[section]

Partially directed paths in a wedge

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May 2, 2006

Abstract

Consider the symmetric wedge in the first and fourth quadrant of the square lattice and formed by the lines $Y = \pm pX$, where $p > 0$ is fixed integer. A partially directed path from the origin is confined to this wedge if it is constrained to step only on vertices inside or on the boundary of the wedge. In this paper we examine recurrences for the generating function of such partially directed paths. We solve the functional equations in the case $p = 1$ using a variation of the kernel method which we call the iterated kernel method. This appears to be similar to the obstinate kernel method used by Bousquet-Mélou.

PACS numbers: 05.50.+q, 02.10.Ab, 05.40.Fb, 82.35.-x

1 Introduction

{ADR: I think we need to start with a more combinatorial introduction — we can use the physics to motivate the problem. We can also make reference to some of the slit-plane walk stuff to talk about the difficulties of putting otherwise

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simple objects into confined geometries. Have to be careful with free-energies and entropies and the such.}

A polymer located in a confined space with physical boundaries loses entropy and so exerts a repulsive entropic force on the boundaries of confining space. This is the mechanism of steric stabilisation of colloidal dispersions by adsorbed polymer chains on the colloidal particles [10]; a phenomenon which is the subject of intensive investigation [5]. Stabilisation of colloids occur when a polymer additive adsorbs on the surface of colloid particles. Approaching particles confine the adsorbed polymers to a smaller interstitial regions between particles, and the resulting loss in entropy induces an entropic repulsion between the particles, thereby stabilising the dispersion.

Colloidal dispersions have been modelled by lattice path and self-avoiding walk models of polymers in confined regions of the square or hypercubic lattice. There is substantial literature on models of lattice paths and self-avoiding walks in slits, wedges or slabs, see for example reference [17] (and citations therein).

Self-avoiding walk models of polymers in confined spaces are (except in the most trivial of cases) not exactly solvable; numerical simulations are often the only means whereby interesting results can be obtained [16]. However, there are some exact results in the literature. For example, the connective constant of self-avoiding walks [8, 9] in a wedge geometry is independent of the angle of the wedge [7]. This implies that the walk does not exert an entropic force on the walls of the wedge in the limit as the length of the walk tends to infinity.

The situation for models of directed lattice paths in wedges is quite different. These combinatorial models of polymers in confined geometries are often more tractable by using generating function techniques, and in some cases the explicit calculation of a free energy (which is simply related to the radius of convergence of the generating function) is possible. In these models the directed path exerts a repulsive entropic force on the walls of the wedge, and the magnitude of this force has been determined for Dyck path and for Motzkin path models of directed polymers [15, 12]. Some results are also known for lattice paths adsorbing on the walls of the wedge; see references [14, 13].

In this paper we consider model of a partially directed path model of a polymer, in some wedge geometries. This model is related to the model solved in reference [13], but it poses a more challenging mathematical question, and is physically more realistic because the path is inhibited by the wedge on two sides, rather than on only one side as in reference [13]. As a result, the solution is harder to determine, and we are not able to give a general solution of all wedges.

A directed walk on the square lattice is a path taking unit steps only in the north and east directions. Such objects are necessarily self-avoiding; they cannot visit the same vertex twice. Partially directed walks are paths that may take unit steps only in the north, south and east directions with the further condition that no vertex is visited twice — *ie* they are self-avoiding. Hence, north steps cannot be followed by south steps and *vice-versa*. The

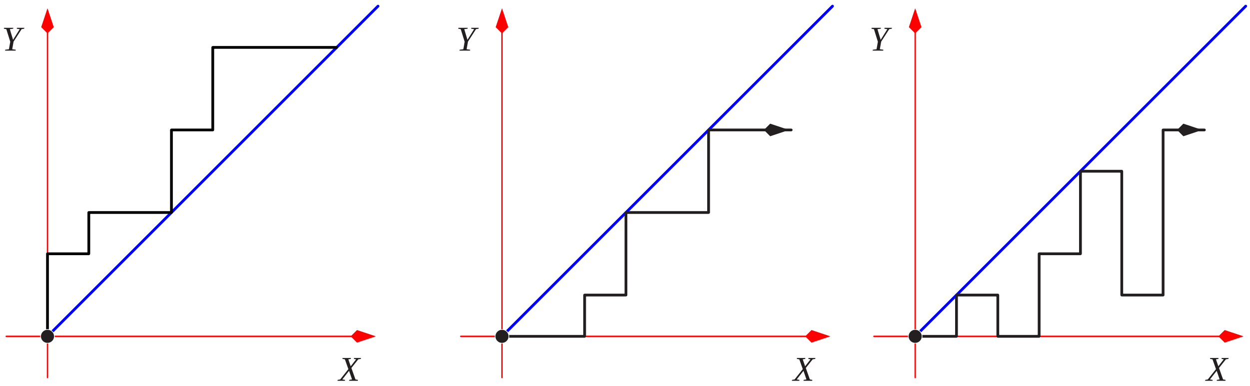

Figure 1: (left) A directed path confined to the wedge formed by the Y-axis and the line $Y = +pX$. (centre) A directed path the wedge formed by the X-axis and the line $Y = +pX$. (right) A partially directed path in a wedge formed by the X-axis and the line $Y = +pX$.

generating function of such walks can be derived using standard techniques:

$$W(t) = \sum_{n\geq 0} c_n t^n = \frac{1+t}{1-2t+t^2}, \tag{1.1}$$

where c_n is the number of walks of length n and t is the length generating variable. This implies that the number of walks is

$$c_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right). \tag{1.2}$$

Hence the growth constant, μ is

$$\mu = \lim_{n\to\infty} c_n^{1/n} = 1+\sqrt{2}. \tag{1.3}$$

The growth constant is simply related to the *limiting free energy*, κ, of the model by

$$\kappa = \log \mu. \tag{1.4}$$

The thermodynamic properties of the model can then be derived from the limiting free energy.

Possible models of directed and partially directed paths in wedge geometries are displayed in Figure 1. The model in Figure 1 (left) has been considered in [6, 14]. In general the growth constant is a (non-trivial) function of the wedge angle. The derivative of the free-energy with respect to the wedge angle gives the entropic force exerted by the polymer on the wedge and this has been computed in [14]. This model may be generalised by introducing an interaction between $Y = +pX$ and the path, or by considering Motzkin or partially directed paths instead [13, 12]. In Figure 1 (centre) a directed path model of a polymer in the wedge

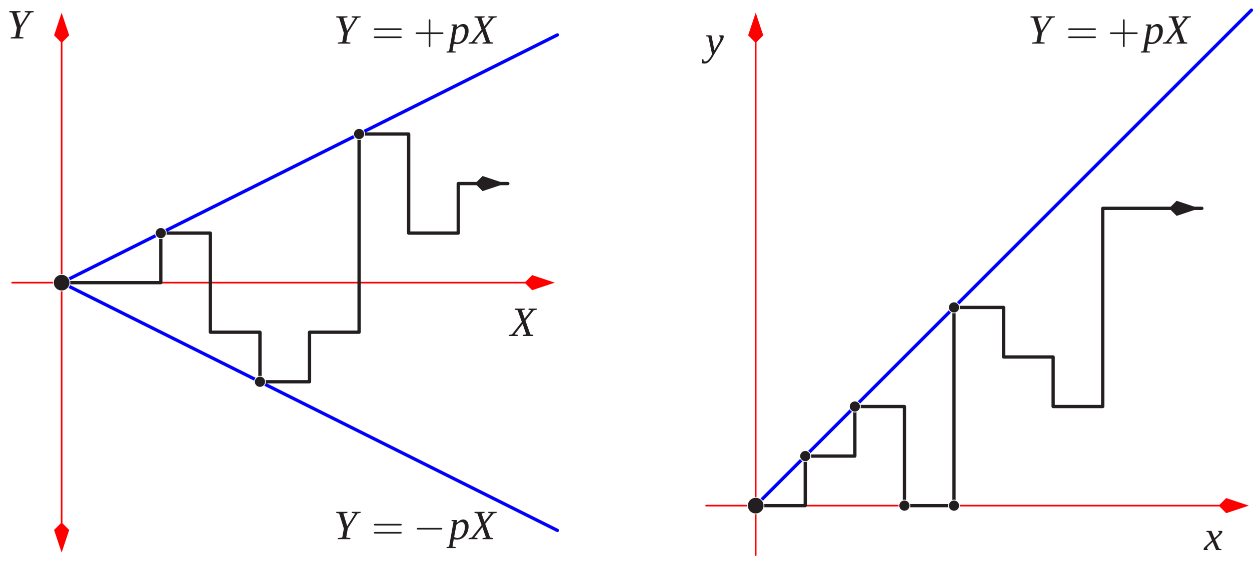

Figure 2: (left) The symmetric model of a partially directed path in a p-wedge formed by the lines $Y = \pm pX$. (right) The asymmetric model of a partially directed path in a p-wedge formed by the line $Y = +pX$ and the X-axis.

formed by the X-axis and the line $Y = +pX$ is proposed instead. Closer inspection of this model shows that it is similar to the model in Figure 1 (left).

Replacing the directed path by a partially directed path in this case gives the model in Figure 1 (right). This model is more general than the model in Figure 1 (centre), and substantially more difficult to treat than the corresponding model with partially directed paths in the wedge formed by the Y-axis and the line $Y = +pX$. In this paper we consider aspects of partially directed path models similar to the model in Figure 1 (right). In particular, we consider the variants illustrated in Figure 2 - firstly a model of a partially directed path in a wedge formed by the lines $Y = \pm pX$ (this is the *symmetric model* - see Figure 2 (left)), and secondly a model of a partially directed path in a wedge formed by the X-axis and the line $Y = +pX$ (this is the *asymmetric model* - see Figure 2 (right)).

The related model of a partially directed path in a wedge is illustrated in Figure 3. In this case the final vertex of the path in constrained to end in the diagonal line $Y = +pX$, where $p > 1$ is an integer, and this model is a *bargraph path* above the line $Y = +pX$. This model was considered in reference [13], and while the generating function $g_p(t)$ is not known explicitly, it is given by

$$g_p(t) = \frac{h(t)}{1 - t^2(1 + h(t))} \tag{1.5}$$

where $h(t)$ is an appropriate solution of the equation

$$h(t) = t^{p+1} \left(1 + h(t)\right)^p \left(1 + \frac{h(t)}{1 - t^2(1 + h(t))}\right), \tag{1.6}$$

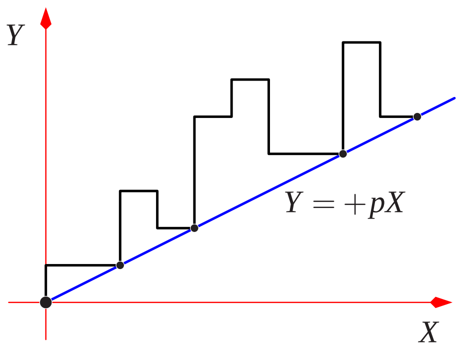

Figure 3: A bargraph path above the line $Y = +pX$. The generating function of this model is not known explicitly. However, a set of equations derived in reference [13] can be solved to determine the radius of convergence of the generating function for integer values of p.

where (as above), t is conjugate to the number of edges in the path. This model was also considered as *adsorbing bargraphs* which interact with the line $Y = +pX$ [6], and an asymptotic expression for the adsorption critical point has been estimated in reference [13] (the location of the singular point on the radius of convergence of the generating function). **{ADR: We might need to explain the critical point more than this}**

The paths in Figure 2 are are constrained on two sides, and solving for their combinatorial properties is challenging. These models are defined as follows. Consider the square lattice $\mathbb{Z}^2$ of points in the plane with integer coordinates. Let $p > 0$ be an integer. The symmetric p-wedge $\mathcal{V}_o$ is defined by

$$\mathcal{V}_p = \left\{ (n,m) \in \mathbb{Z}^2 \,\middle|\, \text{where } n \geq 0 \text{ and } -pn \leq m \leq pn \right\}. \tag{1.7}$$

The (asymmetric) *p-wedge* is defined by

$$\mathcal{W}_p = \left\{ (n,m) \in \mathbb{Z}^2 \,\middle|\, \text{where } n \geq 0 \text{ and } 0 \leq m \leq pn \right\}. \tag{1.8}$$

In the remainder of the paper we will focus on partially directed paths whose vertices are constrained to either the symmetric or asymmetric wedges — see Figure 2.

2 Growth constants in wedges

In this section we prove that the growth constant for partially directed walks is independent of the angle of the wedge and is equal to that of unrestricted partially directed walks. First, let b_n be the number of partially directed walks in the wedge defined by the lines $X = 0$ and $Y = 0$, whose last vertex lies in the line $Y = 0$. These paths are counted by the generating function $g_0(t)$ defined above, and singularity analysis gives the following lemma.

Lemma 2.1. *The growth constant of partially directed paths in the wedge defined by $X = 0$ and $Y = 0$ is*

$$\lim_{n\to\infty} b_n^{1/n} = (1 + \sqrt{2}) = \mu. \tag{2.1}$$

Armed with this result we can proceed to partially directed walks in $\mathcal{V}_p$ and $\mathcal{W}_p$. Let $v_{p,n}$ (resp. $w_{p,n}$) be the number of partially directed walks in $\mathcal{W}_p$ (resp. $\mathcal{W}_p$) of length n.

Lemma 2.2. *For any given $p \in (0, \infty)$ the following limits exist:*

$$\lim_{n\to\infty} v_{n,p}^{1/n} = \mu_p^v \qquad \textit{and} \qquad \lim_{n\to\infty} w_{n,p}^{1/n} = \mu_p^w. \tag{2.2}$$

The limits satisfy

$$\mu_p^w \le \mu_p^v \le \mu = (1 + \sqrt(2)) \tag{2.3}$$

Proof. We have that $w_{n,p} \le v_{n,p} \le c_n$. Hence, if the above limits exist, we must have $\mu_p^w \le \mu_p^v \le \mu = (1 + \sqrt{2})$.

To show existence, we prove that the sequences are supermultiplicative. Take any walk counted by $v_{n,p}$ and append a horizontal step, and any walk counted by $v_{m,p}$. This gives a walk of $n + m + 1$ steps that lies within $\mathcal{V}_p$, and so is counted by $v_{n+m+1,p}$. Hence $v_{n,p} v_{m,p} \le v_{n+m+1,p}$. A standard result (Fekete's lemma) on superadditive sequences (which we can apply by taking logarithms) then implies that μ_p^v exists. The proof for walks in $\mathcal{W}_p$ is identical. $\square$

Lemma 2.3. *For any given $p \in (0, \infty)$ we have*

$$b_n^N \le w_{(\lceil np \rceil + nN + N),p} \tag{2.4}$$

And hence $\lim_{n\to\infty} b_n^{1/n} \le \mu_p^w$.

Proof. Take any walk counted by b_n. By prepending $\lceil np \rceil + 1$ horizontal steps, this walk will fit inside the wedge, $\mathcal{W}_p$. Now append another horizontal step and a walk counted by b_n — repeat this until there are N walks counted by b_n. This now gives a walk counted by $w_{(\lceil np \rceil + nN + N),p}$. Thus we have the first inequality. Massaging this gives

$$\frac{N}{(\lceil np \rceil + nN + N)} \log b_n \le \frac{1}{(\lceil np \rceil + nN + N)} \log w_{(\lceil np \rceil + nN + N),p}$$

Take the limit as $N \to \infty$ to give

$$\frac{1}{n} \log b_n \le \log \mu_p^w.$$

Taking the limit as $n \to \infty$ completes the proof. $\square$

By combining the above lemmas we can prove that the growth constant for partially directed paths is independent of the wedge.

Theorem 2.4. *For any given $p \in (0, \infty)$*

$$\mu_p^v = \mu_p^w = \mu = (1 + \sqrt{2}).$$

3 Functional equations for walks in wedges

3.1 The symmetric wedge model

Consider a model of partially directed paths in a symmetric wedge as illustrated in Figure 2 (right). If p is an integer or a rational number, then the path may touch vertices in the lines $Y = \pm pX$. These vertices are *visits* to the lines $Y = \pm pX$. In the event that p is an irrational number such visits cannot occur, however the path may approach arbitrarily close to the adsorbing lines (for large enough X-ordinate). In this paper we shall only consider the simplest version of this model, and we assume that p is a positive integer. Even in this case the model is apparently intractable, and we have only found the the generating functions when $p = 1$.

We will derive a functional equation satisfied by the generating function of partially directed paths in $\mathcal{V}_p$ (those illustrated in Figure 2 (left)), by finding a recursive construction, similar to those in [2, 3] (and elsewhere).

Let x be the generating variable for horizontal edges in the path and let y be the generating variable for vertical edges in the path. Introduce generating variables a and b to measure the distances between the last vertex in the path and the line $Y = -pX$ and the line $Y = +pX$ respectively. The generating function of the paths are now denoted by $g_p(a, b; x, y) \equiv g_p(a, b)$ where the variables x and y are suppressed.

It turns out that the construction and resulting functional equation is simplified by considering only those partially directed walks that are either a single vertex (no edges) or end in a horizontal step. Let $f_p(a, b; x, y) \equiv f_p(a, b)$ be the generating function of such paths. It is simply related to $g_p(a, b)$ via

$$f_p(a, b) = 1 + x(ab)^p g_p(a, b). \tag{3.1}$$

We now obtain a functional equation satisfied by f_p by recursively constructing the paths column-by-column. Each path is either a single vertex, or can be constructed from a shorter path by appending either a horizontal step, or a sequence of up steps followed by a horizontal step, or a sequence of down steps followed by a horizontal step. See Figure 4.

Consider a path counted by $f_p(a, b)$, and see Figure 4.

- Appending a single horizontal step to its end increases the distance of the end point from both wedge boundary lines by p. Hence the generating function of paths with a horizontal edge appended is $x(ab)^p f_p(a, b)$.

- Appending a up step to the end of such a path, increases the number of vertical steps by 1, increases the distance from the line $Y = -pX$ by 1 and decreases the distances from the line $Y = +pX$ by 1. Hence such a path has g.f. $y(b/a)f_p(a, b)$. Hence appending some positive number of up steps gives $\frac{yb/a}{1-yb/a} f_p(a, b)$. Appending a horizontal step to the end of such a path gives (by the above reasoning) $x(ab)^p \frac{yb/a}{1-yb/a} f_p(a, b)$.

- Similarly appending some positive number of down steps followed by a horizontal step gives $x(ab)^p \frac{ya/b}{1-ya/b} f_p(a, b)$.

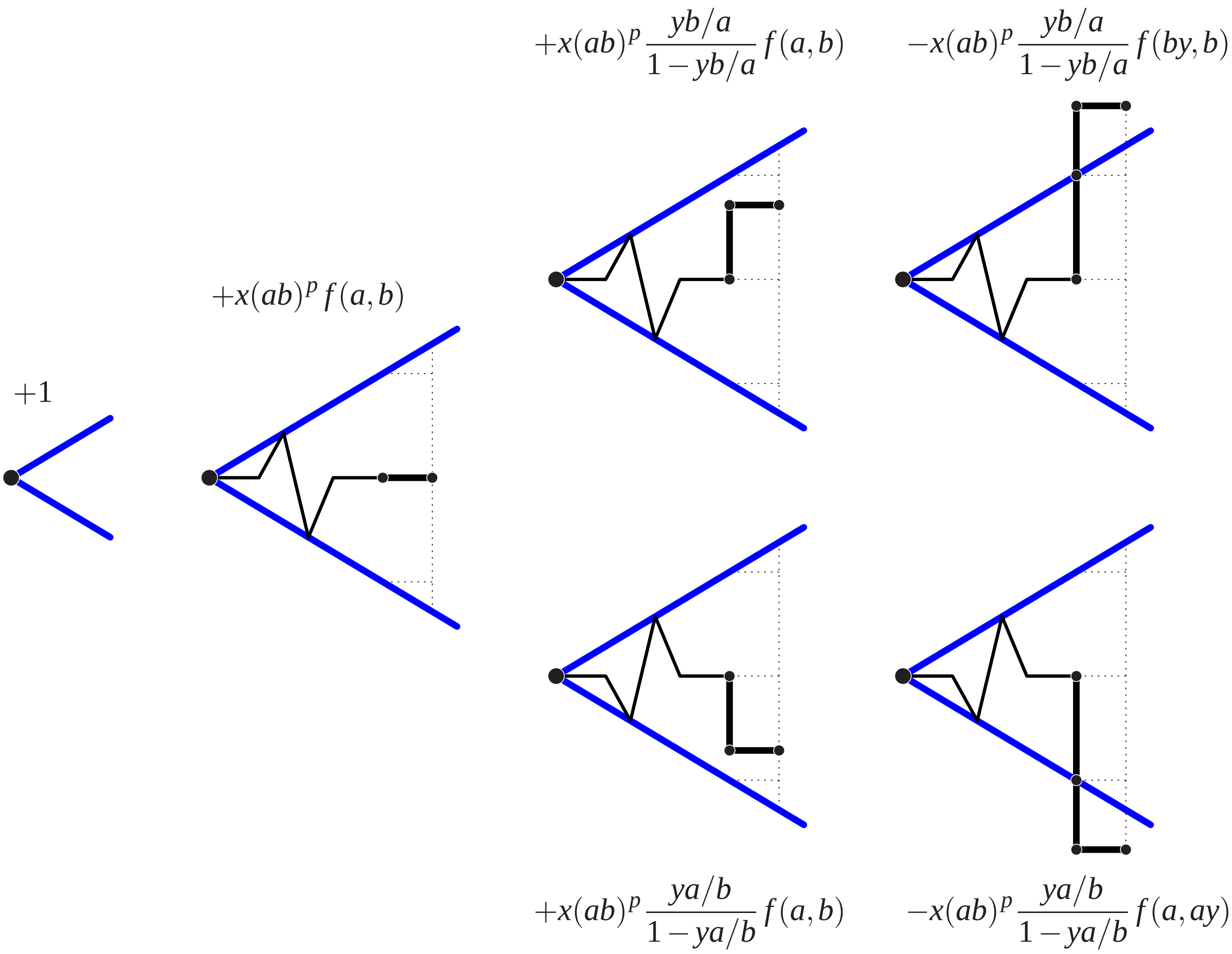

Figure 4: Constructing partially directed walks in the wedge $\mathcal{V}_p$. Every walk is either a single vertex, or can be obtained from a shorter walk by appending a horizontal edge (left), or a run of north steps and a horizontal edge or a run of south steps and a horizontal edge (centre-top and -bottom). Care must be taken to not step outside the wedge when appending north or south steps (right-top and -bottom).

Unfortunately, when appending up or down steps it is possible that the resulting path will step outside of the wedge. Hence we must subtract off the contributions from such paths (Figure 4 right-top and -bottom).

- Consider a path that ends at a distance h_+ from the line $Y = +pX$. It we append $> h_+$ up steps to the path then it will leave the wedge. We can decompose the resulting path into the original path with exactly h_+ up steps appended, and an "overhanging" Γ shaped path which is a sequence of some positive number of up steps and a horizontal step (see Figure 4 top-right).

  Appending exactly h_+ up steps to the path increases the distance from $Y = -pX$

by h_+, decreases the distance from $Y = -pX$ to zero. This gives the generating function $f_p(by, b)$. The overhanging piece is (by the reasoning above) enumerated by $x(ab)^p \frac{yb/a}{1-yb/a}$.

Hence the g.f. of walks that leave the wedge is given by $x(ab)^p \frac{yb/a}{1-yb/a} f(by, b)$.

- Similarly when appending too many down steps we obtain configurations counted by $x(ab)^p \frac{ya/b}{1-ya/b} f_p(a, ay)$.

Using the above construction we arrive at the following theorem

Proposition 3.1. *The generating function $f_p(a, b; x, y) \equiv f_p(a, b)$ of partially directed walks ending in a horizontal step in the wedge $\mathcal{V}_p$ satisfies the following functional equation:*

$$\begin{aligned} f_p(a, b) = 1 &+ x(ab)^p f_p(a, b) \\ &+ x(ab)^p \frac{yb/a}{1 - yb/a} \left(f_p(a, b) - f_p(by, b)\right) \\ &+ x(ab)^p \frac{ya/b}{1 - ya/b} \left(f_p(a, b) - f_p(a, ay)\right) \end{aligned} \tag{3.2}$$

The generating function of all partially directed walks in $\mathcal{V}_p$ is given by

$$g_p(a, b) = x^{-1}(ab)^{-p} \left(f_p(a, b) - 1\right)$$

In the next section we turn to the problem of solving this functional equation.

3.2 The asymmetric wedge model

Let us now turn to the construction of partially directed paths in the asymmetric wedge $\mathcal{W}_p$. Let the generating function of all partially directed walks in this wedge be denoted $k_p(a, b; x, y) \equiv k_p(a, b)$ where the variables x and y are suppressed.

As above, the resulting functional equation satisfied by he generating function is simpler if we consider only those walks that are either a single vertex or end in a horizontal step. Let this generating function be denoted $h_p(a, b; x, y)$. This is simply related back to k_p by

$$h_p(a, b) = 1 + xa^p k_p(a, b). \tag{3.3}$$

We now use the same construction as was used above for the symmetric case — each walk is either a single vertex, or can be constructed from a shorter walk by appending either a horizontal step, or a run of up steps and a horizontal step, or a run of down steps and a horizontal step — see Figure 5. Again care must be taken not to step outside the wedge, and so those walks that do step outside the wedge must be removed. Indeed the argument is *almost* identical to that used above, with the exception that a horizontal step contributes xa^p instead of $x(ab)^p$, since a horizontal step increases the distance from the line $Y = +pX$ by p, but does not change the distance from the line $Y = 0$.

The above construction gives the following theorem:

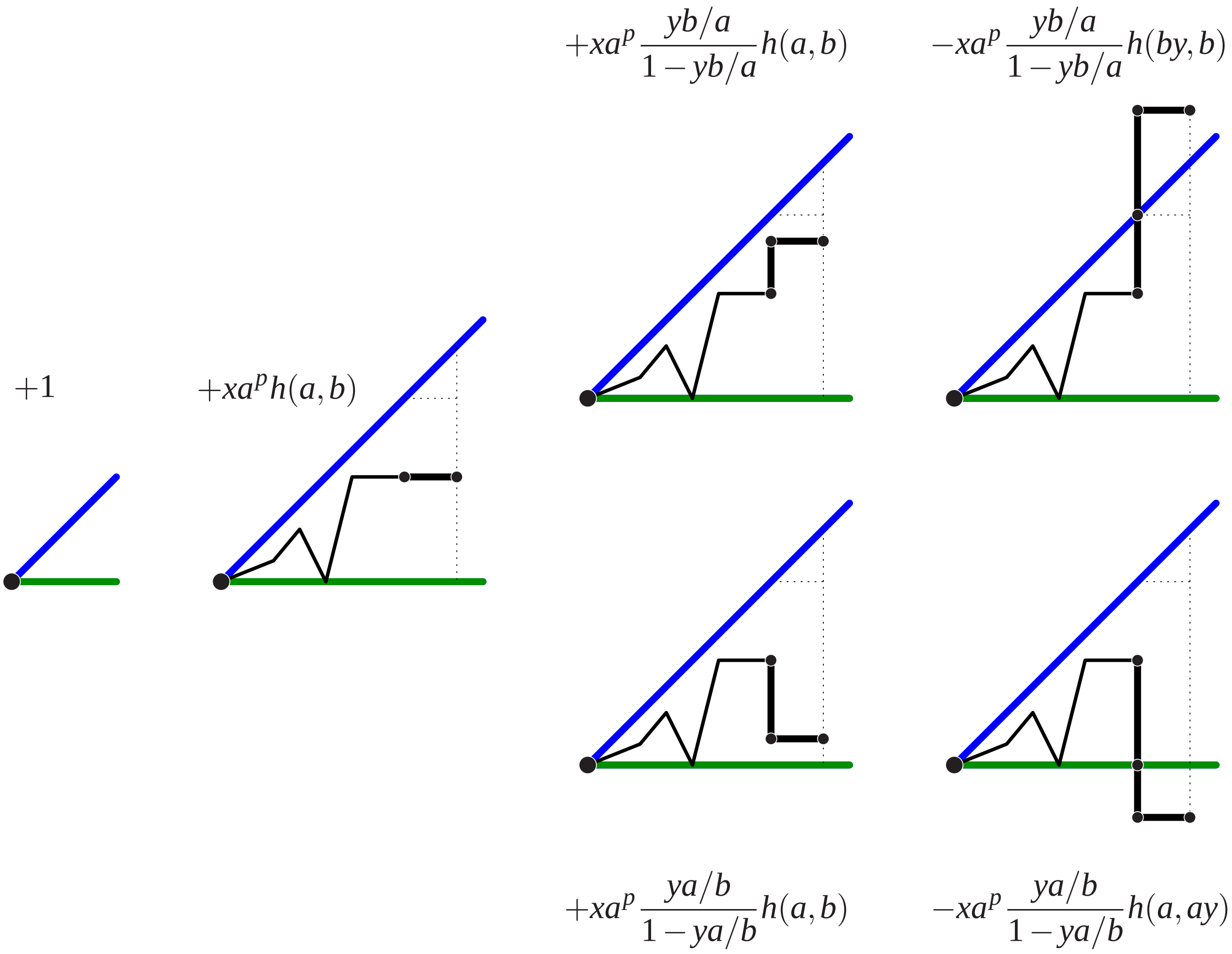

Figure 5: Constructing partially directed walks in the asymmetric wedge $\mathcal{W}_p$. Every walk is either a single vertex, or can be obtained from a shorter walk by appending a horizontal edge (left), or a run of north steps and a horizontal edge or a run of south steps and a horizontal edge (centre-top and -bottom). Care must be taken to not step outside the wedge when appending north or south steps (right-top and -bottom).

Proposition 3.2. *The generating function $h_p(a,b;x,y) \equiv f_p(a,b)$ of partially directed walks ending in a horizontal step in the wedge $\mathcal{W}_p$ satisfies the following functional equation:*

$$\begin{aligned} h_p(a,b) &= 1 + xa^p h_p(a,b) \\ &\quad + xa^p \frac{yb/a}{1-yb/a}\left(h_p(a,b) - h_p(by,b)\right) \\ &\quad + xa^p \frac{ya/b}{1-ya/b}\left(h_p(a,b) - h_p(a,ay)\right) \end{aligned} \tag{3.4}$$

The generating function of all partially directed walks in $\mathcal{V}_p$ is given by

$$k_p(a,b) = x^{-1}a^{-p}\left(h_p(a,b) - 1\right)$$

We solve this equation in Section 5.

4 Solving the symmetric case

At first sight, one might try to solve equation (3.2) by the iteration method used in [2], however the coefficients of the equation are singular when $a = by$ and $b = ay$. Multiplying both sides of the equation by $(a - by)(b - ay)$ gives a non-singular equation, however when we set $a = by$ or $b = ay$ the equation reduces to a tautology.

Instead we apply a variation of the kernel method, which we call the iterated kernel method. This appears to be similar in flavour to the "obstinate kernel method" used by Bousquet-Mélou ([3, 4] for example). We start by collecting all the $f(a,b)$ terms together on the left-hand side of the equation — this gives the *kernel form* of the equation:

$$K(a,b)\, f_p(a,b) = X(a,b) + Y(a,b)\, f_p(a,ya) + Z(a,b)\, f_p(yb,b), \tag{4.1}$$

where the functions $K(a,b)$, $X(a,b)$, $Y(a,b)$ and $Z(a,b)$ are given by

$$K(a,b) = (b - ya)(a - yb)(1 - x(ab)^p) - xy(ab)^p(a^2 + b^2 - 2yab), \tag{4.2a}$$

$$X(a,b) = (b - ya)(a - yb) \tag{4.2b}$$

$$Y(a,b) = -xya^{p+1}b^p(a - yb) \tag{4.2c}$$

$$Z(a,b) = -xya^pb^{p+1}(b - ya) \tag{4.2d}$$

for each integer $p \geq 1$. The function $K(a,b)$ is called the *kernel* of the equation. Note that the equation is symmetric under interchange of a and b:

$$f_p(a,b) = f_p(b,a) \qquad K(a,b) = K(b,a) \qquad X(a,b) = X(b,a) \qquad Y(a,b) = Z(b,a). \tag{4.3}$$

We solve equation (4.1) by substituting an infinite number of pairs of a and b values that set the kernel $K(a,b)$ to zero. While the method we describe below should work for general p, we focus on the (far) more tractable case of $p = 1$.

4.1 Iterated kernel method for $\mathcal{V}_1$

When $p = 1$, the kernel becomes a quadratic function of a and b and we can explicitly write down (simple) expressions for its zeros. A similar situation arises for the asymmetric wedge. For the remainder of this section we only consider $p = 1$. We write $f_1(a,b) \equiv F(a,b)$ and the coefficients in equation (4.1) become

$$K(a,b) = (xy^2a^2 - xa^2 - y)b^2 + (1 + y^2)ab - ya^2 \tag{4.4a}$$

$$X(a,b) = (b - ya)(a - yb) \tag{4.4b}$$

$$Y(a,b) = -xya^2b(a - yb) \tag{4.4c}$$

$$Z(a,b) = -xyab^2(b - ya) \tag{4.4d}$$

Let $\beta_\pm(a;x,y) \equiv \beta_\pm(a)$ be the zeros of $K(a,b)$ with respect to b. Hence

$$K(a,\beta_\pm(a)) = 0. \tag{4.5}$$

Thus, setting $b = \beta_\pm(a)$ removes $F(a,b)$ from equation (4.1). This is the key idea behind the "kernel method" which has been used to solve equations of this type (see [1] for example).

Unfortunately in this case, removing the kernel reduces the recurrence to an equation containing terms $F(a,ya)$ and $F(y\beta_\pm(a),\beta_\pm(a)$, which we cannot use immediately to solve for $F(a,b)$. Similar situations have been studied before using the "obstinate kernel method" ([3, 4] for example).

The method we use appears to be similar to the obstinate kernel method, except that instead of finding a finite number of pairs of values of a and b to set the kernel to zero we must use an infinite sequence of pairs. In this way, our "iterated kernel method" is related both to the kernel method and perhaps also to the iterative scheme used in [2].

The roots $\beta_\pm(a)$ can be determined explicitly:

$$\beta_\pm(a) = \frac{a}{2}\left(\frac{1+y^2 \pm \sqrt{(1-y^2)(1-4xya^2-y^2)}}{y+xa^2-xy^2a^2}\right). \tag{4.6}$$

Define the two roots as

$$\beta_1(a) \equiv \beta_-(a) = ya + O(xy^2a^3), \tag{4.7}$$
$$\beta_{-1}(a) \equiv \beta_+(a) = y/a + O(xy^{-2}a). \tag{4.8}$$

Since a is a variable, we are able to substitute something else for it; substituting $a \mapsto \beta_1(a)$ into equation (4.5) gives

$$K(\beta_1(a),\beta_1(\beta_1(a)) = 0. \tag{4.9}$$

Hence the pair $(a,b) = \big(\beta_1(a),\beta_1(\beta_1(a))\big)$ also sets the kernel to zero. We can continue in this way. Hence we need to define the repeated composition of $\beta_1(a)$ with itself:

$$\beta_n(a) = \beta_1^{(n)}(a) = \underbrace{\beta_1 \circ \beta_1 \circ \ldots \circ \beta_1}_{n}(a). \tag{4.10}$$

Note that

$$\beta_{-1}\circ\beta_1(a) = \beta_1\circ\beta_{-1}(a) = a, \qquad \text{and} \tag{4.11}$$
$$\beta_n(a) = ay^n + O(xy^{n+1}a^3). \tag{4.12}$$

There is no finite value of n such that $\beta_n = \beta_0$. If we further define $\beta_0(a) = a$ and $\beta_{-n}(a)$ by composition of $\beta_{-1}(a)$, then the functions $\{\beta_n \,|\, n \in \mathbb{Z}\}$ form an infinite group with identity β_0 and inverses $\beta_n \circ \beta_{-n} = \beta_0$.

There observations are enough to iterate the functional equation to find a solution. Set $b = \beta_1(a)$ in equation (4.1), and set $a = \beta_n(a)$ for any finite $n \geq 0$. Then since $K(\beta_n(a),\beta_{n+1}(a)) = 0$, we have

$$\begin{aligned} X(\beta_n(a),\beta_{n+1}(a)) + Y(\beta_n(a),\beta_{n+1}(a))\, F(\beta_n(a), y\beta_n(a)) \\ + Z(\beta_n(a),\beta_{n+1}(a))\, F(y\beta_{n+1}(a),\beta_{n+1}(a)) = 0. \end{aligned} \tag{4.13}$$

We can then solve this equation for $F(\beta_n(a), y\beta_n(a))$:

$$\begin{aligned} F(\beta_n(a), y\beta_n(a)) = &- \left[\frac{X(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right] \\ &- \left[\frac{Z(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right] F(y\beta_{n+1}(a), \beta_{n+1}(a)) \end{aligned} \tag{4.14}$$

We can simplify the above by defining

$$\begin{aligned} \mathcal{F}_n(a) &= F(\beta_n(a), y\beta_n(a)) = F(y\beta_n(a), \beta_n(a)), \\ \mathcal{X}_n(a) &= -\left[\frac{X(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right], \qquad (4.15) \\ \mathcal{Z}_n(a) &= -\left[\frac{Z(\beta_n(a), \beta_{n+1}(a))}{Y(\beta_n(a), \beta_{n+1}(a))}\right], \qquad (4.16) \end{aligned}$$

where we have made use of the symmetry $F(a, b) = F(b, a)$. While this symmetry is not essential, it does make the solution substantially simpler. Instead of exploiting this symmetry we could iterate again to find $F(\beta_n(a), y\beta_n(a))$ in terms of $F(\beta_{n+2}(a), y\beta_{n+2}(a))$. Indeed this is what is required to solve walks in the asymmetric wedge $\mathcal{W}_1$ (see Section 5 below).

Equation (4.14) may be written as

$$\mathcal{F}_n(a) = \mathcal{X}_n(a) + \mathcal{Z}_n(a)\mathcal{F}_{n+1}(a). \tag{4.17}$$

Starting at $n = 0$, this can be iterated to get a series solution for $\mathcal{F}_0(a)$:

$$F(a, ya) = \mathcal{F}_0(a) = \sum_{n=0}^{\infty} \mathcal{X}_n(a) \prod_{k=0}^{n-1} \mathcal{Z}_k(a), \tag{4.18}$$

where we have *assumed* that the above sum converges (we will show that this is the case). This is also give $F(yb, b)$:

$$F(yb, b) = F(b, yb) = \mathcal{F}_0(b) = \sum_{n=0}^{\infty} \mathcal{X}_n(b) \prod_{k=0}^{n-1} \mathcal{Z}_k(b). \tag{4.19}$$

This then allows us to write down the solution for $F(a, b)$:

$$f_p(a, b) = \frac{X(a, b)}{K(a, b)} + \frac{Y(a, b)}{K(a, b)} \sum_{n=0}^{\infty} \mathcal{X}_n(a) \prod_{k=0}^{n-1} \mathcal{Z}_k(a) + \frac{Z(a, b)}{K(a, b)} \sum_{n=0}^{\infty} \mathcal{X}_n(b) \prod_{k=0}^{n-1} \mathcal{Z}_k(b). \tag{4.20}$$

Of course, the above "solution" still contains many complicated algebraic functions in the form of the $\beta_n(a)$. It is quite surprising (at least to the authors!) is that these functions can be drastically simplified.

4.2 An explicit expression for $f_1(1,1)$

We have outlined above the iterated kernel method that we shall use to write down the generating function $f_1(a,b) = F(a,b)$. We are primarily interested in the number of paths (and not the location of their endpoints), so we will actually focus on the function $F(1,1)$.

We start by considering the $\beta_n(a)$ functions. It is quite surprising that while $\beta_n(a)$ is (upon superficial inspection for small n) very complicated, its reciprocal appears relatively simple. Examining equations (4.4a) and (4.6) one obtains

$$\frac{1}{\beta_1(a)} + \frac{1}{\beta_{-1}(a)} = \frac{1+y^2}{y}\frac{1}{a}. \tag{4.21}$$

Substituting $a = \beta_{n-1}(a)$ in the above, and using the group properties of β_n leads to the following three term recurrence for β_n:

$$\frac{1}{\beta_n} = \frac{1+y^2}{y}\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{n-2}}. \tag{4.22}$$

Since β_0 is the identity, and β_1 is given explicitly by $\beta_-(a)$ in equation (4.6), the recurrence above can be iterated to get a solution for $\beta_n(a)$:

$$\frac{1}{\beta_n(a)} = \frac{y(1-y^{2n})}{y^n(1-y^2)}\frac{1}{\beta_1(a)} - \frac{y^2(1-y^{2n-2})}{y^n(1-y^2)}\frac{1}{a}. \tag{4.23}$$

By using the expressions for $X(a,b)$, $Y(a,b)$ and $Z(a,b)$ in equation (4.4) to determine $\mathcal{X}(a,b)$ and $\mathcal{Z}(a,b)$, one obtains

$$F(a,ya) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\beta_{n+1}-y\beta_n}{xya\beta_n\beta_{n+1}}\right]\prod_{k=0}^{n-1}\left(\frac{\beta_{k+1}-y\beta_k}{\beta_k-y\beta_{k+1}}\right). \tag{4.24}$$

Substituting the expression for $\beta_n(a)$ given in equation (4.23) and simplifying gives:

$$\frac{\beta_{n+1}-y\beta_n}{xya\beta_n\beta_{n+1}} = y^n\left[\frac{1}{a} - \frac{y}{\beta_1}\right], \qquad \text{and} \tag{4.25}$$

$$\frac{\beta_{k+1}-y\beta_k}{\beta_k-y\beta_{k+1}} = y^{2k+1}\left[\frac{1}{xya^2} - \frac{1}{xa\beta_1} - 1\right]. \tag{4.26}$$

Using these, one can get an explicit expression for the generating function $F(a,ya)$:

$$F(a,ya) = \left[\frac{1}{xya^2} - \frac{1}{xa\beta_1}\right]\sum_{n=0}^{\infty}(-1)^n y^{n(n+1)}\left(\frac{1}{xya^2} - \frac{1}{xa\beta_1} - 1\right)^n. \tag{4.27}$$

By defining

$$Q(a;x,y) = \left(\frac{1}{xa^2} - \frac{y}{xa\beta_1} - y\right). \tag{4.28}$$

the above expression for $F(a, ya)$ can be further simplified to

$$F(a, ya) = \left[1 + \frac{Q(a; x, y)}{y}\right] \sum_{n=0}^{\infty} (-1)^n y^{n^2} Q(a; x, y)^n. \tag{4.29}$$

Using the $a \leftrightarrow b$ symmetry of $F(a, b)$, we can get a similar expression for $F(yb, b)$, and so finally $F(a, b)$.

$$f(a,b) = \frac{X(a,b)}{K(a,b)} + \frac{Y(a,b)}{K(a,b)} \left(1 + \frac{Q(a)}{y}\right) \sum_{n \geq 0} (-1)^n Q(a)^n y^{n^2} \\ + \frac{Z(a,b)}{K(a,b)} \left(1 + \frac{Q(b)}{y}\right) \sum_{n \geq 0} (-1)^n Q(b)^n y^{n^2} \tag{4.30}$$

We can reduce the above equation by considering only the number of walks of length n (by setting $a = b = 1, x = y = t$):

Proposition 4.1. *The generating function of partially directed walks ending in a horizontal step in the wedge $\mathcal{V}_1$ is*

$$f_1(1,1) = \frac{1-t}{1-2t-t^2} - \frac{1 - t^2 - \sqrt{(1-t^2)(1-5t^2)}}{1-2t-t^2} \sum_{n=0}^{\infty} (-1)^n t^{n^2} Q(1; t, t)^n, \tag{4.31}$$

where t counts the number of edges and

$$Q(1; t, t) = (1 - 3t^2 - \sqrt{(1-t^2)(1-5t^2)})/2t. \tag{4.32}$$

The generating function of all paths in $\mathcal{V}_1$ is then found using equation (3.1):

$$g_1(1,1) = \frac{1+t}{1-2t-t^2} - \frac{1 - t^2 - \sqrt{(1-t^2)(1-5t^2)}}{t(1-2t-t^2)} \sum_{n=0}^{\infty} (-1)^n t^{n^2} Q(1; t, t)^n. \tag{4.33}$$

4.3 An aside

Firstly we note that $F(a, ya)$ counts all partially directed paths in the wedge $\mathcal{V}_1$ whose last vertex ends in the line $Y = -pX$. Additionally we note that the generating function $Q(a; x, y)/y$ counts the number of partially directed paths starting at the origin, lying on or above the line $Y = -pX$ and whose last vertex lies in the line $Y = -pX$. Hence $Q(a)/y$ counts a very similar set of paths to $F(a, ya)$, excepting that the paths counted by Q are not confined by the line $Y = pX$.

In light of the above interpretation of the function Q, we expended considerable effort to uncover a more direct combinatorial derivation of the theta-function like alternating sum equation (4.29). Indeed it looks very much like some sort of inclusion-exclusion. Unfortunately we have only made very limited progress towards this goal.

4.4 Asymptotics for $p = 1$

The asymptotics of the number of partially directed paths in the symmetric wedge with $p = 1$ can be analysed by examining the singularities of the generating function $g_1(1,1)$ in equation (4.33). Singularities arise either as zeros of the factor $(1-2t-t^2)$ in equation (4.33), or as singularities in $\sqrt{(1-t^2)(1-5t^2)}$, or as singularities in the series $\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n$.

An examination of $g_1(1,1)$ shows that it has simple poles at the solution of $(1-2t-t^2) = 0$, or when $t = -1 \pm \sqrt{2}$. We note that $\sqrt{(1-t^2)(1-5t^2)}$ has branch-points (square root singularities) at $t = \pm 1$ and again at $t = \pm 1/\sqrt{5}$. The series $\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n$ is a Jacobi θ-function and it is convergent inside the unit circle except at singularities of $Q(1;t,t)$; that is, when $t = \pm 1/\sqrt{5}$.

The dominant singularity is the simple pole at $\sqrt{2}-1$, while the next subdominant contributions to the asymptotics will be given by the singularities at $t = \pm 1/\sqrt{5}$. These two subdominant singularities will give an odd-even effect. The contributions from these singularities allow us to write down the asymptotic form of $v_{n,1}$.

Proposition 4.2. *The number of paths in the wedge $\mathcal{V}_1$ is asymptotic to*

$$v_{n,1} = A_0\left(1+\sqrt{2}\right)^n + \frac{5^{n/2}}{(n+1)^{3/2}}\Big(A_1 + (-1)^n A_2 + O(1/n)\Big). \tag{4.34}$$

Where the constants are

$$A_0 = 0.27730985348603118827\ldots, \tag{4.35a}$$
$$A_1 = 3.71410486533662324953\ldots, \qquad \textit{and} \tag{4.35b}$$
$$A_2 = 0.20697997020804157910\ldots. \tag{4.35c}$$

We note that the constants were derived by expanding the expression for $g_1(1,1)$ about $t = \sqrt{2}-1$ and $t = \pm 1/\sqrt{5}$ (or rather the first 40 or so terms of the sum). These were then checked using both Bruno Salvy's *gdev* package for Maple [11] and by direct examination of $v_{n,1}$ for $n \le 1000$. The above formula is quite precise and it correctly estimates $v_{10,1}, v_{20,1}, v_{30,1}$ and $v_{40,1}$ to within $7\%, 1\%, 0.2\%$ and 0.06% respectively.

Note that the above result implies that walks in the wedge $\mathcal{V}_p$ have the same dominant asymptotic behaviour as walks with no bounding wedge (see equation (1.2)). Further, Since we the number of walks in any wedge $\mathcal{V}_p$ for $1 \le p < \infty$ is bounded between the number of walks in $\mathcal{V}_1$ and partially directed walks with no bounding wedge, we have the following result:

Corollary 4.3. *The number of partially directed walks in the wedge $\mathcal{V}_p$, $c_n^{(p)}$ obeys the following inequality*

$$0.2773\ldots \le \lim_{n\to\infty} \frac{c_n^{(p)}}{(1+\sqrt{2})^n} \le (1+\sqrt{2})/2 = 1.2071\ldots \tag{4.36}$$

for any $1 \le p < \infty$.

5 Partially Directed Paths in the Asymmetric Wedge

In this section we turn our attention to the model in Figure 2 (right). The partially directed path is confined to an asymmetric wedge, $\mathcal{W}_p$, and its generating function does not have the $a \leftrightarrow b$ symmetry we have exploited in solving for $f_1(a,b)$ in the previous section.

Again we start by examining the generating function of walks that end in a horizontal step. The functional equation for these walks is given in Proposition 3.2 and we can arrange equation (3.4) in kernel form:

$$K(a,b)\,h_p(a,b) = X(a,b) + Y(a,b)\,h_p(a,ya) + Z(a,b)\,h_p(yb,b) \tag{5.1}$$

where

$$K(a,b) = (b-ya)(a-yb)(1-xa^p) - xya^p(a^2+b^2-2yab), \tag{5.2a}$$
$$X(a,b) = (b-ya)(a-yb), \tag{5.2b}$$
$$Y(a,b) = -xya^{p+1}(a-yb), \tag{5.2c}$$
$$Z(a,b) = -xya^p b(b-ya). \tag{5.2d}$$

This functional equation is very similar to that of the symmetric wedge given in equation (4.1). However we no longer have $a \leftrightarrow b$ symmetry and this means that we have to work quite a bit harder and we concentrate only on the case $p=1$. We will write $h_1(a,b) \equiv H(a,b)$ for the remainder of this section.

5.1 Solving for $H(a,b)$ when $p=1$

For the remainder of this section we concentrate on the case $p=1$ and walks in the $45°$ wedge $\mathcal{W}_1$. The kernel $K(a,b)$ (given in equation (5.2)) is no longer symmetric in a and b, nor is the desired generating function $H(a,b)$. In order to repeat the iterated kernel method as described in Section 4.1 we must now consider the solutions of the kernel as functions of a and b. These solutions are defined by $K(a,\beta(a))=0$ and $K(\alpha(b),b)=0$:

$$\beta_\pm(a) = \frac{a}{2y}\left[1+y^2-x(1-y^2)a \pm \sqrt{(1-y^2)((1-xa)^2-y^2(1+xa)^2)}\right] \tag{5.3}$$

and

$$\alpha_\pm(b) = \frac{b}{2}\left[\frac{1+y^2 \pm \sqrt{(1-y^2)(1-y^2-4xyb)}}{y+x(1-y^2)b}\right]. \tag{5.4}$$

The "physical roots" which expands as power series in x and y are $\alpha_-(b)$ and $\beta_-(a)$. Write these as $\alpha_1(b)$ and $\beta_1(b)$, and the other roots as $\alpha_{-1}(b)$ and $\beta_{-1}(a)$. In Section 4.1 we considered composing $\beta(a)$ with itself, however due to the asymmetry of the kernel we now need to consider mixed compositions $\beta(\alpha(b))$ and $\alpha(\beta(a))$. Indeed, we find that

$$\alpha_{\pm 1}(\beta_{\mp 1}(a)) = a, \tag{5.5a}$$
$$\beta_{\pm 1}(\alpha_{\mp 1}(b)) = b. \tag{5.5b}$$

We will need the function $\gamma(a) = \alpha_1(\beta_1(a))$, and define its nested composition by $\gamma_n(a) = \gamma(\gamma_{n-1}(a))$ with $\gamma_0(a) = a$. Note that

$$\gamma_n(a) = y^{2n}a + O(xy^{2n}a^2). \tag{5.6}$$

We can now repeat the iterated kernel method in the new asymmetric setting. Setting $b = \beta_1(a)$ in equation (5.1) gives

$$0 = X(a, \beta_1(a)) + Y(a, \beta_1(a))H(a, ya) + Z(a, \beta_1(a))H(y\beta_1(a), \beta_1(a)). \tag{5.7}$$

Since we cannot simply relate $H(by, b)$ to $H(b, by)$ we cannot iterate this equation and we must make use the other roots of the kernel.

Setting $a = \alpha_1(b)$ gives:

$$0 = X(\alpha_1(b), b) + Y(\alpha_1(b), b)H(\alpha_1(b), y\alpha_1(b)) + Z(\alpha_1(b), b)H(yb, b). \tag{5.8}$$

Now set $b = \beta_1(a)$ in the above equation

$$\begin{aligned} 0 = X(\gamma(a), \beta_1(a)) + Y(\gamma_1(a), \beta_1(a))H(\gamma_1(a), y\gamma_1(a)) \\ + Z(\gamma_1(a), \beta_1(a))H(y\beta_1(a), \beta_1(a)). \end{aligned} \tag{5.9}$$

We can now eliminate $H(y\beta_1(a), \beta_1(a))$ between equations (5.7) and (5.9) and solve for $H(a, ya)$. This gives

$$\begin{aligned} H(a, ya) = -\left[\frac{X(a, \beta_1(a))}{Y(a, \beta_1(a))}\right] + \left[\frac{Z(a, \beta_1(a))}{Y(a, \beta_1(a))}\right]\left[\frac{X(\gamma_1(a), \beta_1(a))}{Z(\gamma_1(a), \beta_1(a))}\right] \\ + \left[\frac{Z(a, \beta_1(a))}{Y(a, \beta_1(a))}\right]\left[\frac{Y(\gamma_1(a), \beta_1(a))}{Z(\gamma_1(a), \beta_1(a))}\right] H(\gamma_1(a), y\gamma_1(a)). \end{aligned} \tag{5.10}$$

We can now iterate the above equation by substituting $a = \gamma_{n-1}(a)$. Define

$$\mathcal{X}_n(a) = -\frac{X(\gamma_n(a), \beta_1(\gamma_n(a)))}{Y(\gamma_n(a), \beta_1(\gamma_n(a)))} = \frac{\beta(\gamma_n) - y\gamma_n}{xy\gamma_n^2}, \tag{5.11a}$$

$$\mathcal{Y}_n(a) = \frac{Z(\gamma_n(a), \beta_1(\gamma_n(a)))}{Y(\gamma_n(a), \beta_1(\gamma_n(a)))} = \frac{\beta(\gamma_n)}{\gamma_n}\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)}\right), \tag{5.11b}$$

$$\mathcal{Z}_n(a) = \frac{X(\gamma_{n+1}(a), \beta_1(\gamma_n(a)))}{Z(\gamma_{n+1}(a), \beta_1(\gamma_n(a)))} = -\frac{\gamma_{n+1} - y\beta(\gamma_n)}{xy\gamma_{n+1}\beta(\gamma_n)}, \tag{5.11c}$$

$$\mathcal{A}_n(a) = \frac{Y(\gamma_{n+1}(a), \beta_1(\gamma_n(a)))}{Z(\gamma_{n+1}(a)), \beta_1(\gamma_n(a)))} = \frac{\gamma_{n+1}}{\beta(\gamma_n)}\left(\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\beta(\gamma_n) - y\gamma_{n+1}}\right). \tag{5.11d}$$

And further define

$$\mathcal{B}_n(a) = \mathcal{X}_n(a) + \mathcal{Y}_n(a)\mathcal{Z}_n(a), \qquad \mathcal{C}_n(a) = \mathcal{Y}_n(a)\mathcal{A}_n(a). \tag{5.12}$$

Equation (5.10) now becomes:

$$H(\gamma_n(a), y\gamma_n(a)) = \mathcal{B}_n + \mathcal{C}_n H(\gamma_{n+1}(a), y\gamma_{n+1}(a)). \tag{5.13}$$

We obtain a solution for $H(a, ya)$ by iterating the above equation

$$H(a, ya) = \mathcal{B}_0 + \mathcal{C}_0\mathcal{B}_1 + \mathcal{C}_0\mathcal{C}_1\mathcal{B}_2 + \cdots = \sum_{n=0}^{\infty} \mathcal{B}_n(a) \prod_{m=0}^{n-1} \mathcal{C}_m(a). \tag{5.14}$$

As was the case for the symmetric wedge, we are able to simplify the above expression by rewriting $\gamma_n(a)$ in terms of the original kernel roots and thereby rewrite the expressions for $\mathcal{B}_n$ and $\mathcal{C}_n$.

In order to find $H(a, b)$ from equation (5.1), we need both $H(a, ya)$ $and H(yb, b)$. Equation (5.8) gives $H(b, yb)$ in terms of $H(\alpha_1(b), y\alpha_1(b))$:

$$H(yb, b) = -\frac{X(\alpha_1(b), b)}{Z(\alpha_1(b), b)} - \frac{Y(\alpha_1(b), b)}{Z(\alpha_1(b), b)} H(\alpha_1(b), y\alpha_1(b)) \tag{5.15}$$

So using the above expressions for $H(a, ya)$ and $H(yb, b)$ we have, at least in principle, a solution for $H(a, b)$:

$$\begin{aligned} H(a, b) = \frac{X(a, b)}{K(a, b)} - \frac{Z(a, b)X(\alpha_1(b), b)}{Z(\alpha_1(b), b)K(a, b)} + \frac{Y(a, b)}{K(a, b)} H(a, ya) \\ - \frac{Z(a, b)Y(\alpha_1(b), b)}{Z(\alpha_1(b), b)K(a, b)} H(\alpha_1(b), y\alpha_1(b)) \end{aligned} \tag{5.16}$$

Of course, we would like to be able to simplify the above expression. In particular we would like to rewrite $\gamma_n(a)$ and $\gamma_n(\alpha_1(b))$ in terms of simpler functions, as we did for $\beta_n(a)$ in Section 4.2.

5.2 Simplifying things

In much the same was as for the symmetric case, we can find simple expressions for the $1/\gamma_n(a)$ in terms of the original kernel roots. Consideration of the kernel and its roots gives:

$$\frac{1}{\alpha_{-1}(b)} + \frac{1}{\alpha_{+1}(b)} = \frac{1 + y^2}{yb}, \tag{5.17a}$$

$$\frac{1}{\beta_{-1}(a)} + \frac{1}{\beta_{+1}(a)} = \frac{1 + y^2}{ya} - \frac{x(1 - y^2)}{y}. \tag{5.17b}$$

Since certain compositions of α and β give the identity (see equations (5.5)), we have the additional relations:

$$\frac{1}{\alpha_1(\beta_1(a))} = \frac{1}{\gamma_1(a)} = \frac{1 + y^2}{y\beta_1(a)} - \frac{1}{a}, \tag{5.18a}$$

$$\frac{1}{\beta_1(\alpha_1(b))} = \frac{1 + y^2}{y\alpha_1(b)} - \frac{1}{b} - \frac{x(1 - y^2)}{y}. \tag{5.18b}$$

We note that the last term in equation (5.18b) means that the resulting expressions for $\gamma_n(a)$ are more complicated than those for $\beta_n(a)$ for the symmetric case (see equation (4.23)); this in turn leads to a significantly more complicated solution.

Setting $a = \gamma_{n-1}(a)$ and $b = \beta_1(\gamma_{n-1}(a))$ in the above two equations give:

$$\frac{1}{\gamma_n(a)} = \frac{1+y^2}{y\beta_1(\gamma_{n-1}(a))} - \frac{1}{\gamma_{n-1}(a)}, \tag{5.19a}$$

$$\frac{1}{\beta_1(\gamma_n(a))} = \frac{1+y^2}{y\gamma_n(a)} - \frac{1}{\beta_1(\gamma_{n-1}(a))} - \frac{x(1-y^2)}{y}. \tag{5.19b}$$

These equations can be solved:

$$\begin{aligned}\frac{1}{\gamma_n(a)} &= \frac{1-y^{4n}}{y^{2n-1}(1-y^2)\beta_1(a)} - \frac{1-y^{4n-2}}{y^{2n-2}(1-y^2)a} - \frac{x(1-y^{2n})(1-y^{2n-2})}{y^{2n-2}(1-y^2)}, \\ &= \frac{1}{1-y^2}\left(x(1+y^2) + Q(a)\, y^{2n} + y\bar{Q}(a)\, y^{-2n}\right),\end{aligned} \tag{5.20a}$$

$$\begin{aligned}\frac{1}{\beta_1(\gamma_n(a))} &= \frac{1-y^{4n+2}}{y^{2n}(1-y^2)\beta_1(a)} - \frac{1-y^{4n}}{y^{2n-1}(1-y^2)a} - \frac{x(1-y^{2n})^2}{y^{2n-1}(1-y^2)}, \\ &= \frac{1}{1-y^2}\left(2xy + yQ(a)\, y^{2n} + \bar{Q}(a)\, y^{-2n}\right).\end{aligned} \tag{5.20b}$$

where we have used

$$Q(a) = \frac{1}{a} - \frac{y}{\beta_1(a)} - x \qquad \bar{Q}(a) = \frac{1}{\beta_1(a)} - \frac{y}{a} - xy \tag{5.21}$$

Note that $\bar{Q}(a)Q(a) = x^2 y$. In fact we can reduce the above expressions for γ_n and $\beta(\gamma_n)$ even further using this fact:

$$\frac{1}{\gamma_n(a)} = \frac{(x+y^{2n-2}Q)(x+y^{2n}Q)}{y^{2n-2}(1-y^2)Q} \tag{5.22a}$$

$$\frac{1}{\beta(\gamma_n(a))} = \frac{(x+y^{2n}Q)^2}{y^{2n-1}(1-y^2)Q} \tag{5.22b}$$

The above then lead to to the following expressions that will be useful in writing down our solution:

$$\frac{1}{\gamma_n(a)} - \frac{y}{\beta_1(\gamma_n(a))} = (x+y^{2n}Q), \tag{5.23a}$$

$$\frac{1}{\beta_1(\gamma_n(a))} - \frac{y}{\gamma_n(a)} = \frac{x}{y^{2n-1}Q}\left(x+y^{2n}Q\right), \tag{5.23b}$$

$$\frac{1}{\beta_1(\gamma_n(a))} - \frac{y}{\gamma_{n+1}(a)} = y\left(x+y^{2n}Q\right), \tag{5.23c}$$

$$\frac{1}{\gamma_{n+1}(a)} - \frac{y}{\beta_1(\gamma_n(a))} = \frac{x}{y^{2n}Q}\left(x+y^{2n}Q\right) \tag{5.23d}$$

This in turn lets us write

$$\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)} = \frac{y^{2n-1}}{x}Q \tag{5.24a}$$

$$\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\beta(\gamma_n) - y\gamma_{n+1}} = \frac{y^{2n+1}}{x}Q \tag{5.24b}$$

$$\tag{5.24c}$$

where we have made use of the fact that $\bar{Q} = x^2y/Q$. Hence $\mathcal{C}_n = \mathcal{Y}_n\mathcal{A}_n$ can now be written as

$$\mathcal{C}_n = \frac{\gamma_{n+1}}{\gamma_n}\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)}\right)\left(\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\beta(\gamma_n) - y\gamma_{n+1}}\right) = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{y^{4n}}{x^2}Q^2 \tag{5.25}$$

In a similar way we find that

$$\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n^2} = y(x + y^{2n-2}Q) \tag{5.26a}$$

$$\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\gamma_n\gamma_{n+1}} = y^2(x + y^{2n-2}Q) \tag{5.26b}$$

which allows us to also simplify the expression for $\mathcal{B}_n = \mathcal{X}_n + \mathcal{Y}_n\mathcal{Z}_n$:

$$\mathcal{X}_n = \frac{1}{xy}\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n^2}\right) = \frac{x + y^{2n-2}Q}{x} \tag{5.27a}$$

$$\mathcal{Y}_n\mathcal{Z}_n = -\frac{1}{xy}\left(\frac{\gamma_{n+1} - y\beta(\gamma_n)}{\gamma_n\gamma_{n+1}}\right)\left(\frac{\beta(\gamma_n) - y\gamma_n}{\gamma_n - y\beta(\gamma_n)}\right) = -\frac{y^{2n}Q}{x^2}\left(x + y^{2n-2}Q\right) \tag{5.27b}$$

$$\mathcal{B}_n = \mathcal{X}_n + \mathcal{Y}_n\mathcal{Z}_n = \frac{1}{x^2}\left(x + y^{2n-2}Q\right)\left(x - y^{2n}Q\right) \tag{5.27c}$$

Substituting the above into the expression for $H(a, ya)$ in equation (5.14) gives:

$$\begin{aligned} H(a, ya) &= \sum_{n=0}^{\infty} \mathcal{B}_n(a) \prod_{m=0}^{n-1} \mathcal{C}_m(a). \\ &= \sum_{n=0}^{\infty} \frac{1}{x^2}\left(x + y^{2n-2}Q\right)\left(x - y^{2n}Q\right)\frac{\gamma_n(a)}{a}\left(\frac{Q}{x}\right)^{2n} y^{2n(n-1)} \\ &= \frac{(1-y^2)Q}{ax^2y^2}\sum_{n=0}^{\infty}\frac{(x - y^{2n}Q)}{(x + y^{2n}Q)}\left(\frac{Q}{x}\right)^{2n} y^{2n^2} \end{aligned} \tag{5.28}$$

We can now substitute this into equation (5.16) to obtain $H(a, b)$. This requires us to compute $H(\alpha_1(b), y\alpha_1(b))$ from the above expression. Writing:

$$P(b) = Q(\alpha_1(b)) = \frac{y}{2b}\left(1 - 2xyb - y^2 + \sqrt{(1-y^2)(1-4xyb-y^2)}\right) \tag{5.29}$$

we have

$$H(\alpha, y\alpha) = \frac{(1-y^2)P}{\alpha_1(b)x^2y^2} \sum_{n=0}^{\infty} \frac{(x - y^{2n}P)}{(x + y^{2n}P)} \left(\frac{P}{x}\right)^{2n} y^{2n^2} \tag{5.30}$$

Below we give the length generating function (when $x = t, y = t, a = 1, b = 1$).

Proposition 5.1. *The generating function of partially directed walks ending in a horizontal step in the wedge $\mathcal{W}_1$ is*

$$\begin{aligned} h_1(1,1) = &\frac{(1-t)^2 - \sqrt{(1-t^2)(1-5t^2)}}{2(1-2t-t^2)} \\ &- Q\frac{(1-t^2)}{t^2(1-2t-t)} \sum_{n=0}^{\infty} \frac{(1-t^{2n-1}Q)}{(1+t^{2n-1}Q)} \left(\frac{Q}{t}\right)^{2n} t^{2n^2} \\ &+ \frac{(1-t^2)}{1-2t-t^2} \sum_{n=0}^{\infty} \frac{(1-t^{2n-1}P)}{(1+t^{2n-1}P)} \left(\frac{P}{t}\right)^{2n} t^{2n^2} \end{aligned} \tag{5.31}$$

where t counts the number of edges and

$$Q(1;t,t) = (1 - t - t^2 - t^3 - \sqrt{(1-t^4)(1-2t-t^2)})/2 \tag{5.32a}$$

$$P(1;t,t) = (1 - 3t^2 - \sqrt{(1-t^2)(1-5t^2)})/2t \tag{5.32b}$$

The generating function of all paths in $\mathcal{W}_1$ is then $k_1(1,1;t,t) = (h_1(1,1;t,t) - 1)/t$.

5.3 Another aside

As was the case for the symmetric wedge, the functions P and Q that make up our expression for H have combinatorial interpretations in terms of partially directed walks bounded by a single line. Let $B_-(x,y)$ be the generating function of walks that end with a horizontal step, start and end on the line $Y = 0$ and stay on or above that same line. Then

$$Q(a;x,y) = xy^2(B_-(ax,y) - 1). \tag{5.33}$$

Similarly let $B_/(x,y)$ be the generating function of walks that end with a horizontal step, start and end on the line $Y = X$ and stay on or above that same line. Then

$$P(b;x,y) = xy^2(B_/(bx,y) - 1). \tag{5.34}$$

Again we would like to find a more direct combinatorial derivation of the generating functions $H(1,1)$ and $H(1,t)$. We have been unable to do so.

5.4 Asymptotics for $p = 1$

As was the case for walks in the symmetric wedge, we analyse the asymptotics of partially directed walks in the asymmetric wedge $\mathcal{W}_1$ by singularity analysis. Let us split the expression given in equation (5.31) into 3 pieces and study their dominant singularities:

$$p1 = \frac{(1-t)^2 - \sqrt{(1-t^2)(1-5t^2)}}{2(1-2t-t^2)} \tag{5.35a}$$

$$p2 = -Q\frac{(1-t^2)}{t^2(1-2t-t)}\sum_{n=0}^{\infty}\frac{(1-t^{2n-1}Q)}{(1+t^{2n-1}Q)}\left(\frac{Q}{t}\right)^{2n} t^{2n^2} \tag{5.35b}$$

$$p3 = \frac{(1-t^2)}{1-2t-t^2}\sum_{n=0}^{\infty}\frac{(1-t^{2n-1}P)}{(1+t^{2n-1}P)}\left(\frac{P}{t}\right)^{2n} t^{2n^2} \tag{5.35c}$$

The generating function p_1 appears to have 6 singularities: 2 simple poles from the zeros of the denominators and four square-root singularities at $t = \pm 1, \pm 1/\sqrt{5}$. Closer analysis shows that there are no singularities at the zeros of the denominator and that generating function is dominated by the singularities at $t = \pm 1/\sqrt{5}$.

Inside the unit circle, the generating function p_3 has singularities coming from several different potential sources: the simple poles from the zeros of $1-2t-t^2$, the singularities of P and the singularities from the zeros of $1 + t^{2n-1}P$.

{ADR: WORK IS NEEDED HERE! — sorry lads. I am a bit stuck. I have gone back to try to massage the other bits.}

Proposition 5.2. *The number of paths in the wedge $\mathcal{W}_1$ is asymptotic to*

$$w_{n,1} = \frac{A_0}{n^{1/2}}\left(1+\sqrt{2}\right)^n\left(1+O(n^{-1/2})\right) \tag{5.36}$$

where $A_0 = 0.21869391669430\ldots$

While we have not done so here, subdominant terms in the asymptotic expansion could also be computed in principle. The above proposition has been checked using both numerical analysis of $w_{n,1}$ for $n \leq 1000$, and using Bruno Salvy's *gdev* package for Maple [11].

Note that the above result implies that walks in the wedge $\mathcal{W}_p$ have the same dominant asymptotic behaviour as partially directed walks in the first quadrant. Since we the number of walks in any wedge $\mathcal{W}_p$ for $1 \leq p < \infty$ is bounded between the number of walks in $\mathcal{W}_1$ and partially directed walks inside the first quadrant, we have the following result:

Corollary 5.3. *The number of partially directed walks in the wedge $\mathcal{V}_p$, $c_n^{(p)}$ obeys the following inequality*

$$0.21869\ldots \leq \lim_{n\to\infty}\frac{c_n^{(p)} n^{1/2}}{(1+\sqrt{2})^n} \leq \left(\sqrt{2\pi(5\sqrt{2}-7)}\right)^{-1} = 1.496489\ldots \tag{5.37}$$

for any $1 \leq p < \infty$.

We have made use of the fact that partially directed walks in the first quadrant have generating function

$$\frac{1-t-3t^2-t^3-\sqrt{(1-t^4)(1-2t-t^2)}}{2t^2(1-2t-t^2)}. \tag{5.38}$$

This equation can (and has been) derived by several different techniques, however we derived a functional equation (very similar to that given in Proposition 3.2) and then solved it using the (standard) kernel method. We then obtained the asymptotic behaviour of the coefficients using singularity analysis.

6 Conclusions

Acknowledgements

EJJvR is supported by a grant from NSERC (Canada). AR is partially supported by finding from ARC (Australia) and MASCOS (Australia). TP is partially supported by the DFG. **{ADR: Is this right?}**

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