MTH5105 Differential and Integral Analysis 2009-2010

Solutions 4

1 Exercise for Feedback/Assessment

1) (a) Let $f(x) = \log(1+x)$.

(i) Determine the Taylor polynomials $T_{2,0}$ and $T_{3,0}$ about 0 for f. [7 marks]

(ii) Using the Lagrange form of the remainder, show that $T_{2,0}(x) \le f(x) \le T_{3,0}(x)$ for all $x \ge 0$. [7 marks]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Prove or disprove the following two statements.

(i) 'The Taylor series of f always converges for at least one point.'

(ii) 'The Taylor series of f always converges to the function for at least two points.'

[6 marks]

Solution:

(a) (i) We find

$$f'(x) = 1/(1+x)\ , \quad f''(x) = -1/(1+x)^2\ , \quad f'''(x) = 2/(1+x)^3\ ,$$

[1.5 marks]

and hence

$$f(0) = 0\ , \quad f'(0) = 1\ , \quad f''(0) = -1\ , \quad f'''(0) = 2\ .$$

[2 marks]

Thus

$$T_{2,0}(x) = \frac{1}{1!}x + \frac{(-1)}{2!}x^2 = x - \frac{x^2}{2}\ ,$$

[1.5 marks]

and

$$T_{3,0}(x) = \frac{1}{1!}x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}\ .$$

[2 marks]

(ii) From the Lagrange form of the remainder it follows that:
For $x > 0$ there is a $c \in (0, x)$ such that [1 marks]

$$f(x) = T_{2,0}(x) + R_3$$

[1 mark]

where

$$R_3 = \frac{2/(1+c)^3}{3!}x^3\ .$$

[2 marks]

But

$$0 < \frac{2/(1+c)^3}{3!}x^3 < \frac{x^3}{3}\ ,$$

[1 mark]

so that for $x > 0$

$$T_{2,0}(x) < f(x) < T_{3,0}(x) \ .$$

[1 mark]

For $x = 0$ we have

$$T_{2,0}(0) = f(0) = T_{3,0}(0)$$

so that the claim is true for all $x \geq 0$. [1 mark]

(b) (i) True. [1 mark]

The Taylor series always converges for $x = a$: [1 mark]

For $x = a$, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ simplifies to

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n} 0^n = f(a)$$

which has only one non-zero term. [1 mark]

(ii) False. [1 mark]

Proof by counterexample: [1 mark]

For example, it follows from material in the lectures that the Taylor series of

$$f(x) = \begin{cases} 0 & x = 0 \\ \exp(-1/|x|) & x \neq 0 \end{cases}$$

is identically equal to zero and hence does not converge to $f(x)$ for $x \neq 0$. [1 mark]

2 Extra Exercises

2) Let $f : (-1, \infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0 \ .$$

Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0 \ .$$

Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$.

Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h, $(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)} h^{(k)}$.

Solution:

We find

$$\begin{aligned} f(x) &= \sin(\pi\sqrt{1+x}) \\ f'(x) &= \frac{\cos(\pi\sqrt{1+x})\pi}{2\sqrt{1+x}} \\ f''(x) &= -\frac{\sin(\pi\sqrt{1+x})\pi^2}{4(1+x)} - \frac{\cos(\pi\sqrt{1+x})\pi}{4(1+x)^{3/2}} \end{aligned}$$

From this, the identity immediately follows.

Differentiating $g(x) = 4(1+x)f''(x)$ n times, we find

$$g^{(n)}(x) = 4(1+x)f^{(n+2)}(x) + 4nf^{(n+1)}(x)$$

(this is where the Leibniz formula might be useful, otherwise you might need to use induction). Thus, differentiating the identity gives

$$4(1+x)f^{(n+2)}(x) + 2(2n+1)f^{(n+1)}(x) + \pi^2 f^{(n)}(x) = 0$$

which for $x = 0$ simplifies to the needed formula.

We compute now $f(0) = 0$, $f'(0) = -\pi/2$, and recursively

$$\begin{aligned}
f''(0) &= -\frac{1}{4}\left(2f'(0) + \pi^2 f(0)\right) = \frac{\pi}{4} \\
f'''(0) &= -\frac{1}{4}\left(6f''(0) + \pi^2 f'(0)\right) = \frac{\pi}{8}(\pi^2 - 3) \\
f''''(0) &= -\frac{1}{4}\left(10f''(0) + \pi^2 f''(0)\right) = \frac{3\pi}{16}(5 - 2\pi^2)
\end{aligned}$$

from whence

$$T_{4,0}(x) = -\frac{\pi}{2}x + \frac{\pi}{8}x^2 + \frac{\pi}{48}(\pi^3 - 3)x^3 + \frac{\pi}{128}(5 - 2\pi^2)x^4$$

follows.

3) The number e can be expressed via an alternating series as

$$\frac{1}{e} = \exp(-1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} .$$

Show that remainder term R_n in

$$\frac{n!}{e} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + R_n ,$$

cannot be an integer. Hence deduce that e is irrational.

Hint: look up the convergence criterion for alternating series.

Solution:

As $1/k!$ decreases strictly to zero, the alternating series converges, and

$$\frac{1}{e} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} + r_n$$

where the non-zero remainder r_n is bounded by the first omitted term,

$$0 < |r_n| < \frac{1}{(n+1)!} .$$

Thus the remainder R_n is bounded by

$$0 < |R_n| < \frac{n!}{(n+1)!} = \frac{1}{n+1} < 1 ,$$

and therefore cannot be an integer.

Now $\sum_{k=0}^{n} \frac{(-1)^k n!}{k!}$ is an integer. Therefore we have shown that for all $n \in \mathbb{N}$, $n!/e$ cannot be an integer. It follows that e cannot be a rational number. (If $e = p/q$ was a rational number, then $n!q/p$ would have to be an integer for n sufficiently large.)

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