MTH5105 Differential and Integral Analysis 2009-2010

Exercises 4

There are two sections. Questions in Section 1 will be marked and will form your coursework mark. Questions in Section 2 are voluntary but highly recommended.

1 Exercise for Feedback/Assessment

1) (a) Let $f(x) = \log(1+x)$.

   (i) Determine the Taylor polynomials $T_{2,0}$ and $T_{3,0}$ about 0 for f. [7 marks]

   (ii) Using the Lagrange form of the remainder, show that $T_{2,0}(x) \leq f(x) \leq T_{3,0}(x)$ for all $x \geq 0$. [7 marks]

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Prove or disprove the following two statements.

   (i) 'The Taylor series of f always converges for at least one point.'

   (ii) 'The Taylor series of f always converges to the function for at least two points.'

   [6 marks]

2 Extra Exercises

2) Let $f : (-1, \infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0 \ .$$

Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0 \ .$$

Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$.
Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h, $(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)} h^{(k)}$.

3) The number e can be expressed via an alternating series as

$$\frac{1}{e} = \exp(-1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \ .$$

Show that remainder term R_n in

$$\frac{n!}{e} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + R_n \ ,$$

cannot be an integer. Hence deduce that e is irrational.
Hint: look up the convergence criterion for alternating series.

The deadline is 5.00pm (strict) on Monday 15th February. Please hand in your coursework to the red coursework box on the ground floor.

Thomas Prellberg, February 2010