MAS115 Calculus I

Week 4

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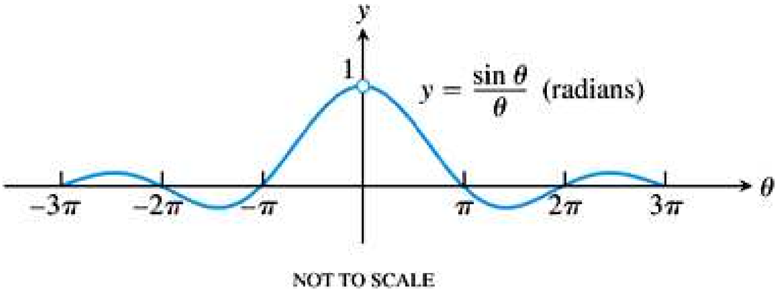
Revision

- $\epsilon - \delta$ definition of limit
- How to find δ for a given ϵ
- One-sided limits

Limits involving $\sin\theta/\theta$

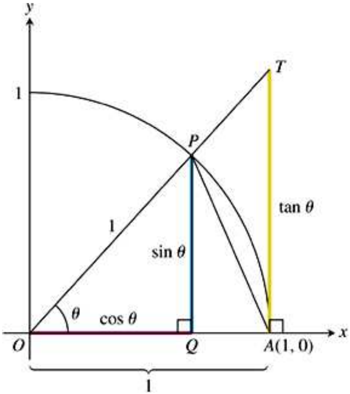


Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Show that both right-hand and left-hand limits are equal to 1.



$$\sin\theta < \theta < \tan\theta$$

this implies

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

by Sandwich theorem (taking the limit as $\theta \to 0$)

$$1 \le \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} \le 1$$

Similarly, $\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$

Applications

Compute

$$
\begin{aligned}
\lim_{h\to 0}\frac{\cos h-1}{h} &= \lim_{h\to 0}\frac{1-2\sin^2(h/2)-1}{h}\\
&= \lim_{h\to 0}\frac{\sin(h/2)}{h/2}(-\sin h)\\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}(-1)\lim_{h\to 0}\sin h\\
&=1(-1)0=0
\end{aligned}
$$

Applications

Compute

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{2}{5}\frac{\sin 2x}{2x}\\ &=\frac{2}{5}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\\ &=\frac{2}{5}1=\frac{2}{5}\end{aligned}$$

Limits as x approaches infinity

- Observation:

  $$x \text{ approaching positive/negative infinity}$$

  is like

  $$1/x \text{ approaching zero from the right/left}$$

- Change terminology in $\epsilon - \delta$ formulation:

  There is a $\delta > 0$ such that for all $0 < 1/x < \delta \ldots$

  translates to

  There is an $M > 0$ such that for all $x > M \ldots$

Limits as x approaches infinity

DEFINITIONS **Limit as x approaches ∞ or $-\infty$**

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

Limit laws as x approaches infinity

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Examples

(a)

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}=5+0=5$$

(b)

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{x^2(5+8/x-3/x^2)}{x^2(3+2/x^2)}$$

$$=\frac{5+\lim_{x\to\infty}8/x-\lim_{x\to\infty}3/x^2}{3+\lim_{x\to\infty}2/x^2}=\frac{5}{3}$$

Examples

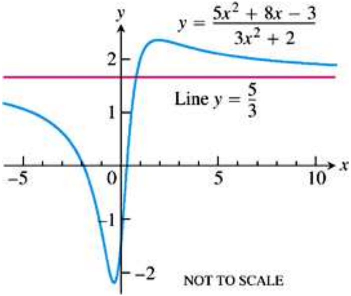
(c)

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{x^3(11/x^2-2/x^3)}{x^3(2-1/x^3)}$$

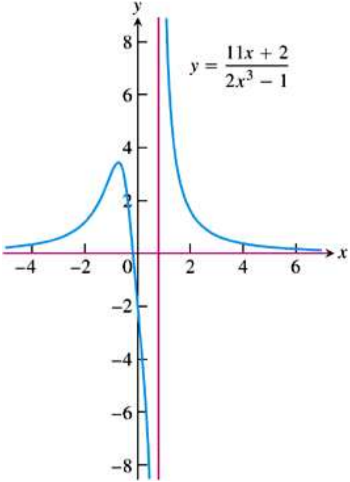
$$= \frac{\lim_{x\to\infty} 11/x^2 - \lim_{x\to\infty} 2/x^3}{2-\lim_{x\to\infty} 1/x^3} = 0$$

Examples

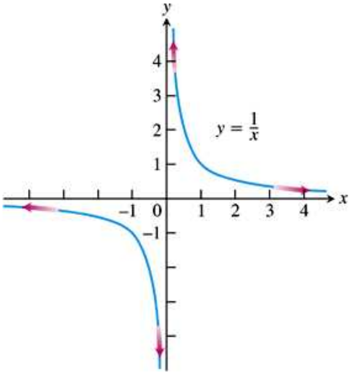
(b)


$y = \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$
Line $y = \dfrac{5}{3}$
NOT TO SCALE
y
x
2
1
−1
−2
−5
0
5
10

(c)


$y = \dfrac{11x + 2}{2x^3 - 1}$
y
x
8
6
4
2
−2
−4
−6
−8
−4
−2
0
2
4
6

Horizontal asymptotes



$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

The graph approaches the line

$$y = 0$$

asymptotically; the line is an asymptote of the graph.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

Examples

(a)

$$f(x) = 5 + \frac{1}{x}\,, \qquad \lim_{x\to\pm\infty} f(x) = 5$$

The curve has the line $y = 5$ as a horizontal asymptote.

(b)

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}\,, \qquad \lim_{x\to\pm\infty} f(x) = \frac{5}{3}$$

The curve has the line $y = 5/3$ as a horizontal asymptote.

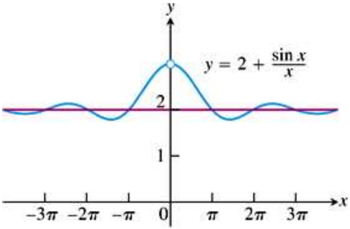
(c)

$$f(x) = \frac{11x + 2}{2x^3 - 1}\,, \qquad \lim_{x\to\pm\infty} f(x) = 0$$

The curve has the line $y = 0$ as a horizontal asymptote.

An application of the Sandwich Theorem

Find the horizontal asymptote to $y = 2 + \frac{\sin x}{x}$:



- $\left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$
- $\lim_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0$
- Therefore, by the Sandwich Theorem,

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

- Hence,

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2$$

Revision

- $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$
- Limits as x approaches infinity
- Horizontal asymptotes

Oblique Asymptotes

If for a rational function $f(x) = p(x)/q(x)$ the degree of $p(x)$ is one greater than the degree of $q(x)$, polynomial division gives

$$f(x) = ax + b + r(x) \qquad \text{with} \lim_{x\to\pm\infty} r(x) = 0$$

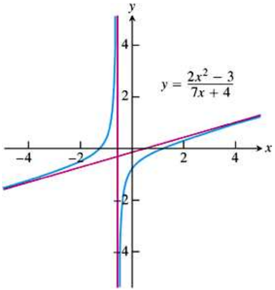
$y = ax + b$ is called an oblique (slanted) asymptote. Example:

$$f(x) = \frac{2x^2-3}{7x+4} = \frac{2}{7}x - \frac{8}{49} + \frac{-115}{49(7x+4)}$$

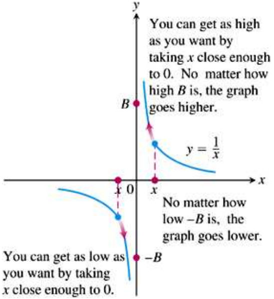
$\lim_{x\to\pm\infty} \frac{-115}{49(7x+4)} = 0$, so that

$$y = \frac{2}{7}x - \frac{8}{49}$$

is an oblique asymptote of $f(x)$.



Infinite limits



$f(x) = \frac{1}{x}$ has no limit as $x \to 0^+$. However, it is convenient to still say that $f(x)$ approaches ∞ as as $x \to 0^+$. We write

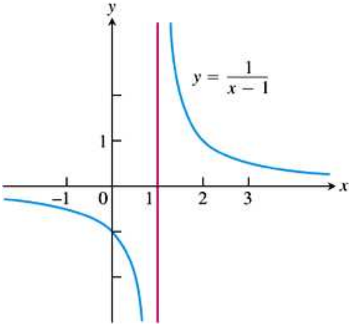
$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

Similarly,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Careful: $\lim_{x \to 0^+} \frac{1}{x} = \infty$ really means that *the limit does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

Example: one-sided infinite limits



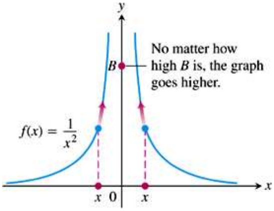
$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

and

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

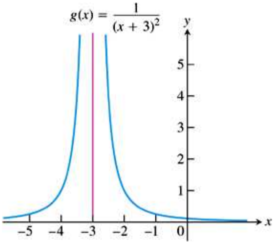
as $y = 1/(x-1)$ is just $y = 1/x$ shifted by one to the right.

Example: two-sided infinite limits



$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

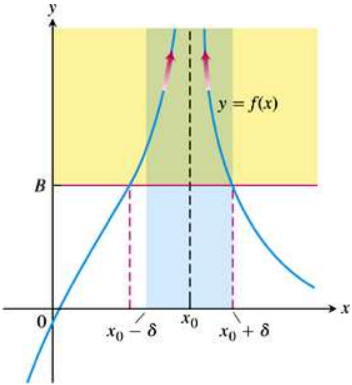
as the values of $1/x^2$ are positive and become arbitrarily large as $x \to 0$.

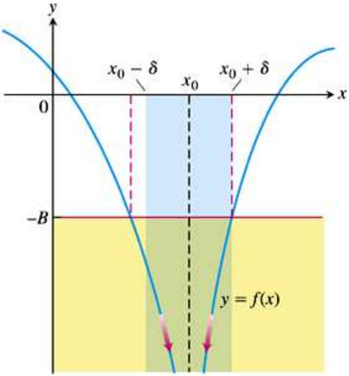


$$\lim_{x\to -3} \frac{1}{(x+3)^2} = \lim_{x\to 0} \frac{1}{x^2} = \infty$$

as $y = 1/(x+3)^2$ is just $y = 1/x^2$ shifted by three to the left.

Towards a precise definition





Precise definition of infinite limits

DEFINITIONS **Infinity, Negative Infinity as Limits**

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write
$$\lim_{x \to x_0} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches x_0**, and write
$$\lim_{x \to x_0} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

Using the definition

Prove that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

- given $B > 0$, find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \Rightarrow \quad \frac{1}{x^2} > B$$

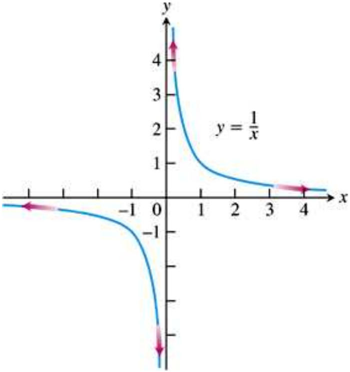
- choose $\delta = \frac{1}{\sqrt{B}}$ so that

$$0 < |x| < \delta \Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = B$$

- Therefore, by definition,

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Vertical asymptotes



$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

The graph approaches the line

$$x = 0$$

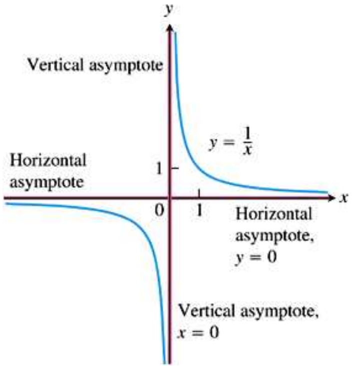
asymptotically; the line is an asymptote of the graph.

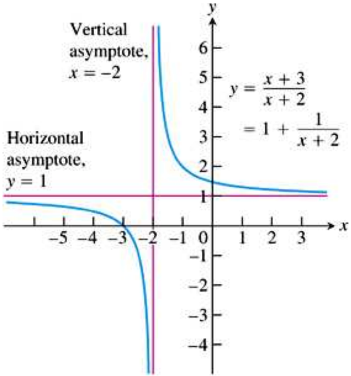
DEFINITION **Vertical Asymptote**

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty.$$

Examples


y
Vertical asymptote
$y = \frac{1}{x}$
Horizontal
asymptote
1
0
1
x
Horizontal
asymptote,
$y = 0$
Vertical asymptote,
$x = 0$


y
Vertical
asymptote,
$x = -2$
6
5
4
3
2
1
$y = \frac{x+3}{x+2}$
$= 1 + \frac{1}{x+2}$
Horizontal
asymptote,
$y = 1$
−5 −4 −3 −2 −1 0 1 2 3
x
−1
−2
−3
−4

Example

Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2-4}$$

- $\lim_{x\to\pm\infty} f(x) = 0$
- division by zero for $x = \pm 2$
  if needed, rewrite:

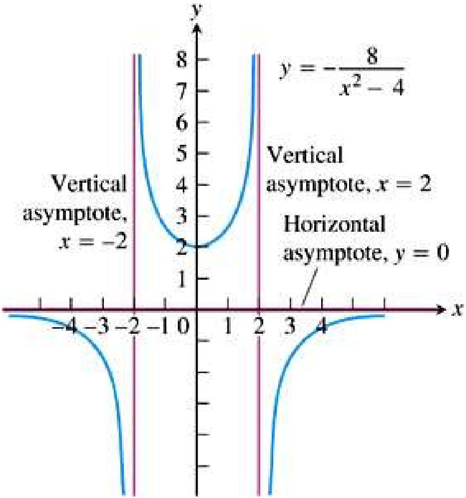
$$-\frac{8}{x^2-4} = \frac{2}{x+2} - \frac{2}{x-2}$$

- $\lim_{x\to-2^-} f(x) = -\infty$, $\lim_{x\to-2^+} f(x) = \infty$
  $\lim_{x\to2^-} f(x) = \infty$, $\lim_{x\to2^+} f(x) = -\infty$

Asymptotes are

$$y = 0\,, \quad x = -2\,, \quad x = 2$$

Example


y
x
8
7
6
5
4
3
2
1
−4 −3 −2 −1 0
1 2 3 4
$y = -\frac{8}{x^2-4}$
Vertical asymptote, $x = 2$
Vertical asymptote, $x = -2$
Horizontal asymptote, $y = 0$

Final example on asymptotes

Find the asymptotes of the graph of

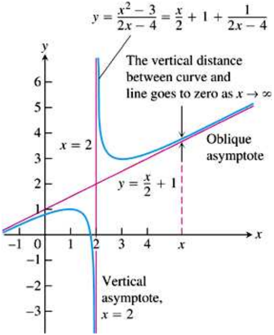
$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- rewrite [polynomial division]:

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

Asymptotes are

$$y = \frac{x}{2} + 1\,, \quad x = 2$$



We say that $x/2 + 1$ dominates when x is large and that $1/(2x - 4)$ dominates when x is near 2.

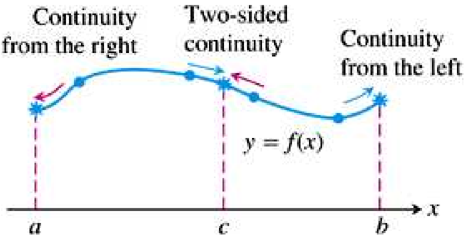
Revision

- Oblique asymptotes
- Infinite limits
- Vertical asymptotes

Continuity

Continuity

- Informally, any function whose graph can be sketched over its domain in one continuous motion, i.e. *without lifting the pen*, is an example of a continuous function.



Continuity

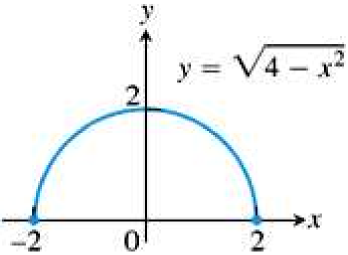
DEFINITION Continuous at a Point

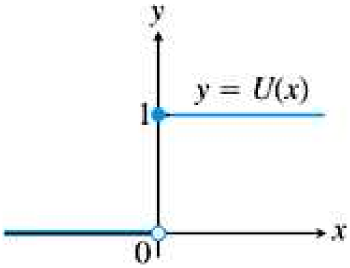
Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

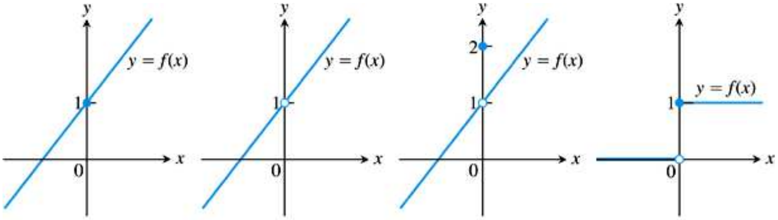
$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

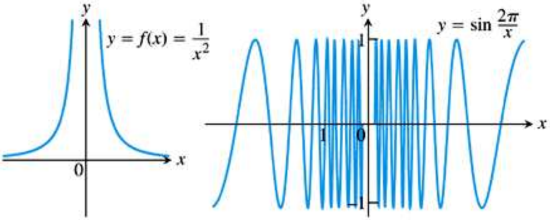




Continuity

- If a function is not continuous at a point c, we say that f is discontinuous at c





Continuity at an interior point

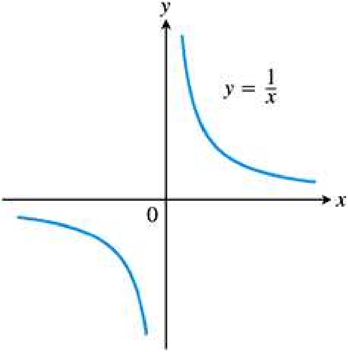
Continuity Test
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

- A function f is right-continuous at a point $x = c$ in its domain if $\lim_{x\to c^+} f(x) = f(c)$
- A function f is left-continuous at a point $x = c$ in its domain if $\lim_{x\to c^-} f(x) = f(c)$
- Therefore: a function f is continuous at a point $x = c$ in its domain if and only if it is both right-continuous and left-continuous at c.

Continuous function

- A function is continuous on an interal if and only if it is continuous at every point of the interval.
- A continuous function is a function that is continuous at every point of its domain.



- $y = 1/x$ is a continuous function.
  (It is continuous at every point of its domain.)
- $y = 1/x$ is not continuous on $[-1, 1]$.

Properties of continuous functions

Limit laws imply:

THEOREM 9 Properties of Continuous Functions

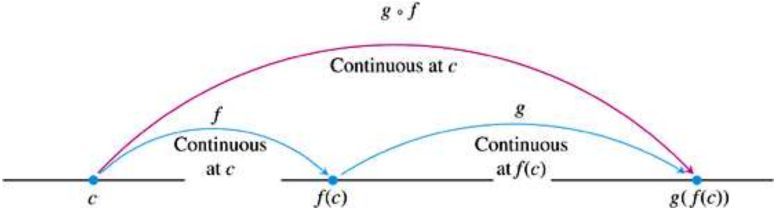
If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Example: Polynomials and rational functions are continuous.

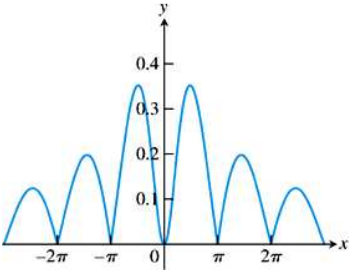
Compositions of continuous functions

THEOREM 10 Composite of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



Example

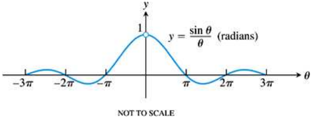
$$y = \left| \frac{x \sin x}{x^2 + 2} \right| \text{ is everywhere continuous}$$



- $f(x) = \frac{x \sin x}{x^2+2}$ is continuous (why?)
- $g(x) = |x|$ is continuous (why?)
- therefore $y = g \circ f(x)$ is continuous

Continuous extension to a point

$$f(x) = \frac{\sin x}{x} \qquad \text{for } x \neq 0$$



is defined and continuous for all $x \neq 0$. As $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it makes sense to *define* a new function

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Definition

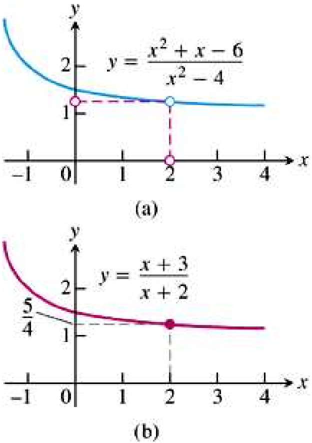
If $\lim_{x\to c} f(x) = L$ exists, but $f(c)$ is not defined, we define a new function

$$F(x) = \begin{cases} f(x) & \text{for } x \neq c \\ L & \text{for } x = c \end{cases}$$

$F(x)$ is continuous at c, and is called the continuous extension of $f(x)$ to c.

Example

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$



For $x \neq 2$, $f(x)$ is equal to

$$F(x) = \frac{x + 3}{x + 2}$$

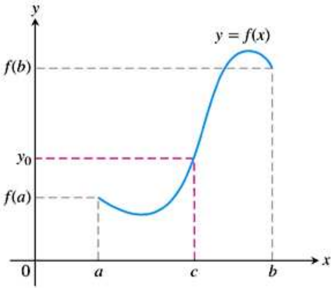
$F(x)$ is the continuous extension of $f(x)$ to $x = 2$, as

$$\lim_{x \to 2} f(x) = \frac{5}{4} = F(2)$$

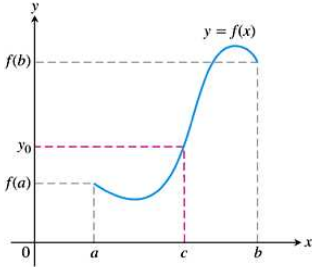
The Intermediate Value Theorem

A function has the intermediate value property if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



The Intermediate Value Theorem



- Geometrical interpretation: any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
- Continuity is essential: if f is discontinuous at any point of the interval, then the function may “jump” and miss some values.

Consequences

- Connectivity: the graph of a continuous function over an interval is *connected*, i.e. a single, unbroken curve without any breaks or jumps.
- Root-finding: A solution of the equation $f(x) = 0$ is called a root of the equation or zero of f.

If $f(x)$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite sign, then $f(x) = 0$ has roots on $[a, b]$.

Application

Show that the equation

$$x^3 - 15x + 1 = 0$$

has three roots in the interval $[-4, 4]$:

- Use $f(x) = x^3 - 15 + 1$ and compute a few values: $f(-4) = -3$, $f(-3) = 19$, $f(-2) = 23$, $f(-2) = 23$, $f(-1) = 15$, $f(0) = 1$, $f(1) = -13$, $f(2) = -21$, $f(3) = -17$, and $f(4) = 5$.
- Notice that

$$\begin{aligned} f(-4) &< 0 < f(-3) \\ f(0) &> 0 > f(1) \\ f(3) &< 0 < f(4) \end{aligned}$$

- Therefore there are three roots in the interval $[-4, 4]$. More precisely, the roots are in the intervals $[-4, -3]$, $[0, 1]$, and $[3, 4]$.

The End