MTH5105 Differential and Integral Analysis 2008-2009

Exercises 6

Exercise 1: Let $f : (0,1) \to \mathbb{R}$ be continuous. Show that f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist. [5 marks]

[Note: the converse is also true, but much harder to show.]

Solution: If $A = \lim_{x\to 0} f(x)$ and $B = \lim_{x\to 1} f(x)$ exist, then the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} A & x = 0 \\ f(x) & 0 < x < 1 \\ B & x = 1 \end{cases}$$

is continuous on $[0,1]$ and therefore uniformly continuous on $[0,1]$. The function f is a restriction of g to the smaller interval $(0,1)$ and therefore also uniformly continuous. [5 marks]

Exercise 2: Let $\alpha \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha & x \in \{1/k; k \in \mathbb{N}\}\ , \\ 0 & \text{else.} \end{cases}$$

For which values of α is f Riemann-integrable? If f is Riemann-integrable, what is the value of $\int_0^1 f(x)\,dx$? [10 marks]

Solution: If $\alpha < 0$ then f is unbounded and therefore not Riemann-integrable.

[2 marks]

Let now $\alpha \geq 0$, so that f is bounded by 1.

As f is zero on all irrational numbers, $L(f,P) = 0$ for all $P \in \mathcal{P}$, and thus

$$\underline{\int_0^1} f(x)\,dx = 0\ .$$

[3 marks]

Consider the partition of $[0,1]$ by

$$P_n = \{0, n/n^2, (n+1)/n^2, \ldots, (n^2-1)/n^2, n^2/n^2\}$$

into one interval of width $1/n$ and $n^2 - n$ intervals of width $1/n^2$. (Many other choices would work here, as well.)

For $x \geq 1/n$, $f(x)$ is non-zero at precisely n points, so that $\sup_{x \in I_i} f(x)$ is non-zero on the left-most interval of width $1/n$ and at most $2n$ intervals of width $1/n^2$. Thus,

$$U(f, P_n) \leq \frac{1}{n} + 2n\frac{1}{n^2} = \frac{3}{n} \; .$$

[3 marks]

We thus have

$$0 = L(f, P_n) \leq U(f, P_n) \leq \frac{3}{n}$$

so that the f is Riemann-integrable and $\int_0^1 f(x)\, dx = 0$.

[2 marks]

Exercise 3: Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable and $c \in \mathbb{R}$.

(a) Given a partition P of $[a, b]$, show that

$$U(cf, P) - L(cf, P) \leq |c|(U(f, P) - L(f, P)) \; .$$

[6 marks]

(b) Deduce from (a) that cf is integrable and

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx \; .$$

[This completes the proof of Theorem 38.] [4 marks]

(c) For a bounded set $\Omega \subset \mathbb{R}$, show that

$$\sup_{y \in \Omega} |y| - \inf_{y \in \Omega} |y| \leq \sup_{y \in \Omega} y - \inf_{y \in \Omega} y \; .$$

[This is needed in the proof of Theorem 41.] [5 marks]

Solution: (a) For $c \geq 0$,

$$\sup_{x \in I_i} cf(x) = c \sup_{x \in I_i} f(x) \quad \text{and} \quad \inf_{x \in I_i} cf(x) = c \inf_{x \in I_i} f(x) \; ,$$

so that

$$\sup_{x \in I_i} cf(x) - \inf_{x \in I_i} cf(x) = c \left(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \right) \; .$$

[2 marks]

For $c \leq 0$ this changes to

$$\sup_{x \in I_i} cf(x) = c \inf_{x \in I_i} f(x) \quad \text{and} \quad \inf_{x \in I_i} cf(x) = c \sup_{x \in I_i} f(x) \; .$$

so that

$$\sup_{x \in I_i} cf(x) - \inf_{x \in I_i} cf(x) = -c \left(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \right) \; .$$

[2 marks]

Taken together, this implies

$$\sup_{x\in I_i} cf(x) - \inf_{x\in I_i} cf(x) = |c| \left(\sup_{x\in I_i} f(x) - \inf_{x\in I_i} f(x) \right) .$$

Multiplying by Δx_i and summing over all i gives the desired result.

[2 marks]

(b) If $U(f,P) - L(f,P) < \epsilon$ for some $\epsilon > 0$, then also

$$U(cf,P) - L(cf,P) \leq |c|(U(f,P) - L(f,P)) < |c|\epsilon .$$

By Riemann's integrability criterion, cf is integrable. [2 marks]

Finally, for $c \geq 0$ we have

$$L(cf,P) = cL(f,P) \leq c\int_a^b f(x)\,dx \leq cU(f,P) = U(cf,P)$$

and for $c \leq 0$ we have

$$L(cf,P) = cU(f,P) \leq c\int_a^b f(x)\,dx \leq cL(f,P) = U(cf,P)$$

so that in both cases

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

follows. [2 marks]

(c) This can be shown using a long chain of transformations:

$$\begin{aligned}
\sup_{y\in\Omega}|y| - \inf_{y\in\Omega}|y| &= \sup_{y\in\Omega}|y| - \inf_{x\in\Omega}|x| && \text{a change of variables} \\
&= \sup_{y\in\Omega}|y| + \sup_{x\in\Omega}(-|x|) && \text{change inf to sup} \\
&= \sup_{x,y\in\Omega}(|y| - |x|) && \text{combine terms} \\
&\leq \sup_{x,y\in\Omega}(|y - x|) && ||y| - |x|| < |y - x| \\
&= \sup_{x,y\in\Omega}(y - x) && \text{rhs is symmetric in } x \text{ and } y \\
&= \sup_{y\in\Omega} y + \sup_{x\in\Omega}(-x) && \text{split terms} \\
&= \sup_{y\in\Omega} y - \inf_{x\in\Omega} x && \text{change sup to inf} \\
&= \sup_{y\in\Omega} y - \inf_{y\in\Omega} y && \text{a change of variables}
\end{aligned}$$

[5 marks]

Thomas Prellberg, February 2009