Define what is meant by the *integral* $\int_C f(z)dz$ of a complex function f along a piecewise-smooth curve C parametrised by a path γ.

Compute $\int_C f(z)dz$ where $f(z) = |z|^2$ and C is

(a) the (straight line) segment of the real axis from $+1$ to -1;
(b) the upper half of the unit circle (centre 0, radius 1) from $+1$ to -1.

A5.

State *Cauchy's Theorem* (for a star-shaped domain). Deduce that when C is a circle with centre at the point -2 and radius 1,

$$\int_C \frac{1}{z(z-2)} dz = 0$$

What is the value of this integral when C is the circle with centre at 0 and radius 1 ? Give reasons for your answer.

Next question on next page

SECTION B *Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B6.

(i) What is meant by a *Möbius transformation*? Show that such a transformation has an *inverse*. Describe (without proof) the possible images of a straight line under a Möbius transformation.

Find the Möbius transformation α having $\alpha(0) = 0$, $\alpha(1) = i$ and $\alpha(-1) = \infty$. What is the image of the upper half-plane under α ?

(ii) Define what it means to say that a function $u : \mathbf{R}^2 \to \mathbf{R}$ is *harmonic* and prove that the real part u of a holomorphic (analytic) function $f = u + iv : \mathbf{C} \to \mathbf{C}$ is harmonic. *(You may assume that the real and imaginary parts u and v of f satisfy the Cauchy-Riemann equations.)*

Let

$$u(x, y) = y + e^x \cos y$$

Show that u is harmonic on $\mathbf{R}^2 - 0$ and find a holomorphic function f on $\mathbf{C} - 0$ having real part u.

B7.

Cauchy's Integral Formula states that if f is holomorphic (analytic) everywhere on and inside a simple closed contour C, then at any point z_0 inside C the value of the nth derivative of f is

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Deduce that if C is a circle with centre z_0 and radius R then *Cauchy's Inequality* holds:

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} M_R$$

where M_R is the maximum value of $|f(z)|$ for z on C.

(i) By applying Cauchy's Inequality with $n = 1$ prove Liouville's Theorem that any bounded entire function is constant.

(ii) State the Fundamental Theorem of Algebra and prove it using Liouville's Theorem.

(iii) By applying Cauchy's Inequality with $n = 2$ show that if f is an entire function and M is a positive real number such that $|f(z)| < M|z|$ for all $z \in \mathbf{C}$ then f has the form $f(z) = \lambda z$ for some constant $\lambda \in \mathbf{C}$.

Next question on next page

B8.

What are meant by an *isolated singularity* z_0 of a function f, the *Laurent series expansion* of f around z_0 and the *residue* of f at z_0 ?

State (without proof) the *Residue Theorem.* Let $g(z) = (z - z_0)^m \phi(z)$ where m is a positive integer, ϕ is holomorphic (analytic) at z_0 and $\phi(z_0) \neq 0$. Prove that $g'(z)/g(z)$ has residue m at z_0, and deduce that if g is holomorphic on and inside a positively oriented simple closed contour C, and if g is non-zero at all points on C, then

$$\int_C \frac{g'(z)}{g(z)} dz = 2\pi i N$$

where N is the number of zeros of g inside C, counted with multiplicities.

State (without proof) *Rouché's Theorem.* Use it to find the number of zeros of the function $h(z) = z^4 + 5z^2 - z - 1$ lying inside the circle $|z| = 2$.

B9.

Evaluate the following integrals (giving justification for your answers):

(i)
$$\int_C \frac{z}{\sin z} dz$$

where C is the circle $|z - 1| = 2$.

(ii)
$$\int_C \frac{z^2 + 3z}{z^2 - 6z + 8} dz$$

where C is the circle $|z - 2| = 1$.

(iii)
$$\int_0^\infty \frac{1}{(x^2+1)^2(x^2+4)} dx$$

End of examination paper

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