MTH5105 Differential and Integral Analysis 2008-2009

Exercises 4

Exercise 1: Let the function $f\colon (0,\pi) \to \mathbb{R}$ be given by $x \mapsto \cos(x)$. Show that f is invertible and that the inverse $g(y) = f^{-1}(y)$ is differentiable. Find the derivative g'. [4 marks]

Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g. What is the remainder term in the Lagrange form? [4 marks]

Hence show that for $|y| \leq 1/2$

$$|g(y) - \pi/2 + y| \leq \sqrt{3}/18 \approx 0.096 \ .$$

[4 marks]

Exercise 2: Let $f : (-1, \infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0 \ .$$

[4 marks]

Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0 \ .$$

[4 marks]

Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h,

$$(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)} h^{(k)} \ .$$

Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$. [4 marks]

Exercise 3: The number e can be expressed via an alternating series as

$$\frac{1}{e} = \exp(-1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \ .$$

Show that remainder term R_n in

$$\frac{n!}{e} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + R_n \ ,$$

cannot be an integer. [3 marks]

Hint: look at the convergence criterion for alternating series.

Hence deduce that e is irrational. [3 marks]

The deadline is 12.15 on Monday, 9th February. Please hand in your coursework at the end of Monday's lecture or to my office MAS113 immediately afterwards.

Thomas Prellberg, February 2009