Counting partially directed paths in a symmetric wedge

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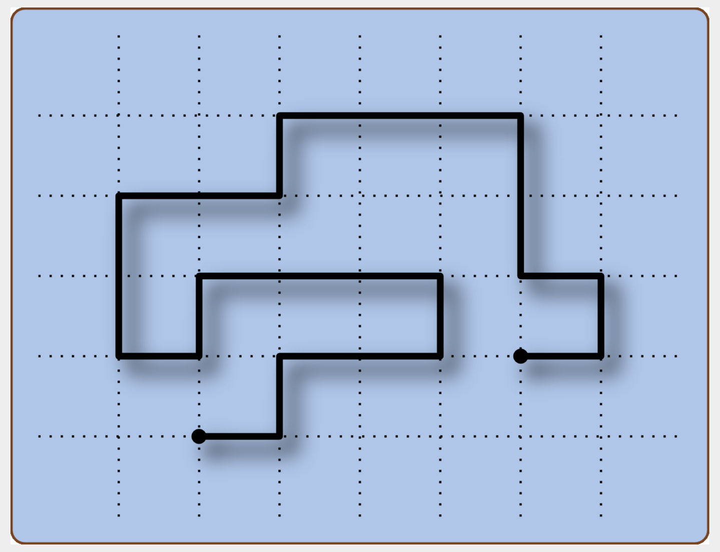
Self-avoiding walk

- SAW = a path on a lattice that does not intersect itself



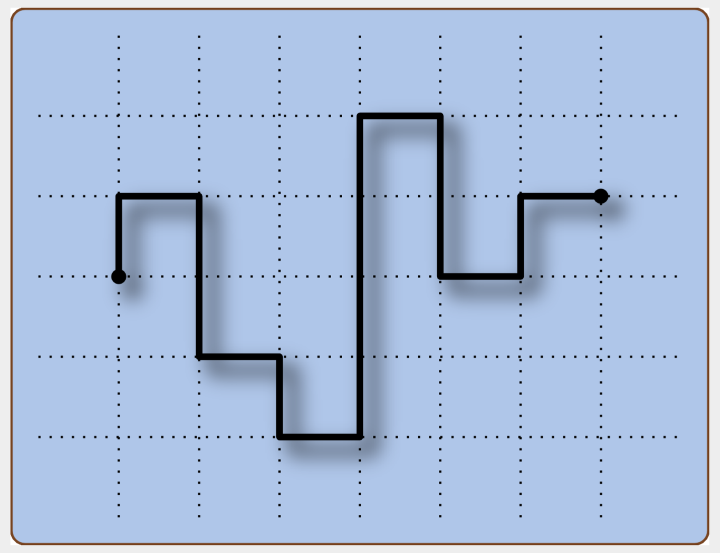
- $c_n = |\{\text{SAWs of } n \text{ steps}\}|$
- Computing c_n is a very hard combinatorial problem

$$c_n \sim A\mu^n n^{\gamma-1}(1+\cdots)$$

  - growth constant $\mu = 2.63815852927(1)$ [Guttmann & Jensen]
  - critical exponent $\gamma = 43/32$ [Nienhuis]

Partially directed self-avoiding walk

- PDSAW = a SAW that cannot step west (and ends with an east step)



- Simple rational generating function

$$P(z) = \sum_{\varphi \in \text{PDSAW}} z^{|\varphi|} = \sum_n p_n z^n = \frac{z(1-z)}{1-2z-z^2}$$

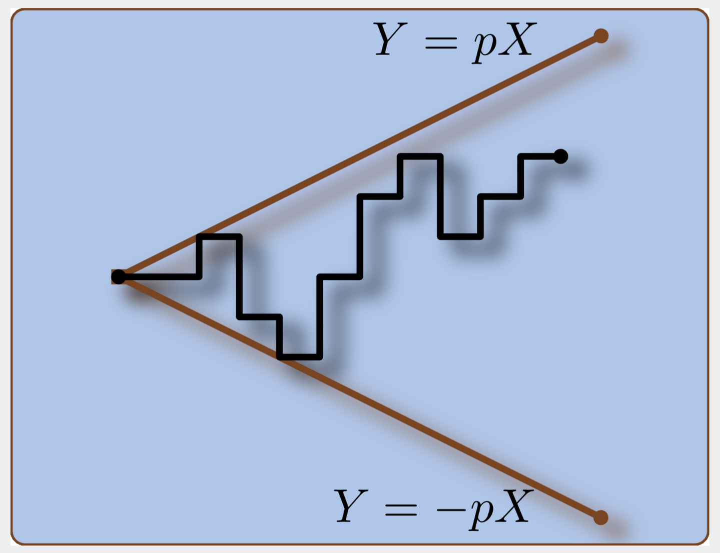
- Dominant singularity gives asymptotics

$$p_n = \frac{\sqrt{2}-1}{2}\left(1+\sqrt{2}\right)^n + o(1)$$

  - Growth constant $\mu = 1+\sqrt{2}$
  - Critical exponent $\gamma = 1$

PDSAW in a symmetric wedge

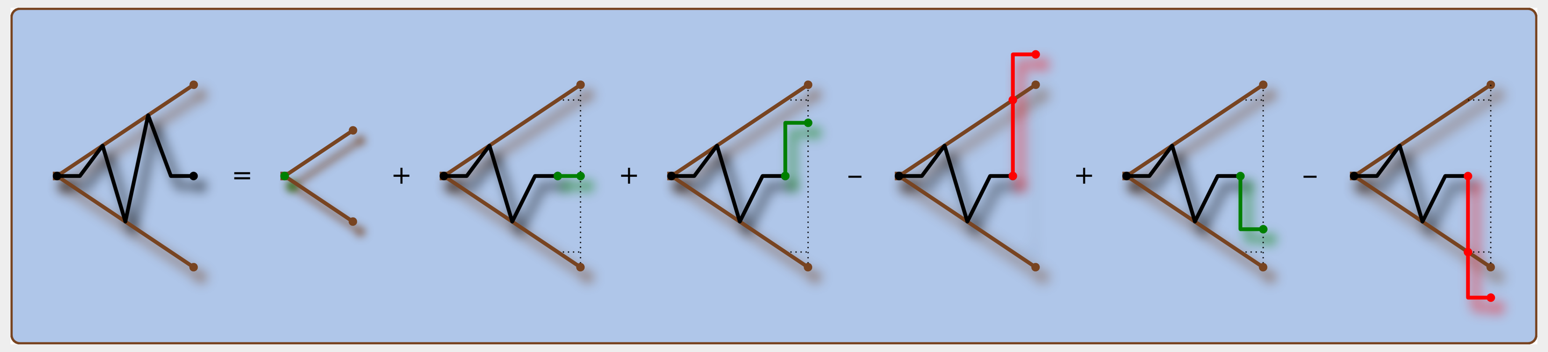
- put PDSAW in the $Y = \pm pX$ wedge



- For $p \geq 1$ the growth constant is $1+\sqrt{2}$
- But what is the critical exponent?

Functional equation in the $Y = \pm pX$ wedge

- Construct PDSAW by adding steps



- Consider the g.f. of PDSAW in the $Y = \pm pX$ wedge

$$f_p(a,b) = f_p(a,b;x,y)$$

where
  - x and y are conjugate to the number of horizontal and vertical steps
  - a and b are conjugate to the distance of the endpoint from the walls
- The construction leads to a functional equation for $f_p(a,b)$

$$f_p(a,b) = 1 + x(ab)^p f_p(a,b) + x(ab)^p \frac{ya/b}{1-ya/b}\Big(f_p(a,b) - f_p(a,ay)\Big) + x(ab)^p \frac{yb/a}{1-yb/a}\Big(f_p(a,b) - f_p(by,b)\Big)$$

- The functional equation is singular when $a = by$ or $b = ay$
  any attempt at a solving for $f_p(a,b)$ by direct iteration fails

Solving the functional equation (for $p = 1$)

- Rewrite

$$f(a,b)K(a,b) = X(a,b) + Y(a,b)\,f(a,ay) + Y(b,a)\,f(b,by)$$

with the kernel

$$K(a,b) = (b-ya)(a-yb)(1-xab) - xyab(a^2+b^2-2yab)$$

- The Iterated Kernel Method
  Solve the functional equation iteratively while satisfying $K(a,b) = 0$
  - Find the roots of the kernel $b = \beta_{\pm 1}(a)$
  - Set $b = \beta_{+1}(a)$ and iterate

$$0 = X(a,\beta(a)) + Y(a,\beta(a))\,f(a,ay) + Y(\beta(a),a)\,f(\beta(a),\beta(a)y)$$

- At first sight very complicated due to composition of

$$\beta_{\pm 1}(a) = \frac{a}{2}\left(\frac{1+y^2 \mp \sqrt{(1-y^2)(1-4xya^2-y^2)}}{y+xa^2-xy^2a^2}\right)$$

- But several "miracles" occur ...

$$f(a,ay) = \left(1+\frac{Q(a)}{y}\right)\sum_{n=0}^{\infty}(-1)^n Q(a)^n y^{n^2}$$

$$Q(a) = \left(\frac{1}{xa^2} - \frac{y}{xa\beta_1} - y\right)$$

Asymptotic results

- The number of PDSAW of length n, $v_n^{(1)}$ in the $Y = \pm X$ wedge grows as

$$v_n^{(1)} = A_0\left(1+\sqrt{2}\right)^n + \frac{5^{n/2}}{(n+1)^{3/2}}\Big(A_1 + (-1)^n A_2 + O(1/n)\Big)$$

where the constants are

$A_0 = 0.277309853486031\ldots$
$A_1 = 3.714104865336623\ldots$ $A_2 = 0.206979970208041\ldots$

- The number of PDSAW of length n, $v_n^{(p)}$ in the $Y = \pm pX$ wedge satisfies

$$0.2773\ldots \leq \lim_{n\to\infty}\frac{v_n^{(p)}}{(1+\sqrt{2})^n} \leq (1+\sqrt{2})/2 = 1.2071\ldots$$

for any $1 \leq p < \infty$

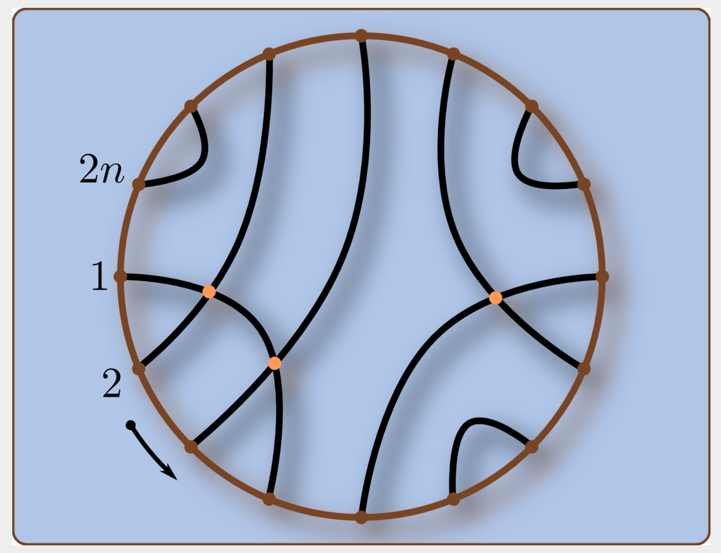
PDSAW ending on one wall

- The g.f. permits the continued fraction expansion

$$f(1,y) = \cfrac{1}{1-\cfrac{xy}{1-\cfrac{xy(1+y^2)}{1-\cfrac{xy(1+y^2+y^4)}{1-\cfrac{xy(1+y^2+y^4+y^6)}{1-\cfrac{xy(1+y^2+y^4+y^6+y^8)}{\ddots}}}}}}$$

- Suggests q-deformation of permutation and involution g.f.s

Chord diagrams $\equiv$ involutions without fixed-points

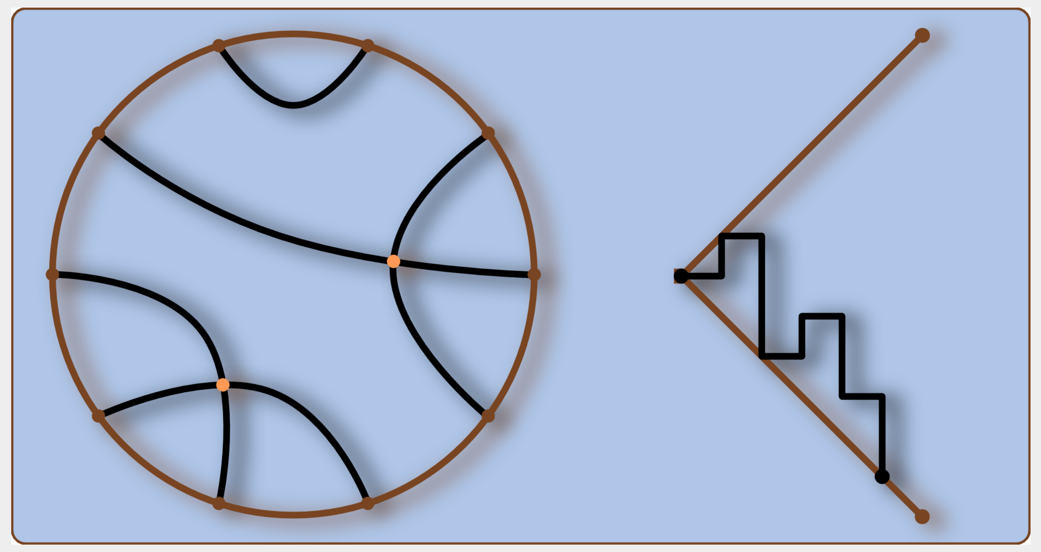


- g.f. of diagrams with n chords in which q counts crossings [Touchard]

$$\Phi_n(q) = \frac{1}{(1-q)^n}\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}q^{\binom{k}{2}}$$

A miraculous connection between PDSAW and chord diagrams

- Two equinumerous sets



The number of chord diagrams with n chords and m crossings = the number of PDSAW ending at $(-n,n)$ with n horizontal edges and $n+2m$ vertical edges

Open problem:

- The two enumeration problems have been solved by entirely different methods

A direct bijective proof would be wonderful!

This and related works

- "Partially directed paths in a wedge," J. Comb. Th. A, in print
- "Partially directed paths in a symmetric wedge," proceedings of FPSAC '07, in print
- "Directed paths in a symmetric wedge," submitted to J. Phys. A