THE EULER-MACLAURIN SUMMATION FORMULA

We derive an expression for the *difference* between a divergent series and an infinite integral:

$$\triangle_0^\infty(f) := \left(\sum_{n=0}^{\infty} f(n)\right) - \int_0^\infty f(x)\,dx.$$

We proceed as Euler [4 , p326 et sui]:
Let $N \in \mathbb{Z}^+$. Then

$$\triangle_0^N(f) := \left(\sum_{n=0}^{N} f(n)\right) - \int_0^N f(x)\,dx =$$

$$1/2[f(N) + f(0)] + \sum_{n=0}^{N-1} 1/2[f(n+1) + f(n)] - \sum_{n=0}^{N-1} \int_n^{n+1} f(x)\,dx =$$

$$1/2[f(N + f(0)] + \sum_{n=0}^{N-1} \{1/2[f(n+1) + f(n)] - \int_n^{n+1} f(x)\,dx\}.$$

Now the $\{\cdots\}$ term above can be re-written by an integration by parts (IBP), so that

$$\{\cdots\} = 1/2[f(n+1) + f(n)] - [xf(x)]_n^{n+1} + \int_n^{n+1} xf'(x)\,dx =$$

$$-(n+1/2)[f(n+1) - f(n)] + \int_n^{n+1} xf'(x)\,dx =$$

$$-(n+1/2)\int_n^{n+1} f'(x)\,dx + \int_n^{n+1} xf'(x)\,dx =$$

$$\int_n^{n+1} (x - n - 1/2)f'(x)\,dx.$$

Hence

$$\triangle_0^N(f) = 1/2[f(N) + f(0)] + \sum_{n=0}^{N-1} \int_n^{n+1} (x - [x] - 1/2)f'(x)\,dx \qquad (1)$$

where $[x]$ is the greatest integer less than x , so that

$$[x] = n \quad \forall\, x \in (n, n+1).$$

Now $S_1(x) := x - [x] - 1/2$ is a *sawtooth* function [5, p280] , and we define the sequence of functions

$$S_0(x),\ S_1(x),\ S_2(x),\ \ldots$$

as follows: we let $S_0(x) := 1$, we let

$$S_k(x) = \int S_{k-1}(x)\,dx \qquad k \in \mathbb{Z}^+ \quad , \qquad (2)$$

and we choose the arbitrary constant of integration in (2) to be such that

$$\int_0^1 S_k(x)\,dx = 0 \quad \forall\, k \geq 1.$$

We defined the functions S_k as above in order to be able to integrate (1) K times by parts, where $K \in \mathbb{Z}^+$, (which assumes f to be K times differentiable),and we chose the constants of integration above so that (S_k) is a DECREASING sequence of functions, in order for our final expression for $\triangle_0^N$ - and therefore for $\triangle_0^\infty$ - to be summable.

We now proceed to integrate (1) once, twice, and by induction go to the integration K times, arriving at a preliminary expression for $\triangle_0^N(f)$ in terms of the functions S_k. The integral in (1) is expanded thus [5]:

$$\int_n^{n+1} S_1(x)f'(x)\,dx = [S_2(x)f'(x)]_n^{n+1} - \int_n^{n+1} S_2(x)f''(x)\,dx$$

where, as in (2),

$$S_2(x) = \int S_1(x)\,dx.$$

S_2 is the integral of 1-periodic S_1 (which has a jump discontinuity at each integer) and

$$\int S_1(x)\,dx = \int x - [x] - 1/2\,dx = x^2/2 - x/2 + c$$

on the interval $[0,1)$ and hence is as above on *each* period of length 1.Hence the behaviour on each period is identical to that on the interval $[0,1]$. So $S_2(0) = S_2(1) = c$ and

$$S_2(n+1) = S_2(n) = S_2(0) = c \qquad \forall\ n \in \mathbb{Z}^+, \qquad (3)$$

ie $S_2(x)$ is *continuous and 1-periodic.* See graphs below (Figure 1, not to scale).

Further integration by parts then gives

$$\int_n^{n+1} S_1(x)f'(x)\,dx = S_2(0)(f'(n+1)-f'(n))-[S_3(x)f''(x)]_n^{n+1}+\int_n^{n+1} S_3 f'''(x)\,dx,$$

due to (3), and where

$$S_3(x) = \int S_2(x)\,dx,$$

so

$$\int_n^{n+1} S_1(x)f'(x)\,dx = S_2(0)(f'(n+1)-f'(n))-S_3(0)(f''(n+1)-f''(n))+\int_n^{n+1} S_3(x)f^{(3)}(x)\,dx,$$

since

$$S_3(n+1) = S_3(n) = S_3(0)\ \forall\ n \in \mathbb{Z}^+, \qquad (4)$$

by a similar argument as for S_2.

Continuing in this way, we obtain, by induction, the following expression for the K-fold integration by parts of (1):

$$\triangle_0^N(f) = 1/2[f(N)+f(0)]+\sum_{n=0}^{N-1}\left(\sum_{k=2}^{K-1}(-1)^k S_k(x)f^{(k)}(x)|_n^{n+1} + (-1)^{K+1}\int_n^{n+1} S_K(x)f^{(K)}(x)\,dx\right),$$

ie

$$\triangle_0^N(f) = 1/2[f(N)+f(0)]+\sum_{k=2}^{K-1}(-1)^k S_k(x)f^{(k)}(x)|_0^N+(-1)^{K+1}\int_0^N S_K(x)f^{(K)}(x)\,dx. \quad (5)$$

(see [4] , p327).

This is our preliminary expression in terms of the functions S_k,as advertised above.We must now ask what explicit form the functions S_k should take.

Suppose ([5] ,p281)

$$\sum_{k=0}^{\infty} S_k(x)t^k = G(x,t) \qquad (0 \le x < 1) \quad (6)$$

Then, since $S'_k(x) = S_{k-1}$ (from (2)), we have

$$\frac{\partial\, G(x,t)}{\partial x} = tG(x,t),$$

since

$$\frac{\partial\, G(x,t)}{\partial x} = \sum_{k=0}^{\infty} S'_k(x)t^k = \sum_{k=0}^{\infty} S_{k-1}(x)t^{k-1}t = tG(x,t).$$

This suggests $G(x,t)$ to be of the form $g(t)e^{xt}$, since

$$\frac{\partial\,(g(t)e^{xt})}{\partial x} = tg(t)e^{xt}.$$

Recalling condition (2), we have

$$\int_0^1 S_k(x)\,dx = 0 \ \ (k \ge 1)$$

so

$$\int_0^1 G(x,t)\,dx = \sum_{k=0}^{\infty}\left(t^k\int_0^1 S_k(x)\,dx\right) =$$

$$t^0\int_0^1 S_0(x)\,dx = \int_0^1 1\,dx = 1.$$

So

$$\int_0^1 g(t)e^{xt}\,dx = 1$$

so

$$\frac{g(t)e^{xt}}{t}|_0^1 = 1$$

ie

$$g(t) = \frac{t}{e^t-1}.$$

Hence

$$\sum_{k=0}^{\infty} S_k(x)t^k = \frac{te^{xt}}{e^t-1}.$$

Now, the *BERNOULLI POLYNOMIALS* are defined by the following expansion

DEFINITION 1

$$\frac{te^{xt}}{e^t - 1} =: \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad \forall x \in \mathbb{R}, |t| < 2\pi. \ (7)$$

Hence

$$S_k(x) = \frac{B_k(x - [x])}{k!}.$$

We have $B_k(x - [x])$ on the rhs since our expansion (7) is then defined for the interval $[0, 1)$ as our construction implies the S_k are polynomials of degree k in the interval [0,1) ([5] , p281).

DEFINITION 2

So let ([4],p327)

$$S_k(x) = \frac{B_k(x)}{k!} \quad , k \in \mathbb{Z}^+ \qquad (8)$$

on $(0, 1)$ *and 1-periodic thereafter.*

We have now defined our functions S_k explicitly in terms of the standard Bernoulli polynomials,whose useful and relevant properties we now explore.

First, we prove the following

PROPOSITION 1

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!} \ \forall \ k \geq 1.$$

Proof

We have

$$\frac{te^{xt}}{e^t - 1} =: \sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!}$$

Differentiating wrt x gives

$$t\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k'(x) t^k}{k!},$$

so

$$t \sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!} = \sum_{k=0}^{\infty} \frac{B_k'(x) t^k}{k!},$$

so

$$\sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!} = \sum_{k=0}^{\infty} \frac{B_k'(x) t^{k-1}}{k!}.$$

But
$$\sum_{k=1}^{\infty}\frac{B_{k-1}(x)t^{k-1}}{(k-1)!}=\sum_{k=0}^{\infty}\frac{B_k(x)t^k}{k!}=\sum_{k=0}^{\infty}\frac{B_k'(x)t^{k-1}}{k!}.$$

Hence equating coefficients of t in the above expression gives us
$$\frac{B_k'(x)}{k!}=\frac{B_{k-1}(x)}{(k-1)!}\ \forall\ k\geq 1\quad QED.$$

Next we introduce the *BERNOULLI NUMBERS* $B_k := B_k(0)$. We prove that

PROPOSITION 2
$$B_k(x)=\sum_{j=0}^{k}\binom{k}{j}B_{k-j}\,x^j\ \ \forall\ k\geq 0.$$

Proof

We have
$$\frac{te^{0t}}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k(0)t^k}{k!}$$
so
$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_kt^k}{k!},$$
so
$$e^{xt}\sum_{k=0}^{\infty}\frac{B_kt^k}{k!}=\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k(x)t^k}{k!}.$$

Proposition 2 is most easy to see if we expand the above expression ([5],p282).We have:
$$\left(1+xt+\frac{(xt)^2}{2!}+\cdots\right)\left(B_0+B_1t+\frac{B_2t^2}{2!}+\cdots\right)=B_0(x)+B_1(x)t+\frac{B_2(x)t^2}{2!}+\cdots$$

We see that *each* $B_k(x)$ on the rhs must be equal to a sum of terms on the lhs where, in *each* such term, k is the *only* power of t, since only then can we cancel this power of t from both sides of the expression for $B_k(x)$. We see that
$$B_0(x)=B_0,\ \ B_1(x)t=B_1t+B_0xt,\ \ B_2(x)t^2=B_0x^2t^2+2B_1xt^2+B_2t^2,\ \ \cdots$$

and for each $k\in\mathbb{Z}^+$
$$\frac{B_k(x)t^k}{k!}=\frac{B_kt^k}{k!}+\frac{B_{(k-1)}t^{(k-1)}}{(k-1)!}(xt)+\frac{B_{(k-2)}t^{(k-2)}}{(k-2)!}\frac{(xt)^2}{2!}+\cdots+\frac{B_0(xt)^k}{k!}$$

remembering our multiplication remarks above.Hence,
$$B_k(x)=B_k+B_{(k-1)}kx+\frac{B_{(k-2)}x^2k!}{(k-2)!2!}+\cdots+B_0x^k$$

ie
$$B_k(x)=\sum_{j=0}^{k}\binom{k}{j}B_{k-j}\,x^j\ \ \forall\ k\geq 0\ \ QED.\qquad(9)$$

So far so good, but (9) is only useful if we know what the Bernoulli numbers are! We now derive a recursion for the Bernoulli numbers:

PROPOSITION 3

$$B_{k-1} = \frac{-1}{k} \sum_{j=0}^{k-2} \binom{k}{j} B_j \qquad (k \geq 2)$$

Proof

(7) is unchanged ([5],282) if we simultaneously replace x and t by $1-x$ and $-t$ respectively, hence

$$B_k(1-x) = (-1)^k B_k(x). \quad (10)$$

Moreover, apart from B_1 , all the Bernoulli numbers with an odd suffix are equal to zero. This is because

$$\frac{t}{e^t - 1} + t/2 = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) + t/2$$

is an *even* function of t , so odd powers of $-t$ must vanish from the series on the rhs of the above expression, (as the function on lhs above is ≥ 0 for positive t) beyond a certain value of k.(10) becomes:

$$B_k(1-0) = (-1)^k B_k(0)$$

ie

$$B_k(1) = B_k \;\; (k \geq 2). \quad (11)$$

So let $x = 1$ in (9) and we have

$$B_k(1) = B_k = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}$$

so

$$B_k = \sum_{j=0}^{k} \binom{k}{k-j} B_{k-j} = \sum_{j=0}^{k} \binom{k}{j} B_j,$$

by first recalling that $\binom{k}{j} = \binom{k}{k-j}$, and secondly just re-labelling $(k-j)$ as j .

So

$$B_k = \binom{k}{0} B_0 + \binom{k}{1} B_1 + \cdots + \binom{k}{k} B_k$$

so

$$B_k - \binom{k}{k} B_k = 0 = \binom{k}{0} B_0 + \cdots + \binom{k}{k-1} B_{k-1},$$

so

$$-\binom{k}{k-1} B_{k-1} = \sum_{j=0}^{k-2} \binom{k}{j} B_j,$$

which gives us the promised recursion for the B_k as

$$B_{k-1} = -1/k \sum_{j=0}^{k-2} \binom{k}{j} B_j \qquad (k \geq 2).\ \ QED \quad (12)$$

Hence,recalling that $B_0 = 1$, we have ([5] p282)

$$B_1 = -1/2,\ \ B_2 = 1/6,\ \ B_3 = 0,\ \ B_4 = -1/30, \cdots$$

So,by (11), we have that the Bernoulli polynomials are 1-periodic,as $B_k(0) = B_k = B_k(1), k \geq 2$.*They are continuous as they are polynomials,and we will now see that successive Bernoulli polynomials have increasing powers of* x,*and so, on* $[0, 1)$,*this means that they also fulfil our requirement that the* S_k *be a* DECREASING *sequence of functions:*

So, from Proposition 2, the first seven Bernoulli polynomials are :

$$B_0(x) = 1,\ \ B_1(x) = x-1/2,\ \ B_2(x) = x^2-x+1/6,\ \ B_3(x) = x^3-(3/2)x^2+(1/2)x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - 1/30,\ \ B_5(x) = x^5 - (5/2)x^4 + (5/3)x^3 - (1/6)x,$$

$$B_6(x) = x^6 - 3x^5 + (5/2)x^4 - (1/2)x^2 + 1/42$$

Hence, from all the above discussion of Bernoulli polynomials, (5) now becomes:

$$\triangle_0^N(f) = 1/2[f(N)+f(0)] + \sum_{k=2}^{K-1} \frac{(-1)^k B_k(x-[x]) f^{(k)}(x)|_0^N}{k!} + (-1)^{K+1} \int_0^N \frac{B_K(x-[x]) f^{(K)}(x)}{K!}\, dx.$$

From our remarks about Bernoulli numbers of odd suffix vanishing for B_3 and beyond, and remembering that the second term above is an integration between *integer* values, recall $B_k(0) := B_k$ so we take the Bernoulli term out as a Bernoulli number, so we have

$$\triangle_0^N(f) = 1/2[f(N)+f(0)] + \sum_{m=1}^{M(K)} \frac{B_{2m}}{(2m)!} \left[f^{(2m-1)}(x)\right]_0^N + (-1)^{K+1} \int_0^N \frac{B_K(x-[x]) f^{(K)}(x)}{K!}\, dx, \qquad (13)$$

where

$$M(K) := K/2\,,\ \ K\ \ even,\ \ and\ \ M(K) := (K-1)/2\,,\ \ K\ \ odd.$$

It will be noticed that we have kept only the odd derivatives of f.This is because, in the above summation, we would lose either all even or all odd derivatives of f , since all odd Bernoulli numbers vanish above B_1. So we choose to keep the odd derivatives. This condition will preserve the first derivative of f.

By the Mean Value Theorem for Integrals ([6], 213) we have

$$(-1)^{K+1} \int_0^N \frac{B_K(x-[x]) f^{(K)}(x)}{K!}\, dx = (-1)^{K+1} \frac{B_K(z-[z]) f^{(K)}(z)}{K!}\, [N-0] := R_K(z)$$

where $z \in (0, N)$ ([4] ,328).

So we have

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$$\triangle_0^N(f) = 1/2[f(N)+f(0)] + \sum_{m=1}^{M(K)} \frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(x)\right]_0^N + (-1)^{K+1} N \frac{B_K(z-[z])f^{(K)}(z)}{K!} \quad , \tag{14}$$

where $M(K)$ and z are as before.

So far so good, but we only have an expression for the difference between a *finite* sum and a *finite* integral. We must now let $N \to \infty$.

Let $f(x) = f_{\lambda,r}(x) = x^r e^{-\lambda x}$ in (13) and (14),where $r \in \mathbb{Z}^+, \lambda \in \mathbb{R}$, and choose $K > r+1$.Recalling the Leibniz rule:

$$f_{\lambda,r}^{(m)}(x) = \sum_{i=0}^{m} \binom{m}{i} \left(\frac{d^i}{dx^i} x^r\right)\left(\frac{d^{m-i}}{dx^{m-i}} e^{-\lambda x}\right)$$

$$= \sum_{i=0}^{m} \binom{m}{i} (r)_i \;\; x^{r-i}(-\lambda)^{m-i} e^{-\lambda x} \tag{15}$$

where

$$(r)_i := r(r-1)\cdots(r-i+1).$$

So in particular

$$f_{\lambda,r}^{(m)}(0) = 0, \;\; m < r,$$

and

$$f_{\lambda,r}^{(m)}(0) = (m)_r(-\lambda)^{m-r}, \;\; m \geq r,$$

since each term of (15) will be zero when $m < r$, for $x = 0$, as each term will include $x = 0$ raised to a non-zero power, and, for $m \geq r$, the only non-zero term will be when $i = r$, so then that term will be

$$\binom{m}{r}(r)_r(-\lambda)^{m-r} = (m)_r(-\lambda)^{m-r}.$$

Now we can let $N \to \infty$ and note that

$$\lim_{N\to\infty} f(N) = \lim_{N\to\infty} N^r e^{-\lambda N} = 0 \quad \forall \lambda > 0 \;\; ,$$

so the limit at infinity of $f(N)$ is zero,hence the same is true of $f'(N), f^3(N), ...f^{(2m-1)}(N), ...$ .Hence (13) becomes

$$\triangle_0^\infty(f_{\lambda,r}) = \frac{1}{2}\delta_{0r} - \sum_{m=1}^{M(K)} \frac{B_{2m}}{(2m)!}(2m-1)_r(-\lambda)^{2m-1-r} + (-1)^{K+1}\int_0^\infty \frac{B_K(x-[x])f^{(K)}(x)}{K!}\,dx. \tag{16}$$

where we have used the Kronecker delta , δ_{ij},which is equal to 1 if $i = j$ and is zero otherwise, since 0^0 is not well-defined.

We need to put a bound on the integral term in (16), to finally ensure that our expression for $\triangle_0^\infty$ converges:

If we call this integral the *remainder term* $R_K(f_{\lambda,r})$,and recall that the integral of the second Bernoulli polynomial (of $x-[x]$) and beyond is zero on every interval of length 1 ,as we saw at the start, we have,by the Cauchy-Schwarz inequality for integrals ([7],559),

$$|R_K(f_{\lambda,r})| \le \frac{|B_K|}{K!}\int_0^\infty |f_{\lambda,r}^{(K)}(x)|\,dx \ \ ,$$

which, recalling the Leibniz rule again, becomes

$$|R_K(f_{\lambda,r})| \le \frac{|B_K|}{K!}\sum_{k=0}^{K}\binom{K}{k}(r)_k\lambda^{K-k}\int_0^\infty x^{r-k}e^{-\lambda x}\,dx \ \ .(17)$$

We are now in a position to prove the following basic lemma, to enable $\triangle_0^\infty$ to be calculated for a wide class of functions.

LEMMA

$$\triangle_0^\infty(x^r) = \frac{-B_{r+1}}{(r+1)} \quad , \quad r\in\mathbb{Z}^+$$

Proof

We use the *GAMMA FUNCTION* ([7],636) :

$$\Gamma(n) := \int_0^\infty y^{n-1}e^{-y}\,dy \qquad n\in\mathbb{R}^+,$$

and we let $y=\lambda x$, so that

$$\Gamma(n) = (\lambda)^n\int_0^\infty x^{n-1}e^{-\lambda x}\,dx \ \ .$$

Hence

$$\Gamma(r-k+1) = \lambda^{r-k+1}\int_0^\infty x^{r-k}e^{-\lambda x}\,dx \ \ .$$

But ([7],636) $\Gamma(n+1)=n!$, so

$$\int_0^\infty x^{r-k}e^{-\lambda x}\,dx = (r-k)!\ \ \lambda^{k-r-1}.$$

Thus, referring back to (17), we see that

$$\lim_{\lambda\to 0}R_K(f_{\lambda,r}) = \lim_{\lambda\to 0}\frac{|B_K|}{(K)!}r!\lambda^{K-r-1}\sum_{k=0}^{K}\binom{K}{k} = 0$$

since we chose $K>r+1$, and hence, in the limit as $\lambda\to 0$, (16) becomes

$$\lim_{\lambda\to 0}\triangle_0^\infty(f_{\lambda,r}) = \frac{1}{2}\delta_{0r} - \sum_{m=1}^{M(K)}\frac{B_{2m}}{(2m)!}(2m-1)_r\delta_{2m-1,r}$$

ie

$$\triangle_0^\infty(x^r) = \frac{-B_{r+1}}{(r+1)} \ ,r\in\mathbb{Z}^+ \qquad QED. \qquad (18)$$

Referring back to our very first expression for $\triangle_0^\infty$,we see that any finite linear combination of the x^r in (18) can be expressed as a finite linear combination of Bernoulli numbers. Hence $\triangle_0^\infty$ for any polynomial $p(x) = \sum_{r=0}^{r=deg(p)} a_r x^r$,where $d = deg(p)$ is the *degree* of the polynomial, can be expressed as a finite series of Bernoulli numbers:

$$\triangle_0^\infty(p(x)) = \sum_{r=0}^{d} a_r \frac{(-B_{r+1})}{(r+1)} \qquad (19)$$

where $a_r \in \mathbb{N}_0 := \{0, 1, 2, 3, ...\}$.

We now determine if we can form an expression for

$$\triangle_0^\infty(f(x)e^{-\lambda x})$$

where

$$f(x) = \sum_{r=0}^{\infty} a_r x^r \quad , r \in \mathbb{N}_0$$

ie $f(x)$ is a *real analytic function*.

We proceed similarly to our derivation for x^r:Let $f_\lambda(x) = (\sum_0^\infty a_r x^r)e^{-\lambda x}$. So we have, by the Leibniz rule again,

$$f_\lambda^{(m)}(x) = \sum_{i=0}^{m} \binom{m}{i} \left(\frac{d^i}{dx^i}\sum_{r=0}^{\infty} a_r x^r\right)\left(\frac{d^{m-i}}{dx^{m-i}} e^{-\lambda x}\right)$$

$$= \sum_{i=0}^{m} \binom{m}{i} \left(\sum_{r=0}^{\infty} (r)_i a_r x^{r-i}\right) (-\lambda)^{m-i} e^{-\lambda x} \quad (20)$$

So

$$f_\lambda^{(m)}(0) = \sum_{i=0}^{m} (m)_i a_i (-\lambda)^{m-i}$$

by similar reasoning as previously, noting that, in (20), for $x = 0$, the only non-zero terms in the infinite series will be when $i = r$.

We require

$$\lim_{N\to\infty} \left(\sum_{r=0}^{\infty} a_r N^r\right) e^{-\lambda N} = 0$$

for the same reason we required $\lim_{N\to\infty} f(N)$ to be zero on p8.

So we simply specify that our function f, although possibly increasing, does NOT increase as fast, or faster than, the exponential function. This will still leave us with an abundant class of real analytic functions for which we will be able to calculate $\triangle_0^\infty$.

We have, by linearity of $\triangle_0^N$,

$$\triangle_0^N(f_\lambda(x)) = \sum_{r=0}^{\infty} a_r \triangle_{0,k(r)}^N (e^{-\lambda x} x^r)$$

where the extra subscript $k(r)$ has been added to $\triangle_0^N$ to register the fact that ,for a real analytic function, the derivation of the expression for $\triangle_0^N$ involves integration of *each and every term of the power series expansion of that function*.That is, k can be chosen as a *function* of r.

So (13) becomes

$$\triangle_0^N(f_\lambda(x)) =$$

$$\sum_{r=0}^{\infty} a_r \left[\frac{1}{2} N^r e^{-\lambda N} + \frac{1}{2}\delta_{0r} + \sum_{k=1}^{K(r)-2} (-1)^k \frac{B_{k+1}}{(k+1)!} \left[f_\lambda^{(k)}(x) \right]_0^N \right] +$$

$$\sum_{r=0}^{\infty} a_r \left[(-1)^{K(r)+1} \int_0^N \frac{B_{K(r)}(x-[x])}{K(r)!} f^{K(r)}(x)\, dx \right] \quad (21)$$

The third,summation, term needs explaining.Our new labelling includes all the odd Bernoulli numbers from B_3 onwards, which will not, of course, affect the outcome as these are all zero.The 2 in the denominator of $M(K)$,in (13), is not included for the same reason, and we see below why we sum to $K(r) - 2$.

We now introduce the summations for the Leibniz expression for the derivatives in the summation and in the integral terms,and in the summation term we sum up to r,by setting $K(r) = r + 2$, since $r = K(r) - 2$ is the maximum number of times we differentiate each $a_r x^r e^{-\lambda x}$ term, by the Leibniz rule, in our power series expansion for $f_\lambda(x)$.We also recall how we required $K > r + 1$ in our basic Lemma ,p9.So (21) becomes

$$\triangle_0^N(f_\lambda(x)) =$$

$$\sum_{r=0}^{\infty} a_r \left[\frac{1}{2} N^r e^{-\lambda N} + \frac{1}{2}\delta_{0r} + \sum_{m=1}^{r} (-1)^m \frac{B_{m+1}}{(m+1)!} \sum_{i=0}^{m} \binom{m}{i} (r)_i x^{r-i} (-\lambda)^{m-i} e^{-\lambda x} \Bigg|_0^N \right] +$$

$$\sum_{r=0}^{\infty} a_r \left[(-1)^{r+1} \int_0^N \frac{B_{r+2}(x-[x])}{(r+2)!} \sum_{i=0}^{r+2} \binom{r+2}{i} (r)_i x^{r-i} (-\lambda)^{r+2-i} e^{-\lambda x}\, dx \right]. \quad (22)$$

So when $i = m = r$,the 'rightmost' term from the middle, summation, portion of (22), for $x = 0$, is

$$-(-1)^r \frac{B_{r+1}}{(r+1)!} (r)_r = -(-1)^r \frac{B_{r+1}}{(r+1)},$$

where the left-most negative signs are since $x = 0$ is the lower limit of integration. The only non-zero Bernoulli numbers above B_1 are *even*, so for non-zero Bernoulli numbers ,$r+1$ must be even, so r must be odd, so $(-1)^r = -1$.So this rightmost term is

$$\frac{B_{r+1}}{(r+1)}$$

We must now see if we can take $\lim N \to \infty$ and $\lim \lambda \to 0$,in (22), in that order. Considering the top line of (22), and recalling our stipulation, p10, that our function does not grow as fast, or faster than, the exponential function, we re-write (22) as

$$\triangle_0^\infty(f_\lambda(x)) = 0 + \frac{a_0}{2} + \lim_{N\to\infty}\sum_{r=0}^{\infty} a_r \sum_{m=1}^{r}(-1)^m \frac{B_{m+1}}{(m+1)!}\sum_{i=0}^{m}\binom{m}{i}(r)_i(-\lambda)^{m-i}x^{r-i}e^{-\lambda x}\Bigg|_0^N +$$
$$\lim_{N\to\infty}\sum_{r=0}^{\infty} a_r\left[(-1)^{r+1}\int_0^N \frac{B_{r+2}(x-[x])}{(r+2)!}\sum_{i=0}^{r+2}\binom{r+2}{i}(r)_i x^{r-i}(-\lambda)^{r+2-i}e^{-\lambda x}\,dx\right]. \quad (23)$$

It will help to write out (23) more explicitly so we can see more clearly what we must do:

$$\triangle_0^\infty(f_\lambda(x)) = \frac{a_0}{2} + \lim_{N\to\infty}\sum_{r=0}^{\infty} a_r\left[(-1)^1\frac{B_2}{2!} + (-1)^2\frac{B_3}{3!} + \cdots + (-1)^r\frac{B_{r+1}}{(r+1)!}\right]$$
$$\left[\binom{m}{0}(r)_0(-\lambda)^m + \binom{m}{1}(r)_1(-\lambda)^{m-1} + \cdots + \binom{m}{m}(r)_m(-\lambda)^0\right]\left[N^{r-i}e^{-\lambda N} - 0^{r-i}\right] +$$
$$\sum_{r=0}^{\infty} a_r\left[(-1)^{r+1}\int_0^\infty \frac{B_{r+2}(x-[x])}{(r+2)!}\sum_{i=0}^{r+2}\binom{r+2}{i}(r)_i x^{r-i}(-\lambda)^{r+2-i}e^{-\lambda x}\,dx\right]. \quad (24)$$

Recalling (7), p4, we see that

$$\frac{1}{e-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!} \quad .$$

Hence

$$\left|(-1)\frac{B_2}{2!} + (-1)^2\frac{B_3}{3!} + \cdots + (-1)^r\frac{B_{r+1}}{(r+1)!}\right| \le \left|\frac{1}{e-1}\right| \quad \forall r \quad .$$

Since we assumed that $f(x)$ does not grow as fast, or faster than, the exponential function, and hence that $\lim_{N\to\infty} f(x) = 0$, we make the *SAME ASSUMPTION ABOUT ALL THE DERIVATIVES OF* f.This then means we can take $\lim_{N\to\infty}$ of the first half of (24), and it is zero: