MAS115 Calculus I

Week 5

Thomas Prellberg

School of Mathematical Sciences
Queen Mary, University of London

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Revision

Continuity

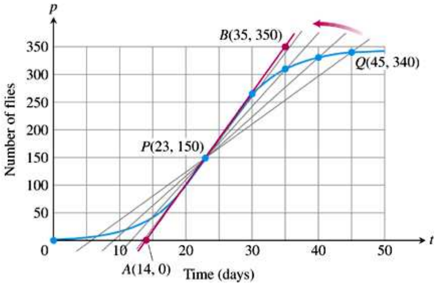
- A function continuous at a point
- A function continuous on an interval
- A continuous function
  (continuous at every point of its domain)
- Continuous extension of a function
- The Intermediate Value Theorem

Differentiation

Tangents and Derivatives

Revisit the question of “Rates of Change”

Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$



Now we can use limits to find slopes of tangents!

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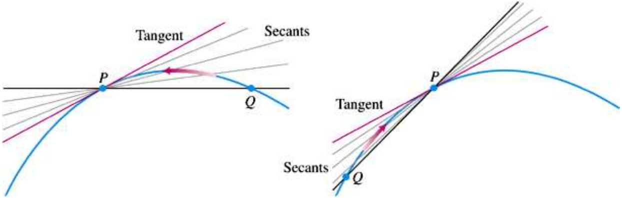
Lecture 13

Lecture 14

Lecture 15

Tangents and Derivatives

- construct a tangent to a curve using a limit of secants



- compute the slope of the tangent as a limit of slopes of secants

Example: Tangent Line to a Parabola

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$:

- choose a point at horizontal distance $h \neq 0$,

$$Q(2 + h, (2 + h)^2)$$

- secant through P and Q has slope

$$\frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{(2 + h) - 2} = 4 + h$$

- tangent through P has slope

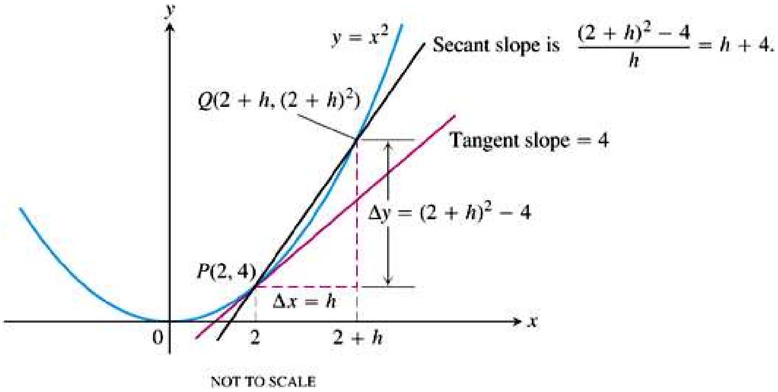
$$m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} (4 + h) = 4$$

- equation of tangent through $P(2, 4)$ is $y = 4 + 4(x - 2)$ or

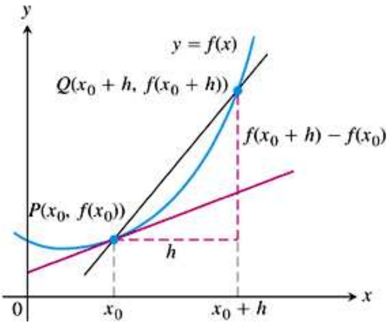
$$y = -4 + 4x$$

Example: Tangent Line to a Parabola

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$:



Slope of a Tangent Line



DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

How to find the tangent to a curve

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

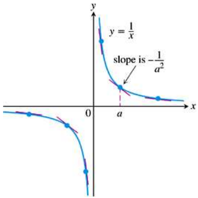
1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

Check the definition:
Show that the line $y = mx + b$ is its own tangent at any point

$$(x_0, mx_0 + b)$$

Example

Tangent line to $y = 1/x$ at $x_0 = a \neq 0$:



1. $f(a) = 1/a$, $f(a+h) = 1/(a+h)$
2. slope

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\
&= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}
\end{aligned}
$$

3. tangent line at $(a, 1/a)$: $y = 1/a + (-1/a^2)(x - a)$ or

$$y = \frac{2}{a} - \frac{x}{a^2}$$

The Derivative

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}$$

is called the difference quotient of f at x_0 with increment h. The limit as h approaches 0, if it exists, is called the derivative of f at x_0.

DEFINITION **Derivative Function**

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h},$$

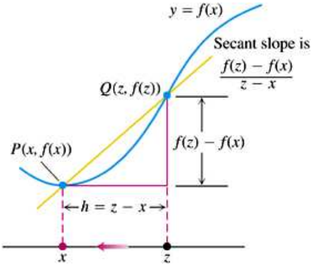
provided the limit exists.

If $f'(x)$ exists, we say that f is differentiable at x.

Alternative formula

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Equivalent notation: if $y = f(x)$,

$$y' = f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx}$$

Note: computing the derivative is called

differentiation

("derivation" is something else!)

Examples

- differentiate

$$f(x) = \frac{x}{x-1}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \dots \\ &= -\frac{1}{(x-1)^2} \end{aligned}$$

[Calculation on whiteboard]

Revision

Differentiation:

- Tangents as limits of secants
- Definition of the derivative

Examples

- differentiate

$$f(x) = \sqrt{x}$$

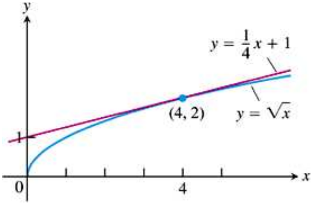
$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \ldots \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

[Calculation on whiteboard]

Examples

$$f(x) = \sqrt{x}\,, \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

Tangent line to the curve at $x = 4$:



- $f(4) = 2$, so the line goes through the point $(4, 2)$
- slope $m = f'(4) = 1/4$
- tangent line $y = 2 + m(x - 4)$, i.e.

$$y = \frac{x}{4} + 1$$

One-sided derivatives

In analogy to one-sided limits, we define one-sided derivatives:

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{right-hand derivative at } a$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \text{left-hand derivative at } b$$

Example: $f(x) = |x|$ is not differentiable at $x = 0$:

- to the right of the origin,

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

- to the left of the origin,

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1$$

so the right-hand and left-hand derivatives differ.

Differentiability implies Continuity

Theorem

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof.

For $h \neq 0$, write

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h} h$$

By assumption, $\frac{f(c+h)-f(c)}{h} \to f'(c)$ as $h \to 0$. Therefore,

$$\lim_{h \to 0} f(c+h) = f(c) + f'(c) \cdot 0 = f(c)$$

This means that f is continuous at $x = c$. $\square$

Caution: the converse is false!

Differentiation Rules

1. **Derivative of a Constant Function:**
   If f has the constant value $f(x) = c$, then
   $$\frac{df}{dx} = \frac{d}{dx}(c) = 0 \; .$$

2. **Power Rule for Positive Integers:**
   If n is a positive integer, then
   $$\frac{d}{dx}x^n = nx^{n-1} \; .$$

$$[z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \ldots + zx^{n-2} + x^{n-1})]$$

Differentiation Rules

3. **Constant Multiple Rule:**
   If u is a differentiable function of x, and c is a constant, then
   $$\frac{d}{dx}(cu) = c\frac{du}{dx} \ .$$

4. **Derivative Sum Rule:**
   If u and v are differentiable functions of x, then
   $$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \ .$$

Example

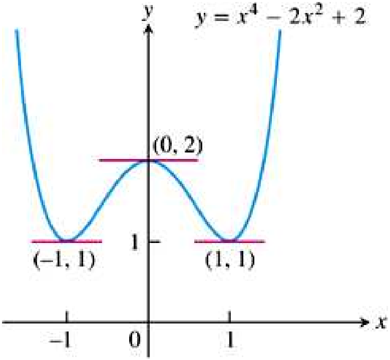
Differentiate $y = x^4 - 2x^2 + 2$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^4 - 2x^2 + 2) \\
\text{Rule 4:} \quad &= \frac{d}{dx}(x^4) + \frac{d}{dx}(-2x^2) + \frac{d}{dx}(2) \\
\text{Rule 3:} \quad &= \frac{d}{dx}(x^4) + (-2)\frac{d}{dx}(x^2) + \frac{d}{dx}(2) \\
\text{Rule 2:} \quad &= 4x^3 + (-2)2x + \frac{d}{dx}(2) \\
\text{Rule 1:} \quad &= 4x^3 - 4x + 0 = 4x^3 - 4x\ .
\end{aligned}$$

Now find, for example, horizontal tangents:

$$y' = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0 \quad \Rightarrow \quad x \in \{0, 1, -1\}$$

Example



$$y = x^4 - 2x^2 + 2\,, \qquad y' = 4x^3 - 4x$$

Differentiation Rules

5. **Derivative Product Rule:**
   If u and v are differentiable functions of x, then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \; .$$

6. **Derivative Quotient Rule:**
   If u and v are differentiable functions of x and $v(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \; .$$

A common mistake:

$$(uv)' = u'v' \qquad (u/v)' = u'/v'$$

is generally WRONG!

Revision

Differentiation

- Differentiation from first principles
- Differentiable functions are continuous
- Rules of Differentiation

Example

- Differentiate $y = (x^2+1)(x^3+3)$:
$$u = x^2+1\ ,\quad v = x^3+3\ ,$$
$$u' = 2x\ ,\quad v' = 3x^2\ ,$$
$$y' = u'v+uv' = 2x(x^3+3)+(x^2+1)3x^2 = 5x^4+3x^2+6x\ .$$
- Differentiate $y = (t^2-1)/(t^2+1)$:
$$u = t^2-1\ ,\quad v = t^2+1\ ,$$
$$u' = 2t\ ,\quad v' = 2t\ ,$$
$$y' = \frac{u'v-uv'}{v^2} = \frac{2t(t^2+1)-(t^2-1)2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}\ .$$

Differentiation Rules

7. **Power Rule for Negative Integers:**
   If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}x^n = nx^{n-1} \ .$$

[use the Quotient Rule]

Examples:

$$\frac{d}{dx}(x^{10}) = 10x^9 \ , \quad \frac{d}{dx}(x^{-11}) = -11x^{-12} \ .$$

Higher Derivatives

- If f' is differentiable, we call

$$f'' = (f')'$$

the second derivative of f.

- Notation:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

- Similarly, we write $f''' = (f'')'$ for the third derivative, and generally for the n-th derivative (for $n \in \mathbb{N}_0$)

$$f^{(n)} = (f^{(n-1)})' \quad \text{with} \quad f^{(0)} = f \ .$$

Example

Differentiate repeatedly $f(x) = x^5$ and $g(x) = x^{-5}$:

$$
\begin{aligned}
f'(x) &= 5x^4 & g'(x) &= -5x^{-6} \\
f''(x) &= 20x^3 & g''(x) &= 30x^{-7} \\
f'''(x) &= 60x^2 & g'''(x) &= -210x^{-8} \\
f^{(4)}(x) &= 120x & g^{(4)}(x) &= 1680x^{-9} \\
f^{(5)}(x) &= 120 & g^{(5)}(x) &= -15120x^{-10} \\
f^{(6)}(x) &= 0 & g^{(6)}(x) &= 151200x^{-11} \\
f^{(7)}(x) &= 0 & g^{(7)}(x) &= \dots
\end{aligned}
$$

$\boxed{\text{Reading Assignment: Section 3.3}}$

Derivatives of Trigonometric Functions

Differentiate $f(x) = \sin x$:

- Start with the definition of $f'(x)$:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

- Use $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$f'(x) = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

- Collect terms:

$$f'(x) = \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

- Use $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$ to conclude

$$f'(x) = \cos x$$

Derivatives of Trigonometric Functions

- We have just shown that $\frac{d}{dx}\sin x = \cos x$ and a very similar derivation gives $\frac{d}{dx}\cos x = -\sin x$.
- We still need

$$
\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\frac{d}{dx}(\sin x)\cos x - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}
\end{aligned}
$$

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Relating Derivatives

An example:

- $y = \frac{3}{2}x$ is the same as

$$y = \frac{1}{2}u \quad \text{and} \quad u = 3x\ .$$

- Writing

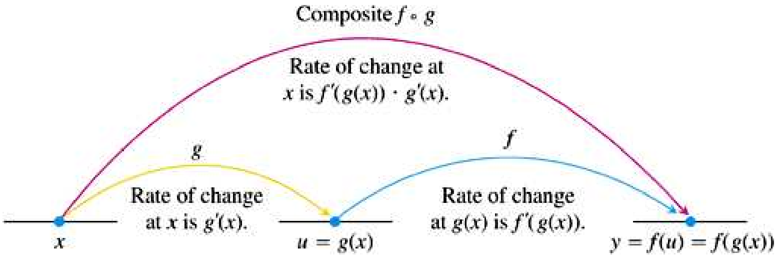
$$\frac{dy}{dx} = \frac{3}{2}\,, \quad \frac{dy}{du} = \frac{1}{2}\,, \quad \frac{du}{dx} = 3$$

  we find

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\ .$$

Accident or general formula? Do rates of change multiply?

The Chain Rule



THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Applying the Chain Rule

- Differentiate $x(t) = \cos(t^2 + 1)$:
  Choose $x = \cos u$ and $u = t^2 + 1$ so that

$$\frac{dx}{du} = -\sin u \quad \text{and} \quad \frac{du}{dt} = 2t$$

  Then

$$\frac{dx}{dt} = (-\sin u)2t = -2t\sin(t^2 + 1)$$

- $\frac{d}{dx} sin(x^2 + x) = (2x + 1)\cos(x^2 + x)$
- A chain with three links:

$$\frac{d}{dt}\tan(5 - \sin 2t) = \frac{-2\cos 2t}{\cos^2(5 - \sin 2t)}$$

[Details on white board]

The End