

Car Parking and Combinatorics

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Colloquium

Department of Mathematics and Statistics
University of Maine, September 11, 2008

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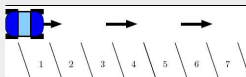
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- Monte Carlo algorithms for polymers (talk in Chemistry)
- Dynamical systems, transfer operators (work with P Kleban)

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- 2 Enumeration (generating functions)
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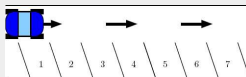
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- Consider n parking spaces in a *one-way street*



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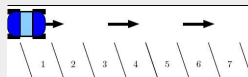
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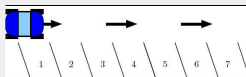
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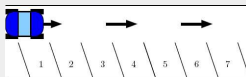
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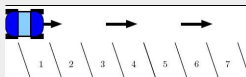
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- If k drivers fail to park, the sequence is called a *defective parking function of degree k*

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Theorem

Every permutation of a defective parking function of degree k is also a defective parking function of degree k .

The counting problem

Enumerate the number

$$cp(n, m, k)$$

of assignments of m drivers to n spaces such that exactly k drivers leave

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The probabilistic question

What is the probability

$$p_{n,m}(k) = \frac{1}{n^m} \text{cp}(n, m, k)$$

that for a randomly chosen assignment exactly k drivers leave? In particular, are there interesting **limiting distributions**?

Many equivalent or related formulations, for example

- Hashing with linear probing

[Konheim and Weiss, 1966]

[Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation

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Connections to other combinatorial objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, inversion in trees

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$cp(n, n, k)$	$k = 0$	1	2	3	4	5	6
$n = 1$	1						
2	3	1					
3	16	10	1				
4	125	107	23	1			
5	1296	1346	436	46	1		
6	16807	19917	8402	1442	87	1	
7	262144	341986	173860	41070	4320	162	1

$cp(n, n, k)$: the number of car parking assignments of n cars to n spaces such that k cars are not parked.

Theorem

For $m \leq n$,

$$\text{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

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- Space $n + 1$ is empty with probability $(n + 1 - m)/(n + 1)$.



Definition

For $r, s, k \in \mathbb{N}_0$ let $a(r, s, k)$ denote the number of choices for which r spaces remain empty, s spaces are occupied in the end and k people drive home.

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- $n = s + r$ parking spaces, $m = s + k$ drivers
- $\text{cp}(n, m, k) = a(n - m + k, m - k, k)$

Lemma (A recursion)

For $r, s, k \in \mathbb{N}_0$, the number of assignments of $s + k$ drivers to $s + r$ spaces such that r spaces remain empty, s spaces are occupied and k drivers leave is recursively defined by

$$a(r, s, k) =$$

Lemma (A recursion)

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$$a(r, s, k) = \begin{cases} 1 & \text{if } r = s = k = 0, \\ \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s-1, i) & \text{if } k > 0, \\ a(r-1, s, 0) + \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s-1, i) & \text{if } k = 0 \text{ and } (r > 0 \text{ or } s > 0). \end{cases}$$

Lemma (A functional equation)

Let A be the formal power series in the three variables u , v , and t defined by

$$A(u, v, t) := \sum_{r, s, k \geq 0} a(r, s, k) \cdot u^r \frac{v^s t^k}{(s+k)!}.$$

Then $A(u, v, t)$ is the unique solution of

$$0 = \left(\frac{v}{t} e^t - 1\right) \cdot A(u, v, t) + \left(u - \frac{v}{t}\right) \cdot A(u, v, 0) + 1$$

in the ring of formal power series in u , v and t .

Solving

$$0 = \left(\frac{v}{t}e^t - 1\right) \cdot A(u, v, t) + \left(u - \frac{v}{t}\right) \cdot A(u, v, 0) + 1$$

with the **Kernel Method**:

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- Setting the *kernel* $K(v, t) = \frac{v}{t}e^t - 1$ equal to zero gives

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$$T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$$

- Now we can solve

$$0 = \left(u - \frac{v}{T(v)}\right) \cdot A(u, v, 0) + 1$$

for $A(u, v, 0)$.

Lemma (The explicit generating function)

The generating function for the car parking problem is given by

$$A(u, v, t) = \frac{1}{1 - \frac{v}{t}e^t} + \frac{u - \frac{v}{t}}{1 - \frac{v}{t}e^t} \cdot \frac{e^{T(v)}}{1 - ue^{T(v)}},$$

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Now extract the coefficients $a(r, s, k) \dots$

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$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}.$$

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- $S(n, m, k) = \sum_{j=k}^m \text{cp}(n, m, j)$

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- $S(n, m, k) = \sum_{j=k}^m \text{cp}(n, m, j)$
- The car parking numbers $\text{cp}(n, m, k)$ are given by

$$\text{cp}(n, m, k) = S(n, m, k) - S(n, m, k+1)$$

- Remembering that $cp(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$, we can rewrite

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} cp(n-m+k+i-1, i, 0) \cdot (m-k-i)^{m-i}.$$

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- If at least k cars do not find a parking space, then there are at least $\ell = n + k - m$ empty parking spaces, and we rewrite

$$S(n, m, \ell + m - n) = \sum_{i=0}^{n-\ell} \binom{m}{i} \cdot cp(\ell + i - 1, i, 0) \cdot (n - \ell - i)^{m-i}.$$

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- This expression leads to a direct combinatorial proof.

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Proof.

- Assuming the ℓ -th empty space occurs at position $\ell + i$, there are i cars successfully parked in the $\ell + i - 1$ spaces to the left of it, which is counted by $\text{cp}(\ell + i - 1, i, 0)$.

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- The remaining $m - i$ cars are assigned to the $n - \ell - i$ rightmost spaces in $(n - \ell - i)^{m-i}$ different ways.

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- The remaining $m - i$ cars are assigned to the $n - \ell - i$ rightmost spaces in $(n - \ell - i)^{m-i}$ different ways.
- Sum over all possible values of i .



Notice that

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}$$

are partial sums occurring in

Lemma (Abel's Binomial Identity, 1826)

For all $a, b \in \mathbb{R}$, $m \in \mathbb{N}_0$,

$$\sum_{i=0}^m \binom{m}{i} \cdot a \cdot (a+i)^{i-1} \cdot (b-i)^{m-i} = (a+b)^m$$

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- In fact, $S(n, m, 0) = n^m$ gives a combinatorial proof for Abel's identity for $a = n - m$ and $b = m$.

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- Consider the distribution of the probability of a parking function with defect k ,

$$p_{n,m}(k) = \frac{1}{n^m} \text{cp}(n, m, k) ,$$

for n (parking spaces) or m (cars) large.

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- Different regimes:
 - $m \ll n$: cars park with probability 1
 - $m \sim \lambda n$, $\lambda < 1$: discrete limit law
 - $(\sqrt{m} \ll m - n \ll m$: exponential distribution)
 - $m - n \sim \lambda \sqrt{m}$: linear-exponential distribution
 - $(\sqrt{m} \ll n - m \ll m$: exponential distribution)
 - $m \sim \lambda n$, $\lambda > 1$: discrete limit law
 - $m \gg n$: cars leave with probability 1

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n, m, k) = \sum_{j=k}^m p_{n,m}(j)$$

• $m \ll n: P_{n,m}(k) \rightarrow 0$

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- $m \sim \lambda n$, $\lambda > 1$: discrete limit law

$$P_{n,m}(n - m + k) \rightarrow k \sum_{l=0}^{\infty} \frac{(l+k)^{l-1}}{l!} \lambda^l e^{-\lambda(l+k)}$$

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n, m, k) = \sum_{j=k}^m p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \rightarrow 0$
- $m \sim \lambda n$, $\lambda < 1$: discrete limit law

$$P_{n,m}(k) \rightarrow (1 - \lambda) \sum_{l=0}^k (-1)^{k-l} \frac{(l+1)^{k-l}}{(k-l)!} \lambda^{k-l} e^{\lambda(l+1)}$$

- $m \sim \lambda n$, $\lambda > 1$: discrete limit law

$$P_{n,m}(n - m + k) \rightarrow k \sum_{l=0}^{\infty} \frac{(l+k)^{l-1}}{l!} \lambda^l e^{-\lambda(l+k)}$$

- $m \gg n$: $P_{n,m}(k) \rightarrow 1$

The interesting scaling regime $m - n \sim \lambda\sqrt{m}$

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Theorem

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n \rightarrow \infty} P_{n, n + \lfloor y\sqrt{n} \rfloor}(\lfloor x\sqrt{n} \rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$

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Therefore we have approximately

$$\frac{\text{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\begin{aligned} \lim_{n \rightarrow \infty} P_{n, n + \lfloor y\sqrt{n} \rfloor}(\lfloor x\sqrt{n} \rfloor) \\ = \int_0^1 \frac{x - y}{\sqrt{2\pi\alpha^3(1-\alpha)}} \cdot \exp\left(-\frac{(x - (1-\alpha)y)^2}{2\alpha(1-\alpha)}\right) d\alpha. \end{aligned}$$

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Under the substitution $\alpha = \frac{u(x-y)}{x+u(x-y)}$ this integral simplifies to

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \sqrt{\frac{x(x-y)}{u^3}} \cdot \exp\left(-x \cdot (x-y) \cdot \frac{(1+u)^2}{2u}\right) du \\ &= \exp(-2x(x-y)). \end{aligned}$$



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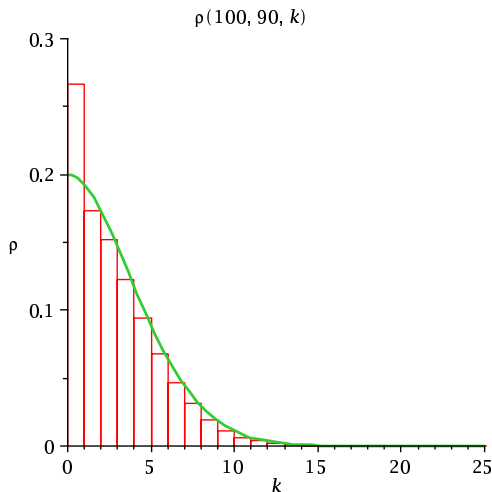
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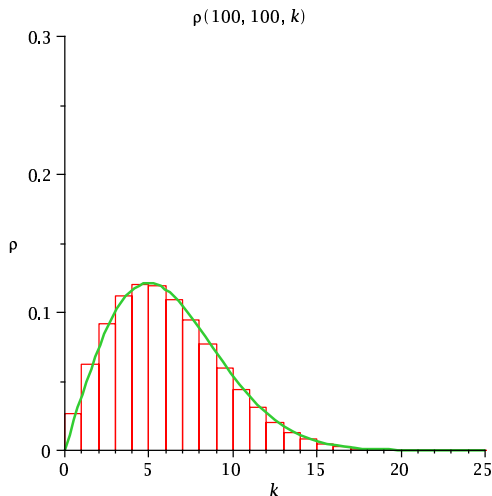
NB: neither Maple nor Mathematica can do the above integral!

Less cars than parking spaces ($n > m$):



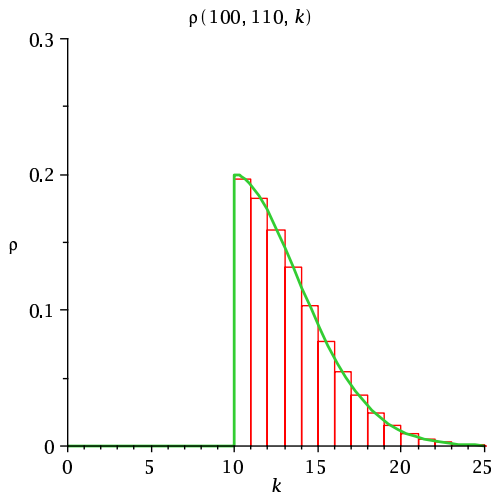
$$\frac{\text{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Equal number of cars and parking spaces ($n = m$):



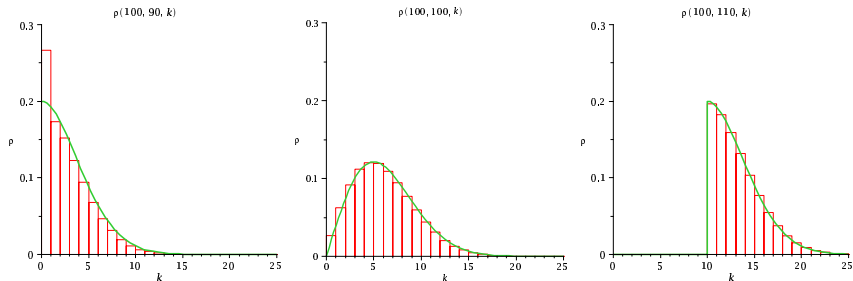
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More cars than parking spaces ($n < m$):



$$\frac{\text{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

The approximation is surprisingly accurate:



$$\frac{cp(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

What is the probability that the car park is full?

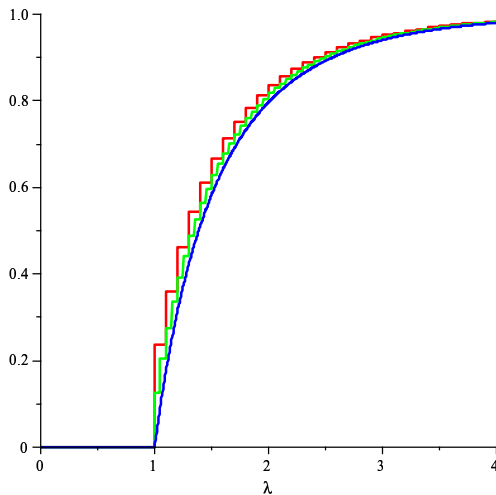
What is the probability that the car park is full?

Theorem

Let $\lambda \in \mathbb{R}^+$. Then the limiting probability that in a random assignment of $\lfloor \lambda n \rfloor$ drivers to n spaces all spaces are occupied is

$$\lim_{n \rightarrow \infty} \frac{\text{cp}(n, \lfloor \lambda n \rfloor, \lfloor \lambda n \rfloor - n)}{n^{\lfloor \lambda n \rfloor}} = \begin{cases} 0 & \text{if } \lambda \leq 1, \\ 1 - \frac{1}{\lambda} \cdot T(\lambda e^{-\lambda}) & \text{if } \lambda > 1. \end{cases}$$

$n = 10, 20, \infty$:



$$\left. \frac{\text{cp}(n, m, m - n)}{n^m} \right|_{m=\lfloor \lambda n \rfloor} \approx 1 - \frac{1}{\lambda} T(\lambda e^{-\lambda})$$

Reference for this work:

- P. J. Cameron, D. Johannsen, T. Prellberg, and P. Schweitzer, "Counting defective parking functions", Electron. J. Combinat. **15** (2008) R92 (arXiv:0803.0302v2) (submitted to EJC on March 3, 2008)

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Related work by Alois Panholzer, TU Wien:

- “Limiting distribution results for a discrete parking problem”, GOCPS 2008 (talk presented on March 5, 2008)
- “On a discrete parking problem”, AofA 2008 (talk presented on April 17, 2008)

- 1 A parking problem (defective parking functions)
- 2 Enumeration (generating functions)
- 3 Asymptotics (limit distributions)
- 4 Conclusion and outlook**

Summary:

- Introduced interesting and apparently new parking problem
- Solved the counting problem (Kernel method)
- Discussed limiting probability distributions

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Outlook:

- So far only “weak limit laws” - can refine analysis
- Extension to several generalised parking problems in the literature possible

The End