MTH5105 Differential and Integral Analysis 2010-2011

Solutions 3

1 Exercises for Feedback

1) The functions sinh and cosh are given by

$$\sinh : \mathbb{R} \to \mathbb{R}\ , \qquad x \mapsto \frac{1}{2}(\exp(x) - \exp(-x))\ ,$$
$$\cosh : \mathbb{R} \to \mathbb{R}\ , \qquad x \mapsto \frac{1}{2}(\exp(x) + \exp(-x))\ .$$

(a) Prove that sinh and cosh are differentiable and that $\sinh' = \cosh$ and $\cosh' = \sinh$.

(b) Prove that the function
$$f(x) = \cosh^2(x) - \sinh^2(x)$$
is constant by considering $f'(x)$.
What is the value of the constant?

(c) Prove that sinh is invertible.

(d) Prove that $\sinh(\mathbb{R}) = \mathbb{R}$. *Hint: show that* $\sinh(2x) > x$ *for* $x > 0$, *and mimic the proof of the statement that* $\exp(\mathbb{R}) = \mathbb{R}^+$.

(e) Prove that $\operatorname{arsinh} = \sinh^{-1}$ is differentiable, and that
$$\operatorname{arsinh}'(x) = \frac{1}{\sqrt{1+x^2}}\ .$$

Solution:

(a) exp is differentiable, therefore sinh and cosh are differentiable. Using $\exp' = \exp$, the derivatives follow immediately.

(b) f is differentiable, and $f'(x) = 2\cosh(x)\sinh(x) - 2\sinh(x)\cosh(x) = 0$. By Theorem 2.5, f is constant. $f(0) = \cosh^2(0) - \sinh^2(0) = 1$, so $\cosh^2(x) - \sinh^2(x) = 1$.

(c) $\sinh'(x) = \cosh(x) > 0$ for all $x \in \mathbb{R}$, therefore sinh is strictly increasing by Theorem 2.4, and therefore invertible by the corollary after Theorem 4.2.

(d) Let $x > 0$. We have $\exp(x) > 1 + x$ (see proof of Theorem 3.3) and $\exp(-x) < 1$ (since exp is strictly increasing, and $\exp(0) = 1$), so that $\sinh(x) > x/2$.
Let $c > 0$. From
$$\sinh(0) = 0 < c < \sinh(2c)$$
it follows by the IVT applied to the interval $[0, 2c]$, that there exists an $x \in (0, 2c)$ such that $\sinh(x) = c$. A similar argument holds for $c < 0$, and for $c = 0$ we have $\sinh(0) = 0 = c$.

(e) $\sinh'(x) = \cosh(x) > 0$ for all $x \in \mathbb{R}$, therefore by Theorem 4.6, arsinh is differentiable and
$$\operatorname{arsinh}'(x) = \frac{1}{\cosh(\operatorname{arsinh}(x))}\ .$$
Now $\cosh(x) = \sqrt{1 + \sinh^2(x)}$ (from (b), and the positive square root is taken as $\cosh(x)$ is positive), so that $\operatorname{arsinh}'(x) = 1/\sqrt{1+x^2}$.

2 Extra Exercises

2) (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continous inverse.

(b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove: If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} f'(x) = 0$.

Solution:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$.
f is differentiable with continuous derivative $f'(x) = 3x^2$. We have $f'(0) = 0$.
The inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{1/3}$.
As f is strictly increasing on $\mathbb{R}$, f is injective. $f(\mathbb{R}) = \mathbb{R}$ implies that f is surjective as well, so f is bijective.
As f is differentiable, it is continuous. Therefore f^{-1} is also continuous.

(b) This can be disproved by a counterexample.
Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be given by $f(x) = -x$.
f is differentiable and $f'(x) = -1$ for $x \geq 0$.
$\lim_{x\to 0} f(x) = 0$, but $\lim_{x\to 0} f'(x) = -1$.

3) Using the Intermediate Value Theorem, prove that a continuous function maps intervals to intervals.

Solution:

We use the following characterisation of an interval: $I \subseteq \mathbb{R}$ is an interval if and only if for all $x_1, x_2 \in I$ with $x_1 < x_2$,

$$x_1 < c < x_2 \Rightarrow c \in I \ .$$

Let $J = f(I)$. We need to show that J is an interval, i.e. for all $y_1, y_2 \in J$ with $y_1 < y_2$, $y_1 < c < y_2 \Rightarrow c \in J$:

Let $y_1, y_2 \in J$ with $y_1 < y_2$. Then there exist $x_1, x_2 \in I$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

As $y_1 \neq y_2$, necessarily $x_1 \neq x_2$ also, so either $x_1 < x_2$ or $x_2 < x_1$.

Consider, without loss of generality, the case $x_1 < x_2$. By assumption, f is a continuous function on I, so it is a continuous function on $[x_1, x_2]$ (or $[x_2, x_1]$, if $x_2 < x_1$).

Hence, by the intermediate value theorem, for all c with $y_1 < c < y_2$ there exists an $a \in [x_1, x_2]$ such that $f(a) = c$.

This implies that $c \in J$.

4*) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 2$ as $x \to 0$. Does it follow that f is differentiable at 0? If yes, give a rigorous proof; if no, provide a counter-example.

Solution:

Given $x > 0$, f is continuous on $[0, x]$ and differentiable on $(0, x)$. Hence, by the Mean Value Theorem applied to f on $[0, x]$, there exists a $c \in (0, x)$ such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$.
Similarly, if $x < 0$ then there exists a $c \in (x, 0)$ such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$. Clearly in both cases $|c| < |x|$, and $c \to 0$ as $x \to 0$. Therefore

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{c\to 0} f'(c) = 2$$

exists.

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