From parabolic fixed points to asymptotic enumeration

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Combinatorial Enumeration

Problems in Combinatorial Enumeration

Examples of recursively definable structures:

- Number of partitions of a set into subsets (Bell, 1934)
  - Bell Numbers
- Number of partitions of a set into ordered subsets (Cayley, 1859)
  - ordered Bell Numbers
- Partition lattice chains (Lengyel, 1984)
  - Lengyel's Constant
- Analysis of a recursive Program (Knuth, 1991)
  - $t(x, y, z) =$ **if** $x \leq y$ **then** y **else**
    $$t(t(x-1, y, z), t(y-1, z, x), t(z-1, x, y))$$

Bell Numbers

- B_n number of partitions of an n-set into subsets

$$B_3 \quad = \quad \binom{2}{0} B_2 \quad + \quad \binom{2}{1} B_1 \quad + \quad \binom{2}{2} B_0$$

$$\begin{array}{ccc} \{\{3\},\{1,2\}\} & \{\{3,1\},\{2\}\} & \{\{3,1,2\}\} \\ \{\{3\},\{1\},\{2\}\} & \{\{3,2\},\{1\}\} & \end{array}$$

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- Recurrence: $$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} \; , \quad B_0 = 1$$

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- Recurrence: $$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} \;, \quad B_0 = 1$$

- Exact expression for B_n from EGF: $b(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$

$$b(z) = \exp(e^z - 1) \quad \Rightarrow \quad B_n = \frac{1}{e} \sum_{m=0}^{\infty} \frac{m^n}{m!}$$

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Bell Numbers (ctd.)

- Asymptotic growth (from saddle point method):

$$B_n \sim \exp\left(e^w(w^2 - w + 1) - \frac{1}{2}\log(w+1) - 1\right)$$

Scale: $w\exp(w) = n$ Lambert W-function: $\boxed{w = W(n)}$

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Scale: $w\exp(w) = n$ Lambert W-function: $\boxed{w = W(n)}$

- Functional equation for OGF: $B(z) = \sum_{n=0}^{\infty} B_n z^n$

$$B(z) = \frac{z}{1-z} B\left(\frac{z}{1-z}\right) + 1$$

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Ordered Bell Numbers

- S_n number of partitions of an n-set into ordered subsets
- Recurrence: $$S_{n+1} = \sum_{k=0}^{n} \binom{n+1}{k+1} S_{n-k} \ , \qquad S_0 = 1$$
- Exact expression for S_n from EGF: $s(z) = 1/(2 - e^z)$

$$S_n = \sum_{m=1}^{n} m! S_{n,m} \ , \qquad S_{n,k} \text{ Stirling numbers 2nd kind}$$

- Asymptotic growth: $$S_n \sim \frac{n!}{2} \frac{1}{(\log 2)^{n+1}}$$
- Functional equation for OGF (Klazar, private comm.):

$$B(z) = \frac{1}{2} \frac{1}{1-z} B\left(\frac{z}{1-z}\right) + \frac{1}{2}$$

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Partition Lattice Chains

- Poset of partitions of an n-set

$$\begin{tikzcd}
 & \{\{1,2,3\}\} & \\
\{\{1,2\},\{3\}\} \arrow[ur] & \{\{1,3\},\{2\}\} \arrow[u] & \{\{2,3\},\{1\}\} \arrow[ul] \\
 & \{\{1\},\{2\},\{3\}\} \arrow[ul] \arrow[u] \arrow[ur] \arrow[uu, bend right] &
\end{tikzcd}$$

- Z_n number of chains from minimal to maximal element

$$Z_1 = 1, \quad Z_2 = 1, \quad , Z_3 = 4, \quad Z_4 = 32, \quad \ldots$$

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Partition Lattice Chains (ctd.)

- Recurrence (Lengyel):

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} Z_k \ , \qquad S_{n,k} \text{ Stirling numbers 2nd kind}$$

- Functional equation for EGF (Lengyel):

$$Z(z) = \frac{1}{2} Z(e^z - 1) + \frac{z}{2}$$

- Asymptotic growth (Babai, Lengyel):

$$Z_n \sim C_{\mathrm{Lengyel}} (n!)^2 (2 \log 2)^{-n} n^{-1-\frac{1}{3}\log 2}$$

- Lengyel's Constant (Flajolet, Salvy): $C_{\mathrm{Lengyel}} = 1.0986858055\ldots$

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Takeuchi Numbers

- Recursive function (Takeuchi):

$$\begin{aligned} t(x,y,z) = &\ \textbf{if } x \le y \textbf{ then } y \textbf{ else} \\ &\quad t(t(x-1,y,z), t(y-1,z,x), t(z-1,x,y)) \end{aligned}$$

- $T(x,y,z)$ number of times the **else** clause is invoked when evaluating $t(x,y,z)$
- $T_n = T(n,0,n+1)$

$$T_1 = 1, \quad T_2 = 4, \quad , T_3 = 14, \quad T_4 = 53, \quad \ldots$$

- Actual value of $t(x,y,z)$ is irrelevant

$$t(x,y,z) = \begin{cases} y & x \le y \\ \left\{ \begin{array}{ll} z & y \le z \\ x & \text{else} \end{array} \right. & \end{cases}$$

Takeuchi Numbers (ctd.)

- Recurrence (Knuth):

$$T_{n+1} = \sum_{k=0}^{n} \left[\binom{n+k}{n} - \binom{n+k}{n+1} \right] T_{n-k} + \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k+1}$$

- Functional equation for OGF (Knuth):

$$T(z) = zC(z)\,T(zC(z)) + \frac{C(z)-1}{1-z}\;, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}$$

- Asymptotic growth (Prellberg):

$$T_n \sim C_{\text{Takeuchi}} B_n \exp \frac{1}{2} W(n)^2\;, \quad C_{\text{Takeuchi}} = 2.2394331040\ldots$$

General Setting

Common Features

- Linear recurrences:

$$X_n = \sum_{k=1}^{n} c_{n,k} X_{n-k} + b_n$$

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- Functional equations for OGF/EGF:

$$X(z) = a(z) X \circ f(z) + b(z)$$

Common Features

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- Parabolic fixed point:

$$f(z) = z + cz^2 + dz^3 + \dots$$

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- Parabolic fixed point:

$$f(z) = z + cz^2 + dz^3 + \dots$$

- Caveat: divergence of GF!

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Generalization: Recursive Structures

- View combinatorial structures as formed of "atoms"
- Substitution operation $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$:
  "substitute elements of $\mathcal{C}$ for atoms of $\mathcal{B}$"

$$\mathcal{B} \circ \mathcal{C} = \sum_{k \geq 0} \mathcal{B}_k \times \overbrace{\mathcal{C} \times \mathcal{C} \times \ldots \times \mathcal{C}}^{k}$$

- The associated OGF satisfies $A(z) = B(C(z))$
- A recursively definable structure $\mathcal{X}$ is defined by

$$\mathcal{X} = \mathcal{A} \times \mathcal{X} \circ \mathcal{F} + \mathcal{B}$$

- The associated OGF satisfies $X(z) = A(z) X \circ F(z) + B(z)$

Ingredients for General Theory

- Formal solution of the functional equation (leads to divergent FPS)
- Cauchy formula
- Analytic iteration theory near parabolic fixed points (Milnor, Beardon)
- Saddle point analysis

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Formal Power Series Solution

Let the FPS $X(z)$ satisfy

$$X(z) = a(z)X \circ f(z) + b(z)$$

with $a(z)$, $f(z)$, and $b(z)$ analytic near $z = 0$ and

$$f(z) = z + cz^2 + dz^3 + \dots\ , \quad c > 0$$

Formal Power Series Solution

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$$f(z) = z + cz^2 + dz^3 + \dots \,, \quad c > 0$$

Then

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z)$$

as FPS (this is a divergent power series!)

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Inversion via Cauchy Formula

From

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z)$$

we compute

$$X_n = [z^n] X(z) = \sum_{m=0}^{\infty} X_{n,m}$$

with

$$X_{n,m} = \frac{1}{2\pi i} \oint \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z) \frac{dz}{z^{n+1}}$$

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Simplification via Homogeneous Eqn.

- Let $Y(z)$ be a solution of the *homogeneous* equation

$$Y(z) = a(z)\,Y \circ f(z)$$

Then $X_{n,m}$ simplifies to

$$X_{n,m} = \frac{1}{2\pi i} \oint \frac{b \circ f^m(z)}{Y \circ f^m(z)} Y(z) \frac{dz}{z^{n+1}}$$

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- Needed: existence of $Y(z)$ and analyticity properties
  - Analytic iteration theory (Milnor, Beardon)

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Analytic Iteration Theory

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"Parabolic Linearization Theorem" $\Rightarrow$ conjugacy of $f(z)$ to a shift

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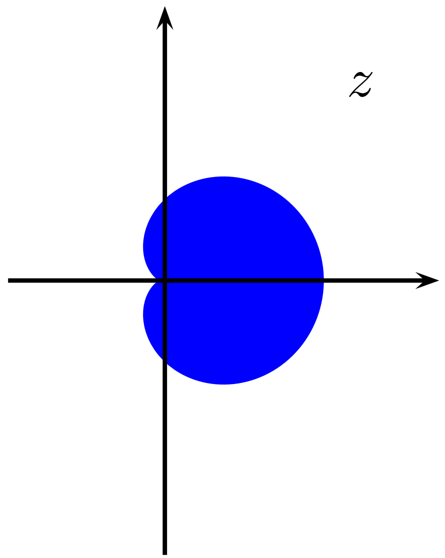
"Parabolic Linearization Theorem" $\Rightarrow$ conjugacy of $f(z)$ to a shift

- $f^{-1}(z)$ exists in cardioid domain and maps contractively into it



Analytic Iteration Theory

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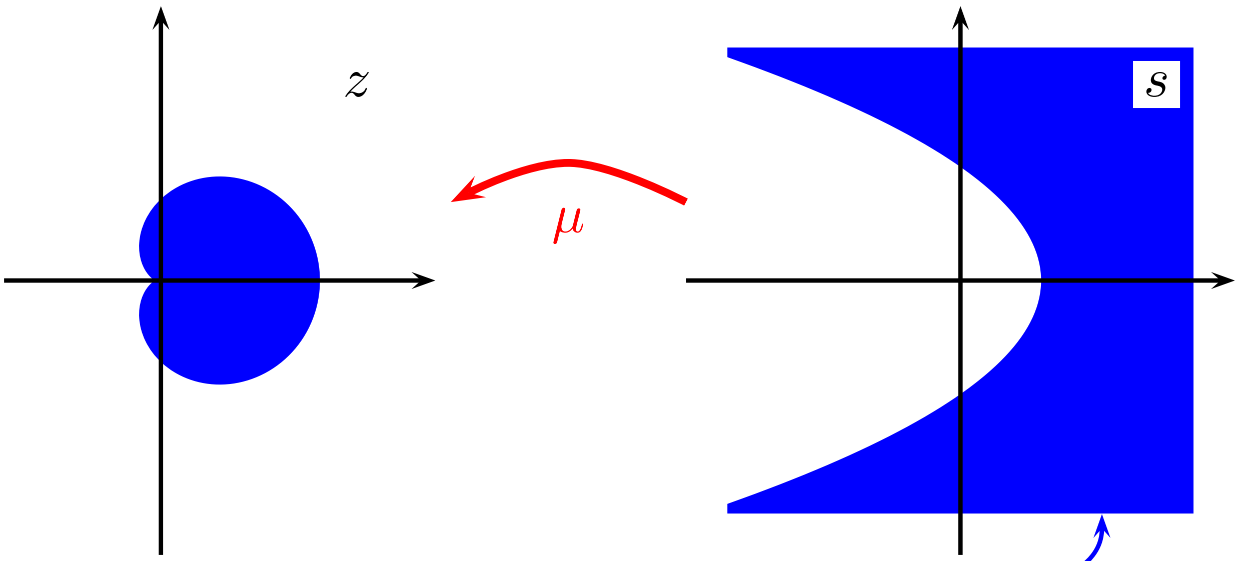

- $f^{-1}(z) = f^{-1} \circ \mu(s) = \mu(s+1)$ for $s \in \mathcal{D}(\mu)$

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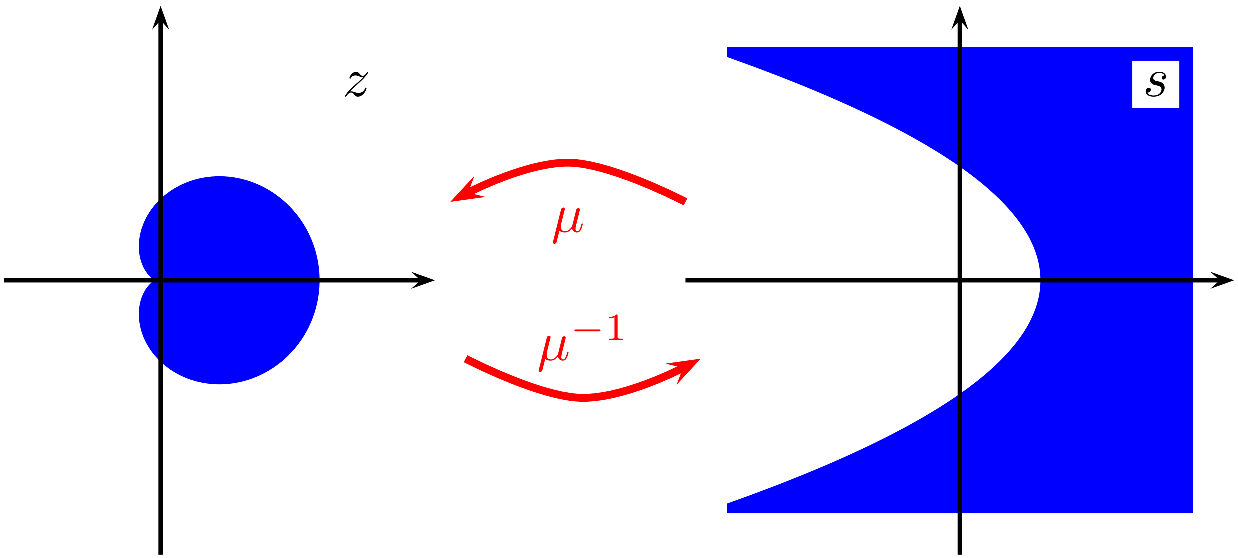

- $f^{-1}(z) = f^{-1} \circ \mu(s) = \mu(s+1)$ for $s \in \mathcal{D}(\mu)$
- $f^k(z) = \mu\left(\mu^{-1}(z) - k\right)$ for z sufficiently small

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Analytic Iteration Theory (ctd.)

- $\mu(s)$ admits a complete asymptotic expansion for $\Re(s) \to \infty$:

$$\mu(s) \sim \frac{1}{cs}\left(1 + \left(1 - \frac{d}{c^2}\right)\frac{\log s}{s} + \sum_{k=2}^{\infty}\sum_{j=0}^{k}\mu_{j,k}\frac{(\log s)^j}{s^k}\right)$$

- $f^{-m} \circ \mu(s) = \mu(s+m)$ admits a complete asymptotic expansion for $m \to \infty$:

$$\mu(s+m) \sim \frac{1}{cm}\left(1 + \left(1 - \frac{d}{c^2} - s\right)\frac{\log m}{m} + \sum_{k=2}^{\infty}\sum_{j=0}^{k}\nu_{j,k}(s)\frac{(\log m)^j}{m^k}\right)$$

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Solution of the Homogeneous Eqn.

- Substitute $z = \mu(s)$:

$$Y(z) = a(z)Y \circ f(z) \quad \Longrightarrow \quad Y \circ \mu(s) = a \circ \mu(s) Y \circ \mu(s-1)$$

- Solution is given by

$$Y \circ \mu(s) = \lim_{n\to\infty} \frac{a \circ \mu(1) a \circ \mu(2) \ldots a \circ \mu(n) \left(a \circ \mu(n)\right)^s}{a \circ \mu(s+1) a \circ \mu(s+2) \ldots a \circ \mu(s+n)}$$

which defines an analytic function in $\mathcal{D}(\mu)$

- Asymptotics as $n \to \infty$:

$$\frac{Y \circ \mu(s+n)}{Y \circ \mu(n)} \sim \left(a \circ \mu(n)\right)^s$$

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The Saddle Point Condition

$$X_n = \sum_{m=0}^{\infty} X_{n,m}\ , \quad X_{n,m} = \frac{1}{2\pi i}\oint \frac{b\circ f^m(z)}{Y\circ f^m(z)} Y(z) \frac{dz}{z^{n+1}}$$

$$\sim \ldots \int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)} \left(a\circ\mu(m)e^{\frac{n}{m}}\right)^s ds$$

- Saddle point at $a\circ\mu(m)e^{\frac{n}{m}} = 1$
- $a(z) = a_k z^k + \ldots,\ \mu(s) \sim (cs)^{-1} \Rightarrow a\circ\mu(m) \sim a_k(cm)^{-k}$
- Different behavior according to
  - $k = 0$:
$$m = -\frac{n}{\log a_0} \qquad 0 < a_0 < 1$$
  - $k \geq 1$:
$$m = \frac{n}{kW(cn/ka_k^{1/k})} \qquad a_k > 0$$

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More tedious calculations...

RESULT 1: Let the FPS $X(z) = \sum_{n=0}^{\infty} X_n z^n$ satisfy

$$X(z) = a(z) X \circ f(z) + b(z)$$

with $f(z) = z + cz^2 + dz^3 + \ldots$, $a(z) = a_0 + a_1 z + \ldots$, and $b(z)$ analytic near zero.

Main Results

RESULT 1: Let the FPS $X(z) = \sum_{n=0}^{\infty} X_n z^n$ satisfy

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If $c > 0$ and $0 < a_0 < 1$ then

$$X_n \sim D n! \, (-c/\log a_0)^n \, n^{(1-\frac{d}{c^2})\log a_0 - 1 + \frac{a_1}{a_0}}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds (-\log a_0)^{-(1-\frac{d}{c^2})\log a_0}$$

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Main Results (ctd.)

RESULT 2: Let the FPS $X(z) = \sum_{n=0}^{\infty} X_n z^n$ satisfy

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Main Results (ctd.)

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with $f(z) = z + cz^2 + dz^3 + \ldots$, $a(z) = a_1 z + \ldots$, and $b(z)$ analytic near zero.
If $c > 0$ and $a_1 > 0$ then

$$X_n \sim D c^n e^{-\frac{1}{2}(1-\frac{d}{c^2}) W(\frac{c}{a_1} n)^2} \sum_{m=0}^{\infty} \frac{m^n}{m!} \left(\frac{a_1}{c}\right)^m$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds e^{\frac{1}{2}(1-\frac{d}{c^2})(\log \frac{a_1}{c})^2}$$

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Applications

Application: Bell Numbers

- Functional equation for OGF:

$$B(z) = \frac{z}{1-z} B\left(\frac{z}{1-z}\right) + 1$$

- $a(z) = \frac{z}{1-z}$, $f(z) = \frac{z}{1-z}$, $b(z) = 1$
- $\mu(s) = 1/s$, $Y \circ \mu(s) = 1/\Gamma(s)$

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insert $c = 1$, $d = 1$, $a_1 = 1$ into

$$B_n \sim D c^n e^{-\frac{1}{2}(1-\frac{d}{c^2}) W(\frac{c}{a_1} n)^2} \sum_{m=0}^{\infty} \frac{m^n}{m!} \left(\frac{a_1}{c}\right)^m$$

with

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- $\mu(s) = 1/s$, $Y \circ \mu(s) = 1/\Gamma(s)$
- Asymptotics:

$$B_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma(s) ds = \frac{1}{e} \quad \text{(sum of residues)}$$

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Application: Ordered Bell Numbers

- Functional equation for OGF:

$$S(z) = \frac{1}{2}\frac{1}{1-z}S\left(\frac{z}{1-z}\right) + \frac{1}{2}$$

- $a(z) = \frac{1}{2}\frac{1}{1-z}$, $f(z) = \frac{z}{1-z}$, $b(z) = \frac{1}{2}$
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insert $c = 1$, $d = 1$, $a_0 = \frac{1}{2}$ $a_1 = \frac{1}{2}$ into

$$S_n \sim D n! \,(-c/\log a_0)^n \, n^{(1-\frac{d}{c^2})\log a_0 - 1 + \frac{a_1}{a_0}}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds (-\log a_0)^{-(1-\frac{d}{c^2})\log a_0}$$

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- $\mu(s) = 1/s$, $Y \circ \mu(s) = 1/\Gamma(s)$
- Asymptotics:

$$S_n \sim Dn!\frac{1}{(\log 2)^n}$$

as $n \to \infty$, where

$$D = \frac{1}{2\log 2}$$

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Application: Partition Lattice Chains

- Functional equation for EGF:

$$Z(z) = \frac{1}{2} Z(e^z - 1) + \frac{z}{2}$$

- $a(z) = \frac{1}{2}$, $f(z) = e^z - 1$, $b(z) = \frac{z}{2}$
- $\mu(s) \sim \frac{2}{s}(1 - \frac{\log s}{3s} + \ldots)$, $Y \circ \mu(s) = 2^s$

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Application: Partition Lattice Chains

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$$Z(z) = \frac{1}{2} Z(e^z - 1) + \frac{z}{2}$$

- $a(z) = \frac{1}{2}$, $f(z) = e^z - 1$, $b(z) = \frac{z}{2}$
- $\mu(s) \sim \frac{2}{s}(1 - \frac{\log s}{3s} + \ldots)$, $Y \circ \mu(s) = 2^s$

insert $c = \frac{1}{2}$, $d = \frac{1}{6}$, $a_0 = \frac{1}{2}$, $a_1 = 0$ into

$$\frac{Z_n}{n!} \sim D n! \, (-c/\log a_0)^n \, n^{(1-\frac{d}{c^2})\log a_0 - 1 + \frac{a_1}{a_0}}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds (-\log a_0)^{-(1-\frac{d}{c^2})\log a_0}$$

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- $a(z) = \frac{1}{2}$, $f(z) = e^z - 1$, $b(z) = \frac{z}{2}$
- $\mu(s) \sim \frac{2}{s}(1 - \frac{\log s}{3s} + \ldots)$, $Y \circ \mu(s) = 2^s$
- Asymptotics:

$$Z_n \sim D(n!)^2 \, (2 \log 2)^{-n} \, n^{-1-\frac{1}{3}\log 2}$$

as $n \to \infty$, where

$$D = \frac{1}{2} (\log 2)^{\frac{1}{3}\log 2} \frac{1}{2\pi i} \int_{\mathcal{C}} 2^s \mu(s) ds = 1.0986858055 \ldots$$

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Application: Takeuchi Numbers

- Functional equation for OGF:

$$T(z) = zC(z)\,T(zC(z)) + \frac{C(z)-1}{1-z}\,, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}$$

- $a(z) = zC(z)$, $f(z) = zC(z)$, $b(z) = \frac{C(z)-1}{1-z}$
- $\mu(s) \sim \frac{1}{s}(1 - \frac{\log s}{s} + \ldots)$, $Y \circ \mu(s) \sim e^{-\frac{1}{2}(\log s)^2}/\Gamma(s)$

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Application: Takeuchi Numbers

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insert $c = 1$, $d = 2$, $a_1 = 1$ into

$$T_n \sim Dc^n e^{-\frac{1}{2}(1-\frac{d}{c^2})W(\frac{c}{a_1}n)^2} \sum_{m=0}^{\infty} \frac{m^n}{m!} \left(\frac{a_1}{c}\right)^m$$

with

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds e^{\frac{1}{2}(1-\frac{d}{c^2})(\log \frac{a_1}{c})^2}$$

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- Asymptotics:

$$T_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!} e^{\frac{1}{2}W(n)^2} = D' B_n e^{\frac{1}{2}W(n)^2}$$

as $n \to \infty$, where

$$D' = \frac{e}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds = 2.2394331040\ldots$$

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- Interesting application of analytic iteration theory and classical complex analysis in the services of asymptotic enumeration

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To be done:

- Computation of the contour integrals determining the constants

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The End