MTH5105 Differential and Integral Analysis 2008-2009

Exercises 3

Exercise 1: The functions sinh and cosh are given by

$$\sinh : \mathbb{R} \to \mathbb{R}\,, \qquad x \mapsto \frac{1}{2}(\exp(x) - \exp(-x))\,,$$
$$\cosh : \mathbb{R} \to \mathbb{R}\,, \qquad x \mapsto \frac{1}{2}(\exp(x) + \exp(-x))\,.$$

(a) Prove that sinh and cosh are differentiable and that $\sinh' = \cosh$ and $\cosh' = \sinh$. [2 marks]

(b) Prove that the function

$$f(x) = \cosh^2(x) - \sinh^2(x)$$

is constant by considering $f'(x)$.
What is the value of the constant? [4 marks]

(c) Prove the identity $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$ by considering the function

$$f(x) = \cosh(x+b) - \cosh(x)\cosh(b) - \sinh(x)\sinh(b)$$

for fixed $b \in \mathbb{R}$. [4 marks]
Hint: you may use that if $g : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $g'' = g$ then there exist $c, d \in \mathbb{R}$ such that $g(x) = c\exp(x) + d\exp(-x)$.

(d) Prove that sinh is invertible. [2 marks]

(e) Prove that $\sinh(\mathbb{R}) = \mathbb{R}$. [4 marks]
Hint: show that $\sinh(2x) > x$ for $x > 0$, and mimic the proof of the statement that $\exp(\mathbb{R}) = \mathbb{R}^+$.

(f) Prove that $\operatorname{arsinh} = \sinh^{-1}$ is differentiable, and that

$$\operatorname{arsinh}'(x) = \frac{1}{\sqrt{1+x^2}}\,.$$

[4 marks]

Solution: (a) exp is differentiable, therefore sinh and cosh are differentiable. Using $\exp' = \exp$, the derivatives follow immediately. [2 marks]

(b) f is differentiable, and $f'(x) = 2\cosh(x)\sinh(x) - 2\sinh(x)\cosh(x) = 0$. By Theorem 11, f is constant. $f(0) = \cosh^2(0) - \sinh^2(0) = 1$, so $\cosh^2(x) - \sinh^2(x) = 1$. [4 marks]

(c) f is differentiable with $f'(x) = \sinh(x+b) - \sinh(x)\cosh(b) - \cosh(x)\sinh(b)$. Similarly, f' is differentiable with $f''(x) = \cosh(x+b) - \cosh(x)\cosh(b) - \sinh(x)\sinh(b) = f(x)$. Therefore $f(x) = c\exp(x) + d\exp(-x)$. From $0 = f(0) = c+d$ and $0 = f'(0) = c-d$ it follows that $c = d = 0$ and thus $f(x) = 0$. [4 marks]

(d) $\sinh'(x) = \cosh'(x) > 0$ for all $x \in \mathbb{R}$, therefore sinh is strictly increasing by Theorem 9, and therefore invertible by the corollary after Theorem 16. [2 marks]

(e) Let $x > 0$. We have $\exp(x) > 1 + x$ (see proof of Theorem 13) and $\exp(-x) < 1$ (since exp is strictly increasing, and $\exp(0) = 1$), so that $\sinh(x) > x/2$.
Let $c > 0$. From

$$\sinh(0) = 0 < c < \sinh(2c)$$

it follows by the IVT applied to the interval $[0, 2c]$, that there exists an $x \in (0, 2c)$ such that $\sinh(x) = c$. A similar argument holds for $c < 0$, and for $c = 0$ we have $\sinh(0) = 0 = c$. [4 marks]

(f) $\sinh'(x) = \cosh(x) > 0$ for all $x \in \mathbb{R}$, therefore by Theorem 20, arsinh is differentiable and

$$\operatorname{arsinh}'(x) = \frac{1}{\cosh(\operatorname{arsinh}(x))} \,.$$

Now $\cosh(x) = \sqrt{1 + \sinh^2(x)}$ (from (b), and the positive square root is taken as $\cosh(x)$ is positive), so that $\operatorname{arsinh}'(x) = 1/\sqrt{1+x^2}$. [4 marks]

Exercise 2: (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$with $f'(0) = 0$ and a continous inverse. [5 marks]

(b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove:
If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} f'(x) = 0$. [5 marks]

Solution: (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$.
f is differentiable with continuous derivative $f'(x) = 3x^2$. We have $f'(0) = 0$.
The inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{1/3}$.
As f is strictly increasing on $\mathbb{R}$, f is injective. $f(\mathbb{R}) = \mathbb{R}$ implies that f is surjective as well, so f is bijective.
As f is differentiable, it is continuous. Therefore f^{-1} is also continuous. [5 marks]

(b) This can be disproved by a counterexample.
Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be given by $f(x) = -x$.
f is differentiable and $f'(x) = -1$ for $x \geq 0$.
$\lim_{x\to 0} f(x) = 0$, but $\lim_{x\to 0} f'(x) = -1$. [5 marks]

Thomas Prellberg, January 2009