MTH5105 Differential and Integral Analysis 2008-2009

Exercises 6

Exercise 1: Let $f : (0,1) \to \mathbb{R}$ be continuous. Show that f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist. [5 marks]

[Note: the converse is also true, but much harder to show.]

Exercise 2: Let $\alpha \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha & x \in \{1/k; k \in \mathbb{N}\} \ , \\ 0 & \text{else.} \end{cases}$$

For which values of α is f Riemann-integrable? If f is Riemann-integrable, what is the value of $\int_0^1 f(x)\, dx$? [10 marks]

Exercise 3: Let $f : [a,b] \to \mathbb{R}$ be Riemann-integrable and $c \in \mathbb{R}$.

(a) Given a partition P of $[a,b]$, show that

$$U(cf,P) - L(cf,P) \le |c|(U(f,P) - L(f,P)) \ .$$

[6 marks]

(b) Deduce from (a) that cf is integrable and

$$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx \ .$$

[This completes the proof of Theorem 38.] [4 marks]

(c) For a bounded set $\Omega \subset \mathbb{R}$, show that

$$\sup_{y\in\Omega} |y| - \inf_{y\in\Omega} |y| \le \sup_{y\in\Omega} y - \inf_{y\in\Omega} y \ .$$

[This is needed in the proof of Theorem 41.] [5 marks]

The deadline is 12.15 on Monday, 9th March. Please hand in your coursework at the end of Monday's lecture or to my office MAS113 immediately afterwards.

Thomas Prellberg, February 2009