

# MTH5105 Differential and Integral Analysis

## 2009-2010

### Solutions 9

## 1 Exercises

- 1) (a) Show that for all  $x \in \mathbb{R}$ , the sum  $\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$  converges.  
[You may use that  $|\sin(t)| \leq |t|$  for all  $t \in \mathbb{R}$ .]  
(b) Show that the sum  $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$  converges uniformly for all  $x \in \mathbb{R}$ .  
(c) Deduce that  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$$

is differentiable.

Solution:

- (a) As  $|\sin t| \leq |t|$  for all  $t \in \mathbb{R}$ , we have

$$\sum_{k=1}^{\infty} \left| \frac{1}{k} \sin\left(\frac{x}{k}\right) \right| \leq |x| \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$\sum \frac{1}{k^2}$  converges, so that the sum converges absolutely (for fixed  $x$ ).

- (b) As  $|\cos t| \leq 1$  for all  $t \in \mathbb{R}$ , we have

$$\left| \sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right) \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Thus the sum converges uniformly by the Weierstraß criterion, as the bound  $\sum \frac{1}{k^2}$  is independent of  $x$ .

- (c) Let  $f_k(x) = \frac{1}{k} \sin\left(\frac{x}{k}\right)$ . Then  $f'_k(x) = \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$ .  
As  $\sum f_k$  converges pointwise and  $\sum f'_k$  converges uniformly,  $f = \sum f_k$  is differentiable and  $f' = \sum f'_k$ .

- 2) Is the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = \sum_{k=1}^{\infty} \sin^2(x/k)$$

differentiable?

Solution:

If  $\sum f_k$  converges pointwise and  $\sum f'_k$  converges uniformly, then  $f = \sum f_k$  is differentiable and  $f' = \sum f'_k$ .

Let  $f_k(x) = \sin^2(x/k)$ . Then  $f'_k(x) = 2 \sin(x/k) \cos(x/k)/k$ .

As  $|\sin t| \leq |t|$  for all  $t \in \mathbb{R}$ , we have

$$\sum_{k=1}^{\infty} |\sin^2(x/k)| \leq x^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

$\sum \frac{1}{k^2}$  converges, so that the sum converges absolutely (for fixed  $x$ ).

*[We could have proven uniform convergence on bounded intervals, but we don't need to. In fact, it would have sufficed to prove convergence at a single point, say at  $x = 0$ , where it is immediately obvious.]*

As also  $|\cos t| \leq 1$  for all  $t \in \mathbb{R}$ , we have

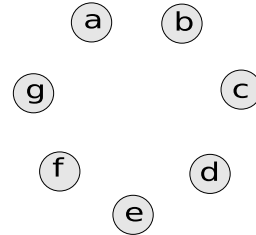
$$\left| \sum_{k=1}^{\infty} 2 \sin(x/k) \cos(x/k)/k \right| \leq |x| \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

If we restrict  $x$  to  $[-A, A]$  for some  $A > 0$ , then the sum converges uniformly on  $[-A, A]$  by the Weierstraß criterion, as the upper bound  $A \sum \frac{1}{k^2}$  is independent of  $x$ .

Thus we can conclude that  $f = \sum f_k$  is differentiable with  $f' = \sum f'_k$  on any interval  $[-A, A]$ , and hence on  $\mathbb{R}$ .

- 3) Let  $f_n : [0, 1] \mapsto \mathbb{R}$  be a sequence of differentiable functions, and let  $f : [0, 1] \mapsto \mathbb{R}$ . Consider the statements

- (a)  $f_n \rightarrow f$  pointwise,
- (b)  $f_n \rightarrow f$  uniformly,
- (c)  $f'_n$  converges pointwise,
- (d)  $f'_n \rightarrow f'$  pointwise,
- (e)  $f$  continuous,
- (f)  $f$  differentiable,
- (g)  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx$ ,



and clearly indicate in the enclosed figure all implications by the appropriate arrows (“ $\implies$ ”).

Solution:

The only valid implications are:

- (b) implies (a),(e),(g)
- (d) implies (c)
- (f) implies (e)