MTH5105 Differential and Integral Analysis 2010-2011

Solutions 2

1 Exercises for Feedback

1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Show that

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

Solution:

For any $x, y \in \mathbb{R}$ with $x < y$, f is continuous on $[x, y]$ and differentiable on (x, y). Therefore we can apply the Mean Value Theorem to f on the interval $[x, y]$.
The MVT implies that there exists a $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) .$$

By (a), $|f'(c)| \leq 1$.
Therefore

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq 1 ,$$

which implies $|f(y) - f(x)| \leq |y - x|$.
This inequality is symmetric in x and y and trivially true if $x = y$, so that we can drop the restriction $x < y$. (This could have been argued earlier: without loss of generality, let $x < y$...)

2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is constant. *Hint: try to compute the derivative of f first.*

Solution:

From the inequality it follows that for all $x, a \in \mathbb{R}$

$$\left| \frac{f(x) - f(a)}{x - a} \right| \leq |x - a| ,$$

so that

$$\left| \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right| = \lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} \right| \leq \lim_{x \to a} |x - a| = 0 .$$

Hence $f'(a)$ exists and equals $f'(a) = 0$. Therefore f is differentiable for all $x \in \mathbb{R}$ with $f'(x) = 0$.

Now we want to apply Theorem 2.5 to show that f is constant, i.e. that $f(x) = f(y)$ for all $x, y \in \mathbb{R}$. Note that the assumption of Theorem 2.5 is that f has zero derivative on a closed and bounded interval. The correct step is therefore to apply Theorem 2.5 to f on the interval $[x, y]$ for $x < y$. Then it follows that f is constant on $[x, y]$ and hence that $f(x) = f(y)$. (Simply to say $f'(x) = 0$, so by Theorem 2.5 f is constant on $\mathbb{R}$ is insufficient.)

2 Extra Exercises

3) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g \qquad \text{and} \qquad g' = -f \ .$$

Show that between every two zeros of f there is a zero of g and between every two zeros of g there is a zero of f.

Solution:

Choose $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b) = 0$.
As f is differentiable on $\mathbb{R}$, the assumptions of Rolle's Theorem are satisfied on $[a, b]$, i.e. f continuous on $[a, b]$ and differentiable on (a, b).

Therefore there exists a $c \in (a, b)$ such that $f'(c) = 0$.

As $f' = g$, $g(c) = f'(c) = 0$.

An analogous argument is valid with f and g exchanged.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable ($f'' = (f')'$) with

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f(1) = 1 \ .$$

Show that there exists a $c \in (0, 1)$ such that $f''(c) > 1$.

Solution:

As f is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, 1]$, i.e. f continuous on $[0, 1]$ and differentiable on $(0, 1)$.

Therefore there exists a $d \in (0, 1)$ such that

$$f'(d) = \frac{f(1) - f(0)}{1 - 0} = 1 \ .$$

As f' is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, d]$, i.e. f' continuous on $[0, d]$ and differentiable on $(0, d)$.

Therefore there exists a $c \in (0, d)$ such that

$$f''(c) = \frac{f'(d) - f'(0)}{d - 0} = \frac{1}{d} \ .$$

As $d \in (0, 1)$, $1/d > 1$.

5*) Suppose that f is continuous on $[0, 1]$, differentiable on $(0, 1)$, and $f(0) = 0$. Prove that if f' is decreasing on $(0, 1)$, then the function $g : (0, 1) \to \mathbb{R}$ given by $g(x) = f(x)/x$ is decreasing on $(0, 1)$.

Solution:

Since g is differentiable on $(0, 1)$ it suffices to show that $g'(x) \leq 0$. As

$$g'(x) = \frac{f'(x)x - f(x)}{x^2} \ ,$$

we only need to show that $f'(x)x - f(x) \leq 0$.

Applying the MVT to f on $[0, x]$, there exists a $c \in (0, x)$ such that $f(x) - f(0) = f'(c)(x - 0)$.

As f' is decreasing and $c < x$, $f'(x) \leq f'(c)$. Therefore

$$f(x) = f'(c)x \geq f'(x)x$$

and hence $f'(x)x - f(x) \leq 0$ for all $x \in (0, 1)$.

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