The Farey Fraction Spin Chain: Effects of an External Field

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We consider the Farey fraction spin chain in an external field h. Using ideas from dynamical systems and an operator analysis, we find that the free energy f in the vicinity of the second-order phase transition is given, rigorously, by

$$f \sim \frac{t}{\log(t)} - \frac{1}{2}\frac{h^2}{t} \quad \text{for} \quad h^2 \ll t \ll 1 \, ,$$

where $t = (1 - \frac{\beta}{\beta_c})/\log(2)\lambda_G$ **[check!]**, so that the temperature deviation from the critical point is scaled by the Lyapunov exponent of the Gauss map, λ_G. It follows that λ_G determines the amplitude of both the specific heat and susceptibility singularities. Our results confirm what was found previously with a cluster approximation, and show that a cluster mechanism is in fact responsible for the transition. However they disagree, in part, with a renormalization group treatment.

Keywords: phase transition, Farey fractions, spin chain, transfer operator

I. INTRODUCTION

The theory of second-order phase transitions has long and well-developed history. However, there are not many rigorous microscopic calculations for the free energy $f(\beta, h)$ as a function of both β *and* h in the vicinity of such a transition. Previous results of this type include [spherical model (includes logs), 1-d Ising at T=0, FF, Kac-vdW]. In this work, by using operator techniques, we calculate $f(\beta, h)$, thus contributing to this canon. Interestingly, we find that both critical amplitudes scale with a Lyapunov exponent.

Phase transitions in one-dimensional systems are unusual, essentially because, as long as the interactions are of finite range and strength, any putative ordered state at finite temperature will be disrupted by thermally induced defects, and a defect in one dimension is very effective at destroying order. On the other hand, long range or infinite interactions generally make the model ordered at all finite temperatures. Despite this, there are a number of examples of one-dimensional systems that exhibit a phase transition. The Farey fraction spin chain [1] is one such case, which has attracted interest from both physicists and mathematicians (see [4] [add more refs] for some recent work and references). This model has a (barely) second-order transition (at finite temperature). For external field $h = 0$, the magnetization goes from completely saturated, below the transition, to zero above it. Despite this unusual behavior, the model does not violate scaling theory, but rather is included as a limiting case [3].

In some recent work, [3, 5], the model has been generalized to finite external field, and analyzed via both renormalization group and with a dynamical system-inspired cluster approximation. Neither method is rigorous, and the results are not quite the same. Specifically, the dependence of $f(\beta, h)$ on h differs. Therefore it was of interest to carry out a rigorous analysis of the model. We find that the cluster picture indeed leads to the correct result for the asymptotic form for $f(\beta, h)$, and, in addition, are able to evaluate the constants. They are determined by λ_G, the Lyapunov exponent of the Gauss map, which arises naturally in this system. It is the return map of the Farey map, which specifies the transfer operator that gives the Farey fraction spin chain partition function [4].

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In this Introduction, we define the model, first in the standard way using matrices, and then via operators. In Section II we derive some operator identities that are necessary for our analysis, and address the important question of specifying the appropriate function space. Section III is the heart of our work. Here we use perturbation theory around the critical point ($\beta = 1, h = 0$) to find the asymptotic behavior of the free energy $f(\beta, h)$. The key to our method is the use of a "cluster operator", which encodes the behavior of clusters of up and down spins while possessing tractable operator properties, thus making rigorous the cluster approximation of [5].

The Farey fraction spin chain may be constructed, for inverse temperature β and magnetic field h, via the two matrices

$$A_\uparrow = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_\downarrow = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{1}$$

The spin chain partition function comes in various guises, all of which have the same free energy (at least for $h = 0$, see [1–4]). In this work it is defined (including a parameter $x \geq 0$, which does not affect the free energy) as

$$Z_N(\beta, h; x) = 1(x) \left| (e^{-\beta h} A_\uparrow + e^{\beta h} A_\downarrow)^N \right. . \tag{2}$$

Equation (2) (see also (5)) extends the "generalized Knauf model", introduced in [4], by including an external field. In (2) we use a notation which (although it may be unfamiliar to physicists) is standard in number theory; the action of a 2×2 matrix $[M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})]$ on a function f is defined to be

$$f(x) \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. = \frac{1}{(cx+d)^{2\beta}} f\left(\frac{ax+b}{cx+d}\right) . \tag{3}$$

In addition, consistent with the group structure of $SL_2(\mathbb{Z})$, any addition and scalar multiplication is performed *after* the matrix action on the function has been computed, i.e. for $M_1, M_2 \in SL_2(\mathbb{Z})$ we have $\phi(x)|(\alpha_1 M_1 + \alpha_2 M_2) = \alpha_1 \phi(x)|M_1 + \alpha_2 \phi(x)|M_2$, but, setting $M = \alpha_1 M_1 + \alpha_2 M_2$, in general $\phi(x)|(\alpha_1 M_1 + \alpha_2 M_2) \neq \phi(x)|M$.

Defining Z_N via (2) means that we are considering the generalized Knauf spin chain [4], not the "trace" model studied in [5]. We make this choice for technical reasons. However, by universality, our results are supposed to apply to any of the Farey spin chains (see [1–4] for definitions of the various chains).

Now define the matrix products that appear in the partition function as

$$M_N := \prod_{i=1}^{N} {A_\uparrow}^{1-\sigma_i} {A_\downarrow}^{\sigma_i}, \qquad \sigma_i \in \{0, 1\}, \tag{4}$$

where the dependence of M_N on the σ_i has been suppressed. Writing a given matrix product as $M_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ our partition function becomes

$$Z_N(\beta, h; x) = \sum_{\{\sigma_i\}} \frac{1}{(cx+d)^{2\beta}} e^{-\beta h \left(2 \sum_{i=1}^{N} \sigma_i - N\right)}. \tag{5}$$

When M_N begins with $A_\uparrow$, c and d are neighboring Farey denominators at level N in the modified Farey sequence (see [1] for further details on this connection).

The free energy is defined as

$$f(\beta, h) = \frac{-1}{\beta} \lim_{N \to \infty} \frac{\log(Z_N(\beta, h; x))}{N} \tag{6}$$

[Omit?? The existence of the free energy for the Farey partition function (for all $\beta \geq 0$ and all $h \in \mathbb{R}$) has been proven at $x = 0$ [check x value] for the closely related Knauf spin chain [doesn't include all matrices] in [3].] The existence of $f(\beta, h)$ and its independence of x follows from the operator considerations below [works for beta lt 1?? or extend JF-PK argument for existence, op args for indep?].

Alternatively, the partition function can be expressed using transfer operators. In fact, our analysis exploits various operator properties. In order to emphasize the difference between matrices and operators, we denote the latter with script letters exclusively. We begin by defining the operator

$$\mathcal{L}_{\beta,h} = e^{-\beta h} \mathcal{L}_\beta^\uparrow + e^{\beta h} \mathcal{L}_\beta^\downarrow \quad \text{where} \quad \mathcal{L}_\beta^\uparrow f(x) = f(x) \left| A_\uparrow \right. \quad \text{and} \quad \mathcal{L}_\beta^\downarrow f(x) = f(x) \left| A_\downarrow \right. . \tag{7}$$

Thus we obtain, as in [4]

$$Z_N(\beta,h;x) = \mathcal{L}_{\beta,h}^N 1(x) \ . \tag{8}$$

[fix the following–need to prove it] It follows that the free energy $f(\beta,h)$ is given by the logarithm of the spectral radius of $\mathcal{L}_{\beta,h}$, which demonstrates its independence of x.

In the disordered (high-temperature) phase, we expect that there is a leading eigenvalue $\lambda(\beta,h)$ of $\mathcal{L}_{\beta,h}$ which satisfies $\lambda(\beta,h) > 1$, is non-degenerate, and belongs to the discrete spectrum. Since the corresponding eigenvector is of definite sign, it has a non-zero projection onto $1(x)$. Then the free energy is given by

$$f(\beta,h;x) = \frac{-1}{\beta} \ln \lambda(\beta,h) \ , \tag{9}$$

which is independent of x.

We have not, so far, distinguished between the "slash" operator defined in (3) and the transfer operators introduced in (7), since we have not specified the function space on which the latter operate. However, further analysis, which hinges on their spectra, is delicately sensitive to this choice, which is specified below.

There is an obvious symmetry in our model. Since

$$SA_\uparrow S = A_\downarrow \quad \text{with} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{10}$$

it is natural to define the corresponding operator $\mathcal{S}_\beta$ as

$$\mathcal{S}_\beta f(x) = f(x)|S = x^{-2\beta} f(1/x) \ . \tag{11}$$

Note that $S^{-1} = S$ and $\mathcal{S}_\beta^{-1} = \mathcal{S}_\beta$, i. e. both S and $\mathcal{S}_\beta$ are involutions. For the transfer operators we find

$$\mathcal{L}_\beta^\uparrow = \mathcal{S}_\beta \mathcal{L}_\beta^\downarrow \mathcal{S}_\beta \quad \text{and} \quad \mathcal{L}_{\beta,h} = \mathcal{S}_\beta \mathcal{L}_{\beta,-h} \mathcal{S}_\beta \ . \tag{12}$$

The operator $\mathcal{L}_{\beta,0}$ has a nice interpretation as the Ruelle-Perron-Frobenius transfer operator for the dynamical system given by iteration of the map

$$T(x) = \begin{cases} x/(1-x) \ , & 0 \le x < 1 \\ x-1 \ , & x \ge 1 \ . \end{cases} \tag{13}$$

Note that the map T has the symmetry $T(1/x) = 1/T(x)$. (This map differs from the Farey map used in, say, [4], but is related to a symmetrized Farey map shown in Fig. 1 below.)

Now consider the generating function

$$G(\beta,h,z;x) = \sum_{N=0}^{\infty} z^N Z_N(\beta,h;x) \ . \tag{14}$$

Examination of G helps motivate the operator relations found below and makes a connection with the treatment in [5]. One may rewrite G as

$$\begin{aligned} G(\beta,h,z;x) &= 1(x) \left| [1 - z(e^{-\beta h} A_\uparrow + e^{\beta h} A_\downarrow)]^{-1} \right. && (15) \\ &= [1 - z\mathcal{L}_{\beta,h}]^{-1} 1(x) \ . && (16) \end{aligned}$$

The free energy is then given as

$$\beta f(\beta,h) = \log z_c(\beta,h) \ , \tag{17}$$

where $z_c(\beta,h)$ is the smallest singularity of $G(\beta,h,z;x)$ for z on the positive real axis. If $\mathcal{L}_{\beta,h}$ has a largest eigenvalue $\lambda(\beta,h)$ with eigenfunction that has a non-zero projection on $1(x)$ we see that (17) and (9) coincide. Thus, in principle, we *could* find the free energy by analyzing G. However, it is very difficult to do this directly, since $\mathcal{L}_{\beta,h}$ is not sufficiently well-behaved. In order to make progress we resort, below, to a more nuanced treatment.

II. IDENTITIES AND SPECTRAL RELATIONS

In this Section we derive two operator Lemmas that are the basis for our analysis. They allow us to avoid dealing directly with $\mathcal{L}_{\beta,h}$, which is difficult to control at the critical point $(\beta, h) = (1, 0)$.

Notice that a formal expansion of $[1 - z\mathcal{L}_{\beta,h}]^{-1} = [1 - ze^{-\beta h}\mathcal{L}_\beta^{\uparrow} - ze^{\beta h}\mathcal{L}_\beta^{\downarrow}]^{-1}$ gives a weighted sum over terms, each of which corresponds to a particular configuration of spins. Alternatively, we can group spins into clusters of identically oriented spins. This leads to

$$[1 - z\mathcal{L}_{\beta,h}]^{-1} = [1 - ze^{-\beta h}\mathcal{L}_\beta^{\uparrow}]^{-1}\sum_{n=0}^{\infty}\left(ze^{\beta h}\mathcal{L}_\beta^{\downarrow}[1 - ze^{\beta h}\mathcal{L}_\beta^{\downarrow}]^{-1}ze^{-\beta h}\mathcal{L}_\beta^{\uparrow}[1 - ze^{-\beta h}\mathcal{L}_\beta^{\uparrow}]^{-1}\right)^n[1 - ze^{\beta h}\mathcal{L}_\beta^{\downarrow}]^{-1} , \qquad (18)$$

which may be illustrated by

$$\underbrace{\uparrow\cdots\uparrow}_{\geq 0}\overbrace{\underbrace{\downarrow\cdots\downarrow}_{\geq 1}\ \uparrow\cdots\downarrow\ \underbrace{\uparrow\cdots\uparrow}_{\geq 1}}^{n \text{ pairs}, \ n\geq 0}\underbrace{\downarrow\cdots\downarrow}_{\geq 0} .$$

Notice that configurations starting and ending with either spin orientation are included. We now introduce the operators

$$\mathcal{M}_{\beta,\tau}^{\uparrow} = \tau\mathcal{L}_\beta^{\uparrow}[1 - \tau\mathcal{L}_\beta^{\uparrow}]^{-1} \quad \text{and} \quad \mathcal{M}_{\beta,\tau}^{\downarrow} = \tau\mathcal{L}_\beta^{\downarrow}[1 - \tau\mathcal{L}_\beta^{\downarrow}]^{-1} . \qquad (19)$$

Notice that as a formal power series in τ,

$$\mathcal{M}_{\tau,\beta}^{\uparrow} = \sum_{n=1}^{\infty}\tau^n\mathcal{L}_\beta^{\uparrow\,n} \quad \text{and} \quad \mathcal{M}_{\tau,\beta}^{\downarrow} = \sum_{n=1}^{\infty}\tau^n\mathcal{L}_\beta^{\downarrow\,n} . \qquad (20)$$

These operators exist whenever the resolvents $[1 - \tau\mathcal{L}_\beta^{\uparrow}]^{-1}$ and $[1 - \tau\mathcal{L}_\beta^{\downarrow}]^{-1}$ exist, i.e. for $\tau^{-1} \notin \sigma(\mathcal{L}_\beta^{\uparrow})$ or $\tau \notin \sigma(\mathcal{L}_\beta^{\downarrow})$, respectively. We are thus led to the identity:

Lemma 1 *For $z^{-1}e^{\beta h} \notin \sigma(\mathcal{L}_\beta^{\uparrow})$ and $z^{-1}e^{-\beta h} \notin \sigma(\mathcal{L}_\beta^{\downarrow})$,*

$$[1 + \mathcal{M}_{\beta,ze^{\beta h}}^{\downarrow}][1 - z\mathcal{L}_{\beta,h}][1 + \mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow}] = [1 - \mathcal{M}_{\beta,ze^{\beta h}}^{\downarrow}\mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow}] . \qquad (21)$$

Proof: The proof is a straightforward calculation using the definition (19). $\square$

As above, we have the symmetry

$$\mathcal{M}_{\beta,\tau}^{\uparrow} = \mathcal{S}_\beta\mathcal{M}_{\beta,\tau}^{\downarrow}\mathcal{S}_\beta . \qquad (22)$$

It is helpful to take advantage of this symmetry by defining $\mathcal{M}_{\beta,\tau} = \mathcal{M}_{\beta,\tau}^{\downarrow}\mathcal{S}_\beta$ so that

$$\mathcal{M}_{\beta,ze^{\beta h}}^{\downarrow}\mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow} = \mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}} . \qquad (23)$$

Note that the rhs is a square when $h = 0$. Utilising (23) and Lemma 1, we arrive at the following characterisation of eigenvalues and eigenfunctions of $\mathcal{L}_{\beta,h}$:

Lemma 2 *Let $z^{-1}e^{\beta h} \notin \sigma(\mathcal{L}_\beta^{\uparrow})$ and $z^{-1}e^{-\beta h} \notin \sigma(\mathcal{L}_\beta^{\downarrow})$. If f is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1, then $[1 + \mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow}]f$ is an eigenfunction of $\mathcal{L}_{\beta,h}$ with eigenvalue $\lambda = 1/z$. Conversely, if g is an eigenfunction of $\mathcal{L}_{\beta,h}$ with eigenvalue $\lambda = 1/z$, then $[1 - ze^{\beta h}\mathcal{L}_\beta^{\downarrow}]g$ is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1.*

Proof: If f is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1, then by (23) the rhs of (21) acting on f is $[1 - \mathcal{M}_{\beta,ze^{\beta h}}^{\downarrow}\mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow}]f = 0$. Due to the assumption on z, the kernels of both $[1 + \mathcal{M}_{\beta,ze^{\beta h}}^{\downarrow}] = [1 - ze^{\beta h}\mathcal{L}_\beta^{\downarrow}]^{-1}$ and $[1 + \mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow}] = [1 - ze^{-\beta h}\mathcal{L}_\beta^{\uparrow}]^{-1}$ are zero, so it follows from (21) that $[1 - z\mathcal{L}_{\beta,h}]g = 0$ with $g = [1 + \mathcal{M}_{\beta,ze^{-\beta h}}^{\uparrow}]f$. The second assertion follows similarly. $\square$

Lemma 2 motivates the definition of the set

$$\Omega_{\beta,h} = \left\{\lambda \in \mathbb{C} : \lambda = 1/z \text{ with } 1 \in \sigma(\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}})\right\} , \qquad (24)$$

which we will use below.

Using the operator relations of Lemma 1 and Lemma 2, we can characterize the spectral properties of $\mathcal{L}_{\beta,h}$ precisely. For this, we need to fix the function space the operator acts on. In fact, it will be advantageous to consider a function space which is invariant under the action of $\mathcal{S}_\beta$, so that we can make use of the symmetry between $\mathcal{L}_\beta^\uparrow$ and $\mathcal{L}_\beta^\downarrow$ given in (12).

Consider the generalized Laplace transform

$$\mathrm{L}_\beta\psi(x) = \int_0^\infty \exp(-sx)\psi(s)s^{2\beta-1}ds \; . \tag{25}$$

Then, $\mathcal{S}_\beta$ is related by the conjugacy

$$\mathcal{S}_\beta \mathrm{L}_\beta = \mathrm{L}_\beta \mathcal{T}_\beta \tag{26}$$

to the integral operator

$$\mathcal{T}_\beta\psi(s) = \int_0^\infty K(s,t)\psi(t)t^{2\beta-1}dt \; . \tag{27}$$

The integral kernel $K(s,t)$ is given as

$$K(s,t) = (st)^{\frac{1}{2}-\beta} J_{2\beta-1}(2\sqrt{st}) \; . \tag{28}$$

We have chosen the weight $s^{2\beta-1}$ to make $K(s,t)$ symmetric in s and t, so that $\mathcal{T}_\beta$ is a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}_+, s^{2\beta-1}ds)$. The fact that $\mathcal{T}_\beta$ is an involution is less obvious than for $\mathcal{S}_\beta$, but can be shown directly by using well-known orthogonality properties of Bessel functions.

We now define the Hilbert space $\mathcal{H}_\beta = \mathrm{L}_\beta L^2(\mathbb{R}_+, s^{2\beta-1}ds)$, i.e.

$$\mathcal{H}_\beta = \left\{ f(x) = \int_0^\infty e^{-sx}\psi(s)s^{2\beta-1}ds : \psi \in L^2(\mathbb{R}_+, s^{2\beta-1}ds) \right\} \tag{29}$$

with appropriately induced inner product.

Lemma 3 *Let β be real. The spectrum of the operators $\mathcal{L}_\beta^\uparrow$ and $\mathcal{L}_\beta^\uparrow$ acting on $\mathcal{H}_\beta$ is continuous and $\sigma(\mathcal{L}_\beta^\uparrow) = \sigma(\mathcal{L}_\beta^\downarrow) = [0,1]$.*

Proof: The argument is analogous to the analysis of [12, 13, 17]; first, by (12), $\mathcal{L}_\beta^\uparrow$ and $\mathcal{L}_\beta^\downarrow$ are related by conjugacy via $\mathcal{S}_\beta$, and therefore have identical spectral properties on $\mathcal{H}_\beta$. Furthermore, $\mathcal{L}_\beta^\downarrow$ is a shift operator which conjugates under L_β to multiplication by e^{-s} on $L^2(\mathbb{R}_+, s^{2\beta-1}ds)$. The spectrum of this multiplication operator is continuous and equal to the closure of the range of the multiplying function, i.e. $[0,1]$. $\square$

Theorem 4 *Let β and h be real. The operator $\mathcal{L}_{\beta,h}$ acting on $\mathcal{H}_\beta$ has continuous spectrum $\sigma_c(\mathcal{L}_{\beta,h}) \subset [0, e^{|\beta h|}]$. Outside $[0, e^{|\beta h|}]$, the spectrum coincides with $\Omega_{\beta,h}$ and consist of isolated eigenvalues with finite multiplicity.*

Proof: By Lemma 3, $e^{-\beta h}\mathcal{L}_\beta^\uparrow$ has spectrum $[0, e^{-\beta h}]$ and $e^{\beta h}\mathcal{L}_\beta^\uparrow$ has spectrum $[0, e^{\beta h}]$. It follows that $\mathcal{M}^\uparrow_{\beta, ze^{-\beta h}}$ is defined for complex $z^{-1} \notin [0, e^{-\beta h}]$ and $\mathcal{M}^\downarrow_{\beta, ze^{\beta h}}$ is defined for complex $z^{-1} \notin [0, e^{+\beta h}]$.

Lemma 1 implies that the spectrum of $\mathcal{L}_{\beta,h}$ satisfies

$$\sigma(\mathcal{L}_{\beta,h}) \in \sigma(e^{-\beta h}\mathcal{L}_\beta^\uparrow) \cup \sigma(e^{\beta h}\mathcal{L}_\beta^\downarrow) \cup \Omega_{\beta,h} = [0, e^{|\beta h|}] \cup \Omega_{\beta,h} \; .$$

Moreover, Lemma 2 shows that $\Omega_{\beta,h}$ contains eigenvalues of finite multiplicity, so that the continuous spectrum must lie in $[0, e^{|\beta h|}]$, and that all of $\Omega_{\beta,h}$ is point spectrum. $\square$

We conclude this section by providing explicit representations of the operators defined above. For $|\tau| < 1$, we can write explicitly

$$\mathcal{M}^\uparrow_{\beta,\tau} f(x) = \sum_{n=1}^\infty \frac{\tau^n}{(1+nx)^{2\beta}} f\left(\frac{x}{1+nx}\right) \quad , \quad \mathcal{M}^\downarrow_{\beta,\tau} f(x) = \sum_{n=1}^\infty \tau^n f(x+n) \; , \tag{30}$$

and

$$\mathcal{M}_{\beta,\tau} f(x) = \sum_{n=1}^{\infty} \frac{\tau^n}{(n+x)^{2\beta}} f\left(\frac{1}{n+x}\right) . \tag{31}$$

Note that $\mathcal{M}_{\beta,1}$ **[check!]** is the transfer operator of the Gauss map $x \mapsto 1/x \mod 1$, which has branches

$$T_n(x) = \frac{1}{x} - n . \tag{32}$$

$\mathcal{M}_{\beta,\tau}$ may be regarded as an extended transfer operator of the Gauss map, with the branches having individual weights.

One can obtain a symmetric version of the operators under conjugacy with the map $f_0(x) = x/(1-x)$, which corresponds to an operator conjugacy with $\mathcal{L}_\beta^\uparrow$ itself. This conjugacy maps operators defined on functions of the positive real line to operators defined on functions of the unit interval. $\mathcal{L}_\beta^\uparrow$ is formally invariant under this conjugacy, and we find that $\mathcal{L}_\beta^\downarrow$ is related by this conjugacy to the transformation $f_1(x) = 1 - f_0(x)$, which follows immediately from the "spin-flip"-symmetry given by (12). Figure 1 shows this map, together with the first return map on the interval $[1/3, 2/3]$.

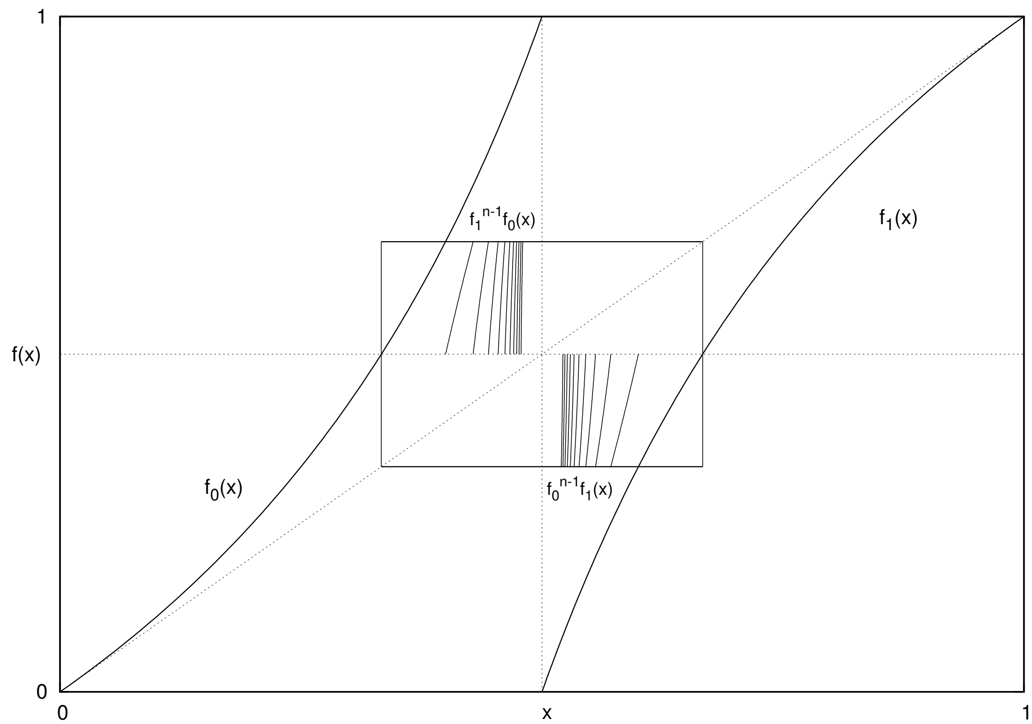

FIG. 1: Symmetric Farey map and first-return map on the interval $[1/3, 2/3]$. The first-return map is given explicitly by the branches ${f_0}^{n-1} f_1$ and ${f_1}^{n-1} f_0$ for $n \in \mathbb{N}$.

As discussed above, $\mathcal{L}_{\beta,h}$ has a continuous spectrum with $\sigma^c(\mathcal{L}_{\beta,h}) \subset \left[0, e^{|\beta h|}\right]$. In order to find eigenvalues (and eigenfunctions) for $\mathcal{L}_{\beta,h}$ outside $\left[0, e^{|\beta h|}\right]$, it suffices to find eigenvalues of $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$. We exploit this method in the next section.

III. PERTURBATION THEORY

In this section we come to the central point of our analysis. We exploit Lemma 2 to make a perturbation expansion around the critical point $\beta = 1, h = 0$. More explicitly, we use the fact that for β close to 1, we can pick z and h so that $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$ has eigenvalue 1. Then, the properties of the various quantities near that point give the leading corrections to the eigenvalue. This results in an equation for $z(\beta, h)$ which, via Lemma 2 and (9), leads to the asymptotic form of the free energy $f(\beta, h)$.

At the point $h = 0$ and $\beta = 1$ the spectral radius of $\mathcal{L} := \mathcal{L}_{1,0}$ is one, and we find

$$\mathcal{L}f(x) = f(x) \quad \text{with} \quad f(x) = \frac{1}{x} \; . \tag{33}$$

Moreover, as can be most easily seen from the dynamical systems perspective, the dual operator (which has the same spectrum as $\mathcal{L}$) leaves the Lebesgue measure on $\mathbb{R}^+$ invariant, i.e. for all functions f that are integrable on $\mathbb{R}^+$ we have

$$\mu_L(\mathcal{L}f) = \mu_L(f) = \int_0^\infty f(x)dx \tag{34}$$

However, the function $f(x) = 1/x$ of (33) is not normalizable, and therefore not a proper eigenfunction. The "eigenvalue" $\lambda = 1$ is simply the upper limit of the continuous spectrum $[0, 1]$. A perturbation argument involving expansions of $\mu(\mathcal{L}_{\beta,h}f)$, and $\mu_L(\mathcal{L}f)$ gives infinite results.

We can rescue the situation by considering the dynamical systems interpretation of the product operator that appears in Lemma 2. Let

$$\mathcal{P}_{\beta,ze^{\beta h},ze^{-\beta h}} = \mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}} \; . \tag{35}$$

We refer to $\mathcal{P}_{\beta,\vec{\tau}} := \mathcal{P}_{\beta,ze^{\beta h},ze^{-\beta h}} = \mathcal{M}^{\downarrow}_{\beta,ze^{\beta h}}\mathcal{M}^{\uparrow}_{\beta,ze^{-\beta h}}$ as the "cluster operator" since it is a sum of terms, each of which involves two spin clusters only. Further, it is sufficiently well-behaved to be useful. In fact, it is the central quantity in our analysis.

In a slight abuse of notation, we replace the variables $(ze^{\beta h}, ze^{-\beta h})$ with the placeholder $\vec{\tau}$. Occasionally we will refer to these variables as $\vec{\tau} = (\tau_\downarrow, \tau_\uparrow) = (ze^{\beta h}, ze^{-\beta h})$, and abbreviate $(z = 1, h = 0)$ as $\vec{\tau} = (1, 1)$ (i.e. $(\tau_\downarrow, \tau_\uparrow) = (1, 1)$). This avoids a proliferation of indices, and emphasizes the separate treatment of the dependence on β and $(ze^{\beta h}, ze^{-\beta h})$ in what follows. Further, we omit any of the variables $(\beta, \tau_\downarrow, \tau_\uparrow)$ when it takes on its value at the critical point (1). Thus, for example, $\mathcal{P} = \mathcal{P}_{(\beta=1),(\vec{\tau}=1)} = \mathcal{P}_{(\beta=1),(\tau_\downarrow=1),(\tau_\uparrow=1)}$ and $\mathcal{P}_{\vec{\tau}} = \mathcal{P}_{(\beta=1),\vec{\tau}} = \mathcal{P}_{(\beta=1),ze^{\beta h},ze^{-\beta h}}$. The analogous convention is used for $\mathcal{M}$ and other quantities (e. g. λ) as well.

Our strategy is to do perturbation theory by expanding around the point $(\beta = 1, h = 0)$. To implement this, we first collect a few facts about the situation at the critical point. Then we derive the analogue of the familiar first-order perturbation formula from quantum mechanics ($\delta E =< \psi|\delta H|\psi >$) for the eigenvalue $\lambda_{\beta,\vec{\tau}}$ of the cluster operator. This is done at fixed $\vec{\tau}$, by making use of analyticity in β. Interestingly, the Lyapunov exponent of the map defined by the cluster operator enters here, as a scaling factor. The dependence on $\vec{\tau}$ near $(1, 1)$ is then found in a second step. Combining the two results gives our final formula.

At $z = 1$, $h = 0$, and $\beta = 1$, the cluster operator $\mathcal{P}$ is the Perron-Frobenius operator of an expanding map

$$T_{m,n}(x) = T_n(T_m(x)) = \frac{x}{1 - mx} - n \; , \tag{36}$$

on the interval $[1, \infty)$, where we have used (32). It follows, after taking a conjugacy to a compact interval, and considering the proper function space [specify], that $\mathcal{P}$ has point spectrum with a simple leading eigenvalue $\lambda = 1$. The corresponding eigenfunction is explicitly known,

$$\mathcal{P}g(x) = g(x) \quad \text{with} \quad g(x) = \frac{1}{\log(2)x(x+1)} \; . \tag{37}$$

The dual operator leaves the Lebesgue measure on $[1, \infty)$ invariant, i.e. for all functions f that are integrable on $[1, \infty)$ we have

$$\mu_L(\mathcal{P}f) = \mu_L(f) = \int_1^\infty f(x)dx \; . \tag{38}$$

Hence $\mu = \mu_{1,1} = \mu_L$.

Note that the factor $\log(2)$ in (37) serves to normalize g

$$\mu_L(\mathcal{P}g) = \mu_L(g) = \int_1^\infty g(x)dx = 1 \; . \tag{39}$$

From Lemma 2, the leading eigenvalue $\lambda_{\beta,h}$ of $\mathcal{L}_{\beta,h}$ can be obtained from the leading eigenvalue $\lambda_{\beta,\vec{\tau}}$ of $\mathcal{P}_{\beta,\vec{\tau}}$ by solving

$$1 = \lambda_{\beta,\vec{\tau}} \tag{40}$$

for $z(\beta,h)$. This follows since for $z>0$ and $\beta\in\mathbb{R}$, $\mathcal{P}_{\beta,\vec{\tau}}$ is a positive operator, which means that its leading eigenvalue is simple and the leading eigenfunction is positive. Thus solving (40) for $z=z(\beta,h)$, by Lemma 2 one has that $1/z(\beta,h)$ is the corresponding eigenvalue of $\mathcal{L}_{\beta,h}$. The free energy then follows immediately from (9).

We now proceed to calculate the perturbation theory. First we determine $\lambda_{\beta,\vec{\tau}}$ for $\vec{\tau}=(1,1)$ and β near 1.

Denoting the left and right eigenfunctions of $\mathcal{P}_{\beta,\vec{\tau}}$ by $\mu_{\beta,\vec{\tau}}$ and $g_{\beta,\vec{\tau}}$, respectively, with normalization $\mu_{\beta,\vec{\tau}}(g_{\beta,\vec{\tau}})=1$, gives

$$\lambda_{\beta,\vec{\tau}} = \mu_{\beta,\vec{\tau}}\left(\mathcal{P}_{\beta,\vec{\tau}}\, g_{\beta,\vec{\tau}}\right) \ . \tag{41}$$

For β close to 1, $\mathcal{P}_{\beta,\vec{\tau}}$ is jointly continuous in β and $\vec{\tau}$, (or more explicitly $\tau_\downarrow,\tau_\uparrow\in[0,1]$ with $\tau_\downarrow=ze^{\beta h}$ and $\tau_\uparrow=ze^{-\beta h}$), and analytic in β [explain why]. By standard perturbation theory, its leading eigenvalue $\lambda_{\beta,\vec{\tau}}$ and the corresponding left and right eigenfunctions $\mu_{\beta,\vec{\tau}}$ and $g_{\beta,\vec{\tau}}$ are jointly continuous in β and $\tau_\downarrow,\tau_\uparrow\le 1$, and analytic in β. We write

$$\mathcal{P}_{\beta,\vec{\tau}} = \mathcal{P}_{\vec{\tau}} + \sum_{n=1}^{\infty}(1-\beta)^n\mathcal{P}_{\vec{\tau}}^{(n)} \qquad\qquad \lambda_{\beta,\vec{\tau}} = \lambda_{\vec{\tau}} + \sum_{n=1}^{\infty}(1-\beta)^n\lambda_{\vec{\tau}}^{(n)} \ . \tag{42}$$

To lowest order, we find

$$\lambda_{\beta,\vec{\tau}} = \lambda_{\vec{\tau}} + (1-\beta)\left[\mu_L(\mathcal{P}^{(1)}g) + o(1-\tau_\downarrow) + o(1-\tau_\uparrow)\right] + O((1-\beta)^2) \ . \tag{43}$$

Now, as mentioned in section II, $\mathcal{P}_\beta=\mathcal{M}_\beta^2$, where $\mathcal{M}_\beta=\mathcal{M}_{\beta,1}$ is the transfer operator for the Gauss map. Therefore, $\mu_L(\mathcal{P}^{(1)}g)=2\mu_L(\mathcal{M}^{(1)}\mathcal{M}\,g)=2\mu_L(\mathcal{M}^{(1)}\,g)$, where $\mathcal{M}^{(1)}=-\partial/\partial\beta|_{\beta=1}\mathcal{M}_\beta$. But, by a standard result, this is just twice the Lyapunov exponent λ_G of the Gauss map $\lambda_G=\zeta(2)/\log(2)$. It follows that $\mu_L(\mathcal{P}^{(1)}g)=\pi^2/3\log(2)$. Thus to leading order, as expected from the analogy to first-order perturbation theory,

$$\lambda_{\beta,\vec{\tau}} \sim \lambda_{\vec{\tau}} + (1-\beta)2\lambda_G \ , \tag{44}$$

where the error is as in (43).

Next, we consider the $\vec{\tau}$ dependence of $\lambda_{\beta,\vec{\tau}}$ at $\beta=1$.

First,

$$\lambda_{\vec{\tau}} = \mu_L(\mathcal{P}_{\vec{\tau}}\, g_{\vec{\tau}}), \tag{45}$$

so that

$$\mu_L(\mathcal{P}_{\vec{\tau}}\, g_{\vec{\tau}}) - \mu_L(\mathcal{P}\, g_{\vec{\tau}}) = (\lambda_{\vec{\tau}}-1)\mu_L(g_{\vec{\tau}}), \tag{46}$$

hence

$$\lambda_{\vec{\tau}} = 1 + \frac{\mu_L(\mathcal{P}_{\vec{\tau}}-\mathcal{P})g_{\vec{\tau}}}{\mu_L(g_{\vec{\tau}})} = 1 + \frac{\mu_L(\mathcal{P}_{\vec{\tau}}-\mathcal{P})g + \mu_L(\mathcal{P}_{\vec{\tau}}-\mathcal{P})(g_{\vec{\tau}}-g)}{1+\mu_L(g_{\vec{\tau}}-g)} \ , \tag{47}$$

where we have used the normalization condition $\mu_L(g)=1$. Now

$$\mathcal{P}_{\vec{\tau}}-\mathcal{P} = \mathcal{M}_{\tau_\downarrow}\mathcal{M}_{\tau_\uparrow} - \mathcal{M}^2 = (\mathcal{M}_{\tau_\downarrow}-\mathcal{M})\mathcal{M} + \mathcal{M}(\mathcal{M}_{\tau_\uparrow}-\mathcal{M}) + (\mathcal{M}_{\tau_\downarrow}-\mathcal{M})(\mathcal{M}_{\tau_\uparrow}-\mathcal{M}) \ . \tag{48}$$

Recall that $\mathcal{M}$ is the Gauss map transfer operator (at $\beta=1$), and that $\mathcal{M}_\tau$ may be regarded as an extended Gauss map, with the branches weighted differently from each other. We now claim that

$$||\mathcal{M}_\tau-\mathcal{M}|| \le C\eta(\tau) \ , \tag{49}$$

where C is a constant and $\eta(\tau)=\sum_{n=1}^\infty\frac{1-\tau^n}{n^2}=\mathrm{Li}_2(1)-\mathrm{Li}_2(\tau)$, with Li_2 the dilogarithm.

Proof: $-(\mathcal{M}_\tau-\mathcal{M})\psi(x)=\sum_{n=1}^\infty\frac{1-\tau^n}{(n+x)^2}\psi(\frac{1}{n+x})$ so that $||(\mathcal{M}_\tau-\mathcal{M})\psi(x)||\le\sum_{n=1}^\infty(1-\tau^n)||\frac{1}{(n+x)^2}\psi(\frac{1}{n+x})||\le\sum_{n=1}^\infty(1-\tau^n)||\frac{1}{(n+x)^2}||\;||\psi(\frac{1}{n+x})||$, where the norm is L_1. Hence (since $0\le x$), $||(\mathcal{M}_\tau-\mathcal{M})||\le C\eta(\tau)$, where C is a constant. $\square$

Furthermore, it follows immediately that $||(\mathcal{M}_{\tau_\downarrow} - \mathcal{M})(\mathcal{M}_{\tau_\uparrow} - \mathcal{M})|| \leq C^2\eta(\tau_\downarrow)\eta(\tau_\uparrow)$. Hence, $\mathcal{P}_{\vec{\tau}} - \mathcal{P} = (\mathcal{M}_{\tau_\downarrow} - \mathcal{M})\mathcal{M} + \mathcal{M}(\mathcal{M}_{\tau_\uparrow} - \mathcal{M}) + O(\eta(\tau_\downarrow)\eta(\tau_\uparrow)) = O(\eta(\tau_\downarrow)) + O(\eta(\tau_\uparrow))$, in operator norm. Hence, by (47),

$$\lambda_{\vec{\tau}} = 1 + \mu_L(\mathcal{P}_{\vec{\tau}} - \mathcal{P})g + O(\eta(\tau_\downarrow)) + O(\eta(\tau_\uparrow)) \sim 1 + \mu_L(\mathcal{M}_{\tau_\downarrow} - \mathcal{M})g + \mu_L(\mathcal{M}_{\tau_\uparrow} - \mathcal{M})g \; . \tag{50}$$

An explicit calculation then gives the exact expression

$$\mu_L(\mathcal{M}_\tau - \mathcal{M})g = -\frac{(1-\tau)^2}{\tau^2\log(2)}\sum_{n=1}^{\infty}\tau^n\log(n). \tag{51}$$

The asymptotic form follows on writing

$$\log(n) = \sum_{k=1}^{n}\frac{1}{k} - \gamma - \frac{1}{2n} + O(n^{-2}) \; , \tag{52}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. Inserting this in (51) gives immediately

$$\sum_{n=1}^{\infty}\tau^n\log(n) = \frac{1}{1-\tau}\log\frac{1}{1-\tau} - \gamma\frac{1}{1-\tau} - \frac{1}{2}\log\frac{1}{1-\tau} + O(1) \; , \tag{53}$$

uniformly for $|\tau| < 1$.

Finally, we are in a position to obtain the asymptotic value of the free energy. Equation (40) is the condition that gives the leading eigenvalue of $\mathcal{L}_{\beta,h}$, and hence (via (9)) the free energy. Combining it with (44) and (50), which determine the asymptotic value of $\lambda_{\beta,\vec{\tau}}$, gives to leading order

$$\frac{1}{2}[-(1-\tau_\downarrow)\log(1-\tau_\downarrow) - (1-\tau_\uparrow)\log(1-\tau_\uparrow)] \sim (1-\beta)\log(2)\lambda_G \; . \tag{54}$$

Substituting $\tau_\downarrow = e^{\beta(f+h)}$ and $\tau_\uparrow = e^{\beta(f-h)}$ and expanding for small f and h gives to leading order

$$2\log(2)\lambda_G t \sim (f+h)\log(-(f+h)) + (f-h)\log(-(f-h)) \; . \tag{55}$$

Setting $\beta_c = 1$ and $C = \log(2)\lambda_G$, we see that this is, aside from constants, the same as equation (40) in [5], which was found using a cluster approximation. The analysis therein then immediately gives (cf. equation (46) in [5])

$$f \sim \frac{t}{\log(t)} - \frac{1}{2}\frac{h^2}{t} \quad \text{for} \quad h^2 \ll t \ll 1 \; , \tag{56}$$

where $\log(2)\lambda_G t = (1 - \frac{\beta}{\beta_c}) = (1-\beta)$, so that the temperature deviation from the critical point is scaled by the Lyapunov exponent of the Gauss map, $\lambda_G = \frac{\zeta(2)}{\log(2)} = \frac{\pi^2}{6\log(2)}$. Note that, in addition, (56) implies that λ_G determines the amplitude of both the specific heat and susceptibility singularities.

IV. DISCUSSION

[Analogue of the "cluster generating function" $\Lambda(\beta, z)$ in [5]. Relation of present treatment to cluster approx. Results if we linearize the Gauss map] [cf. (56) with general scaling form]

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As mentioned, the renormalization group result for $f(\beta, h)$ found in [3] does not quite agree with (56). Specifically, the second term has the form $\frac{h^2\log(t)}{t}$, which as $t \to 0$, is larger than the corresponding term in (56). As discussed in [3], there does not seem to be any consistent way to remove this term in the renormalization group framework. However, this is perhaps not so surprising, since the Farey model is known to have long-range forces [refs!], which renders the renormalization group treatment a bit suspect.

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