[Reading assignment: section 3.3] password: attended

Derivatives of Trigonometric Functions

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

use $\sin(x+h) = \sin x \cos h + \cos x \sin h$

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{=0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{=1}$$

done earlier:

$$= \cos x$$

[3-46]

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \text{(derivation similar)}$$

$f(x) = \tan x$:

write $f(x) = \dfrac{\sin x}{\cos x}$ $\qquad u = \sin x$, $v = \cos x$

$$f'(x) = \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

Example: Find $f^{(4)}(x)$ for $f(x) = \sin x$:

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$$f^{(4)}(x) = -(-\sin x) = \sin x$$

The chain rule

Example $y = \frac{3}{2} x$ [3-53]

is the same as: $y = \frac{1}{2} u$ and $u = 3x$

- $\frac{dy}{dx} = \frac{3}{2}$ $\frac{dy}{du} = \frac{1}{2}$, $\frac{du}{dx} = 3$

we find $\underline{\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}}$

Accident or general formula?

Towards a general rule: [3-54]

Rates of change multiply?

<u>Theorem</u> [3-55]

If $f(u)$ differentiable at $u = g(x)$

and $g(x)$ differentiable at x

then $f \circ g(x)$ differentiable at x and

$$\underline{(f \circ g)'(x) = f'(g(x))\, g'(x)}$$

[Proof later]

<u>Example</u> $\quad x(t) = \cos(t^2+1)$

Write $x = \cos u$, $u = t^2+1$

$$\frac{dx}{du} = -\sin u, \quad \frac{du}{dt} = 2t$$

so $\quad \dfrac{dx}{dt} = (-\sin u)\, 2t$

$$= -2t \sin(t^2+1)$$

<u>Example</u>

$$\frac{d}{dx} \sin(\underbrace{x^2+x}_{u = x^2+x}) = \cos(\underbrace{x^2+x}_{u}) \underbrace{(2x+1)}_{u'}$$

<u>3-link-chain:</u> $\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dt}$

$$\frac{d}{dx} \tan(\underbrace{5 - \sin 2t}_{u = 5-\sin 2t})$$

$$= \frac{1}{\cos^2(\underbrace{5 - \sin 2t}_{u})} \underbrace{(-\cos 2t)}_{u'} (2)$$

$$= \frac{-2 \cos 2t}{\cos^2(5 - \sin 2t)}$$

Parametric equations

[3-58]

A Point moving in xy-plane traces a path, which may be the graph of a function $y = h(x)$, or it may not.

- The position of the point depends on a parameter t ("time"), so that in general we can write
$$x = f(t)$$
$$y = g(t)$$

- We use this to define a parametric curve [3-57]

t is called a parameter for the curve.

If $t \in [a,b]$, then

$(f(a), g(a))$ is the initial point

$(f(b), g(b))$ is the terminal point

Example. Motion on a circle

$$x = a \cos t \qquad 0 \le t \le 2\pi$$

$$y = a \sin t \qquad [3-59]$$

Since $x^2 + y^2 = (a \cos t)^2 + (a \sin t)^2$

$$= a^2,$$

the parametrisation describes a motion that starts at initial point $(a, 0)$ and traverses the circle $x^2+y^2=a^2$ counterclockwise once, ending at the terminal point $(a, 0)$.

Moving along a parabola: [3-60]

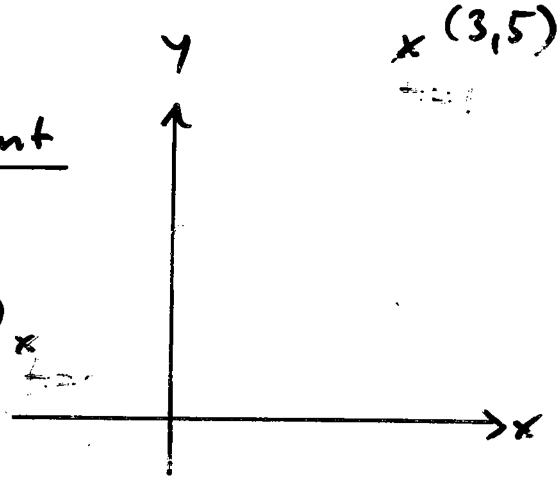
$$x = \sqrt{t} \quad , \quad y = t \quad , \quad t \geq 0$$

We can solve this as $y = f(x)$:

$$y = t = (\sqrt{t})^2 = x^2$$

Note that the domain of f is $\underline{[0, \infty)}$

Parametrising a line segment



- start at $(-2,1)$ for $t=0$

$$x = -2 + at$$

$$y = 1 + bt$$

- end at $(3,5)$ for $t=1$

$$3 = -2 + a \cdot 1 \quad \Rightarrow \quad a = 5$$

$$5 = 1 + b \cdot 1 \quad \Rightarrow \quad b = 4$$

- $x = -2 + 5t, \quad y = 1 + 4t \quad , \quad 0 \leq t \leq 1$

Slopes of parametrised curves

$x = f(t)$, $y = g(t)$ is differentiable at t if f and g are differentiable at t. If y is a differentiable function of x, say $y = h(x)$, then $y(t) = h(x(t))$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \text{(Chain rule!)}$$

and therefore, if $\frac{dx}{dt} \neq 0$,

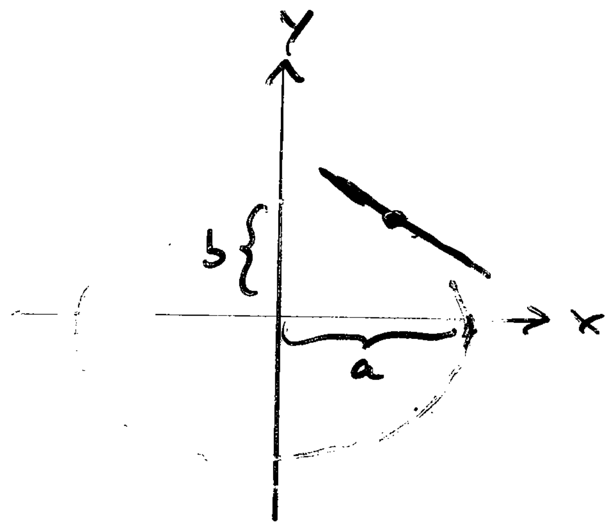
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad [3-61]$$

Careful: dx, dy, dt are NOT numbers!

It looks like it here, but it is wrong to write $\frac{dy/dt}{dx/dt} = \frac{dy}{\cancel{dt}}\frac{\cancel{dt}}{dx} = \frac{dy}{dx}$!!!

<u>Example</u> Moving along an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



($a = b = r$: circle)

- $x = a \cos t \quad , \quad y = b \sin t \quad , \quad 0 \le t \le 2\pi$

$$\frac{dx}{dt} = -a \sin t \quad , \quad \frac{dy}{dt} = b \cos t$$

- Slope $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

$$= \frac{b \cos t}{-a \sin t} = -\frac{b^2}{a^2}\frac{x}{y}$$

- The slope at the point (x, y) is

$$m = -\frac{b^2}{a^2}\frac{x}{y}$$

$t = \frac{\pi}{4}$: $x = a\frac{\sqrt{2}}{2}$, $y = b\frac{\sqrt{2}}{2}$

$$m = -\frac{b^2}{a^2}\frac{a\frac{\sqrt{2}}{2}}{b\frac{\sqrt{2}}{2}} = -\frac{b}{a}$$

What about higher derivatives?

$$y' = \frac{dy}{dt} \Big/ \frac{dx}{dt} \quad , \quad y'' = ?$$

- Remember $y'' = (y')'$:

$$\underline{\underline{y'' = \frac{dy'}{dt} \Big/ \frac{dx}{dt}}} = \frac{\frac{d}{dt}\left[\frac{dy}{dt} \Big/ \frac{dx}{dt}\right]}{\frac{dx}{dt}} = \ldots$$

- <u>Example continued:</u> $\quad y' = -\frac{b}{a}\frac{\cos t}{\sin t}$

$$y'' = \frac{\frac{d}{dt}\left[-\frac{b}{a}\frac{\cos t}{\sin t}\right]}{-a\sin t} = -\frac{b}{a^2}\frac{1}{\sin^3 t}$$

$$= -\frac{b^4}{a^2}\frac{1}{y^3}$$

[Summary 3-64]

Implicit differentiation

Compute y' if we don't have

$$y = f(x)$$

- but $F(x,y) = 0$, an <u>implicit</u> relation between x and y.

- One way can be parametrisation:

- $x = \cos t$, $y = \sin t$ gives

$$F(x,y) = x^2 + y^2 - 1 = 0$$

- Another way, if $F(x,y) = 0$ is given:

Differentiate the relation directly!

$F(x,y) = y^2 - x \qquad F(x,y) = 0$

<u>Example:</u> $y^2 = x$, compute y'

[Of course we know that we have two solutions $y_1 = +\sqrt{x}$, $y_2 = -\sqrt{x}$ with derivatives $y_1' = \frac{1}{2\sqrt{x}}$, $y_2' = -\frac{1}{2\sqrt{x}}$]

New method: differentiate directly $h \circ g(x)$ with $h(y) = y^2$, $g(x) = y(x)$

$$y^2 = x \qquad \Big| \ \frac{d}{dx}$$

$$2yy' = 1 \qquad \Big| \ \text{solve for } y'$$

$$y' = \frac{1}{2y}$$

Substituting $y_1 = +\sqrt{x}$, $y_2 = -\sqrt{x}$ gives the above result. [3-71]

General recipe : [3-69]

<u>Example</u> $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

differentiate:

$$\frac{2x}{a^2} + \frac{2y\,y'}{b^2} = 0 \qquad (*)$$

- solve for y':

$$\underline{\underline{y' = -\frac{b^2}{a^2}\,\frac{x}{y}}}$$

[looks familiar ?]

differentiate $(*)$ again :

- $$\frac{2}{a^2} + \frac{2\,(y'y' + y\,y'')}{b^2} = 0$$

solve for y'': $$y'' = -\frac{1}{y}\left(\frac{b^2}{a^2} - (y')^2\right)$$

insert y' and simplify (several steps)

gives again $$\underline{\underline{y'' = -\frac{b^4}{a^2}\,\frac{1}{y^3}}}$$

Power rule for rational powers [3-76]

$$y = x^{p/q} \qquad \frac{p}{q} \text{ rational}$$

$$y' = ?$$

- Use implicit differentiation:

$$y^q = x^p \qquad \Big| \frac{d}{dx}$$

$$q\, y^{q-1} y' = p\, x^{p-1} \qquad \Big| \text{ solve for } y'$$

$$y' = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}}$$

-

$$= \frac{p}{q} \underbrace{\cancel{\frac{x^p}{y^q}}}_{=1} \frac{y}{x}$$

$$= \frac{p}{q} \frac{x^{p/q}}{x}$$

$$y' = \frac{p}{q} x^{\frac{p}{q}-1}$$

Linearisation

"Close to" the point $(a; f(a))$,

the tangent

$$y = f(a) + f'(a)\,(x-a)$$

is a "good" approximation

for $y = f(x)$ [3-83, 84]

- Definition of linearisation

  on slide [3-85]

Example :

$$f(x) = \sqrt{1+x} \quad , \quad a = 0 \qquad [3-86]$$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$f(0) = 1 \qquad f'(0) = \frac{1}{2}$$

$$L(x) = 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x$$

So "near" $x = 0$ we have

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x \qquad [3-87]$$

for example, $x = 0.05$ gives

$$\sqrt{1.05} = 1.024695\ldots \quad , \quad 1 + \frac{1}{2}0.05 = 1.025$$

[3-88]

<u>Differentials</u> The derivative

$$y' = \frac{dy}{dx}$$ is not a ratio!

We introduce two new variables

dx and dy with the property

that if their ratio exists, it

will be equal to the derivative

$$dy = f'(x)\, dx \qquad [3\text{-}90]$$

Geometrically, dy is the change

in the linearisation of f if x

changes by dx [3-91]

Estimating with differentials

True value:

$$f(a+\Delta x) = f(a) + \underline{\Delta f}$$

differential approximation

$$f(a+\Delta x) \approx f(a) + \Delta y$$

$$= f(a) + \underline{f'(a)\ \Delta x}$$

The approximation error is

$$\Delta f - f'(a)\ \Delta x = f(a+\Delta x) - f(a) - f'(a)\ \Delta x$$

$$= \left[\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right] \Delta x$$

$$= \varepsilon\ \Delta x \qquad \text{with } \varepsilon \to 0$$

$$\text{as } \Delta x \to 0 \qquad [3\text{-}93]$$

Proof of the chain rule

$y = f(u)$ differentiable at $u = g(x)$, and

$u = g(x)$ differentiable at x

We have

$$\Delta u = g'(x)\,\Delta x + \varepsilon_1\,\Delta x = (g'(x) + \varepsilon_1)\,\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Also

$$\Delta y = f'(u)\,\Delta u + \varepsilon_2\,\Delta u = (f'(u) + \varepsilon_2)\,\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. Together,

$$\Delta y = (f'(u) + \varepsilon_2)(g'(x) + \varepsilon_1)\,\Delta x$$

so

$$\frac{\Delta y}{\Delta x} = (f'(u) + \varepsilon_2)(g'(x) + \varepsilon_1)$$

and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u)\,g'(x)$$

$\square$

Extreme Values of Functions

Definitions [4-4]

f function with domain D.

Absolute (global) maximum of f on D at c if

$$f(x) \leq f(c) \quad \text{for all } x \in D$$

Absolute (global) minimum of f on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \in D$$

Absolute (global) extremum of f on D at c if either of the above.

Example [4-5]

Example [4-6] $y = x^2$

	Domain	Abs. max.	Abs. min.
(a)	$(-\infty, \infty)$	NO	0, at 0
(b)	$[0,2]$	4, at 2	at 0
(c)	$(0,2]$	4, at 2	NO
(d)	$(0,2)$	NO	NO

When does a global max/min exist?

Extreme Value Theorem [4-7]

If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum and absolute minimum on $[a,b]$.

Examples [4-8] [4-9]

Local (Relative) Extreme Values

Definitions [4-10]

A function f has a local $\underset{\text{maximum}}{\text{minimum}}$

at an interior point c of its domain

if $$f(x) \overset{\geq}{\leq} f(c)$$

for all x in some open interval containing c.

A function f has a local $\underset{\text{maximum}}{\text{minimum}}$

at an endpoint c of its domain

if $$f(x) \overset{\geq}{\leq} f(c)$$

for all x in some half-open interval containing c.

A function f has a local extremum at c

if either of the above holds. [4-11]

Finding Extrema

[4-12]

Theorem If f has a local extremum at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0$$

[4-13]

Proof If at a local maximum c

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then the one-sided limits exist and

$$f'(c) = \lim_{h \to 0^+} \frac{\overbrace{f(c+h) - f(c)}^{\leq 0}}{h} \leq 0$$

and

$$f'(c) = \lim_{h \to 0^-} \frac{\overbrace{f(c+h) - f(c)}^{\geq 0}}{h} \geq 0$$

so that $f'(c) = 0$. (Similarly for minimum)

Finding extreme values

Where can a function f possibly have an extreme value ?

1)

2)

3)

<u>Definition</u> An interior point of the domain of a function f where f' is zero or undefined is a <u>critical point</u> of f

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This leads to a recipe: [4-14]

int, $f'=0$; int, f' not def; endpoints of D

<u>Example</u> $f(x) = x^2$ on $[-2, 1]$

- f is differentiable on $[-2, 1]$, $f'(x) = 2x$
- critical point : $f'(x) = 0 \Rightarrow x = 0$
- endpoints $x = -2$, $x = 1$
- $f(0) = 0$, $f(-2) = 4$, $f(1) = 1$
- f has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Note: the result <u>depends on the domain</u> of the function f !

<u>Example</u> $g(t) = 8t - t^4$ on $[-2,1]$

- $g'(t) = 8 - 4t^3$, critical point

  $8 - 4t^3 = 0 \Rightarrow t = \sqrt[3]{2}$ (>1)

- $g(-2) = -32$ absolute minimum

  $g(1) = 7$ absolute maximum

[4-15]

<u>Example</u> $f(x) = x^{2/3}$ on $[-2, 3]$

- $f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3\sqrt[3]{x}}$, critical point

  $f'(x) = 0$ or $f'(x)$ undefined $\Rightarrow x = 0$

- $f(0) = 0$, $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

  $f(3) = (3)^{2/3} = \sqrt[3]{9}$

[4-16]

Rolle's Theorem [4-19]

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). If

$f(a) = f(b)$ then there exists a $c \in (a,b)$

with $f'(c) = 0$

Geometrically, a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line [4-20]

Proof • f is continuous on $[a,b]$, so it has absolute maximum and minimum values.

• These occur only if $f'(x)=0$ on (a,b) or at the endpoints a and b.

• If an absolute maximum or minimum occurs on $c \in (a,b)$, then necessarily $f'(c)=0$ (and we're done).

• Assume the absolute maximum and the absolute minimum occur at a or b. As $f(a)=f(b)$, it follows that $f(x)$ must be constant on $[a,b]$, so that $f'(x)=0$ for all $x \in [a,b]$. $\square$

All the assumptions in the Theorem are necessary [4-21]

<u>Example</u> $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$

$f(-3) = 0$, $f(3) = 0$ [4-22]

by Rolle's Theorem there exists a

$c \in [-3, 3]$ with $f'(c) = 0$.

We find indeed $f'(x) = x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

<u>Example</u> Show that $x^3 + 3x + 1 = 0$ has only one real solution: [4-23]

$f(x) = x^3 + 3x + 1$ has derivative

$f'(x) = 3x^2 + 3 > 0$ for all $x \in (-\infty, \infty)$

If there were two solutions with $f(x) = 0$,

then by Rolle's Theorem $f'(c) = 0$ for some c.

The Mean Value Theorem [4-24]

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically, a differentiable curve has at least one tangent between any two points with the same slope as the secant through these points [4-25]

<u>Proof</u> [4-26]

- The straight line through

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

is given by $y = g(x)$ where

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

- consider $h(x) = f(x) - g(x)$ [4-27]

We find $h(a) = f(a) - g(a) = 0$

$$h(b) = f(b) - g(b) = 0$$

- Using Rolle's Theorem, there is a $c \in (a,b)$

with $h'(c) = 0$

- $$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

- $h'(c) = 0$ implies $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ □

Examples: [4-28,29]

- $f(x) = \sqrt{1-x^2}$

is continuous on $[-1,1]$ and

differentiable on $(-1,1)$ (but not at ± 1)

Therefore there is a c with

$$f'(c) = \frac{f(1)-f(-1)}{1-(-1)} = 0$$

(We find easily $c=0$)

- $f(x) = x^2$ is continuous and

differentiable on $[0,2]$

Therefore there is a c with

$$f'(c) = \frac{f(2)-f(0)}{2-0} = 2$$

(We find easily $c=1$)

Consequences of the Mean Value Theorem

Corollary 1 If $f'(x) = 0$ on (a,b)

then $f(x) = C$ for all $x \in (a,b)$.

- Geometrically, functions with zero derivatives are constant. [4-31]

Corollary 2 If $f'(x) = g'(x)$ on (a,b)

- then $f(x) = g(x) + C$ for all $x \in (a,b)$

Geometrically, functions with the same derivative differ by a constant.

[4-32]

<u>Proof</u> (of Corollary 1) :

For any $x_1 \neq x_2$ with $x_1, x_2 \in (a,b)$

there is a $c \in (x_1, x_2)$ with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

but as $f'(c) = 0$ by assumption,

$f(x_2) = f(x_1)$ [for any x_1, x_2 in (a,b) !!]

Thus, $f(x)$ is constant. $\square$

<u>Proof</u> (of Corollary 2) :

Consider $h(x) = f(x) - g(x)$. Then

$h'(x) = f'(x) - g'(x) = 0$ on (a,b)

By corollary 1, $h(x) = C$, so

$$f(x) = g(x) + C \qquad \square$$

<u>Example</u> Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through $(0,2)$:

- $g(x) = -\cos x$ satisfies

  $g'(x) = \sin x = f'(x)$

- Therefore $f(x) = g(x) + C_1$

  $f(x) = -\cos x + C_1$

- $f(0) = 2$ gives $2 = -\cos 0 + C_1$

  so that $C_1 = 3$

- $f(x) = 3 - \cos x$