MTH5105 Differential and Integral Analysis

Lecture Notes 2009-2010, Week 6 to Week 12

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Contents

6 Definition of the Riemann Integral **2**

7 Properties of the Riemann Integral **12**

8 The Fundamental Theorem of Calculus **19**

9 Sequences and Series of Functions **23**

10 Power Series **35**

6 Definition of the Riemann Integral

Lecture 16: 15/02/10

Let $I = [a, b]$ for $a < b$ be an interval. Given

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\ ,$$

we call

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

a partition of I. We denote the set of all partitions of I by $\mathcal{P}$.

We denote $I_i = [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$. A partition is called equidistant, if all I_i have equal length Δx_i.

P_2 is called a refinement of P_1 if $P_1 \subseteq P_2$. Two partitions P_1 and P_2 have a common refinement, for example $P = P_1 \cup P_2$ is such a refinement. The notion of refinement defines a partial order on $\mathcal{P}$.

$\sigma(P) = \max\{\Delta x_i | i = 1, 2, \ldots, n\}$ is called the mesh of P. $P_1 \subseteq P_2$ implies $\sigma(P_1) \geq \sigma(P_2)$, i.e. a refinement has a smaller mesh.

Examples.

1) $P = \left\{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \ldots, a + n\frac{b-a}{n} = b\right\}$ is an equidistant partition of $[a, b]$ with $\sigma(P) = \frac{b-a}{n}$.

2) $P_2 = \left\{0, \frac{1}{2n}, \frac{2}{2n}, \ldots, \frac{2n}{2n}\right\}$ is a refinement of $P_1 = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$. $\sigma(P_2) = \frac{1}{2n} < \sigma(P_1) = \frac{1}{n}$. Note that $P_3 = \left\{0, \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n+1}{n+1}\right\}$ is not a refinement of P_1.

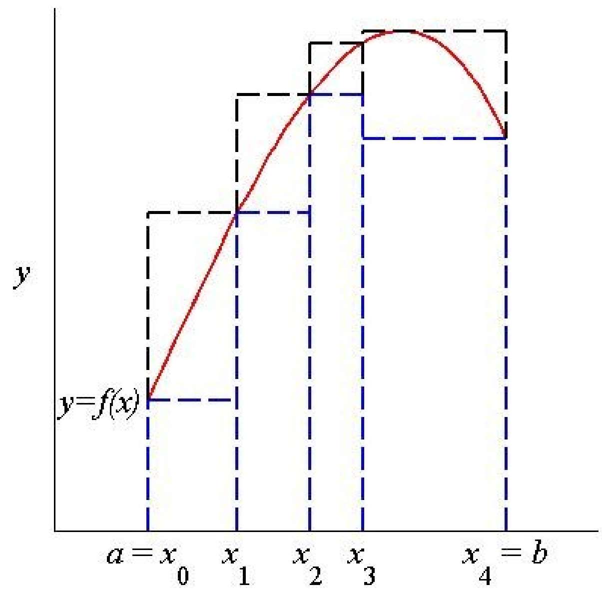
Definition 6.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded and $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We define the upper sum of f with respect to P*

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

and the lower sum of f with respect to P

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i\ ,$$

where $M_i = \sup\{f(x) : x \in I_i\}$ and $m_i = \inf\{f(x) : x \in I_i\}$.



Remark: Geometrically, if f is positive then the area A between the x-axis and the graph of $f(x)$ from a to b should satisfy

$$L(f, P) \leq A \leq U(f, P) \ .$$

Lecture 17: 18/02/10

Theorem 6.2. *Let $f : [a, b] \to \mathbb{R}$ be bounded. If P_2 is a refinement of the partition P_1 then*

(1) $U(f, P_2) \leq U(f, P_1)$, and

(2) $L(f, P_2) \geq L(f, P_1)$.

Proof. Let $P_1 = \{x_0, x_1, \ldots, x_n\}$ and $P_2 = P_1 \cup \{y\}$. If $x_{i-1} < y < x_i$ then

$$\begin{aligned} M' =& \sup\{f(x) : x \in [x_{i-1}, y]\} \leq M_i \quad \text{and} \\ M'' =& \sup\{f(x) : x \in [y, x_i]\} \leq M_i \ . \end{aligned}$$

Therefore $M_i \Delta x_i = M_i(y - x_{i-1}) + M_i(x_i - y) \geq M'(y - x_{i-1}) + M''(x_i - y)$, so that

$$\begin{aligned} U(f, P_1) =& \sum_{\substack{j=1 \\ j \neq i}}^{n} M_j \Delta x_j + M_i \Delta x_i \\ \geq& \sum_{\substack{j=1 \\ j \neq i}}^{n} M_j \Delta x_j + M'(y - x_{i-1}) + M''(x_i - y) \\ =& U(f, P_2) \ . \end{aligned}$$

Now let P_2 be an arbitrary refinement of P_1. Then P_2 is obtained from P_1 by adding a finite number of points y_j, creating a chain of partitions

$$P_1 = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_r = P_2$$

and

$$U(f, Q_0) \geq U(f, Q_1) \geq \ldots \geq U(f, Q_r) \ .$$

A similar argument leads to $L(f, P_2) \geq L(f, P_1)$. □

Corollary. *Let P_1, P_2 be partitions of $[a, b]$. Then*

$$L(f, P_1) \leq U(f, P_2) \ .$$

Proof. Let $P = P_1 \cup P_2$ be a common refinement of P_1 and P_2. Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2) \ .$$

□

Corollary. $\{U(f, P) : P \in \mathcal{P}\}$ *is bounded below and* $\{L(f, P) : P \in \mathcal{P}\}$ *is bounded above.*

Definition 6.3. *Let $f : [a, b] \to \mathbb{R}$ be bounded. We call*

$$\int_a^{*b} f(x)\,dx = \inf\{U(f, P) : P \in \mathcal{P}\}$$

the <u>upper integral</u> of f and

$$\int_{*a}^{b} f(x)\,dx = \sup\{L(f, P) : P \in \mathcal{P}\}$$

the <u>lower integral</u> of f.

Remark. Clearly,

$$\int_a^{*b} f(x)\,dx \geq \int_{*a}^{b} f(x)\,dx \ .$$

Definition 6.4. *A bounded function $f : [a, b] \to \mathbb{R}$ is <u>Riemann integrable</u> if the upper and lower integral of f agree. The quantity*

$$\int_a^b f(x)\,dx = \int_a^{*b} f(x)\,dx = \int_{*a}^{b} f(x)\,dx$$

is called the <u>Riemann integral</u> of f over $[a, b]$.

Theorem 6.5. *Let $f : [a, b] \to \mathbb{R}$ be bounded. f is Riemann integrable if and only if*

$$\forall \varepsilon > 0 \, \exists P \in \mathcal{P} : U(f, P) - L(f, P) < \varepsilon \; .$$

Proof. "$\Rightarrow$" Let f be Riemann integrable and

$$A = \sup\{L(f, P) : P \in \mathcal{P}\} = \inf\{U(f, P) : P \in \mathcal{P}\} \; .$$

Then for a given $\varepsilon > 0$ there exist $P_1, P_2 \in \mathcal{P}$ such that

$$A - \frac{\varepsilon}{2} < L(f, P_1) \quad \text{and} \quad U(f, P_2) < A + \frac{\varepsilon}{2} \; .$$

For $P = P_1 \cup P_2$ we have

$$U(f, P) - L(f, P) \le U(f, P_2) - L(f, P_1) < A + \frac{\varepsilon}{2} - \left(A - \frac{\varepsilon}{2}\right) = \varepsilon \; .$$

"$\Leftarrow$" If for any $\varepsilon > 0$ there is a $P \in \mathcal{P}$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

then

$$\int_a^{*b} f(x)\, dx - \int_{*a}^{b} f(x)\, dx \le U(f, P) - L(f, P) < \varepsilon \; .$$

As $\varepsilon > 0$ can be arbitrarily small,

$$\int_a^{*b} f(x)\, dx = \int_{*a}^{b} f(x)\, dx \; ,$$

so f is Riemann integrable.

$\square$

Lecture 18: 19/02/10

Examples.

1) Let $f : [a, b] \to \mathbb{R}$, $x \mapsto c$ be the constant function.

For $P = \{x_0, x_1, \ldots, x_n\}$ we find $m_i = M_i = c$ and thus

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = c \sum_{i=1}^{n} \Delta x_i = c(b - a)$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = c \sum_{i=1}^{n} \Delta x_i = c(b-a) \; .$$

Therefore f is Riemann integrable with

$$\int_a^b f(x)\, dx = c(b-a) \; .$$

2) Let $f : [a,b] \longrightarrow \mathbb{R}$, $x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \; , \\ 0 & x \notin \mathbb{Q} \; . \end{cases}$

For $P = \{x_0, x_1, \ldots, x_n\}$ we find $m_i = 0$ and $M_i = 1$ and thus

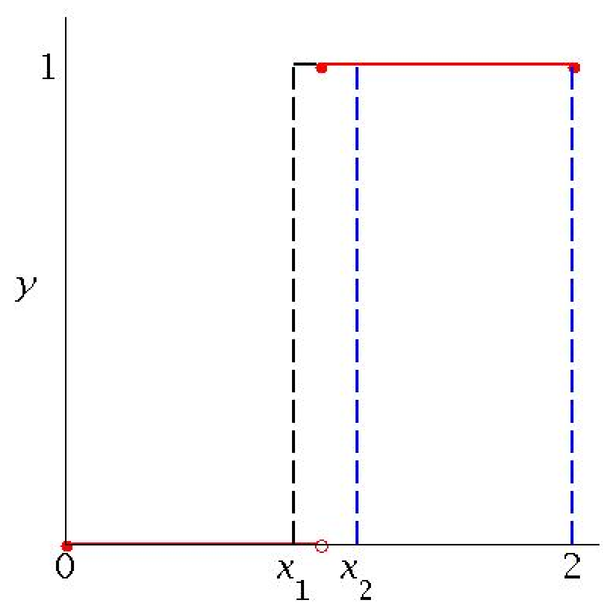
$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = (b-a)$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = 0 \; .$$

Therefore f is <u>not</u> Riemann integrable.

3) Let $f : [0,2] \longrightarrow \mathbb{R}$, $x \mapsto \begin{cases} 0 & x \in [0,1) \; , \\ 1 & x \in [1,2] \; . \end{cases}$



Choose $0 < x_1 < 1 < x_2 < 2$ with $x_2 - x_1 < \varepsilon$ and $P = \{0, x_1, x_2, 2\}$. Then

$$M_1 = m_1 = 0 \ , \quad M_2 = 1 \ , \quad m_2 = 0 \ , \quad M_3 = m_3 = 1 \ ,$$

and thus

$$U(f,P) = 0 \cdot (x_1 - 0) + \varepsilon \cdot (x_2 - x_1) + 1 \cdot (2 - x_2)$$

and

$$L(f,P) = 0 \cdot (x_1 - 0) + 0 \cdot (x_2 - x_1) + 1 \cdot (2 - x_2) \ ,$$

so that

$$U(f,P) - L(f,P) = x_2 - x_1 < \varepsilon \ .$$

Therefore f is Riemann integrable with

$$\int_0^2 f(x)\,dx = 1 \ .$$

Theorem 6.6. *Every increasing or decreasing function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Assume without loss of generality that f is increasing. Then $f(a) \leq f(x) \leq f(b)$ for $x \in [a,b]$, so f is bounded.

Let $\varepsilon > 0$. Choose a partition P with a mesh

$$\sigma(P) \leq \frac{\varepsilon}{f(b) - f(a) + 1} \ .$$

As f is increasing, $M_i = f(x_i)$ and $m_i = f(x_{i-1}$, so that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_i - 1)\Delta x_i \\ &\leq \sum_{i=1}^{n} (f(x_i) - f(x_i - 1)\sigma(P) = (f(b) - f(a)\sigma(P) \\ &\leq (f(b) - f(a)) \frac{\varepsilon}{1 + f(b) - f(a)} < \varepsilon \ . \end{aligned}$$

By Theorem 6.5, f is Riemann integrable. □

Lecture 19: 01/03/10

Definition 6.7. *A function $f : \mathcal{D} \to \mathbb{R}$ is uniformly continuous if*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall c \in \mathcal{D} \ \forall x \in \mathcal{D} : \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \ .$$

Remark. This means that δ is chosen independently of c. The statement that a function $f : \mathcal{D} \to \mathbb{R}$ is merely *continuous* is equivalent to

$$\forall c \in \mathcal{D}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} :\ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon\ .$$

Note how the statement "$\forall c \in \mathcal{D}$" has moved places. Clearly a uniformly continuous function is continuous, but a continuous function need not be uniformly continuous.

Example.

$f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is continuous, but not uniformly continuous:

To show this, assume that f is uniformly continuous. Then for $\varepsilon = 1$, say, there exists a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |x^2 - c^2| < \varepsilon = 1$ for all $x, c \in \mathbb{R}$. As δ is independent of c, this should be true for all c, for example if $c = 1/\delta$. But then, for $x = c + \delta/2$, we find $|x - c| = \delta/2 < \delta$ and

$$|x^2 - c^2| = |(c + \delta/2)^2 - c^2| = |c\delta + \delta^2/4| = 1 + \delta^2/4 > 1$$

which is a contradiction.

This example works because the domain is not closed and bounded. Continuous functions on closed and bounded domains are in fact uniformly continuous. We shall see below that this is an important ingredient in proving Riemann integrability of continuous functions.

Theorem 6.8. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

Lecture 20: 04/03/10

Proof. Suppose f is continuous on $[a, b]$ but *not* uniformly continuous. Then

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists c \in \mathcal{D}\ \exists x \in \mathcal{D} :\ |x - c| < \delta \Rightarrow |f(x) - f(c)| \geq \varepsilon\ .$$

So there exists $\varepsilon > 0$ such that for $\delta = 1/n$ there exist $c_n, x_n \in \mathcal{D}$ with

$$|x_n - c_n| < \delta \quad \text{but} \quad |f(x_n) - f(c_n)| \geq \varepsilon\ .$$

Now (and this is the key step!) using Bolzano-Weierstraß, (c_n) contains a convergent subsequence. Therefore there exist $(n_r)_{r \in \mathbb{N}}$ such that

(a) $\lim\limits_{r \to \infty} c_{n_r} = d$ for some $d \in [a, b]$,

(b) $\lim_{r\to\infty} x_{n_r} = d$ (as $|x_{n_r} - d| \le |x_{n_r} - c_{n_r}| + |c_{n_r} - d|$), and

(c) $\lim_{r\to\infty} f(c_{n_r}) = f(d)$ and $\lim_{r\to\infty} f(x_{n_r}) = f(d)$.

But by assumption for all n, $|f(x_n) - f(c_n)| \ge \varepsilon$, which is a contradiction. □

Theorem 6.9. *Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. By Theorem 6.8, f is uniformly continuous on $[a, b]$, so that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall c, c' \in [a, b] :\ |c - c'| < \delta \Rightarrow |f(c) - f(c')| < \frac{\varepsilon}{b-a}\ .$$

Now choose a partition P with $\sigma(P) < \delta$. Then on each interval I_i, f assumes its minimum m_i at some c_i and its maximum M_i at some c_i', so that $m_i = f(c_i)$ and $M_i = f(c_i')$. As $|c_i - c_i'| \le \sigma(P) < \delta$,

$$M_i - m_i = |f(c_i') - f(c_i| < \frac{\varepsilon}{b-a}\ .$$

Therefore

$$U(f, P) - L(f, P) = \sum_{i=0}^{n} (M_i - m_i)\Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \varepsilon\ .$$

By Theorem 6.5, f is Riemann integrable. □

Lecture 21: 05/03/10

Examples.

1) $f : [a, b] \to \mathbb{R}$, $f(x) = x$:

   f is increasing, therefore Riemann integrable. To compute the Riemann integral, choose

   $$P_n = \{a, a + \Delta, a + 2\Delta, \dots, a + n\Delta = b\}$$

   where $\Delta = \frac{b-a}{n}$. The mesh of the partition is given by $\sigma(P_n) = \Delta = \frac{b-a}{n}$. We find

   $$m_i = a + (i - 1)\Delta\ , \quad \text{and} \quad M_i = a + i\Delta\ .$$

Therefore

$$
\begin{aligned}
L(f, P_n) =& \sum_{i=1}^{n} (a + (i-1)\Delta)\Delta \\
=& an\Delta + \frac{n(n-1)}{2}\Delta^2 \\
=& a(b-a) + \frac{1}{2}(b-a)^2 \left(1 - \frac{1}{n}\right) .
\end{aligned}
$$

Therefore

$$
\int_{*a}^{b} f(x)\, dx = \lim_{n\to\infty} L(f, P_n) = a(b-a) + \frac{1}{2}(b-a)^2 = \frac{b^2}{2} - \frac{a^2}{2} .
$$

As we already know that f is Riemann integrable, we now conclude that

$$
\int_{a}^{b} f(x)\, dx = \int_{*a}^{b} f(x)\, dx = \frac{b^2}{2} - \frac{a^2}{2} .
$$

If we didn't know that f was Riemann integrable, a computation of the upper sums shows that

$$
U(f, P_n) = a(b-a) + \frac{1}{2}(b-a)^2 \left(1 + \frac{1}{n}\right) .
$$

Just as we should, we find that $U(f, P_n) - L(f, P_n) = (b-a)^2 \dfrac{1}{n} \to 0$ as $n \to \infty$, and that

$$
\int_{a}^{*b} f(x)\, dx = \frac{b^2}{2} - \frac{a^2}{2} = \int_{*a}^{b} f(x)\, dx .
$$

2) $f : [1, a] \to \mathbb{R}$, $f(x) = 1/x$:

f is decreasing, therefore Riemann integrable. To compute the Riemann integral, choose

$$
P_n = \{1 = q^0, q^1, q^2, \ldots, q^n = a\}
$$

where $q = \sqrt[n]{a}$. We find

$$
\Delta x_i = q^i - q^{i-1} = (q-1)q^{i-1} ,
$$

so that the mesh of the partition is given by $\sigma(P_n) = (q-1)q^{n-1}$. We find

$$
m_i = \frac{1}{q^i} , \quad \text{and} \quad M_i = \frac{1}{q^{i-1}} .
$$

Therefore

$$
\begin{aligned}
L(f, P_n) &= \sum_{i=1}^{n} \frac{1}{q^i}(q-1)q^{i-1} \\
&= \sum_{i=1}^{n} \frac{1}{q}(q-1) = n\left(1 - \frac{1}{q}\right) = n\left(1 - \frac{1}{\sqrt[n]{a}}\right) \ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_{*1}^{a} f(x)\,dx &= \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} n\left(1 - a^{-1/n}\right) \\
&= \lim_{n\to\infty} n\left(1 - \exp\left(-\frac{1}{n}\log(a)\right)\right) \\
&= \lim_{t\to 0} \frac{1 - \exp(-t\log(a))}{t} \\
&= \lim_{t\to 0} \frac{\log(a)\exp(-t\log(a))}{1} = \log(a) \ .
\end{aligned}
$$

As we already know that f is Riemann integrable, we now conclude that

$$
\int_{1}^{a} f(x)\,dx = \int_{*1}^{a} f(x)\,dx = \log(a) \ .
$$

If we didn't know that f was Riemann integrable, a computation of the upper sums shows that

$$
U(f, P_n) = n(q-1) \ .
$$

Just as we should, we find that $U(f, P_n) - L(f, P_n) = n(q-1)^2/q \to 0$ as $n \to \infty$, and that

$$
\int_{1}^{*a} f(x)\,dx = \log(a) = \int_{*1}^{a} f(x)\,dx \ .
$$

7 Properties of the Riemann Integral

Theorem 7.1. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. If $[c, d] \subseteq [a, b]$ then f is Riemann integrable on $[c, d]$.*

Proof. Let $\varepsilon > 0$. Then there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. If we let

$$P' = P \cup \{c, d\} = \{x_0, x_1, \dots, x_k = c, x_{k+1}, \dots, x_{k+r} = d, x_{k+r+1}, \dots, x_n\}$$

then

$$U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \varepsilon$$

. Now let

$$P'' = \{x_k, x_{k+1}, \dots, x_{k+r}\} .$$

This is a partition of $[c, d]$ with

$$\begin{aligned} U(f, P'') - L(f, P'') &= \sum_{i=k+1}^{k+r} (M_i - m_i)\Delta x_i \\ &\leq \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= U(f, P') - L(f, P') < \varepsilon . \end{aligned}$$

Thus f is Riemann integrable on $[c, d]$. □

Lecture 22: 08/03/10

Theorem 7.2. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, c]$ and $[c, b]$ where $a < c < b$. Then f is Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx .$$

Proof. Let $\varepsilon > 0$ and let P_1 and P_2 be partitions of $[a, c]$ and $[c, b]$, respectively, with

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad \text{and} U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2} .$$

Then $P = P_1 \cup P_2$ is a partition of $[a, b]$ with

$$U(f, P) - L(f, P) = U(f, P_1) + U(f, P_2) - L(f, P_1) - L(f, P_2) < \varepsilon$$

and hence f is Riemann integrable on $[a, b]$. Moreover, as

$$L(f, P_1) \leq \int_a^c f(x)\,dx \leq U(f, P_1) \quad \text{and} \quad L(f, P_2) \leq \int_c^b f(x)\,dx \leq U(f, P_2)$$

we have

$$L(f, P) \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \leq U(f, P)\ .$$

Clearly we also have

$$L(f, P) \leq \int_a^b f(x)\,dx \leq U(f, P)\ ,$$

and taking differences leads to

$$L(f, P) - U(f, P) \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx \leq U(f, P) - L(f, P)$$

or, equivalently,

$$\left| \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx \right| \leq U(f, P) - L(f, P)\ .$$

Therefore. we have shown that for all $\varepsilon > 0$

$$\left| \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx \right| < \varepsilon$$

so that

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx\ .$$

□

Remark. Because of Theorem 7.2 it makes sense to define for $a > b$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\ .$$

Then, if f is Riemann integrable on a closed and bounded interval I, and $a, b, c \in I$, we have

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx\ .$$

Theorem 7.3. *Let $f, g : [a, b] \to \mathbb{R}$ be bounded and P be a partition of $[a, b]$. Then*

(a) $U(f + g, P) \leq U(f, P) + U(g, P)$, and

(b) $L(f+g,P) \geq L(f,P) + L(g,P)$.

Proof. For a subinterval I_i of the partition P, we write $M_i(h) = \sup\{h(x) : x \in I_i\}$ and $m_i(h) = \inf\{h(x) : x \in I_i\}$.

(a) On a subinterval I_i of the partition P we have

$$\begin{aligned} M_i(f+g) &= \sup\{f(x)+g(x) : x \in I_i\} \\ &\leq \sup\{f(x) : x \in I_i\} + \sup\{g(x) : x \in I_i\} = M_i(f) + M_i(g) \ . \end{aligned}$$

Thus

$$\begin{aligned} U(f+g,P) &= \sum_{i=1}^{n} M_i(f+g)\Delta x_i \\ &\leq \sum_{i=1}^{n} M_i(f)\Delta x_i + \sum_{i=1}^{n} M_i(g)\Delta x_i = U(f,P) + U(g,P) \ . \end{aligned}$$

(b) Similarly,

$$\begin{aligned} L(f+g,P) &= \sum_{i=1}^{n} m_i(f+g)\Delta x_i \\ &\geq \sum_{i=1}^{n} m_i(f)\Delta x_i + \sum_{i=1}^{n} m_i(g)\Delta x_i = L(f,P) + L(g,P) \ . \end{aligned}$$

□

Theorem 7.4. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable and $c \in \mathbb{R}$. Then $f+g$ and cf are Riemann integrable, and*

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \quad \textit{and}$$
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \ .$$

Proof. (a) Let $\varepsilon > 0$. There exist partitions P_1 and P_2 such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2} \quad \text{and} U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2} \ .$$

Let $P = P_1 \cup P_2$. Then

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2} \quad \text{and} \\ U(g,P) - L(g,P) &\leq U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2} \ . \end{aligned}$$

By Theorem 7.3 it follows that

$$U(f+g,P) - L(f+g,P) \le U(f,P) + U(g,P) - L(f,P) - L(g,P) < \varepsilon \ ,$$

so $f+g$ is Riemann integrable on $[a,b]$.

We proceed now as in the proof of Theorem 7.2. As

$$L(f,P) \le \int_a^b f(x)\,dx \le U(f,P) \quad \text{and} \quad L(g,P) \le \int_a^b g(x)\,dx \le U(g,P)$$

we have

$$L(f,P) + L(g,P) \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le U(f,P) + U(g,P) \ .$$

Clearly we also have

$$\begin{aligned} L(f,P) + L(g,P) \le L(f+g,P) \le \int_a^b f(x) + g(x)\,dx & \\ \le U(f+g,P) \le U(f,P) + U(g,P) \ , & \end{aligned}$$

and taking differences leads to

$$\begin{aligned} \left| \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \int_a^b f(x) + g(x)\,dx \right| & \\ \le U(f,P) + U(g,P) - L(f,P) - L(g,P) \ . & \end{aligned}$$

Therefore we have shown that for all $\varepsilon > 0$

$$\left| \int_a^b f(x) + g(x)\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < \varepsilon \ ,$$

so that

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \ .$$

(b) This is an exercise. The key step is to show that

$$U(cf,P) - L(cf,P) \le |c|(U(f,P) - L(f,P)) \ .$$

$\square$

Theorem 7.5. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. If $g : [a,b] \to \mathbb{R}$ differs from f at finitely many points then g is also Riemann integrable, and*

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx\ .$$

Proof. For $c \in [a,b]$, define

$$\chi_c(x) = \begin{cases} 1 & x = c\ , \\ 0 & x \neq c\ . \end{cases}$$

If g differs from f at $\{c_1, c_2, \ldots, c_n\}$, then

$$g(x) = f(x) + \sum_{i=1}^{n} (g(c_i) - f(c_i))\chi_{c_i}(x)\ ,$$

and it suffices to show that $\chi_c(x)$ is Riemann integrable with $\int_a^b \chi_c(x)\,dx = 0$. We shall show this by choosing suitable partitions. Lecture 23: 11/03/10

If $a < c < b$, choose $P = \{a, x_1, x_2, b\}$ with $a < x_1 < x_2 < b$ and $x_2 - x_1 < \varepsilon$. It follows that

$$0 = L(\chi_c, P) < U(\chi_c, P) < \varepsilon\ .$$

If $c = a$, choose $P = \{a, x_1, b\}$ with $a < x_1 < b$ and $x_1 - a < \varepsilon$. It follows that

$$0 = L(\chi_a, P) < U(\chi_a, P) < \varepsilon\ .$$

If $c = b$, choose $P = \{a, x_1, b\}$ with $a < x_1 < b$ and $b - x_1 < \varepsilon$. It follows that

$$0 = L(\chi_b, P) < U(\chi_b, P) < \varepsilon\ .$$

Thus, for all $\varepsilon > 0$ there exists a partition P with $U(\chi_c, P) - L(\chi_c, P) < \varepsilon$. Therefore χ_c is Riemann integrable. As $L(\chi_c, P) = 0$ for any partition P,

$$\int_a^b \chi_c(x)\,dx = 0\ .$$

□

Theorem 7.6. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable. If $f(x) \leq g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx\ .$$

Proof. As $g(x) - f(x) \geq 0$, we find

$$0 \leq L(g - f, P) \leq \int_a^b g(x) - f(x)\, dx = \int_a^b g(x)\, dx - \int_a^b f(x)\, dx \ .$$

□

Theorem 7.7. *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then $|f|$ is Riemann integrable, and*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)| dx \ .$$

Proof. For a partition P of $[a, b]$, we define

$$M_i = \sup\{f(x) : x \in I_i\} \ , \qquad M_i^* = \sup\{|f(x)| : x \in I_i\} \ ,$$
$$m_i = \inf\{f(x) : x \in I_i\} \ , \qquad m_i^* = \inf\{|f(x)| : x \in I_i\} \ .$$

Starting with

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|$$

we can show (exercise problem) that

$$M_i^* - m_i^* \leq M_i - m_i \ .$$

Therefore

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{i=1}^n (M_i^* - m_i^*)\Delta x_i \\ &\leq \sum_{i=1}^n (M_i - m_i)\Delta x_i = U(f, P) - L(f, P) \ . \end{aligned}$$

As f is Riemann integrable, it follows that $|f|$ is Riemann integrable. Furthermore,

$$-|f(x)| \leq f(x) \leq |f(x)|$$

implies by Theorem 7.6 that

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)| dx \ .$$

□

Theorem 7.8. *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable then f^2 is Riemann integrable.*

Proof. As f is bounded on $[a,b]$, there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Given a partition P of $[a,b]$, we have

$$M_i(f^2) = (M_i(|f|))^2 \quad \text{and} \quad m_i(f^2) = (m_i(|f|))^2 \; .$$

Therefore

$$M_i(f^2) - m_i(f^2) = (M_i(|f|) + m_i(|f|))(M_i(|f|) - m_i(|f|)) \leq 2M(M_i(|f|) - m_i(|f|)) \; .$$

Thus

$$U(f^2, P) - L(f^2, P) \leq 2M(U(|f|, P) - L(|f|, P));,$$

and hence f^2 is Riemann integrable. □

Remark. The above proof shows also that

$$\int_a^b f^2(x)\, dx \leq 2M \int_a^b |f(x)| dx \; .$$

Theorem 7.9. *If $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable then fg is Riemann integrable.*

Proof. We write

$$f(x)g(x) = \frac{1}{4}\left((f(x)+g(x))^2 - (f(x)-g(x))^2\right) \; .$$

Now $f+g$ and $f-g$ are Riemann integrable by Theorem 7.4, and thus $(f+g)^2$ and $(f-g)^2$ are Riemann integrable by Theorem 7.8. By Theorem 7.4 it follows that $fg = \dfrac{1}{4}\left((f+g)^2 - (f-g)^2\right)$ is Riemann integrable. □

8 The Fundamental Theorem of Calculus

Lecture 24: 12/03/10

Definition 8.1. *Let I be an interval and $f : I \to \mathbb{R}$ be a function. A differentiable function $F : I \to \mathbb{R}$ is called an antiderivative of f if $F'(x) = f(x)$ for all $x \in I$.*

Theorem 8.2. *If F and G are antiderivatives of f, then $G = F + c$ for some $c \in \mathbb{R}$. Also, $F + c$ is an antiderivative for all $c \in \mathbb{R}$.*

Proof. $(G - F)' = G' - F' = f - f = 0$, so $G - F$ is constant. Also $(F + c)' = F' = f$ for all $c \in \mathbb{R}$. □

Theorem 8.3 (The Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable. If F is an antiderivative of f then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \ .$$

Proof. Let P be a partition of $[a, b]$. Applying the Mean Value Theorem to F on I_i, there exists a $c_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x_i \ .$$

As

$$m_i = \inf\{f(x) : x \in I_i\} \le f(c_i) \le \sup\{f(x) : x \in I_i\} = M_i \ ,$$

it follows that

$$L(f, P) \le \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \le U(f, P) \ .$$

Therefore

$$\int_{*a}^{b} f(x)\,dx \le F(b) - F(a) \le \int_a^{*b} f(x)\,dx \ ,$$

and as f is Riemann integrable, it follows that

$$\int_a^b f(x)\,dx = F(b) - F(a) \ .$$

□

Example. An antiderivative of $f(x) = 1/x$ is $F(x) = \log(x)$, as $F'(x) = f(x)$. We use this to compute

$$\int_1^a \frac{dx}{x} = \log(x)|_1^a = \log(a) - \log(1) = \log(a) \ .$$

For further examples, see Calculus I.

Theorem 8.4. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and define $F : [a,b] \to \mathbb{R}$ by*

$$F(t) = \int_a^t f(x)\,dx \;.$$

Then

(a) F is continuous on $[a,b]$.

(b) If f is continuous at $c \in [a,b]$ then F is differentiable at c and $F'(c) = f(c)$.

Proof. (a) f is Riemann integrable, hence bounded, i.e. there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Given $t, t_0 \in [a,b]$, we have

$$|F(t) - F(t_0)| = \left| \int_a^t f(x)\,dx - \int_a^{t_0} f(x)\,dx \right| = \left| \int_{t_0}^t f(x)\,dx \right| \leq M|t - t_0| \;.$$

If $|t - t_0| < \delta = \dfrac{\varepsilon}{M}$ then $|F(t) - F(t_0)| < \varepsilon$, implying continuity of F.

(b) Let f be continuous at c, i.e. $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in [a,b] : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$. Hence, if $0 < |t - c| < \delta$ then

$$\left| \frac{F(t) - F(c)}{t - c} - f(c) \right| = \left| \frac{\int_c^t f(x)\,dx - \int_c^t f(c)\,dx}{t - c} \right| \leq \left| \frac{\int_c^t |f(x) - f(c)| dx}{t - c} \right| < \varepsilon \;.$$

Thus $F'(c) = \lim_{t \to c} \dfrac{F(t) - F(c)}{t - c}$ exists and $F'(c) = f(c)$.

□

Lecture 25: 15/03/10

Example. Let $f : [-1,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & x \in [-1,0] \;, \\ 1 & x \in (0,1] \;. \end{cases}$$

Then

$$F(t) = \int_{-1}^t f(x)\,dx = \begin{cases} 0 & t \in [-1,0] \;, \\ t & t \in (0,1] \;. \end{cases}$$

The function F is continuous on $[-1,1]$ and differentiable on $[-1,0) \cup (0,1]$, but not differentiable at $t = 0$.

Corollary. *Every continuous function $f : [a, b] \to \mathbb{R}$ has an antiderivative.*

Proof. By Theorem 8.4, $F(t) = \int_a^t f(t)\,dt$ is an antiderivative of f. □

Definition 8.5. *If F is an antiderivative of f, we define*

$$\int f(x)\,dx = F(x) + c\;,$$

the indefinite integral of f.

Theorem 8.6. *If f and g have antiderivatives on I, then so do $f + g$ and cf for $c \in \mathbb{R}$. Moreover,*

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx \quad \textit{and} \quad \int cf(x)\,dx = c\int f(x)\,dx\;.$$

Proof. $F' = f$ and $G' = g$ imply $(F + G)' = F' + G' = f + g$. Therefore

$$\int f(x) + g(x)\,dx = F(x) + G(x) = \int f(x)\,dx + \int g(x)\,dx\;.$$

Similarly, $(cF)' = cF'$, so that

$$\int cf(x)\,dx = cF(x) = c\int f(x)\,dx\;.$$

□

Theorem 8.7. *Let $f, g : I \to \mathbb{R}$ be differentiable. If fg' has an antiderivative, then so does $f'g$, and*

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx;\,.$$

Proof. Let H be the antiderivative of $h = fg'$, i.e. $H' = h = fg'$. Then $(fg)' = f'g + fg'$ implies that

$$f'g = (fg)' - fg' = (fg)' - H' = (fg - H)'\;.$$

Therefore $fg - H$ is an antiderivative of $f'g$, and

$$\int f'(x)g(x)\,dx = f(x)g(x) - H(x) = f(x)g(x) - \int f(x)g'(x)\,dx\;.$$

□

Theorem 8.8. *Let $g : I \to \mathbb{R}$ be differentiable and let F be an antiderivative of $f : g(I) \to \mathbb{R}$. Then $F \circ g$ is an antiderivative of $(f \circ g)g'$, i.e.*

$$F(g(x)) = \int f(g(x))g'(x)\,dx\ .$$

Proof. We verify that $(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$. $\square$

Corollary. *Let $g : [a, b] \to \mathbb{R}$ be continuously differentiable and let $f : g([a, b]) \to \mathbb{R}$ be continuous. Then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)du\ .$$

Proof. f and $(f \circ g)g'$ are both continuous on $[a, b]$, hence Riemann integrable. As f is continuous, it has an antiderivative, F. By Theorem 8.8, $F \circ g$ is an antiderivative of $(f \circ g)g'$, and

$$\int f(g(x))g'(x) = F(g(x))\ .$$

By the Fundamental Theorem of Calculus,

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du\ .$$

$\square$

9 Sequences and Series of Functions

Lecture 26: 18/03/10

Let $\mathcal{D} \subseteq \mathbb{R}$ be a domain. Unless stated otherwise, in this section all functions map $D \to \mathbb{R}$.

Definition 9.1. *Let (f_n) be a sequence of functions.*

(1) f_n converges pointwise to a function f if

$$\forall x \in \mathcal{D}\ \forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : |f_n(x) - f(x)| < \varepsilon\ .$$

(2) f_n converges uniformly to a function f if

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \forall x \in \mathcal{D} : |f_n(x) - f(x)| < \varepsilon\ .$$

Remark. In (1) n_0 depends on x and ε, whereas in (2) n_0 depends on ε, but not on x. In both cases, we can write

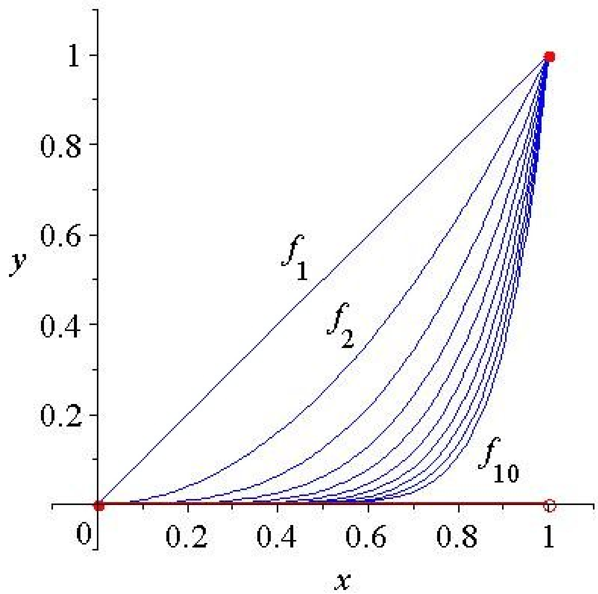
$$f(x) = \lim_{n\to\infty} f_n(x)\ .$$

Note that the limit notation does not indicate whether the convergence is uniform or pointwise.

Clearly uniform convergence implies pointwise convergence, but the converse is not true.

Examples.

(1) $f_n : [0,1] \to \mathbb{R}$, $x \mapsto x^n$.



We find

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \le x < 1\ , \\ 1 & x = 1\ . \end{cases}$$

Thus f_n converges pointwise to the discontinuous function

$$f : [0,1] \to \mathbb{R}\ , \quad x \mapsto \begin{cases} 0 & 0 \le x < 1\ , \\ 1 & x = 1\ . \end{cases}$$

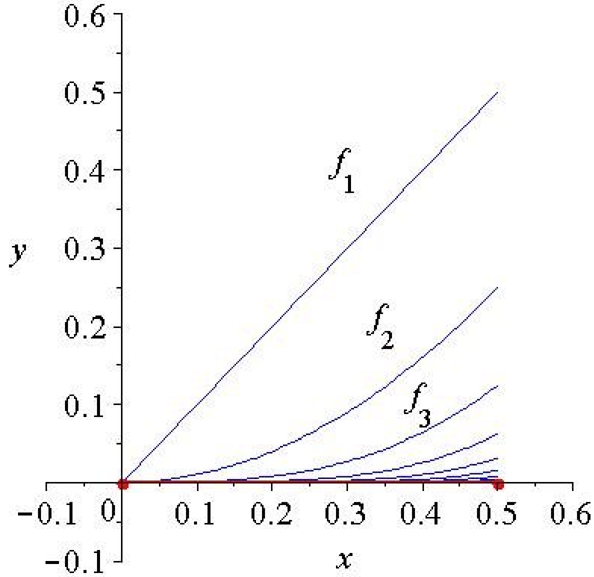
This convergence is <u>not</u> uniform: we need to show

$$\exists \varepsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists n \ge n_0\ \exists x \in [0,1] : |f_n(x) - f(x)| \ge \varepsilon\ .$$

Take $\varepsilon = 1/2$ and consider $x = 2^{-1/n}$:

$$|f_n(2^{-1/n}) - f(2^{-1/n})| = |(2^{-1/n})^n - 0| = \frac{1}{2} \ge \varepsilon\ .$$

(2) $f_n : [0, 1/2] \to \mathbb{R}$, $x \mapsto x^n$.



For $0 \le x \le 1/2$ we find $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$. Thus f_n converges to

$$f : [0, 1/2] \to \mathbb{R}\ , \quad x \mapsto 0\ .$$

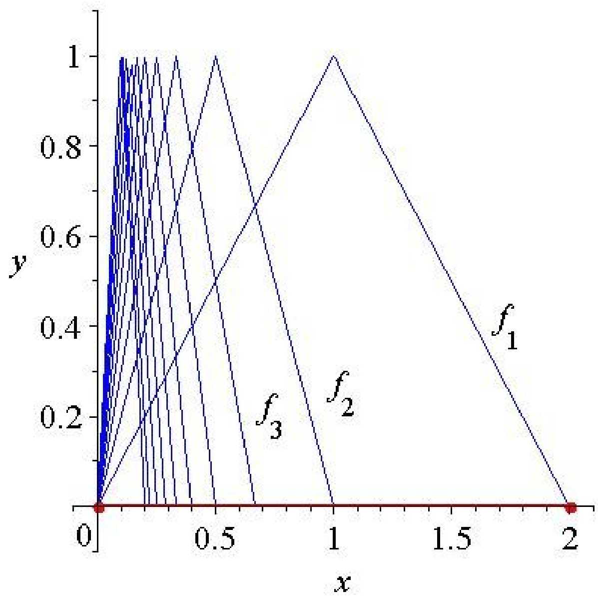
This convergence is uniform:

The difference between $f_n(x)$ and $f(x)$ is largest at $x = 1/2$. Therefore, if we pick an integer n_0 such that $n_0 > -\log(\varepsilon)/\log(2)$ to ensure $(1/2)^{n_0} < \varepsilon$, then for all $n \ge n_0$,

$$|f_n(x) - f(x)| = |x^n - 0| \le (1/2)^n \le (1/2)^{n_0} < \varepsilon\ .$$

(3) $f_n : [0,2] \to \mathbb{R}$,

$$x \mapsto \begin{cases} nx & 0 \le x \le 1/n\ , \\ 2 - nx & 1/n < x \le 2/n\ , \\ 0 & 2/n < x \le 2\ . \end{cases}$$



$f_n(0) = 0$, and if $0 < x \le 2$ then $f_n(x) = 0$ if $n \ge 2/x$, so that

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for all } 0 \le x \le 2.$$

Thus f_n converges to

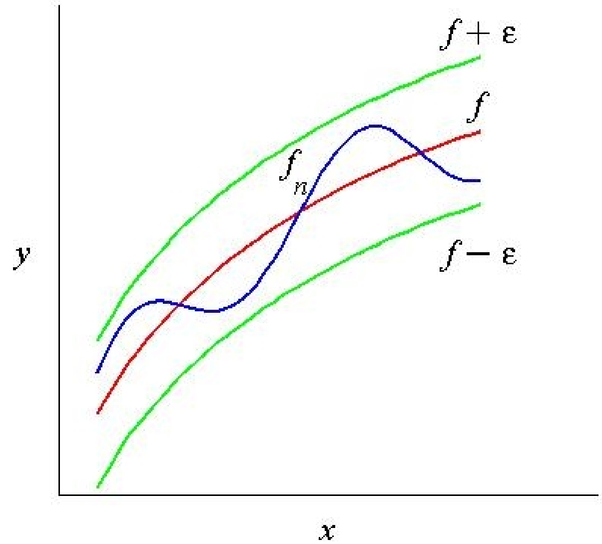
$$f : [0,2] \to \mathbb{R}\ , \quad x \mapsto 0\ .$$

This convergence is <u>not</u> uniform: take $\varepsilon = 1$ and consider $x = 1/n$:

$$|f_n(1/n) - f(1/n)| = |1 - 0| = 1 \ge \varepsilon\ .$$

Remark. The following figures indicate the idea of an "ε-tube" around the limiting function f.

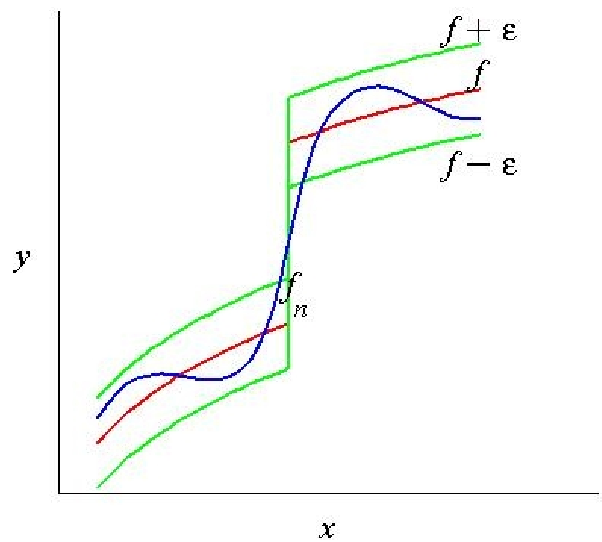
Lecture 27: 19/03/10



ε − *tube of uniform convergence*

In the case of uniform convergence, given $\varepsilon > 0$, the graph of $y = f_n(x)$ must lie entirely within the ε-tube of f for all sufficiently large n.

When the limiting function f is discontinuous, the ε-tube is "broken".



ε − *tube of a discont. function is broken*

If f is a limit of continuous f_n, no f_n can lie entirely within the ε-tube of f if ε is sufficiently small.

Theorem 9.2. *Let $f_n : \mathcal{D} \to \mathbb{R}$ converge uniformly to $f : \mathcal{D} \to \mathbb{R}$. If f_n are continuous at $a \in \mathcal{D}$ then f is continuous at a.*

Proof. We need to show

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\ .$$

By assumption we have

(a) $\forall \varepsilon' > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \forall x \in \mathcal{D} : |f(x) - f_n(x)| < \varepsilon'$, and

(b) $\forall \varepsilon'' > 0\ \exists \delta > 0\ \forall x \in \mathcal{D} : |x - a| < \delta \Rightarrow |f_n(x) - f_n(a)| < \varepsilon''$.

We start estimating the distance between $f(x)$ and $f(a)$ by splitting $|f(x) - f(a)|$ into three parts:

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|\ .$$

First, given $\varepsilon > 0$, we choose $\varepsilon' = \varepsilon/3$. By (a) there is an n_0 such that for all $n \geq n_0$ and for all $x \in \mathcal{D}$:

$$|f(x) - f_n(x)| < \varepsilon/3$$

(so that clearly also $|f(a) - f_n(a)| < \varepsilon/3$). Next, fix an $n > n_0$ and choose $\varepsilon'' = \varepsilon/3$. By (b) there exists a $\delta > 0$ such that for all $x \in \mathcal{D}$:

$$|x - a| < \delta \Rightarrow |f_n(x) - f_n(a)| < \varepsilon/3\ .$$

Thus, given $\varepsilon > 0$ we have shown that there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for $|x - a| < \delta$. □

Remark. This theorem implies that *under the assumption of uniform convergence of the functions* we can exchange limits as follows:

$$\lim_{x \to a} \underbrace{\lim_{n \to \infty} f_n(x)}_{f(x)} = \lim_{n \to \infty} \underbrace{\lim_{x \to a} f_n(x)}_{f_n(a)}\ .$$

If the convergence of f_n to f is not uniform, this is generally not correct. For example $\lim_{x \to 1^-} \lim_{n \to \infty} x^n = 0$ but $\lim_{n \to \infty} \lim_{x \to 1^-} x^n = 1$ (see example (1) above).

An immediate consequence of Theorem 9.2 is the next theorem.

Theorem 9.3. *If a sequence of continuous functions converges uniformly, then the limiting function is continuous.*

Remark. If the limiting function is discontinuous, the convergence cannot be uniform.

Examples (continued).

(1) f_n are continuous, the limiting function is not continuous. Therefore the convergence of f_n to f cannot be uniform.

(2) f_n are continuous, and the convergence is uniform. Therefore the limiting function is continuous.

(3) f_n are continuous, the limiting function is continuous. However, this does not imply uniform convergence.

Lecture 28: 22/03/10

Theorem 9.4. *Let $f_n : [a, b] \to \mathbb{R}$ be Riemann integrable. If f_n converges uniformly to $f : [a, b] \to \mathbb{R}$ then f is Riemann integrable and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx \ .$$

Remark. This theorem implies that *under the assumption of uniform convergence of the functions* we can exchange limits as follows:

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx \ .$$

Proof. Let $\varepsilon > 0$. We want to show that there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$. We shall do this in three steps.

(a) We know that f_n converges uniformly to f:

$$\exists n \in \mathbb{N}\ \forall x \in [a, b] : |f(x) - f_n(x)| < \frac{\varepsilon}{3(b-a)} \ .$$

(b) Once n is chosen, we use Riemann integrability for f_n:

$$\exists P : U(f_n, P) - L(f_n, P) < \frac{\varepsilon}{3} \ .$$

(c) Now we constrain upper and lower sums $U(f,P)$ and $L(f,P)$: f_n is bounded, and (a) implies that $f-f_n$ is bounded, so that

$$\begin{aligned} M_i = \sup\{f(x) : x\in I_i\} &\le \sup\{f_n(x) : x\in I_i\} + \sup\{f(x)-f_n(x) : x\in I_i\} \\ &\le M_i^{(n)} + \frac{\varepsilon}{3(b-a)} \text{ , and} \\ m_i = \inf\{f(x) : x\in I_i\} &\ge \inf\{f_n(x) : x\in I_i\} + \inf\{f(x)-f_n(x) : x\in I_i\} \\ &\ge m_i^{(n)} - \frac{\varepsilon}{3(b-a)} \ . \end{aligned}$$

Therefore

$$\begin{aligned} U(f,P)-U(f_n,P) &\le \sum_{i=1}^{n}(M_i - M_i^{(n)})\Delta x_i \le \frac{\varepsilon}{3(b-a)}\sum_{i=1}^{n}\Delta x_i = \frac{\varepsilon}{3} \text{ , and} \\ L(f,P)-L(f_n,P) &\ge \sum_{i=1}^{n}(m_i - m_i^{(n)})\Delta x_i \ge -\frac{\varepsilon}{3(b-a)}\sum_{i=1}^{n}\Delta x_i = -\frac{\varepsilon}{3} \ . \end{aligned}$$

Thus

$$\begin{aligned} &U(f,P)-L(f,P) = \\ &\quad (U(f,P)-U(f_n,P)) + (U(f_n,P)-L(f_n,P)) + (L(f_n,P)-L(f,P)) \\ &\qquad \le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon \ . \end{aligned}$$

Therefore f is Riemann integrable.

Moreover

$$\begin{aligned} \left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| &= \left|\int_a^b f(x)-f_n(x)\,dx\right| \\ &\le \int_a^b |f(x)-f_n(x)|\,dx \le (b-a)\sup\{|f(x)-f_n(x)| : x\in[a,b]\} < \frac{\varepsilon}{3} \ , \end{aligned}$$

so

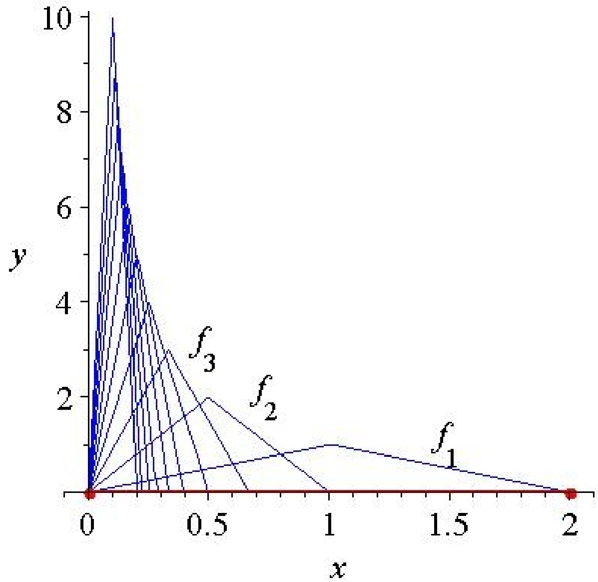
$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \ .$$

□

Example.

(4) Consider

$$f_n : [0,2] \to \mathbb{R}\ , \quad x \mapsto \begin{cases} n^2 x & 0 \le x \le 1/n\ , \\ 2n - n^2 x & 1/n < x \le 2/n\ , \\ 0 & 2/n < x \le 2\ . \end{cases}$$



As in Example (3), as $n \to \infty$, $f_n \to f(x) = 0$ pointwise, but not uniformly.

We compute

$$\int_0^2 f_n(x)\, dx = \int_0^{1/n} n^2 x\, dx + \int_{1/n}^{2/n} (2n - n^2 x)\, dx = 1$$

which is not equal to

$$\int_0^2 f(x)\, dx = 0\ .$$

Theorem 9.5. *Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable. If f_n converges pointwise to $f : [a,b] \to \mathbb{R}$ and f_n' converges uniformly to $g : [a,b] \to \mathbb{R}$, then f is differentiable and $f' = g$.*

Remark.

This theorem implies that *under the assumption of uniform convergence of the derivative of the functions* we can exchange limits as follows:

$$\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} (f_n')\ .$$

Lecture 29: 25/03/10

Proof. Consider $g_n = f_n'$. By assumption, g_n converges uniformly to g on $[a,b]$. Hence, by Theorem 9.3, g is continuous.

Moreover, g_n is Riemann integrable on $[a,b]$. Restricting to the interval $[a,x]$ for $a < x \le b$, we apply Theorem 9.4 to g on $[a,x]$. It follows that g is Riemann integrable on $[a,x]$ and that

$$\int_a^x g(t)\,dt = \lim_{n\to\infty} \int_a^x g_n(t)\,dt \ .$$

Now $f_n(x) = f_n(a) + \int_a^x g_n(t)\,dt$ is an antiderivative of $g_n = f_n'$, and as f_n converges pointwise to f, we compute

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(f_n(a) + \int_a^x g_n(t)\,dt \right) \\ &= \lim_{n\to\infty} f_n(a) + \lim_{n\to\infty} \int_a^x g_n(t)\,dt = f(a) + \int_a^x g(t)\,dt \ . \end{aligned}$$

As g is continuous, by Theorem 8.4 f is differentiable. This implies that f is an antiderivative of g and, hence, that $f' = g$.

□

Remarks.

(1) We only need convergence of f_n to f at *one* point x_0. Moreover, it follows that f_n converges uniformly to f.

Proof. By the Mean Value Theorem, $(f_n - f)(x) = (f_n - f)(x_0) + (x - x_0)(f_n' - f')(c_n)$ for some $c_n \in (a,b)$. Hence

$$|f_n(x) - f(x)| \le |f_n(x_0) - f(x_0)| + (b-a)|f_n'(c_n) - f'(c_n)| \ .$$

The first term tends to zero because $f_n(x_0)$ converges to $f(x_0)$, and the second term tends to zero because f_n' converges to f' uniformly. □

(2) It suffices for f_n to be differentiable, i.e. f_n' need not be continous (without proof).

(3) Even if f_n is differentiable and $f_n \to f$ uniformly, the limiting function need not be differentiable.

Definition 9.6. *(a)* $\sum_{n=1}^{\infty} f_n(x)$ *converges pointwise if*

$$s_k(x) = \sum_{n=1}^{k} f_n(x)$$

converges pointwise as $k \to \infty$.

(b) $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly if*

$$s_k(x) = \sum_{n=1}^{k} f_n(x)$$

converges uniformly as $k \to \infty$.

Lecture 30: 26/03/10

Example. $\sum_{n=1}^{\infty} \frac{1}{(2+x^2)^n}$ converges uniformly: we compute

$$s_k(x) = \sum_{n=1}^{k} \frac{1}{(2+x^2)^n} = \frac{1}{2+x^2} \cdot \frac{1 - \frac{1}{(2+x^2)^k}}{1 - \frac{1}{2+x^2}} = \frac{1}{1+x^2}\left(1 - \frac{1}{(2+x^2)^k}\right) .$$

As $\frac{1}{2+x^2} \leq \frac{1}{2}$ for all $x \in \mathbb{R}$, $\frac{1}{(2+x^2)^k} \to 0$ as $k \to \infty$, which implies (pointwise) convergence

$$\sum_{n=1}^{\infty} \frac{1}{(2+x^2)^n} = \frac{1}{1+x^2} .$$

We estimate

$$\left| \frac{1}{1+x^2} - s_k(x) \right| = \frac{1}{1+x^2} \cdot \frac{1}{(2+x^2)^k} \leq \frac{1}{2^k} .$$

The bound $1/2^k$ tends to zero as $k \to \infty$ *independently* of x, so convergence is uniform.

Theorem 9.7 (Weierstraß M-Test)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *be convergent. If* $|f_n(x)| \leq a_n$ *for all* $x \in \mathcal{D}$ *then* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* $\mathcal{D}$.

Proof. We estimate

$$\left| \sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{k} f_n(x) \right| = \left| \sum_{n=k+1}^{\infty} f_n(x) \right| \leq \sum_{n=k+1}^{\infty} |f_n(x)| \leq \sum_{n=k+1}^{\infty} a_n .$$

As $\sum_{n=1}^{\infty} a_n$ converges, the bound $\sum_{n=k+1}^{\infty} a_n \to 0$ as $k \to \infty$ independently of $x \in \mathcal{D}$. $\square$

Example (continued). For $f_n(x) = \dfrac{1}{(2+x^2)^n}$ we estimate

$$|f_n(x)| \le \frac{1}{2^n} = a_n \ ,$$

and as $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n} = 1$ converges, by the Weierstraß M-Test $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly for $x \in \mathbb{R}$.

Theorem 9.8. *(a) Let f_n be continuous. If $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent then $f = \sum\limits_{n=1}^{\infty} f_n$ is continuous.*

(b) Let f_n be continuously differentiable. If $\sum\limits_{n=1}^{\infty} f_n$ is convergent and $\sum\limits_{n=1}^{\infty} f_n'$ is uniformly convergent then $f = \sum\limits_{n=1}^{\infty} f_n$ is differentiable and $f' = \sum\limits_{n=1}^{\infty} f_n'$.

(c) Let f_n be Riemann integrable on $[a,b]$. If $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent then $f = \sum\limits_{n=1}^{\infty} f_n$ is Riemann integrable and $\int\limits_a^b f(x)\,dx = \sum\limits_{n=1}^{\infty} \int_a^b f_n(x)\,dx$.

Proof. This is an immediate consequence of Theorems 9.3, 9.4, and 9.5. □

10 Power Series

Definition 10.1. $\sum_{n=0}^{\infty} a_n x^n$ *with* $a_n \in \mathbb{R}$ *is called a* *power series.*
Its *radius of convergence* r *is given by*

$$r = \sup\left\{ |x| : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\} .$$

(a finite r *may not exist if* $\sum_{n=0}^{\infty} a_n x^n$ *converges for all* $x \in \mathbb{R}$*.)*

Theorem 10.2. *(a) If* $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $x = c$*, then* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for all* $x \in \mathbb{R}$ *with* $|x| < |c|$*.*

(b) If $\sum_{n=0}^{\infty} a_n x^n$ *diverges for* $x = c$*, then* $\sum_{n=0}^{\infty} a_n x^n$ *diverges for all* $x \in \mathbb{R}$ *with* $|x| > |c|$*.*

Proof. (a) Convergence of $\sum_{n=0}^{\infty} a_n c^n$ implies that $\lim_{n\to\infty} a_n c^n = 0$. Thus for $|x| < |c|$ there exists a $n_0 \in \mathbb{N}$ such that

$$|a_n x^n| = |a_n c^n| \cdot \left|\frac{x}{c}\right|^n \leq \left|\frac{x}{c}\right|^n \quad \text{for } n \geq n_0.$$

Therefore $\sum_{n=n_0}^{\infty} |a_n x^n|$ is majorised by $\sum_{n=n_0}^{\infty} \left|\frac{x}{c}\right|^n$, which converges absolutely.

(b) If $\sum_{n=0}^{\infty} a_n x^n$ converged for some x with $|x| > |c|$, then by (a) $\sum_{n=0}^{\infty} a_n y^n$ would converge for all y with $|y| < |x|$, in particular for $y = c$, which is a contradiction.

□

Corollary. $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for all* $x \in \mathbb{R}$ *with* $|x| < r$ *and diverges for all* $x \in \mathbb{R}$ *with* $|x| > r$*, where* r *is the radius of convergence of* $\sum_{n=0}^{\infty} a_n x^n$*.*

Remark. Convergence for $x = \pm r$ must be considered separately. Lecture 31: 29/03/10

Theorem 10.3. *Let* $r > 0$ *be the radius of convergence of* $\sum_{n=0}^{\infty} a_n x^n$ *and let* $0 < \rho < r$*. Then* $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly on* $\mathcal{D} = \{x : |x| \leq \rho\}$*.*

Proof. As $\rho < r$, $\sum\limits_{n=0}^{\infty} a_n x^n$ converges absolutely. As $|a_n x^n| \leq |a_n \rho^n|$ for $x \in \mathcal{D}$, the Weierstraß M-Test implies uniform convergence of $\sum\limits_{n=0}^{\infty} a_n x^n$ on $\mathcal{D}$. □

Theorem 10.4. *Let $r > 0$ be the radius of convergence of $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Then for all $x \in \mathbb{R}$ such that $|x| < r$,*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}\ .$$

Proof. Choose $\rho \in \mathbb{R}$ such that $0 < \rho < r$. Then $\sum\limits_{n=0}^{\infty} a_n x^n$ converges uniformly on $\mathcal{D} = \{x \in \mathbb{R} : |x| \leq \rho\}$. As $f_n(x) = a_n x^n$ is Riemann integrable, Theorem 9.8(c) implies that $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is Riemann integrable on $\mathcal{D}$ and that

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}\ .$$

□

Theorem 10.5. *Let $r > 0$ be the radius of convergence of $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Then for all $x \in \mathbb{R}$ such that $|x| < r$,*

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}\ .$$

Proof. Choose $\rho \in \mathbb{R}$ such that $0 < \rho < r$. Then $\sum\limits_{n=0}^{\infty} a_n x^n$ converges uniformly on $\mathcal{D} = \{x \in \mathbb{R} : |x| \leq \rho\}$. To apply Theorem 9.8(b), we need to show that $\sum\limits_{n=0}^{\infty} n a_n x^n$ also converges uniformly on $\mathcal{D}$. Once this is established, it follows that f is differentiable on $\mathcal{D}$ and that $f'(x) = \sum\limits_{n=0}^{\infty} n a_n x^{n-1}$.

Now pick ρ' such that $\rho < \rho' < r$. Then $\sum\limits_{n=0}^{\infty} a_n \rho'^n$ converges absolutely, and

$$|n a_n x^n| \leq |n a_n \rho^n| = |a_n \rho'^n| \underbrace{\left| n \left(\frac{\rho}{\rho'}\right)^n \right|}_{\leq 1 \text{ for } n \geq n_0} \leq |a_n \rho'^n|$$

implies by the Weierstraß M-Test uniform convergence of $\sum\limits_{n=0}^{\infty} n a_n x^n$ for $|x| \leq \rho$, as needed. □

Corollary. *$f(x)=\sum\limits_{n=0}^{\infty} a_n x^n$ is for $|x|<r$ infinitely often differentiable, and $f^{(k)}(x)=\sum\limits_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n x^{n-k}$.*

Remark. We find $f^{(k)}(0)=k!a_k$, so that $f(x)=\sum\limits_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n$, the Taylor series of f about zero.

Examples.

(1) For $|x|<1$ we have

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots=\sum_{n=0}^{\infty}(-1)^n x^n\ ,$$

and integration gives by Theorem 10.4

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

for $|x|<1$ (we had only proved this earlier for $0\le x<1$).

Note that for $x=1$ the first sum diverges $(1-1+1-1+\ldots)$ but the second sum converges $(1-1/2+1/3-1/4+\ldots)$, whereas for $x=-1$ both sums diverge.

(2) For $|x|<1$ we have

$$\frac{1}{1-x^2}=\sum_{n=0}^{\infty}x^{2n}\ .$$

As $\dfrac{1}{1-x^2}=\dfrac{1}{2}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}\right)$, we have for $|x|<1$

$$\frac{1}{2}\log\frac{1+x}{1-x}=\int_0^x\frac{dx}{1-x^2}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\ .$$

Thus, for example, $x=1/2$ gives

$$\log 3=2\left(\frac{1}{2}+\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}+\ldots\right)\ .$$

(3) $\exp(-x^2)=\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{n!}$ for all $x\in\mathbb{R}$, so that

$$\int_0^x\exp(-t^2)\,dt=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\quad\text{for all } x\in\mathbb{R}.$$

(4) $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|x|<1$, so that

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ for } |x|<1.$$

We shall now connect power series to Taylor series. We note that Lecture 32: 01/04/10

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

converges for $|x-a|<r$, where $r>0$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$. We identify $f^{(k)}(a) = k! a_k$, so that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n ,$$

which is just the Taylor series of f about a.

Theorem 10.6 (Taylor's Theorem with Integral Form of the Remainder)**.** *Let $f:[a,x]\to\mathbb{R}$ be n times continuously differentiable on $[a,x]$ and $(n+1)$ times differentiable on (a,x). Then*

$$f(x) = T_{n,a}(x) + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt .$$

Proof. As in the proof of Taylor's Theorem (Theorem 5.3), we write

$$F(t) = T_{n,t}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

and compute

$$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n .$$

Therefore by the Fundamental Theorem of Calculus

$$F(x) - F(a) = \int_a^x F'(t)\, dt = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt ,$$

and with $F(x) = T_{n,x}(x) = f(x)$ and $F(a) = T_{n,a}(x)$ we have

$$f(x) = T_{n,a}(x) + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt .$$

$\square$

Remark. An analogous result holds if $[a, x]$ is replaced by $[x, a]$ for $x < a$.

Theorem 10.7. *For $\alpha \in \mathbb{R}$ we have*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \text{ for } |x| < 1,$$

where $\displaystyle\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$.

Proof. We need only consider $x \neq 0$. We apply Theorem 10.6 to $f(x) = (1+x)^\alpha$. From

$$f^{(k)}(x) = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}$$

we see that $f^{(k)}(0) = \alpha(\alpha-1)\dots(\alpha-k+1)$. Therefore

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + \int_0^x \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}(1+t)^{\alpha-n-1}(x-t)^n \, dt \, .$$

We need to estimate the remainder term

$$\int_0^x \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}(1+t)^{\alpha-n-1}(x-t)^n \, dt = \alpha\binom{\alpha-1}{n} \int_0^x (1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n dt$$

If $x > 0$ we have $0 \leq t \leq x < 1$, so that

$$0 \leq \frac{x-t}{1+t} = x - t\frac{1+x}{1+t} \leq x \, .$$

Similarly, if $x < 0$ we have $0 \geq t \geq x > -1$, so that

$$0 \geq \frac{x-t}{1+t} = x - t\frac{1+x}{1+t} \geq x \, .$$

Taken together, we conclude that inside the integral we can estimate

$$\left|\frac{x-t}{1+t}\right| \leq |x| \, .$$

Moreover, for $|x| < 1$, $M = \max\{|1+t|^{\alpha-1} : |t| \leq |x|\}$ is finite. Putting this together, we arrive at

$$\left|\alpha\binom{\alpha-1}{n} \int_0^x (1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n dt\right| \leq M \left|\alpha\binom{\alpha-1}{n}\right| |x|^n \, .$$

Applying the quotient test, we find that

$$\frac{M\left|\alpha\binom{\alpha-1}{n+1}\right||x|^{n+1}}{M\left|\alpha\binom{\alpha-1}{n}\right||x|^{n}} = \left|1-\frac{\alpha}{n+1}\right||x| \to |x| < 1 \text{ as } n\to\infty,$$

and thus $M\left|\alpha\binom{\alpha-1}{n}\right||x|^n \to 0$ as $n\to\infty$. This proves that

$$\int_0^x \frac{\alpha(\alpha-1)\dots(\alpha-n)}{n!}(1+t)^{\alpha-n-1}(x-t)^n\,dt \to 0$$

as $n\to\infty$, as required. $\square$

Examples. For $|x|<1$,

$$\frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty}\binom{-1/2}{k}x^k\ ,$$

so that (also for $|x|<1$)

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k x^{2k}\ .$$

Term-by-term intergration gives

$$\begin{aligned}\arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-x^2}} &= \sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{(-1)^k}{2k+1}x^{2k+1}\\ &= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots\ .\end{aligned}$$