Counting Defective Parking Functions

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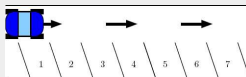
Combinatorics Seminar

Dartmouth College
September 29, 2008

1. A parking problem (defective parking functions)

2. Enumeration (generating functions)

3. Asymptotics (limit distributions)
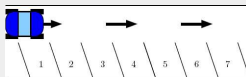
4. Conclusion and outlook

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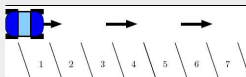
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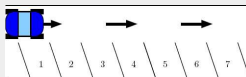- n^m sequences of choices

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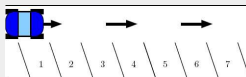 *parking function*
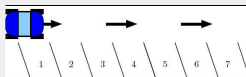
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- If all drivers park successfully, the sequence is called a

  parking function

- If *k* drivers fail to park, the sequence is called a

  defective parking function of degree k

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Theorem

Every permutation of a defective parking function of degree k is also a defective parking function of degree k.

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The probabilistic question

What is the probability

$$p_{n,m}(k) = \frac{1}{n^m} \, \mathrm{cp}(n, m, k)$$

that for a randomly chosen assignment exactly k drivers leave? In particular, are there interesting limiting distributions?

Many equivalent or related formulations, for example

- Hashing with linear probing

  [Konheim and Weiss, 1966]

  [Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation

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Connections to other combinatorial objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, inversion in trees

$\mathrm{cp}(n, n, k)$	$k = 0$	1	2	3	4	5	6
$n = 1$	1						
2	3	1					
3	16	10	1				
4	125	107	23	1			
5	1296	1346	436	46	1		
6	16807	19917	8402	1442	87	1	
7	262144	341986	173860	41070	4320	162	1

$\mathrm{cp}(n, n, k)$: the number of car parking assignments of n cars to n spaces such that k cars are not parked.

Theorem

For $m \leq n$,

$$\mathrm{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

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- Space $n + 1$ is empty with probability $(n + 1 - m)/(n + 1)$.

□

Definition

For $r, s, k \in \mathbb{N}_0$ let $a(r, s, k)$ denote the number of choices for which r spaces remain empty, s spaces are occupied in the end and k people drive home.

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- $n = s + r$ parking spaces, $m = s + k$ drivers
- $\mathrm{cp}(n, m, k) = a(n - m + k, m - k, k)$

Lemma (A recursion)

For $r, s, k \in \mathbb{N}_0$, the number of assignments of $s + k$ drivers to $s + r$ spaces such that r spaces remain empty, s spaces are occupied and k drivers leave is recursively defined by

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$$
a(r, s, k) = \begin{cases} 1 & \text{if } r = s = k = 0, \\[2ex] \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s-1, i) & \text{if } k > 0, \\[2ex] a(r-1, s, 0)+ & \text{if } k = 0 \text{ and} \\ \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s-1, i) & (r > 0 \text{ or } s > 0). \end{cases}
$$

Lemma (A functional equation)

Let A be the formal power series in the three variables u, v, and t defined by

$$A(u, v, t) := \sum_{r,s,k \geq 0} a(r, s, k) \cdot u^r \frac{v^s t^k}{(s+k)!} .$$

Then $A(u, v, t)$ is the unique solution of

$$0 = (\tfrac{v}{t} e^t - 1) \cdot A(u, v, t) + (u - \tfrac{v}{t}) \cdot A(u, v, 0) + 1$$

in the ring of formal power series in u, v and t.

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$$T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$$

- Now we can solve

$$0 = (u - \tfrac{v}{T(v)}) \cdot A(u, v, 0) + 1$$

for $A(u, v, 0)$.

Lemma (The explicit generating function)

The generating function for the car parking problem is given by

$$A(u, v, t) = \frac{1}{1 - \frac{v}{t}e^t} + \frac{u - \frac{v}{t}}{1 - \frac{v}{t}e^t} \cdot \frac{e^{T(v)}}{1 - ue^{T(v)}} \, ,$$

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Now extract the coefficients $a(r, s, k) \ldots$

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The number of car parking assignments of m cars on n spaces such that at least k cars do not find a parking space is given by

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i} .$$

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- $S(n, m, k) = \sum_{j=k}^{m} \mathrm{cp}(n, m, j)$
- The car parking numbers $\mathrm{cp}(n, m, k)$ are given by

$$\boxed{\mathrm{cp}(n, m, k) = S(n, m, k) - S(n, m, k+1)}$$

- Remembering that $cp(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$, we can rewrite

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- This expression leads to a direct combinatorial proof.

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Proof.

- Assuming the ℓ-th empty space occurs at position $\ell + i$, there are i cars successfully parked in the $\ell + i - 1$ spaces to the left of it, which is counted by $\mathsf{cp}(\ell + i - 1, i, 0)$.

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- The remaining $m - i$ cars are assigned to the $n - \ell - i$ rightmost spaces in $(n - \ell - i)^{m-i}$ different ways.

- Sum over all possible values of i.

Notice that

$$S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}$$

are partial sums occurring in

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> For all $a, b \in \mathbb{R}, m \in \mathbb{N}_0$,
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- In fact, $S(n, m, 0) = n^m$ gives a combinatorial proof for Abel's identity for $a = n - m$ and $b = m$.

- Consider the distribution of the probability of a parking function with defect k,

$$p_{n,m}(k) = \frac{1}{n^m} \, cp(n, m, k) \; ,$$

for n (parking spaces) or m (cars) large.

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- Different regimes:

  - $m \ll n$: cars park with probability 1
  - $m \sim \lambda n$, $\lambda < 1$: discrete limit law
  - ($\sqrt{m} \ll m - n \ll m$: exponential distribution)
  - $m - n \sim \lambda\sqrt{m}$: linear-exponential distribution
  - ($\sqrt{m} \ll n - m \ll m$: exponential distribution)
  - $m \sim \lambda n$, $\lambda > 1$: discrete limit law
  - $m \gg n$: cars leave with probability 1

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n,m,k) = \sum_{j=k}^{m} p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \to 0$

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$$P_{n,m}(k) \to (1 - \lambda) \sum_{l=0}^{k} (-1)^{k-l} \frac{(l+1)^{k-l}}{(k-l)!} \lambda^{k-l} e^{\lambda(l+1)}$$

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- $m \gg n$: $P_{n,m}(k) \to 1$

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Theorem

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n \to \infty} P_{n, n + \lfloor y\sqrt{n} \rfloor}(\lfloor x\sqrt{n} \rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$

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Therefore we have approximately

$$\frac{\text{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\lim_{n \to \infty} P_{n, n + \lfloor y\sqrt{n} \rfloor}(\lfloor x\sqrt{n} \rfloor)$$
$$= \int_0^1 \frac{x - y}{\sqrt{2\pi\alpha^3(1-\alpha)}} \cdot \exp\left(-\frac{(x - (1-\alpha)y)^2}{2\alpha(1-\alpha)}\right) d\alpha.$$

Proof.

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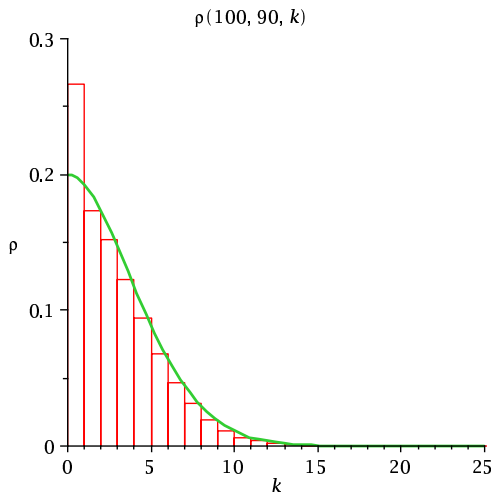
Under the substitution $\alpha = \frac{u(x-y)}{x + u(x-y)}$ this integral simplifies to

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \sqrt{\frac{x(x - y)}{u^3}} \cdot \exp\left(-x \cdot (x - y) \cdot \frac{(1 + u)^2}{2u} \right) du$$

$$= \exp(-2x(x - y)).$$

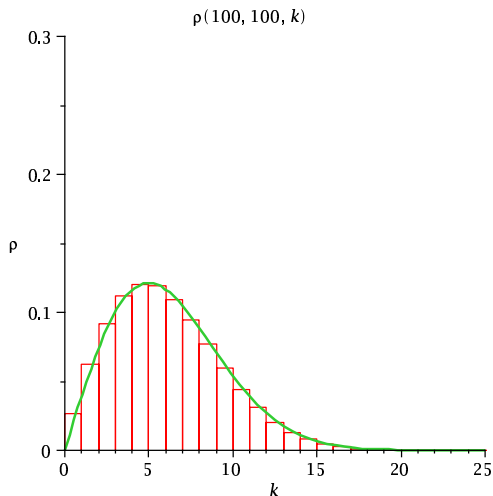$\square$

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NB: neither Maple nor Mathematica can do the above integral!

Less cars than parking spaces ($n > m$):
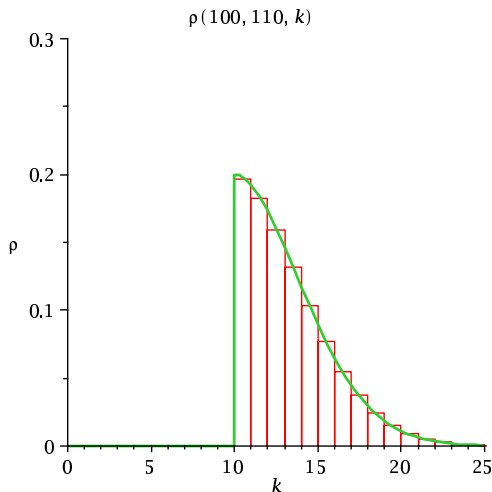

$\rho(100, 90, k)$

$$\frac{\mathrm{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

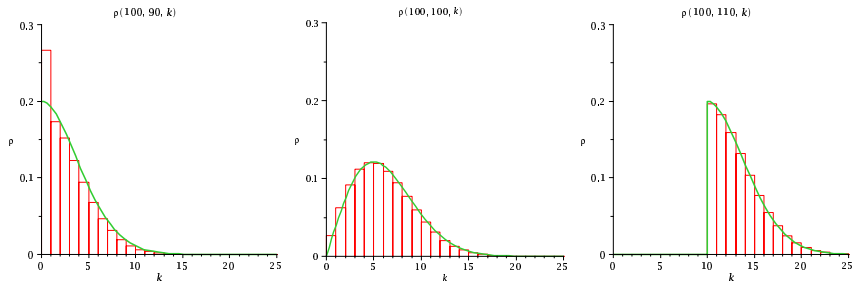Equal number of cars and parking spaces ($n = m$):

$$\frac{\mathrm{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

More cars than parking spaces ($n < m$):

$$\frac{\mathrm{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

The approximation is surprisingly accurate:

$$\frac{\mathrm{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$
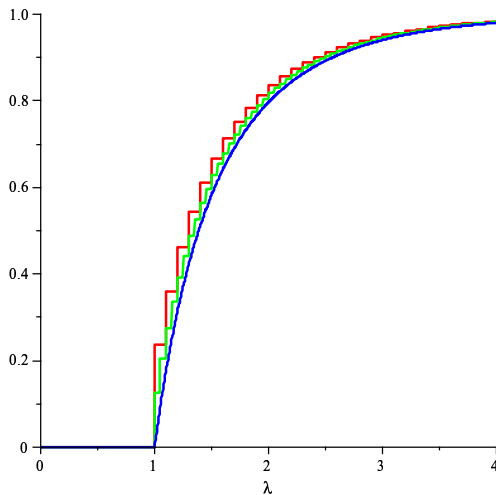
What is the probability that the car park is full?

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> ## Theorem
>
> *Let $\lambda \in \mathbb{R}^+$. Then the limiting probability that in a random assignment of $\lfloor \lambda n \rfloor$ drivers to n spaces all spaces are occupied is*
>
> $$\lim_{n \to \infty} \frac{\mathsf{cp}(n, \lfloor \lambda n \rfloor, \lfloor \lambda n \rfloor - n)}{n^{\lfloor \lambda n \rfloor}} = \begin{cases} 0 & \text{if } \lambda \leq 1, \\ 1 - \frac{1}{\lambda} \cdot T\left(\lambda e^{-\lambda}\right) & \text{if } \lambda > 1. \end{cases}$$

$n = 10, 20, \infty$:

$$\left.\frac{\text{cp}(n, m, m - n)}{n^m}\right|_{m = \lfloor \lambda n \rfloor} \approx 1 - \frac{1}{\lambda}\, T(\lambda e^{-\lambda})$$

Reference for this work:

- P. J. Cameron, D. Johannsen, T. Prellberg, and P. Schweitzer, "Counting defective parking functions", Electron. J. Combinat. **15** (2008) R92 (arXiv:0803.0302v2) (submitted to EJC on March 3, 2008)

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Related work by Alois Panholzer, TU Wien:

- "Limiting distribution results for a discrete parking problem", GOCPS 2008 (talk presented on March 5, 2008)
- "On a discrete parking problem", AofA 2008 (talk presented on April 17, 2008)

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- Introduced interesting and apparently new parking problem
- Solved the counting problem (Kernel method)
- Discussed limiting probability distributions

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Outlook:

- So far only "weak limit laws" - can refine analysis
- Extension to several generalised parking problems in the literature possible

The End