MTH5105 Differential and Integral Analysis 2009-2010

Solutions 1

1 Exercise for Feedback/Assessment

1) Using the definition of the derivative of a function, investigate for which values of x each of the following two functions is differentiable, and find the derivatives, if they exist.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2|x|$, [10 marks]

   (b) $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto x|x-1|$, [10 marks]

   Solution:

   (a) We need to distinguish three cases: (1) $a > 0$, (2) $a < 0$, and (3) $a = 0$:
   (1) For $a > 0$, we find

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2|x| - a^2|a|}{x-a} \\ &= \lim_{x\to a} \frac{x^2 x - a^2 a}{x-a} = \lim_{x\to a} (x^2 + ax + a^2) = 3a^2 \ . \end{aligned}$$

[2 marks]

Some argument is needed as to why we can replace $|x|$ by x when calculating the limit. It suffices to say that x becomes positive as $x \to a$ when $a > 0$. [1 mark]
(More formally, in the definition of the limit one can replace δ by $\delta' = \min\{\delta, a\}$ as then $|x-a| < \delta'$ implies $|x-a| < a$ and thus $x > 0$. However, I don't require this degree of formality.)
(2) For $a < 0$, we find

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2|x| - a^2|a|}{x-a} \\ &= \lim_{x\to a} \frac{x^2(-x) - a^2(-a)}{x-a} = \lim_{x\to a} (-x^2 - ax - a^2) = -3a^2 \ . \end{aligned}$$

[2 marks]

Some argument is needed as to why we can replace $|x|$ by $-x$ when calculating the limit. It suffices to say that x becomes negative as $x \to a$ when $a < 0$. [1 mark]
(3) For $a = 0$, we find

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2|x| - 0}{x-0} = \lim_{x\to 0} x|x| = 0 \ .$$

[2 marks]

Taken together, this shows that f is differentiable for all $x \in \mathbb{R}$ [1 mark]
and that

$$f'(x) = \begin{cases} 3x^2 & x > 0 \\ 0 & x = 0 \\ -3x^2 & x < 0 \end{cases}$$

or simply $f'(x) = 3x|x|$. [1 mark]

(b) We need to distinguish three cases: (1) $a > 1$, (2) $a < 1$, and (3) $a = 1$:

(1) For $a > 1$, we find

$$\begin{aligned} g'(a) &= \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x|x-1| - a|a-1|}{x - a} \\ &= \lim_{x \to a} \frac{x(x-1) - a(a-1)}{x - a} = \lim_{x \to a} (x + a - 1) = 2a - 1 \,. \end{aligned}$$

[2 marks]

Some argument is needed as to why we can replace $|x - 1|$ by $x - 1$ when calculating the limit. It suffices to say that $x - 1$ becomes positive as $x \to a$ when $a > 1$. [1 mark]

(2) For $a < 1$, we find

$$\begin{aligned} g'(a) &= \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x|x-1| - a|a-1|}{x - a} \\ &= \lim_{x \to a} \frac{x(1-x) - a(1-a)}{x - a} = \lim_{x \to a} (-x - a + 1) = -2a + 1 \,. \end{aligned}$$

[2 marks]

Some argument is needed as to why we can replace $|x - 1|$ by $1 - x$ when calculating the limit. It suffices to say that $x - 1$ becomes negative as $x \to a$ when $a < 1$. [1 mark]

(3) For $a = 1$, we find

$$\frac{g(x) - g(1)}{x - 1} = \frac{x|x-1| - 0}{x - 1} = \begin{cases} x & x > 1 \\ -x & x < 1 \end{cases}$$

so that the limit $\lim_{x \to 1} \frac{g(x)-g(1)}{x-1}$ does not exist. [2 marks]

Taken together, this shows that g is differentiable for all $x \in \mathbb{R} \setminus \{1\}$ [1 mark]

and that

$$g'(x) = \begin{cases} 2x - 1 & x > 1 \\ \text{undefined} & x = 1 \\ 1 - 2x & x < 1 \end{cases}$$

[1 mark]

2 Extra Exercises

2) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at zero and find $f'(0)$.

Find $f'(x)$ for $x \neq 0$ assuming that $\sin' = \cos$.

Give a rough sketch of the curve $f'(x)$ for small x and mark $f'(0)$ clearly on your sketch.

Solution:

Consider the difference quotient

$$\frac{g(x) - g(0)}{x - 0} = \frac{x^2 \sin(1/x^2) - 0}{x} = x \sin(1/x^2) \,.$$

Since $|\sin(1/x^2)| \leq 1$,

$$\frac{g(x) - g(0)}{x - 0} \to 0$$

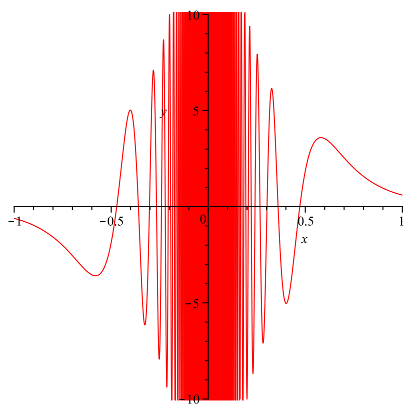
as $x \to 0$. Therefore f is differentiable at zero with $f'(0) = 0$.

For $x \neq 0$ differentiation gives

$$f'(x) = 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2) \; .$$

Graph of $f'(x)$:

For $x \to 0$, $2x \sin(1/x^2) \to 0$ and the second term dominates. The graph of f' oscillates rapidly with increasing amplitude as $x \to 0$. At zero, the derivative is zero.



3) Let $f : [-1, 1] \to \mathbb{R}$ be continuous on $[-1, 1]$, differentiable at zero and $f(0) = 0$. Show that the function

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

is continuous at zero.

Is g continuous for $x \neq 0$?

Deduce that there is some number M such that

$$f(x)/x \leq M \quad \text{for all} \quad x \in [-1, 1] \setminus \{0\} \; .$$

Solution:

A function g is continuous at a if $\lim_{x \to a} g(x) = g(a)$.

With $a = 0$, this gives

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0)$$

so g is continuous at 0.

For $x \neq 0$, g is continuous since it is a quotient of continuous functions.

By the boundedness principle, a continuous function on a closed interval attains it maximum and minimum.

Therefore there exists a number M such that $g(x) \leq M$ for all $x \in [-1, 1]$.

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