MTH5105 Differential and Integral Analysis 2008-2009

Exercises 4

Exercise 1: Let the function $f\colon (0,\pi) \to \mathbb{R}$ be given by $x \mapsto \cos(x)$. Show that f is invertible and that the inverse $g(y) = f^{-1}(y)$ is differentiable. Find the derivative g'. [4 marks]

Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g. What is the remainder term in the Lagrange form? [4 marks]

Hence show that for $|y| \leq 1/2$

$$|g(y) - \pi/2 + y| \leq \sqrt{3}/18 \approx 0.096\ .$$

[4 marks]

Solution: As $f'(x) = -\sin(x) < 0$ on $(0,\pi)$, f is strictly decreasing and therefore invertible, with differentiable inverse $g : (-1,1) \mapsto \mathbb{R}$. (Of course we recognize from Calculus that $g = \arccos$.) [2 mark]

To compute the inverse, note that $f'(x) = -\sqrt{1-\cos^2(x)}$, and thus $g'(y) = 1/f'(x) = -1/\sqrt{1-y^2}$. [2 marks]

We have $g(0) = \pi/2$ and $g'(0) = -1$, so that $T_{1,0}(y) = \pi/2 - y$. [2 marks]

From $g''(y) = -y(1-y^2)^{-3/2}$, the remainder term in the Lagrange form is given by

$$R = \frac{1}{2} g''(c) y^2 = -\frac{cy^2}{2(1-c^2)^{3/2}}\ .$$

[2 marks]

There exists a $|c| < |y|$ such that $g(y) - T_{1,0}(y) = R$. For $|y| \leq 1/2$ we can get an explicit bound on $|R|$ by estimating

$$|R| \leq \frac{|y|^3}{2(1-y^2)^{3/2}} \leq \frac{(1/2)^3}{2(1-1/4)^{3/2}} = \frac{1}{18}\sqrt{3}\ .$$

[4 marks]

Exercise 2: Let $f : (-1, \infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0\ .$$

[4 marks]

Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0\ .$$

[4 marks]

Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h,

$$(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)} h^{(k)}\ .$$

Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$. [4 marks]

Solution: We find

$$\begin{aligned}
f(x) &= \sin(\pi\sqrt{1+x}) \\
f'(x) &= \frac{\cos(\pi\sqrt{1+x})\pi}{2\sqrt{1+x}} \\
f''(x) &= -\frac{\sin(\pi\sqrt{1+x})\pi^2}{4(1+x)} - \frac{\cos(\pi\sqrt{1+x})\pi}{4(1+x)^{3/2}}
\end{aligned}$$

From this, the identity immediately follows. [4 marks]

Differentiating $g(x) = 4(1+x)f''(x)$ n times, we find

$$g^{(n)}(x) = 4(1+x)f^{(n+2)}(x) + 4nf^{(n+1)}(x)$$

(this is where the Leibniz formula might be useful, otherwise you might need to use induction). Thus, differentiating the identity gives

$$4(1+x)f^{(n+2)}(x) + 2(2n+1)f^{(n+1)}(x) + \pi^2 f^{(n)}(x) = 0$$

which for $x = 0$ simplifies to the needed formula. [4 marks]

We compute now $f(0) = 0$, $f'(0) = -\pi/2$, and recursively

$$\begin{aligned}
f''(0) &= -\frac{1}{4}\left(2f'(0) + \pi^2 f(0)\right) = \frac{\pi}{4} \\
f'''(0) &= -\frac{1}{4}\left(6f''(0) + \pi^2 f'(0)\right) = \frac{\pi}{8}(\pi^2 - 3) \\
f''''(0) &= -\frac{1}{4}\left(10f''(0) + \pi^2 f''(0)\right) = \frac{3\pi}{16}(5 - 2\pi^2)
\end{aligned}$$

from whence

$$T_{4,0}(x) = -\frac{\pi}{2}x + \frac{\pi}{8}x^2 + \frac{\pi}{48}(\pi^3 - 3)x^3 + \frac{\pi}{128}(5 - 2\pi^2)x^4$$

follows. [4 marks]

Exercise 3: The number e can be expressed via an alternating series as

$$\frac{1}{e} = \exp(-1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \ .$$

Show that remainder term R_n in

$$\frac{n!}{e} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + R_n \ ,$$

cannot be an integer. [3 marks]

Hint: look at the convergence criterion for alternating series.

Hence deduce that e is irrational. [3 marks]

Solution: As $1/k!$ decreases strictly to zero, the alternating series converges, and

$$\frac{1}{e} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} + r_n$$

where the non-zero remainder r_n is bounded by the first omitted term,

$$0 < |r_n| < \frac{1}{(n+1)!} \ .$$

Thus the remainder R_n is bounded by

$$0 < |R_n| < \frac{n!}{(n+1)!} = \frac{1}{n+1} < 1 \ ,$$

and therefore cannot be an integer. [3 marks]

Now $\sum_{k=0}^{\infty} \frac{(-1)^k n!}{k!}$ is an integer. Therefore we have shown that for all $n \in \mathbb{N}$, $n!/e$ cannot be an integer. It follows that e cannot be a rational number. (If $e = p/q$ was a rational number, then $n!q/p$ would have to be an integer for n sufficiently large.) [3 marks]