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Examiner: Dr T Prellberg

MTH5105 Differential and Integral Analysis
MID-TERM TEST

Date: 25 Feb 2011 *Time:* 12:10–12:50

Complete the following information:

Name	
Student Number (9 digit code)	

The test has THREE questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish to be marked.

Question	Marks
1	
2	
3	
Total Marks	

Nothing on this page will be marked!

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Question 1.

(a) State the formula for the Taylor polynomial $T_{n,a}$ of degree n of a function f at a. [*10 marks*]

Let $g(x) = -\log(1-x)$.

(b) Prove by mathematical induction that $g^{(n)}(x) = \dfrac{(n-1)!}{(1-x)^n}$ for $n \in \mathbb{N}$. [*10 marks*]

Hence compute the Taylor polynomial $T_{4,0}$ of degree 4 of g at $a = 0$. [*10 marks*]

Let $h(x) = \log(1+x+x^2)$.

(c) By factorising $1-x^3$, or otherwise, determine the Taylor polynomial $T_{4,0}$ of degree 4 of h at $a=0$. [*10 marks*]

Answer 1.

(a)

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \ .$$

[*10 marks, -2/mistake, NO MARKS if no polynomial*]

(b) Base case $n=1$: $g'(x) = -\dfrac{1}{1-x}(-1) = \dfrac{1}{1-x} = \dfrac{0!}{(1-x)^1}$. [*2 marks*]

Inductive step $n \to n+1$:

$$g^{(n+1)}(x) = (g^{(n)})'(x) = \left[\frac{(n-1)!}{(1-x)^n}\right]' = (n-1)!\frac{(-n)}{(1-x)^{n+1}}(-1) = \frac{n!}{(1-x)^{n+1}}$$

[*2+2+2+2 marks*]

As $g(0) = -\log(1) = 0$ and $g^{(n)}(0) = (n-1)!$, [*2+2 marks*]

$$T_{4,0}(x) = \sum_{k=1}^{4} \frac{(k-1)!}{k!}x^k = \sum_{k=1}^{4} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \ .$$

[*2+2+2 marks*]

(c) Using $1-x^3 = (1-x)(1+x+x^2)$ we write [*2 marks*]

$$\log(1+x+x^2) = \log[(1-x^3)/(1-x)] = \log(1-x^3) - \log(1-x)$$

[*2 marks*]

Recognising that

$$\log(1-x^3) = -\left[x^3 + \frac{(x^3)^2}{2} + \dots\right] = -x^3 - \frac{x^6}{2} - \dots$$

[*3 marks*]

leads to

$$T_{4,0}(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - x^3 = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \ .$$

[*3 marks*]

Answer 1. (*Continue*)

Question 2.

(a) Give the definition of $f : \mathcal{D} \to \mathbb{R}$ being differentiable at a point $a \in \mathcal{D}$. [*10 marks*]

(b) Using this definition, show that $g(x) = 1/x$ is differentiable at $a = 1$ and find its derivative. [*10 marks*]

(c) Suppose that the function f is continuous at 0. Show that the function h defined by

$$h(x) = xf(x)$$

is differentiable at zero and find its derivative. [*10 marks*]

Answer 2.

(a) f is differentiable at $a \in \mathcal{D}$ if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. [*10 marks, -2/mistake, -4 for wrong lim*]

(b) We compute

$$\lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{1/x - 1}{x - 1} = \lim_{x \to 1} \left(-\frac{1}{x} \right) = -1 \ .$$

[*2+2+2 marks*]

Hence g is differentiable at 1 and [*2 marks*]

$$g'(1) = -1 \ .$$

[*2 marks*]

(c) We compute

$$\lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0} \frac{xf(x) - 0}{x} = \lim_{x \to 0} f(x) = f(0) \ ,$$

[*2+2+2 marks*]

as f is continuous at zero. Hence h is differentiable at zero and [*2 marks*]

$$h'(0) = f(0) \ .$$

[*2 marks*]

Answer 2. (*Continue*)

Question 3.

(a) State the Mean Value Theorem. [*15 marks*]

Let g be differentiable on $[0, 1]$ with

$$g'(x) = 0$$

for all $x \in [0, 1]$, and let $g(0) = 1$.

(b) Using the Mean Value Theorem, or otherwise, prove that

$$g(x) = 1$$

for all $x \in [0, 1]$. [*15 marks*]

Answer 3.

(a) MVT: Let f be continuous on $[a, b]$ and differentiable on (a, b).

[*4 marks*]

Then there is a $c \in (a, b)$ such that

[*3 marks*]

$$f'(c) = \frac{f(b) - f(a)}{b - a} \, .$$

[*8 marks, -2/mistake*]

(b) Let $x \in (0, 1]$. [*2 marks*]

g is continuous on $[0, x]$ and differentiable on $(0, x)$. [*2 marks*]

We can apply the MVT to g on $[0, x]$. [*3 marks*]

There exists a $c \in (0, x)$ such that [*2 marks*]

$$g'(c) = \frac{g(x) - g(0)}{x - 0} \, .$$

[*3 marks*]

As $g'(c) = 0$, it follows that $0 = g(x) - g(0)$, and hence

$$g(x) = g(0) = 1$$

for all $x \in (0, 1]$. [*3 marks*]

Answer 3. (*Continue*)