MTH5105 Differential and Integral Analysis 2010-2011

Solutions 6

1 Exercise for Feedback

1) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Show that f is uniformly continuous.

[Hint: Use that if $|f'(x)| \leq M$ for all $x \in \mathbb{R}$ then $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$ (similar to Exercise sheet 2).]

(b) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ and $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$. Prove or disprove:

(i) f is uniformly continuous.

(ii) g is uniformly continuous.

(iii) fg is uniformly continuous.

(iv) $x \mapsto \begin{cases} g(x)/f(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$ is uniformly continuous.

Solution:

(a) To say f' is bounded means that there exists an $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Hence, as shown in exercise sheet 2,

$$\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq M|x - y| \ .$$

Now given $\epsilon > 0$ choose $\delta = \epsilon/M$. Then

$$\forall x, y \in \mathbb{R} : |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon \ .$$

(b) (i) TRUE: f is uniformly continuous.
As $f'(x) = 1$, $|f'(x)| \leq M$ with $M = 1$, so f has bounded derivative, and the assertion follows from (a).

(ii) TRUE: g is uniformly continuous.
As $g'(x) = \cos(x)$, $|g'(x)| \leq M$ with $M = 1$, so g has bounded derivative, and the assertion follows from (a).

(iii) FALSE: fg is not uniformly continuous.
As $(fg)'(x) = x\cos(x) + \sin(x)$, g' is not bounded. This alone is no proof, but it indicates that the reason for non-uniformity is that at $x = 2n\pi$ we find $(fg)'(x) = 2n\pi$ which is arbitrarily large.
To turn this into a proof, we repeat the strategy from the example in the lecture: Given $\delta > 0$ we need to pick $x_n, y_n \in \mathbb{R}$ with $|x_n - y_n| < \delta$ but $|x_n \sin(x_n) - y_n \sin(y_n)| \geq 1$. Taking $x_n = 2n\pi$ and $y_n = 2n\pi + \delta'$, we estimate

$$|x_n \sin(x_n) - y_n \sin(y_n)| = |(2n\pi + \delta'/2)\sin(\delta'/2)| \geq 4n\delta' \ .$$

In the last step we need $\delta' \leq \pi/2$. Therefore, if we choose $\delta' = \min(1/4n, \pi/2)$ then $|x_n - y_n| < \delta$ and $|x_n \sin(x_n) - y_n \sin(y_n)| \geq 1$.

(iv) TRUE: $x \mapsto h(x) = \begin{cases} f(x)/g(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$ is uniformly continuous.

For $x \neq 0$ the quotient rule gives $h'(x) = (g/f)'(x) = (x\cos(x) - \sin(x))/x^2$, and for $x = 0$ we get

$$h'(0) = \lim_{x\to 0} \frac{\sin(x)/x - 1}{x} = \lim_{x\to 0} \frac{\sin(x) - x}{x^2} = \lim_{x\to 0} \frac{\cos(x) - 1}{2x} = \lim_{x\to 0} \frac{-\sin(x)}{2} = 0 .$$

As $\lim_{x\to 0} h'(x) = \lim_{x\to 0} \dfrac{x\cos(x) - \sin(x)}{x^2} = \lim_{x\to 0} \dfrac{\cos(x) + x\sin(x) - \cos(x)}{2x} = 0 = h'(0)$, h' is continuous and hence bounded on $[-L, L]$ for any $L > 0$. Additionally, if $|x| > L$ we estimate $|h'(x)| \leq |\cos(x)/x| + |\sin(x)/x^2| < 1/L + 1/L^2$. Therefore h' is bounded on $\mathbb{R}$, and the assertion follows from (a).

2 Extra Exercises

2) Let $f : (0, 1) \to \mathbb{R}$ be continuous. Show that

a) f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

*b) If f is uniformly continuous then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

Solution:

a) If $A = \lim_{x\to 0} f(x)$ and $B = \lim_{x\to 1} f(x)$ exist, then the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} A & x = 0 \\ f(x) & 0 < x < 1 \\ B & x = 1 \end{cases}$$

is continuous on $[0, 1]$ and therefore uniformly continuous on $[0, 1]$. The function f is a restriction of g to the smaller interval $(0, 1)$ and therefore also uniformly continuous.

*b) This part is considerably harder. We start by showing that f is bounded. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any two points $x, y \in (0, 1)$ that are less than distance δ apart. Now two arbitrary points $u, v \in (0, 1)$ are less than distance one apart and can therefore be connected by a chain of $n = \lfloor 1/\delta \rfloor$ points such that two consecutive points are less than distance δ apart. Therefore, $|f(u) - f(v)| < (n + 1)\varepsilon$ is finite and f must be bounded.

We now show that $\lim_{x\to 0} f(x)$ exists (the case of $x \to 1$ is completely analogous). As we have established that f is bounded, we know that for $0 < \delta < 1$

$$a(\delta) = \inf\{f(x) : 0 < x < \delta\} \quad \text{and} \quad b(\delta) = \sup\{f(x) : 0 < x < \delta\}$$

are well-defined, bounded functions of δ. Moreover, $a(\delta)$ increases as $\delta \to 0$ and $b(\delta)$ decreases as $\delta \to 0$. As $a(\delta) \leq b(\delta)$, both

$$a = \lim_{\delta\to 0} a(\delta) \quad \text{and} \quad b = \lim_{\delta\to 0} b(\delta)$$

exist. If we can show that $a = b$ then it follows that $\lim_{x\to 0} f(x) = a$. We bound

$$\begin{aligned} b(\delta) - a(\delta) &= \sup\{f(x) : 0 < x < \delta\} + \sup(-f(y) : 0 < y < \delta\} \\ &= \sup\{f(x) - f(y) : 0 < x, y < \delta\} \leq \varepsilon , \end{aligned}$$

so that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$b(\delta) \leq a(\delta) + \varepsilon .$$

But this implies that $b(\delta) \leq a(\delta)$, whence equality follows.

3) Let $\alpha \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha & x \in \{1/k; k \in \mathbb{N}\} \ , \\ 0 & \text{else.} \end{cases}$$

For which values of α is f Riemann-integrable? If f is Riemann-integrable, what is the value of $\int_0^1 f(x)\,dx$?

Solution:

If $\alpha < 0$ then f is unbounded and therefore not Riemann-integrable.

Let now $\alpha \geq 0$, so that f is bounded by 1.

As f is zero on all irrational numbers, $L(f,P) = 0$ for all $P \in \mathcal{P}$, and thus

$$\int_{*0}^1 f(x)\,dx = 0 \ .$$

Consider the partition of $[0,1]$ by

$$P_n = \{0, n/n^2, (n+1)/n^2, \ldots, (n^2-1)/n^2, n^2/n^2\}$$

into one interval of width $1/n$ and $n^2 - n$ intervals of width $1/n^2$. (Many other choices would work here, as well.)

For $x \geq 1/n$, $f(x)$ is non-zero at precisely n points, so that $\sup_{x\in I_i} f(x)$ is non-zero on the left-most interval of width $1/n$ and at most $2n$ intervals of width $1/n^2$. Thus,

$$U(f,P_n) \leq \frac{1}{n} + 2n\frac{1}{n^2} = \frac{3}{n} \ .$$

We thus have

$$0 = L(f,P_n) \leq U(f,P_n) \leq \frac{3}{n}$$

so that the f is Riemann-integrable and $\int_0^1 f(x)\,dx = 0$.

4) Let $f : [a,b] \to \mathbb{R}$ be Riemann-integrable and $c \in \mathbb{R}$.

(a) Given a partition P of $[a,b]$, show that

$$U(cf,P) - L(cf,P) \leq |c|(U(f,P) - L(f,P)) \ .$$

(b) Deduce from (a) that cf is integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$. *[This completes the proof of Theorem 7.4.]*

Solution:

(a) For $c \geq 0$,

$$\sup_{x\in I_i} cf(x) = c \sup_{x\in I_i} f(x) \quad \text{and} \quad \inf_{x\in I_i} cf(x) = c \inf_{x\in I_i} f(x) \ ,$$

so that

$$\sup_{x\in I_i} cf(x) - \inf_{x\in I_i} cf(x) = c\left(\sup_{x\in I_i} f(x) - \inf_{x\in I_i} f(x)\right) \ .$$

For $c \leq 0$ this changes to

$$\sup_{x\in I_i} cf(x) = c \inf_{x\in I_i} f(x) \quad \text{and} \quad \inf_{x\in I_i} cf(x) = c \sup_{x\in I_i} f(x) \ .$$

so that

$$\sup_{x\in I_i} cf(x) - \inf_{x\in I_i} cf(x) = -c\left(\sup_{x\in I_i} f(x) - \inf_{x\in I_i} f(x)\right) \ .$$

Taken together, this implies

$$\sup_{x\in I_i} cf(x) - \inf_{x\in I_i} cf(x) = |c| \left(\sup_{x\in I_i} f(x) - \inf_{x\in I_i} f(x) \right) .$$

Multiplying by Δx_i and summing over all i gives the desired result.

(b) If $U(f,P) - L(f,P) < \epsilon$ for some $\epsilon > 0$, then also

$$U(cf,P) - L(cf,P) \le |c|(U(f,P) - L(f,P)) < |c|\epsilon .$$

By Riemann's integrability criterion, cf is integrable.
Finally, for $c \ge 0$ we have

$$L(cf,P) = cL(f,P) \le c\int_a^b f(x)\,dx \le cU(f,P) = U(cf,P)$$

and for $c \le 0$ we have

$$L(cf,P) = cU(f,P) \le c\int_a^b f(x)\,dx \le cL(f,P) = U(cf,P)$$

so that in both cases

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

follows.

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