MTH5105 Differential and Integral Analysis 2008-2009

Exercises 9

Exercise 1: Is the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sum_{k=1}^{\infty} \sin^2(x/k)$$

differentiable?

[10 marks]

Exercise 2: Let $f_n : [0, \infty) \mapsto \mathbb{R}$ be a sequence of continuous functions that converge uniformly to $f(x) = 0$. Show that if

$$0 \leq f_n(x) \leq e^{-x}$$

for all $x \geq 0$ and for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)\, dx = 0 \ .$$

[Recall from Calculus I the definition of the improper integral $\int_0^{\infty} f(x)\, dx = \lim_{A\to\infty} \int_0^A f(x)\, dx$.]

[10 marks]

Exercise 3: Let $f_n : [0, 1] \mapsto \mathbb{R}$ be a sequence of differentiable functions, and let $f : [0, 1] \mapsto \mathbb{R}$. Consider the statements

(a) $f_n \to f$ pointwise,

(b) $f_n \to f$ uniformly,

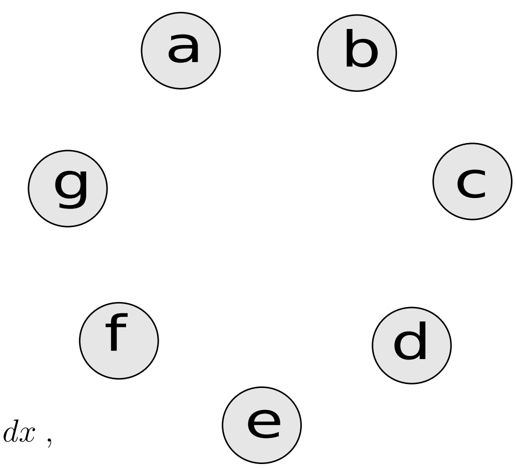
(c) f_n' converges pointwise,

(d) $f_n' \to f'$ pointwise,

(e) f continuous,

(f) f differentiable,

(g) $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx$,



and cleary indicate in the enclosed figure all implications by the appropriate arrows ("$\Longrightarrow$").

[10 marks]

These exercises do *not* constitute coursework. Model solutions will be made available on the course webpage by the last day of term.

Thomas Prellberg, March 2009