A Proof of the Monotonicity Conjecture by Friedman, Joichi, and Stanton

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Monotonicity of Partitions

- Let $\mathcal{A} \subset \mathbb{N}$ be a set of positive integers.
  Let $a_n(\mathcal{A})$ be the number of partitions of n into parts from $\mathcal{A}$.
- Example: $\mathcal{A} = \{3, 4, 5\}$

$$\begin{array}{rcl} 7 = 4 + 3 & & a_7(\{3,4,5\}) = 1 \\ 8 = 5 + 3 = 4 + 4 & \Rightarrow & a_8(\{3,4,5\}) = 2 \\ 9 = 5 + 4 = 3 + 3 + 3 & & a_9(\{3,4,5\}) = 2 \end{array}$$

- Problem (Bateman and Erdös, 1956):

For which sets $\mathcal{A}$ is $a_n(\mathcal{A})$ monotonically increasing?

Bateman and Erdös (1956)

- Criterion for asymptotic monotonicity:

$$a_{n+1}(\mathcal{A}) > a_n(\mathcal{A}) \quad \text{for sufficiently large } n$$

if and only if

  - $\mathcal{A}$ contains more than one element and
  - if one removes an arbitrary element from $\mathcal{A}$ then the remaining elements have $\gcd$ of unity

- Example: $\mathcal{A} = \{3, 4, 5\}$
  - $|\{3, 4, 5\}| = 3$
  - $\gcd(\{3, 4\}) = \gcd(\{3, 5\}) = \gcd(\{4, 5\}) = 1$

Generating Functions

- The generating function for the sequence $a_n(\mathcal{A})$ is

$$F_{\mathcal{A}}(q) = \sum_{n=0}^{\infty} a_n(\mathcal{A}) q^n = \prod_{i \in \mathcal{A}} \frac{1}{1-q^i}$$

- Monotonicity of $a_n(\mathcal{A})$ is equivalent to non-negativity of the coefficients in

$$(1-q)F_{\mathcal{A}}(q) + q$$

- Example: $\mathcal{A} = \{3,4,5\}$

$$\begin{aligned} F_{\{3,4,5\}}(q) &= [(1-q^3)(1-q^4)(1-q^5)]^{-1} \\ &= 1 + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + \dots \\ (1-q)F_{\{3,4,5\}}(q) + q &= 1 + q^3 + q^8 + q^{12} + q^{15} + q^{18} + \dots \end{aligned}$$

Friedman, Joichi, and Stanton (1994)

Considering $\mathcal{A}_n = \{n, n+1, \ldots, 2n-1\}$, we define

$$f_n(q) = (1-q)F_{\mathcal{A}_n}(q) = \frac{1-q}{\prod_{i=0}^{n}(1-q^{n+i})}$$

Monotonicity Conjecture: Let $n \geq 3$ be an odd integer.

- The power series expansion of $f_n(q) + q$ has non-negative coefficients.
- The power series expansion of $f_n(q) + q$ has strictly positive coefficients past q^{3n+4}.

Assuming validity of this conjecture, all sets $\mathcal{A}$[a] with monotonically increasing $a_n(\mathcal{A})$ can be classified.

[a] whose minimum value is not equal to 2, 5, or 7

Andrews (2001)

We define the usual q-product and the q-binomial as

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} .$$

Lemma 1: For positive integers n and m, let

$$A(n,m) = \frac{1-q}{1-q^n}\begin{bmatrix} n \\ m \end{bmatrix}_q .$$

- If $\gcd(n,m)=1$, then $A(n,m)$ is a reciprocal polynomial of degree $(m-1)(n-m-1)$ with non-negative coefficients.
- If $\gcd(n,m)>1$, then $A(n,m)$ is a not a polynomial.

Andrews (2001) ctd.

Theorem 1: The Monotonicity Conjecture holds for $n \geq 3$ prime.

Proof of Theorem 1: apply Lemma 1 to the individual terms in

Lemma 2:

$$f_n(q) = \frac{1}{1 - q^{n(3n-1)/2}} \left(1 - q + \sum_{j=0}^{\frac{n-3}{2}} \frac{q^{(j+1)(n+j)} A(n-j-1, j+1)}{(q^{n+1};q)_j (q^{2n-j-1};q)_{j+1}} \right)$$

Proof of Lemma 2: interpret RHS as basic hypergeometric series ${}_4\phi_3$ and simplify ...

Prellberg and Stanton (2002)

Theorem 2: The Monotonicity Conjecture holds.

Proof of Theorem 2: apply Lemma 1 to the individual terms in

Lemma 3:

$$f_n(q) = \frac{1}{1-q^{4n^2-6n+2}} \left(1-q+\sum_{m=0}^{n-2} q^{(n+m)(2m+1)} \frac{1-q}{(q^n;q)_{m+1}} \begin{bmatrix} n+m \\ 2m+1 \end{bmatrix}_q \right.$$

$$\left. + \sum_{m=0}^{n-3} q^{(n+m+1)(2m+2)} \frac{1-q}{(q^n;q)_{m+1}} \begin{bmatrix} n+m \\ 2m+2 \end{bmatrix}_q \right)$$

Proof of Lemma 3: iterative application of the q-Binomial theorem to

$$f_n(q) = \frac{1-q}{(q^n;q)_n}$$

Remarks

- Generalization

$$\begin{aligned}\frac{1}{(x;q)_n} = &\sum_{m=0}^{\infty} \frac{q^{rm^2} x^{rm}}{(x;q)_m} \begin{bmatrix} n+(r-1)m-1 \\ rm \end{bmatrix}_q + \\ &\sum_{m=0}^{\infty} \frac{q^{(rm+1)m} x^{rm+1}}{(x;q)_{m+1}} \begin{bmatrix} n+(r-1)m \\ rm+1 \end{bmatrix}_q + \\ &\sum_{i=2}^{r-1} \sum_{m=0}^{\infty} \frac{q^{(rm+i)(m+1)} x^{rm+i}}{(x;q)_{m+1}} \begin{bmatrix} n+(r-1)m+i-2 \\ rm+i \end{bmatrix}_q .\end{aligned}$$

- $r = 2$ and $x = q^n$ implies Lemma 3.
- $r > 2$ and $x = q^n$ does not give non-negative terms for $f_n(q)$.

Prellberg and Stanton (2002) ctd.

Lemma 4:

$$f_n(q) = \frac{1}{1-q^{n(2n-1)}}\left(1-q+\sum_{m=1}^{n-1} q^{m(n+m-1)}\frac{1-q}{(q^n;q)_m}\begin{bmatrix} n \\ m \end{bmatrix}_q\right)$$

The Monototicity Conjecture would follow from

Conjecture:

$$\frac{1-q}{(q^n;q)_m}\begin{bmatrix} n \\ m \end{bmatrix}_q$$

has non-negative power series coefficients

- if $n > 0$ is odd and $0 < m < n$, or
- if $n > 0$ is even and $0 < m < n$ with $m \neq 2, n-2$.

($m = n$ odd reduces to Monotonicity Conjecture.)

Summary and Outlook

- We have proved a *key conjecture* needed to solve a 50 year old open problem.
- Identities used can be generalized.

  "Summable Sums of Hypergeometric Series," D. Stanton, preprint

- Drawback: positivity of terms in Lemma 3 "accidental".
- New conjecture regarding positivity of

$$\frac{1-q}{(q^n;q)_m} \begin{bmatrix} n \\ m \end{bmatrix}_q \ .$$

- Open problem: Find a combinatorial proof (injection).

"Proof of a Monotonicity Conjecture," T. Prellberg and D. Stanton, J. Comb. Th. A, in print