Techniques of Integration

- Basic properties (chapter 5)
- Rules (substitution, integration by parts - today)
- Basic formulas, integration tables, book pages T1-T6 [8-4]
- Procedures to simplify integrals (bag of tricks, methods) [8-6]

this needs practice, practice, practice, ...

exercise class 9, coursework 10,
end-of-term test

Integration by parts

chain rule $\leftrightarrow$ substitution

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du \quad , u = g(x)$$

product rule $\leftrightarrow$?

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$$

integrate:

$$\int \frac{d}{dx}\left[f(x)\,g(x)\right]dx = \int\left[f'(x)\,g(x) + f(x)\,g'(x)\right]dx$$

therefore

$$f(x)\,g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

[8-8]

rewrite to get

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

or, with $u = f(x)$ and $v = g(x)$

$$\int u\, v'\, dx = u\, v - \int u'\, v\, dx$$

and, as $dv = v'\, dx$ and $du = u'\, dx$,

$$\underline{\int u\, dv = u\, v - \int v\, du}$$

(easiest to remember?)

For definite integrals, this becomes

$$\underline{\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x) \Big|_a^b - \int_a^b f'(x)\, g(x)\, dx}$$

Example: evaluate $\int x \cos x \, dx$

- choose $u = x$, $dv = \cos x \, dx$

  then $du = dx$, $v = \sin x$

  therefore $\int u \, dv = uv - \int v \, du$

  gives $\int x \cos x \, dx = x \sin x - \int \sin x \, dx$

  $= x \sin x + \cos x + C$

- other choices of u ?

  e.g. $u = 1$, $dv = x \cos x \, dx$

  $u = \cos x$, $dv = x \, dx$

  $u = x \cos x$, $du = dx$

Which choice is best?

- $u = 1$, $dv = x \cos x \, dx$

  computing v is the same as original problem

  no good!

- $u = \cos x$, $dv = x \, dx$

  $du = -\sin x \, dx$, $v = \frac{x^2}{2}$

  $$\int x \cos x \, dx = \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x \, dx$$

  this makes the problem worse!

- $u = x \cos x$, $du = dx$

  $du = (\cos x - x \sin x) dx$, $u = x$

  $$\int x \cos x \, dx = x^2 \cos x - \int x (\cos x - x \sin x) \, dx$$

  again, this is worse!

<u>Example</u>: evaluate $\int \ln x \, dx$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x} dx \qquad v = x$$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Could we have obtained this by guessing?

$$\frac{d}{dx}[x \ln x] = 1 \cdot \ln x + x \frac{1}{x}$$

$$= \ln x + 1$$

$$= \ln x + \frac{d}{dx} x$$

well, maybe ...

<u>Example</u> repeated integration by parts

$$\int x^2 e^x \, dx =$$

$u = x^2 \qquad dv = e^x dx$

$du = 2x\,dx \qquad v = e^x$

$$= x^2 e^x - 2\int x e^x \, dx =$$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$= x^2 e^x - 2\, x e^x + 2\int e^x dx$$

$$= x^2 e^x - 2x e^x + 2 e^x + C_1$$

<u>Example</u> repeated integration with a "twist"

$$\int e^x \cos x \, dx =$$

$u = e^x \qquad dv = \cos x \, dx$

$du = e^x dx \qquad v = \sin x$

$$= e^x \sin x - \int e^x \sin x \, dx =$$

$u = e^x \qquad dv = \sin x \, dx$

$du = e^x dx \qquad v = -\cos x$

$$= e^x \sin x + e^x \cos x$$

$$- \int e^x \cos x \, dx$$

what now?

We find

$$2\int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

so that

$$\underline{\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C_1}$$

<u>note:</u> don't forget the constant of integration!

Careful: don't forget that

$$\int f(x)\, g(x)\, dx = \int f(x)\, dx \int g(x)\, dx$$

is <u>wrong</u>!!!

<u>Example</u>: a reduction formula

for $\int \cos^n x \, dx$:

$$u = \cos^{n-1} x \qquad dv = \cos x \, dx$$

$$du = (n-1) \cos^{n-2} x \, (-\sin x) \, dx \, , \quad v = \sin x$$

so that

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x$$

$$+ (n-1) \int \cos^{n-2} x \, \sin^2 x \, dx =$$

$$= \cos^{n-1} x \sin x + \overbrace{1 - \cos^2 x}$$

$$+ (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

same trick as above!

$$n\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

This reduces the exponent from n to n-2:

$$\underline{\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx}$$

[similar exam question, 2006]

Application:

$$\int \cos^3 x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\int \cos x\,dx$$

$$= \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C_1$$

<u>Example</u> Finding area [8-9]

between x-axis and $y = xe^{-x}$ from $x=0$ to $x=4$:

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx$$

$$\left[u = x\ ,\ dv = e^{-x}dx\ ;\ du = dx\ ,\ v = -e^{-x}\right]$$

$$= -xe^{-x}\Big|_0^4 - e^{-x}\Big|_0^4$$

$$= -4e^{-4} + \cancel{0e^{-0}} - e^{-4} + e^{-0}$$

$$= 1 - 5e^{-4} \approx 0.91$$

What about area between $x=0$ and $x=\infty$?

$$\int_0^\infty xe^{-x}\,dx = \left(-xe^{-x} - e^{-x}\right)\Big|_0^\infty = 1$$

A more careful treatment will follow shortly!

The method of partial fractions

- If you know that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3} \qquad (*)$$

then you can integrate easily

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

- To obtain $(*)$ we use the method of partial fractions

Let $\frac{f(x)}{g(x)}$ be a rational function, say
$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}$$

- If $\deg(f) \geq \deg(g)$, we first need to use polynomial division and consider remainder term

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

- We also have to know the factors of $g(x)$:

$$x^2-2x-3 = (x+1)(x-3)$$

- Now we can write

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

and obtain $A = 2$, $B = 3$

- General method: [8-12]

<u>Example</u> for distinct linear factors:

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

- $$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

- multiply by $(x-1)(x+1)(x+3)$ to get

$$\begin{aligned} x^2+4x+1 &= A(x+1)(x+3) + B(x-1)(x+3) \\ &\quad + C(x-1)(x+1) \\ &= (A+B+C)x^2 \\ &\quad + (4A+2B+0C)x \\ &\quad + (3A-3B-C) \end{aligned}$$

- equate coefficients:

$$\underset{(1)}{A+B+C=1}\,, \quad \underset{(2)}{4A+2B=4}\,, \quad \underset{(3)}{3A-3B-C=1}$$

- Solving for A, B, C:

$$
\begin{array}{rl}
3A - 3B - C = 1 & (3) \\
A + B + C = 1 & (1) \\
\hline
4A - 2B \quad = 2 & \\
4A + 2B \quad = 4 & (2) \\
\hline
8A \quad = 6 &
\end{array}
$$

$A = 3/4$, $B = 2A - 1 = 1/2$, $C = 1 - A - B = -1/4$

- $\displaystyle\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx$

$$= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+3}$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3|$$

$$+ C_1$$

<u>Example</u> for a repeated linear factor

$$\int \frac{6x+7}{(x+2)^2}\,dx$$

- $\dfrac{6x+7}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$

- $6x+7 = A(x+2) + B$

- $6 = A \quad , \quad 7 = 2A + B$

- $A = 6, \quad B = -5$

- $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx = 6\int \frac{dx}{x+2} - 5\int \frac{dx}{(x+2)^2}$

$$= 6 \ln|x+2| + 5(x+2)^{-1} + C$$

<u>Example</u> for a quadratic factor

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

[x^2+1 is <u>irreducible</u>, cannot be factored in $\mathbb{R}$]

- $$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

- $0 = A+C$, $0 = -2A+B-C+D$,
  $-2 = A-2B+C$, $4 = B-C+D$

- $A=2$, $B=1$, $C=-2$, $D=1$

- $$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1}\,dx - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$
  $$= \ln(x^2+1) + \tan^{-1}(x) - 2\ln|x-1| - (x-1)^{-1} +$$

Example for a repeated quadratic factor

$$\int \frac{dx}{x(x^2+1)^2}$$

- $$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

- $A=1,\ B=-1,\ C=0,\ D=-1,\ E=0$

- $$\int \frac{dx}{x(x^2+1)^2} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \frac{1}{2}\int \frac{2x\,dx}{(x^2+1)^2}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}(x^2+1)^{-1} + C$$

Conceptually easy, but it gets quickly cumbersome!

Trigonometric Integrals

$$\int \sin^m x \cos^n x \, dx$$

for m, n non-negative integers

Method [8-15]

Example

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int \left[1 + \cos 2x - \cos^2 2x - \cos^3 2x\right] dx$$

$$= \ldots = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C_1$$

For the integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx,$$

$$\int \cos mx \cos nx \, dx$$

there is a nice trick: write products of sin, cos as a <u>sum</u> of sin, cos:

$$\sin(m+n)x = \sin mx \cos nx + \cos mx \sin nx$$

$$\sin(m-n)x = \sin mx \cos nx - \cos mx \sin nx$$

$$\sin(m+n)x + \sin(m-n)x = 2 \sin mx \cos nx$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

Similarly,

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

<u>Example</u>:

$$\int \sin 3x \cos 5x \, dx$$

$$= \frac{1}{2} \int \left[\sin (3-5)x + \sin (3+5)x \right] dx$$

$$= \frac{1}{2} \int \sin (-2x) \, dx + \frac{1}{2} \int \sin 8x \, dx$$

$$= -\frac{1}{2} \int \sin 2x \, dx + \frac{1}{2} \int \sin 8x \, dx$$

$$= \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + C$$

For other tricks, see 8.4 and coursework

Trigonometric substitutions

for Integrals containing $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

$x = a\tan\theta$:

$$a^2+x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$$

$x = a\sin\theta$:

$$a^2-x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$$

$x = a\sec\theta$:

$$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$$

Careful with signs when taking the square-root!!

<u>Example</u> Finding the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

[8-24]

In the first quadrant, $y = \frac{b}{a}\sqrt{a^2-x^2}$

$$A = \underset{\substack{\uparrow \\ \text{symmetry}}}{4} \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$x = a \sin\theta \rightsquigarrow a^2 - x^2 = a^2\cos^2\theta$

$dx = a\cos\theta\, d\theta$

$$A = 4 \int_0^{\pi/2} \frac{b}{a}\sqrt{a^2\cos^2\theta}\; a\cos\theta\, d\theta$$

$$= 4ab \underbrace{\int_0^{\pi/2} \cos^2\theta\, d\theta}_{\pi/4} = \underline{\underline{\pi ab}}$$

Improper Integrals

Can we compute areas under infinite curves? [8-50]

Example [8-51]

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b$$

$$= 2 - 2e^{-b/2}$$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty}\left(2 - 2e^{-b/2}\right) = 2$$

Assign to the area the value 2:

$$\int_0^\infty e^{-x/2}\,dx = 2$$

Definitions of Type I improper integrals (infinite limits of integration) [8-52]

Example [8-53]

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

integration by parts

with $$\int_1^b \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1$$

we get

l'Hôspital rule

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = 1$$

Example [8-54]

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1}x \Big|_{a}^{0} + \lim_{b\to\infty} \tan^{-1}x \Big|_{0}^{b}$$

$$= \tan^{-1}0 - \lim_{a\to-\infty} \tan^{-1}a + \lim_{b\to\infty} \tan^{-1}b - \tan^{-1}0$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0$$

$$= \pi$$

For which values of p does

$$\int_1^\infty \frac{dx}{x^p} \text{ converge ?}$$

$$p = 1: \quad \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b = \infty$$

$$p \neq 1: \quad \int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1} \Big|_1^b = \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & p<1 \end{cases}$$

Thus, the integral converges if and only if

$$p > 1$$

Example [8-56]

$$A(a) = \int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_a^1 = 2 - 2\sqrt{a}$$

$$\lim_{a \to 0^+} A(a) = \lim_{a \to 0^+} (2 - 2\sqrt{a}) = 2$$

Assign to the area the value 2:

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2$$

Definition of Type II improper integrals

(functions become infinite at a point) [8-57]

Example [8-58]

$$\int_0^1 \frac{dx}{1-x} = \lim_{b\to 1^-} \int_0^b \frac{dx}{1-x}$$

$$= \lim_{b\to 1^-} -\ln|1-x| \Big|_0^b = \lim_{b\to 1^-} \left(-\ln(1-b)\right)$$

$$= \infty$$

Example [8-59]

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

split up, vertical asymptote at $x=1$!

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big|_0^b$$

$$= \lim_{b\to 1^-} \left[3(b-1)^{1/3} + 3\right] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{a\to 1^+} 3(x-1)^{1/3}\Big|_a^3$$

$$= \lim_{a\to 1^+} \left[3(3-1)^{1/3} - 3(a-1)^{1/3}\right] = 3\cdot 2^{1/3}$$

Therefore $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}$

<u>Careful: a wrong calculation:</u>

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big|_0^3 = \ln 2 - \ln 1$$
$$= \ln 2$$

The integral is <u>improper</u> due to the discontinuity at $x=1$. Using

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

we would have gotten

$$\int_1^3 \frac{dx}{x-1} = \lim_{a\to 1^+} \ln|x-1|\Big|_a^3 = \infty$$

and

$$\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-} \ln|x-1|\Big|_0^b = -\infty$$

Tests for convergence / divergence

Example: does $\int_1^\infty e^{-x^2}\,dx$ converge?

Use that for $x \geq 1$, $e^{-x^2} < e^{-x}$ [8-61]

Therefore

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = e^{-1} - e^{-b} < e^{-1}$$

for all $b > 1$

and $$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx \leq e^{-1}$$

1st Comparison Test [8-62]

Example: (a) $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges, as

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}, \quad \int_1^\infty \frac{dx}{x^2} \text{ converges}$$

(b) $\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}}$ diverges, as

$$\frac{1}{x} \le \frac{1}{\sqrt{x^2-0.1}}, \quad \int_1^\infty \frac{dx}{x} \text{ diverges}$$

2nd Comparison Test [8-63]

Example: (a) $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges, as

$\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges and

$$\lim_{x\to\infty} \frac{1/x^2}{1/(x^2+1)} = \lim_{x\to\infty}\left(\frac{1}{x^2}+1\right) = 1$$

(b) $\displaystyle\int_1^\infty \frac{3}{e^x+5}\,dx$ converges, as

$\displaystyle\int_1^\infty \frac{dx}{e^x}$ converges and

$$\lim_{x\to\infty} \frac{1/e^x}{3/(e^x+5)} = \lim_{x\to\infty}\left(\frac{1}{3}+\frac{5}{3e^x}\right) = \frac{1}{3}$$

Polar coordinates

How can we describe a point P in the plane?

- give x and y coordinates (x, y)

  Cartesian coordinates

  to be precise: we need to first

  1) fix the origin O

  2) fix +ve x-direction

  [details: go back to page 9]

- Alternatively, instead of (x,y), give

  r : distance from origin O

  θ : angle between OP and +ve x-direction [10-60]

(r, θ) polar coordinates

Slight complication: polar coordinates are not unique:

1) angle θ can vary by multiples of 2π

$$\ldots = (r, \theta - 2\pi) = (r, \theta) = (r, \theta + 2\pi) = \ldots \qquad [10\text{-}62]$$

2) if we also allow <u>negative</u> r, then

$$(r, \theta) = (-r, \theta \pm \pi) \qquad [10\text{-}63]$$

Note: sometimes, negative r is excluded, but we will find it useful in calculations.

<u>Example:</u> find all polar coordinates of the point $\left(2, \frac{\pi}{6}\right)$: [10-64]

$r=2$: $\theta = \frac{\pi}{6}, \ \frac{\pi}{6} \pm 2\pi, \ \frac{\pi}{6} \pm 4\pi, \ldots$

$r=-2$: $\theta = \frac{7\pi}{6}, \ \frac{7\pi}{6} \pm 2\pi, \ \frac{7\pi}{6} \pm 4\pi, \ldots$

Some graphs have simple equations in polar coordinates:

1) circle about O: $r = a$

2) line through O: $\theta = \theta_0$

[10-66]

<u>Example</u>

- circle centred at O with radius 1:

$$r = 1 \quad \text{or} \quad r = -1 \qquad (!!)$$

- inequalities [10-67]

(a) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$

(c) $r \le 0$ and $\theta = \frac{\pi}{4}$

(d) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$

Relating polar and cartesian coordinates

[10-68] shows:

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

and $r^2 = x^2 + y^2$

- given (r, θ), we can uniquely compute (x, y)
- given (x, y), we have to choose one of many polar coordinates.

Usual convention: $r \geq 0$

$$0 \leq \theta < 2\pi$$

(if $r = 0$, choose also $\theta = 0$ for uniqueness)

Examples:

Polar	Cartesian
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$ (a lot simpler!)	$x^4 + y^4 + 2x^2y^2 + 2x^3 - 2xy^2 - y^2 = 0$

Converting cartesian to polar

$$x^2 + (y-3)^2 = 9 \qquad [10-71]$$

$$\Leftrightarrow \quad \underbrace{x^2 + y^2}_{} - 6y = 0$$

$$\Leftrightarrow \quad r^2 - 6r\sin\theta = 0$$

$$\Leftrightarrow \quad r = 0 \quad \text{or} \quad \underline{r = 6\sin\theta}$$

Converting polar to cartesian

$$r = \frac{4}{2\cos\theta - \sin\theta} \qquad \Big| \cdot (2\cos\theta - \sin\theta)$$

$$2\underbrace{r\cos\theta}_{x} - \underbrace{r\sin\theta}_{y} = 4$$

$$y = 2x - 4 \qquad \text{straight line!}$$

Graphing in polar coordinates

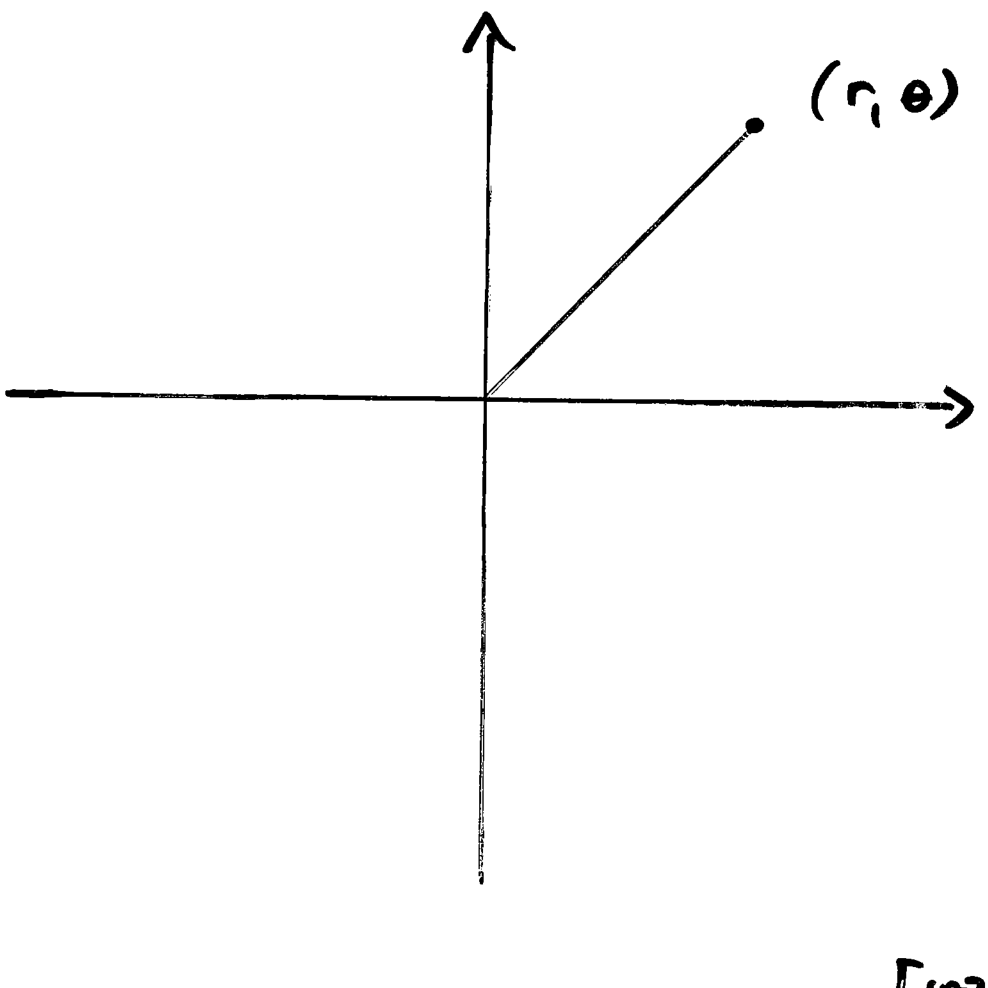
Symmetry:

For (r, θ), find

(i) $(r, -\theta)$

(ii) $(-r, \theta)$

(iii) $(-r, -\theta)$



The slope of a polar curve

- for $r = f(\theta)$, compute the slope:

The slope is still $\frac{dy}{dx}$, so

think of x and y as given by the parameter θ:

$$x(\theta) = f(\theta)\cos\theta$$
$$y(\theta) = f(\theta)\sin\theta$$

then
$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$$

$$\left.\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta\end{aligned}\right\} \quad [10\text{-}75]$$

Example: a cardioid

Graph $r = 1 - \cos\theta$:

- Symmetry:

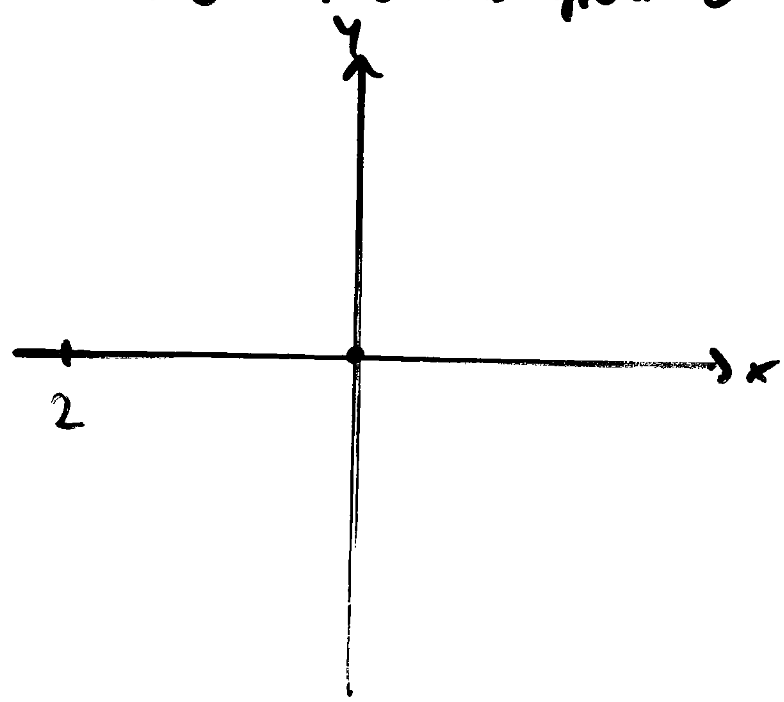
  $\cos\theta = \cos(-\theta)$

  so both (r, θ) and $(r, -\theta)$ on graph

  symmetric about the x-axis

- as θ increases from 0 to π,

  $r = 1 - \cos\theta$ increases from 0 to 2



- horizontal tangents

$$r = f(\theta) = 1 - \cos\theta$$
$$f'(\theta) = \sin\theta$$

$$\frac{dy}{dx} = 0 : \quad f'(\theta)\sin\theta + f(\theta)\cos\theta = 0$$

$$\Leftrightarrow \quad \sin^2\theta + (1-\cos\theta)\cos\theta = 0$$

$$\Leftrightarrow \quad 1 + \cos\theta - 2\cos^2\theta = 0$$

$$\Leftrightarrow \quad \cos\theta = 1 \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$

$$\Leftrightarrow \quad \theta = 0 \quad \text{or} \quad \theta = \pm\frac{2}{3}\pi$$

- vertical tangents:

$$\theta = \pi \quad \text{or} \quad \theta = \pm\frac{\pi}{3}$$