I think it may be time to revisit the following puzzle which has bothered me for several years. We begin with an observation by Boston *et al.* [1].

> Let G be a finite permutation group. Let F_n be the number of orbits of G on n-tuples of distinct elements; let $F(x) = \sum_{n\geq 0} F_n x^n/n!$; and let p_0 be the probability that a random element of G is a derangement. Then
>
> $$p_0 = F(-1).$$

Problem Suppose that $(G^{(m)})$ is a sequence of finite permutation groups which, in some sense, converge to an infinite permutation group G. For my purposes, it will suffice that, for fixed n, the sequence $(F_n(G^{(m)})$ is eventually constant, the final value being $F_n(G)$. Suppose further that the sequence $p_0(G^{(m)})$ tends to a limit c. Then

- what does the constant c tell us about G?
- is there a sense in which $F(G)(-1) = c$?

Example Let $G^{(m)}$ be the symmetric group of degree m. Then $F_n(G^{(m)}) = 1$ if $n \leq m$, and 0 if $n > m$; so for fixed n the sequence is eventually equal to 1. So, as is well known, $F(G^(m))(x)$ is the exponential series truncated to degree m. So the "limit group" G can be taken to be an infinite symmetric group; we have $F(G)(x) = \mathrm{e}^x$ and

$$F(G)(-1) = \mathrm{e}^{-1} = \lim_{m\to\infty} F(G^{(m)})(-1).$$

Example This is the example that I am really interested in. Let $G^{(m)}$ denote the permutation group induced by the symmetric group of degree m on the set of 2-element subsets of $\{1, \ldots, m\}$. Then the sequence $F_n(G^{(m)})$ is equal to the number of graphs with n labelled edges and no isolated vertices for $m \geq 2n$. Thus we can take the limiting group G to be the group induced on 2-sets by an infinite symmetric group. We have

$$\lim_{m\to\infty} p_0(G^{(m)}) = 2\mathrm{e}^{-3/2}.$$

Now the power series $F(G)(x)$ has radius of convergence zero. The problem is, is there a notion of summability for which we can sensibly give the value $2\mathrm{e}^{-3/2}$ to $F(G)(-1)$? And what does this mean?

I think this question is timely because of recent improvements in the asymptotic estimates both for $F_n(G)$ [2] and the proportion of derangements in $G^{(m)}$ [3]. In particular, the number $F_n(G)$ is asymptotically

$$\frac{B_{2n}}{2^n\sqrt{n}}\,\mathrm{e}^{-\left(\frac{1}{2}\log(2n/\log n)\right)^2}$$

where B_{2n} is the Bell number (the number of partitions of $1,\ldots,2n$).

References

[1] N. Boston, W. Dabrowski, T. Foguel, P. J. Gies, J. Leavitt, D. T. Ose and D. A. Jackson, The proportion of fixed-point-free elements of a transitive permutation group, *Commun. Algebra* **21** (1993), 3259–3275.

[2] P. Cameron, T. Prellberg and D. Stark, Asymptotic enumeration of 2-covers and line graphs, preprint.

[3] P. Diaconis, J. Fulman and R. Guralnick, On fixed points of permutations, preprint.