MTH5105 Differential and Integral Analysis 2010-2011

Solutions 8

1 Exercise for Feedback

1) Let the sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be given by

$$g_n(x) = \frac{x}{1+nx^2} \ .$$

(a) Compute $g(x) = \lim_{n\to\infty} g_n(x)$.

(b) Show that g_n converges to g uniformly.

(c) Compute $h(x) = \lim_{n\to\infty} g_n'(x)$.

(d) Does $g'(x) = h(x)$ hold?

(e) Why does Theorem 9.5 not apply here?

Solution:

(a) We have $g_n(0) = 0$, and for $x \neq 0$ we estimate

$$|g_n(x)| = \frac{|x|}{1+nx^2} \leq \frac{|x|}{nx^2} = \frac{1}{n|x|} \ .$$

The right-hand side converges to zero as $n \to \infty$, hence

$$g(x) = \lim_{n\to\infty} g_n(x) = 0 \ .$$

(b) Here we have to work a bit harder (we could have done so immediately in part (a)): From

$$g_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$$

we can determine the extrema of g_n by solving $g_n'(x) = 0$. We find $x = \pm 1/\sqrt{n}$. As $\lim_{x\to\pm\infty} g_n(x) = 0$, we can conclude that

$$|g_n(x)| \leq g_n(1/\sqrt{n}) = \frac{1}{2\sqrt{n}} \ .$$

The right-hand side converges to zero as $n \to \infty$ independently of x, hence the convergence is uniform.

An alternative argument goes as follows: For $\varepsilon > 0$ we have

$$|g_n(x)| < \varepsilon \quad \text{for } |x| \leq \varepsilon,$$

and

$$|g_n(x)| \leq \frac{1}{n\varepsilon} \quad \text{for } |x| > \varepsilon.$$

Now, given $\varepsilon > 0$ choose $n_0 = \lceil 1/\varepsilon^2 \rceil$. Then if $n > n_0$ it follows that $|g_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$, i.e. g_n converges uniformly to zero.

(c) From

$$g_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

it follows that

$$|g_n'(x)| = \frac{|1 - nx^2|}{(1 + nx^2)^2} \le \frac{1 + nx^2}{(1 + nx^2)^2} = \frac{1}{1 + nx^2} \ .$$

For $x \neq 0$, this implies that $\lim_{n \to \infty} g_n(x) = 0$. If $x = 0$ then $g_n'(x) = 1$, so that

$$h(x) = \begin{cases} 0 & x \neq 0 \ , \\ 1 & x = 0 \ . \end{cases}$$

(d) No: $g'(0) = 0$ but $h(0) = 1$.

(e) For Theorem 9.5 to apply, g_n' must converge to h uniformly, which is not the case here. (This can be seen from the fact that if the convergence was uniform then h would be continuous, which it is not.)

2 Extra Exercises

2) For $x \in \mathbb{R}$, compute

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{(1 + x^2)^n} \ .$$

Show that the convergence is not uniform.

Solution:

We have a geometric series with terms of the form aq^n where $a = x$ and $q = 1/(1 + x^2)$. For $|q| < 1$ the sum is therefore $aq/(1 - q)$.

$|q| < 1$ is equivalent to $x \neq 0$, in which case we find

$$f(x) = \frac{x}{(1 + x^2)\left(1 - \frac{1}{1+x^2}\right)} = \frac{1}{x} \ .$$

For $x = 0$, $f(x) = \sum_{n=1}^{\infty} 0 = 0$. Thus,

$$f(x) = \begin{cases} 0 & x = 0 \ , \\ 1/x & x \neq 0 \ . \end{cases}$$

The convergence cannot be uniform, as the limiting function is discontinuous.

[Alternatively, to directly show lack of uniform convergence you would need to consider the partial sums

$$f_N(x) = \sum_{n=1}^{N} \frac{x}{(1 + x^2)^n} = \frac{1}{x} - \frac{1}{x(1 + x^2)^N} \ .$$

Clearly something goes wrong with uniform convergence near zero. For instance, if you check what happens for $x = 1/N$ you will find that $f(1/N) - f_N(1/N)$ actually diverges as $N \to \infty$ (in fact, $f_N(1/N) \to 1$).]

3) (a) Show that the following sequences of functions converge uniformly on the given intervals.

(i) $u_n(x) = (1 - x)x^n \ ,$ $\qquad [0, 1] \ ;$

(ii) $v_n(x) = \dfrac{x^2}{1 + nx^2} \ ,$ $\qquad \mathbb{R} \ .$

(b) Which of the following sequences of functions converge uniformly to $s(x) = 1$ on the interval $[0, 1]$?

$$\begin{aligned} \text{(i)} \quad & f_n(x) = (1 + x/n)^2 \ , \\ \text{(ii)} \quad & g_n(x) = 1 + x^n(1-x)^n \ , \\ \text{(iii)} \quad & h_n(x) = 1 - x^n(1-x^n) \ . \end{aligned}$$

Solution:

(a) On $[0, 1]$, $u_n(x) = (1 - x)x^n$ is non-negative and maximal at $x = n/(1 + n)$ (compute u_n' to find this value), so that

$$0 \le u_n(x) \le u_n(n/(1+n)) = \frac{1}{n}\left(1 - \frac{1}{n+1}\right)^{n+1} < \frac{1}{n} \ .$$

Therefore $|u_n(x)| < e/n$ which tends to zero independent of x.

On $\mathbb{R}$, $v_n(x) = x^2/(1 + nx^2)$ is non-negative and bounded above by $1/n$, as

$$0 \le v_n(x) = \frac{1}{n} - \frac{1}{n(1+nx^2)} < \frac{1}{n} \ .$$

Therefore $|v_n(x)| < 1/n$ which tends to zero independent of x.

(b) On $[0, 1]$, $0 \le f_n(x) - s(x) = x^2/n^2 + 2x/n \le 3/n$. Therefore $|f_n(x) - s(x)| < 3/n$ which tends to zero independent of x.

Hence f_n converges uniformly to s.

On $[0, 1]$, $0 \le g_n(x) - s(x) = (x(1 - x))^n$. This is maximal at $x = 1/2$, and therefore $|g_n(x) - s(x)| \le 1/4^n$ which tends to zero independent of x.

Hence g_n converges uniformly to s.

On $[0, 1]$, $0 \le s(x) - h_n(x) = x^n(1 - x^n)$. However, this is maximal at $x_n = 2^{-1/n}$, and therefore $s(x_n) - h_n(x_n) = 1/4$ which does *not* tend to zero as n becomes large.

Hence h_n does not converge uniformly to s.

*4) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to a function f. Show that if $\lim_{n\to\infty} x_n = x$ then

$$\lim_{n\to\infty} f_n(x_n) = f(x) \ .$$

Solution:

We need to show that for all $\epsilon > 0$ there exists an n_0 such that $|f_n(x_n) - f(x)| < \epsilon$ for all $n \ge n_0$.

The key step is to use the triangle inequality

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \ .$$

f_n converges uniformly to f, so for given $\epsilon_1 > 0$ there is an n_1 such that

$$|f_n(x) - f(x)| < \epsilon_1$$

for all $n \ge n_1$ *independently* of the value of x, so in particular

$$|f_n(x_n) - f(x_n)| < \epsilon_1$$

for all $n \ge n_1$.

As f is a uniform limit of continuous functions f_n, f is continuous. Therefore, for given $\epsilon_2 > 0$ there is an n_2 such that

$$|f(x_n) - f(x)| < \epsilon_2$$

for all $n \geq n_2$.

Now, for given ϵ choose $\epsilon_1 = \epsilon_2 = \epsilon/2$. Then for $n_0 = \max(n_1, n_2)$ we find that

$$|f_n(x_n) - f(x)| \leq \epsilon/2 + \epsilon/2 = \epsilon \ .$$

Thomas Prellberg, March 2011