Matrix Ansatz, lattice paths and rook placements

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We give two combinatorial interpretation of the Matrix Ansatz of the PASEP in terms of lattice paths and rook placements. This gives two (mostly) combinatorial proofs of a new enumeration formula of the partition function of the PASEP. This formula gives also for example the generating function for permutations of given size with respect to the number of ascents and occurrences of pattern 13-2.

Résumé.

Nous donnons deux interprétations combinatoires du Matrix Ansatz du PASEP en termes de chemins et de placements de tours. Cela donne deux preuves (presque) combinatoires d'une nouvelle formule pour la fonction de partition du PASEP. Cette formule donne aussi par exemple la fonction génératrice des permutations de taille donnée par rapport au nombre de montées et d'occurrences du motif 13-2.

Keywords: Enumeration, Permutation tableaux, Rook placements, Lattice paths

1 Introduction

In the recent work of Postnikov [15], permutations were showed with a quite new description, as pattern-avoiding fillings of Young diagrams. More precisely, he made a correspondance between positive grassman cells, these pattern-avoiding fillings called ⅃-diagrams, and decorated permutations (which are permutations with a weight 2 on each fixed point). In particular, the usual permutations are in bijection with permutation tableaux, a subclass of ⅃-diagrams. Permutation tableaux have then been studied by Steingrìmsson, Williams, Burstein, Corteel, Nadeau [4, 7, 8, 17], and revealed themselves very useful for working on permutations.

Rather surprisingly, Corteel and Williams observed, and explained, a link between these permutation tableaux and the stationary distribution of a classical process of statistical physics, the Partially Asymmetric Self-Exclusion Process (PASEP). This model is described in [9, 8]. A statement doing this link

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is that the stationary probability of a given state in the process is proportional to the sum of weights of permutation tableaux of a given shape. The factor behind this proportionality is the partition function, which is the sum of weights of permutation tableaux of a given half-perimeter.

Another way of finding the steady distribution of the PASEP is the matrix ansatz [9]. Suppose that we have operators D and E, a row vector $\langle W|$ and a column vector $|V\rangle$ such that:

$$DE - qED = D + E, \qquad \langle W|E = \langle W|, \qquad D|V\rangle = |V\rangle, \qquad \text{and} \quad \langle W||V\rangle = 1. \tag{1}$$

Then, coding any state of the process by a word w of length n in D and E, the probability of the state w is given by $\langle W|w|V\rangle$, divided by the factor $\langle W|(D+E)^n|V\rangle$ which is the partition function.

We briefly describe how the matrix ansatz is related to permutation tableaux[8]. First, notice that there are unique polynomials $n_{i,j} \in \mathbb{Z}[q]$ such that:

$$(D+E)^n = \sum_{i,j\geq 0} n_{i,j} E^i D^j$$

This sum is called the normal form of $(D+E)^n$. It is particularly useful, since for example the sum of coefficients $n_{i,j}$ give an evaluation of $\langle W|(D+E)^n|V\rangle$. Each coefficient $n_{i,j}$ is a generating function for permutation tableaux satisfying certain conditions. Equivalently this can be done with the *alternative tableaux* defined by Viennot [24].

We give here two combinatorial interpretation of the Matrix Ansatz thanks to lattice paths and rook placements and get two semi-combinatorial proofs of the following theorem :

Theorem 1 *For any $n > 0$, we have:*

$$\langle W|(yD+E)^{n-1}|V\rangle = \tfrac{1}{y(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\sum_{j=0}^{n-k} y^j \left(\tbinom{n}{j}\tbinom{n}{j+k} - \tbinom{n}{j-1}\tbinom{n}{j+k+1} \right) \right) \left(\sum_{i=0}^{k} y^i q^{i(k+1-i)} \right).$$

The combinatorial interpretation of this polynomial, in terms of permutations, is given in Proposition 1. When $y = 1$, this can be specialised in:

Corollary 1 *For any $n > 0$, we have:*

$$\langle W|(D+E)^{n-1}|V\rangle = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\tbinom{2n}{n-k} - \tbinom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right).$$

Besides references earlier mentioned, we have to point out the work of Williams [26], where we find a formula giving the coefficients of y^{m-1} in $\langle W|(yD+E)^n|V\rangle$ is :

$$E_{m,n}(q) = \sum_{i=0}^{m-1} (-1)^i [m-i]^n q^{mi-m^2} \left(\tbinom{n}{i} q^{m-i} + \tbinom{n}{i-1} \right). \tag{2}$$

It was obtained by a more direct approach, via the enumeration of Ⅎ-diagrams on a given shape and then a sum of all the possible shapes. It was the only known polynomial formula for the distribution of a permutation pattern of length greater than 2 (See Proposition 1). Whereas the article [26] is rather focused on Ⅎ-diagrams, Williams and her coauthors sketched in Section 4 of [14] how this could have been done directly on permutation tableaux. Moreover Williams's formula has also been recently obtained by Kasraoui, Stanton and Zeng in their work on orthogonal polynomials [12]. We will show in the last Section how our formula can be applied to prove and extend a conjecture presented in [26].

Recently this polynomial $\langle W|y(yD+E)^{n-1}|V\rangle$ was heavily studied. Indeed :

Proposition 1 *For any $n \geq 1$ the following polynomials are equal:*

- $\langle W|y(yD+E)^{n-1}|V\rangle$,
- *the generating function for permutation tableaux of size n, the number of lines counted by y and the number of superfluous 1's counted by q [8, 25],*
- *the generating function for permutations of size n, the number of ascents counted by y and the number of 13-2 patterns counted by q [7, 17],*
- *the generating function for permutations of size n, the number of weak excedances counted by y and the number of crossings counted by q [6, 17],*
- *the nth moment of the q-Laguerre polynomials [12],*
- *the generating function of PDSAWs (partially directed self avoiding walks) in the asymmetric wedge of length n where the number of North steps is counted by q and the number of descents plus one is counted by y [20].*

Remark. We can view this formula as a variation of the Touchard-Riordan formula [21]. It gives the q-enumeration of involutions of size $2n$ without fixed points with respect to the number of crossings, and is also the $2n$th moment of the q-Hermite polynomials. This formula is:

$$\sum_{I\in\mathrm{Inv}(2n)} q^{\mathrm{cr}(I)} = \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)q^{\frac{k(k+1)}{2}}.$$

There is a combinatorial proof given by Penaud in [19]. It was in the goal of generalising this method that we conjectured our Theorem 1 and hoped for a totally combinatorial proof. At the time of writing the last step of this combinatorial proof is still unsolved.

This article is organized as follows. We first show how the matrix ansatz is naturally related to lattice paths. Then we give two proofs of our main Theorem. One based on the lattice paths and the other one based on rook placements. We end by a discussion and some applications.

2 A first proof using lattice paths and functional equations

2.1 Matrix Ansatz and lattice paths

This is very close to ideas developed in [2, 3]. We look for a solution of the system defined in equation (1). One solution is the following :

Proposition 2 *Let $D = (D_{i,j})_{i,j\geq 0}$ and $E = (E_{i,j})_{i,j\geq 0}$ such that*

$$\begin{aligned}
D_{i,j} &= \begin{cases} 1+\ldots+q^i & \text{if } i=j \text{ or } j-1 \\ 0 & \text{otherwise} \end{cases} \\
E_{i,j} &= \begin{cases} 1+\ldots+q^i & \text{if } i=j \text{ or } j+1 \\ 0 & \text{otherwise} \end{cases} \\
\langle W| &= (1,0,0,\ldots) \\
|V\rangle &= (1,0,0,\ldots)^T
\end{aligned}$$

then these matrices and vectors follow the ansatz of equation (1)*.*

One can interpret $y\langle W|(yD+E)^{n-1}|V\rangle$ as the generating polynomials of paths of length $n-1$ in the first quadrant using unitary steps such that the weight of each path is the product of the weight of its steps and the weight of the starting and ending points. If a path starts (resp. ends) at $(0,i)$ (resp. $(n-1,i)$) the weight of the starting (resp. ending) point is W_i (resp. V_i). The weight of a step going from (x,i) to $(x+1,j)$ is $D_{i,j}+E_{i,j}$. We call i the starting height of the step. See [3, 2] for details.

The proposition implies that the paths we are dealing with here are bicolored Motzkin paths, which are paths in the first quadrant that start and end at height 0 and use north-east, south-east and two types of east steps. We now change those paths of length $n-1$ to paths of length n (using a classical argument changing a bicolored Motzkin path of length $n-1$ into a bicolored Moztkin path of length n where the east steps of type 2 can not appear at height 0).

Proposition 3 $y\langle W|(yD+E)^{n-1}|V\rangle$ *is the generating polynomial of weighted bicolored Motzkin paths of length n such that the weight of the steps starting at height i can be*

- $y, yq, \ldots, yq^i$ *for the north east steps or east steps of type 1.*
- $1, q\ldots, q^{i-1}$ *for the south-east steps or east steps of type 2.*

This can also be done combining results in [6, 8, 17].

2.2 The proof

The ideas developed in this Subsection are inspired by the work of Penaud [19]. We now know that $y\langle W|(yD+E)^{n-1}|V\rangle$ is the generating polynomial of weighted bicolored Motzkin paths of length n.

We will first divide the polynomial by $(1-q)^n$ and get that

$$y\langle W|(yD+E)^{n-1}|V\rangle = \frac{1}{(1-q)^n}\sum_{p\in P(n)} w(p),$$

where $P(n)$ is the set of weighted bicolored Motzkin paths of length n such that the weight of the steps starting at height i can be

- 1 or $-q^i$ for the south-east steps or east steps of type 2.
- y or $-yq^{i+1}$ for the north east steps or east steps of type 1.

This follows from the fact that $1 + \ldots + q^{i-1} = (1-q^i)/(1-q)$. Following Penaud, we call those paths "Tout ou rien".

Let $M(n)$ be the subset of the paths in $P(n)$ such that the weight of any East step and the weight of any peak (a North East step followed by a South-East step) can not be y or 1. Let $\mathcal{M}_{n,k,j}$ be the number of left factors of bicolored Motzkin paths of length n, final height k, and with j South-east steps and East steps of type 1.

Theorem 2 *There exists a bijection between paths "Tout ou rien" of length n and pairs of paths such that for some k between 0 and n :*

- *the first one is a left factor of a bicolored Motzkin path of length n that ends at height k ;*
- *the second one is in $M(k)$.*

This bijection is such that

$$\sum_{p \in P(n)} w(p) = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor (n-k)/2 \rfloor} \mathcal{M}_{n,k,j} y^j \sum_{p \in M(k)} w(k).$$

Proof: (Sketch.) Each time, that a path has a factor that is a bicolored Motzkin path (i.e. all steps are weighted by 1 or y), and this factor is maximal, it is deleted from the path. The left-over steps form a path in $M(k)$. The deleted factors are put together and give a bicolored Motzkin path of length $n-k$ and the k extra north-east steps are inserted to tell where these factors were in the original path. □

We now need to compute $\mathcal{M}_{n,k,j}$ and $M_k = \sum_{p \in M(k)} w(k)$.

Proposition 4 *The number $\mathcal{M}_{n,k,j}$ of left factors of bicoloured Motzkin paths of length n, final height k, and with j South-East steps and East steps of type 1, is :*

$$\binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1}. \tag{3}$$

Proof: We note that the formula can be seen as a 2×2 determinant. By the Lindström-Gessel-Viennot lemma, this equals the number of pairs of non-intersecting lattice paths taking north and east steps from $(1,0)$ to $(n-j, j)$ and $(0,1)$ to $(n-j-k, j+k)$ respectively.

We transform such a pair of paths into a single Motzkin path according to the following translation table:

i^{th} step of	lower path	upper path	Motzkin path
	north	north	east type 1
	east	east	east type 2 east
	north	east	north east
	east	north	south east.

It is easy to see that the condition that the two lattice paths do not intersect is transformed into the condition that the Motzkin path does not run below the x-axis. Furthermore, we see that the number of east and south-east steps equals j, the number of north steps of the lower path.

□

Proposition 5 *The generating polynomial M_k is equal to $\sum_{i=0}^{k} y^i q^{i(k+1-i)}$.*

Proof: We add an extra parameter on the paths in $M(n)$. The weight of the steps starting at height i are now

- 1 or $-zq^i$ for the south-east steps or east steps of type 2.
- y or $-yzq^{i+1}$ for the north east steps or east steps of type 1.

Let $M(z) = \sum_{n\geq 0} t^n \sum_{p\in M(n)} w(p)$. It is easy to see that $M(z)$ follows the following functional equation :

$$0 = 1 - (1 + qyzt + zt + yt^2)M(z) + yt^2(1 - qz)^2 M(z)M(qz) .$$

The problem is to find the right linearising Ansatz of the form

$$M(z) = A(z)\frac{H(qz)}{H(z)} + B(z),$$

and we want to solve for the coefficients c_n of $H(z) = \sum_{n=0}^{\infty} c_n z^n$. As there is no linear term in $M(qz)$, it follows that $B(z) = 0$ to obtain a linear q-difference equation for $H(z)$. Next observe that the coefficients are quadratic in z, and that the Ansatz $A(z) = 1/(1 - z)$ leads to

$$H(z) - (1 + yt^2)H(qz) + yt^2 H(q^2z) = z\left(H(z) + (1 + qy)tH(qz) + qyt^2 H(q^2z)\right) \ ; ,$$

which has coefficients linear in z.

After some calculations, we find that

$$H(z) = \sum_{n=0}^{\infty} \frac{(-t, -tqy; q)_n}{(t^2qy, q; q)_n} z^n \ ,$$

or, equivalently, with the usual ${}_2\phi_1$ notation, $H(z) = {}_2\phi_1(-t, -tqy; t^2qy; q, z)$.

Note that we are dealing with series of the type ${}_2\phi_1(a, b; ab; q, z)$ where $a = -t$ and $b = -tqy$. In order to take the limit $z \to 1$, we need to transform using Heine's transformation

$${}_2\phi_1(a, b, ab; q, z) = \frac{(az, b; q)_\infty}{(ab, z; q)_\infty} {}_2\phi_1(a, z; az; q, b) \ .$$

and find that

$$M(z) = \frac{1}{1 - az} \frac{{}_2\phi_1(a, qz; aqz; q, b)}{{}_2\phi_1(a, z, az; q, b)}$$

and therefore

$$M(1) = \frac{1}{1 - a} {}_2\phi_1(a, q; aq; q, b) = \sum_{n=0}^{\infty} \frac{b^n}{1 - aq^n} \ .$$

Changing back to $a = -t$ and $b = -tqy$,

$$M_k = (-1)^k [t^k] M(1) = \sum_{m+n=k} y^n q^{n(m+1)} = \sum_{i=0}^{k} y^i q^{i(k-i+1)}.$$

□

Combining the previous results, we get a proof of Theorem 1.

3 A second proof using the matrix Ansatz and rook placements

For further details about material in this section, see [11]. One of the ideas at the origin of this proof is the following. From D and E of the matrix ansatz, we define new operators:

$$\hat{D} = \frac{q-1}{q}D + \frac{1}{q}, \qquad \hat{E} = \frac{q-1}{q}E + \frac{1}{q}.$$

An immediate consequence is that:

$$\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1-q}{q^2}, \qquad \langle W|\hat{E} = \langle W|, \qquad \text{and} \quad \hat{D}|V\rangle = |V\rangle. \tag{4}$$

This new commutation relation is in a way much more simple than the one satisfied by D and E. It is close to the relation between creation and annihilation operators studied by physicists. Moreover from these definitions we have $q(y\hat{D} + \hat{E}) + (1-q)(yD + E) = 1 + y$, for any parameter y. By isolating one term of the left-hand side and rising to the n with the binomial rule, we have the following inversion formulas between $(yD + E)^n$ and $(y\hat{D} + \hat{E})^n$:

$$(1-q)^n(yD+E)^n = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k}(-1)^k q^k (y\hat{D}+\hat{E})^k, \quad \text{and} \tag{5}$$

$$q^n(y\hat{D}+\hat{E})^n = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k}(-1)^k (1-q)^k (D+E)^k. \tag{6}$$

In particular, the first formula means that if we want to compute the coefficients of the normal form of $(yD + E)^n$, it is enough to compute the ones of $(y\hat{D} + \hat{E})^n$ for all n (taking the normal form is a linear operation).

Up to a factor which only depends on q, these operators $\hat{D}$ and $\hat{E}$ are also defined in [22] and [1]. In the first reference, Uchiyama, Sasamoto and Wadati used the new relation between $\hat{D}$ and $\hat{E}$ to find explicit matrices for these operators. They derive the eigenvalues and eigenvectors of $\hat{D}+\hat{E}$, and consequently the ones of $D + E$, in terms of orthogonal polynomials. In the second reference, Blythe, Evans, Colaiori and Essler also use these eigenvalues and obtain an integral form for $\langle W|(D+E)^n|V\rangle$. They also provide an exact integral-free formula of this quantity, although quite complicated since it contains three sum signs and several q-binomial coefficients.

In this article, instead of working on representations of $\hat{D}$ and $\hat{E}$ and their eigenvalues, we study the combinatorics of the rewriting in normal form of $(\hat{D} + \hat{E})^n$, and more generally $(y\hat{D} + \hat{E})^n$ for some parameter y. In the case of $\hat{D}$ and $\hat{E}$, the objects that appear are the *rook placements in Young diagrams*, long-known by combinatorists since the results of Kaplansky, Riordan, Goldman, Foata and Schützenberger (see [16] and references therein). This method is described in [23], and is the same that the one leading to permutation tableaux or alternative tableaux in the case of D and E.

Definition 1 *Let λ be a Young diagram. A rook placement of shape λ is a partial filling of the cells of λ with rooks (denoted by a circle $\circ$), such that there is at most one rook per row (resp. per column).*

For convenience, we distinguish with a cross ($\times$) each cell of the Young diagram that is not below (in the same column) or to the left (in the same row) of a rook (we are using the French convention). The number of crosses is an important statistic on rook placements, which was introduced in [10] as a generalisation of the inversion number for permutations. Indeed, if λ is a square of side length n, a rook placements R with n rooks may be seen as the graph of a permutation $\sigma \in \mathfrak{S}_n$, and in this view the number of crosses in R is the inversion number of σ.

Definition 2 *The weight of a rook placement R with r rooks, s crosses and t columns is $w(R) = p^r q^s y^t$, where $p = \frac{1-q}{q^2}$.*

With the definition of rook placements and their weights we can give the combinatorial interpretation of $\langle W|(y\hat{D} + \hat{E})^n|V\rangle$.

Proposition 6 *For any n, $\langle W|(y\hat{D} + \hat{E})^n|V\rangle$ is equal to the sum of weights of all rook placements of half-perimeter n.*

The enumeration of rook placements leads to an evaluation of $\langle W|(y\hat{D}+\hat{E})^{n-1}|V\rangle$, hence of $\langle W|(yD+E)^{n-1}|V\rangle$ via the inversion formula (5).

3.1 Rook placements and involutions

Given a rook placement R of half-perimeter n, we define an involution $\alpha(R)$ by the following construction: label the north-east boundary of R with integers from 1 to n. This way, each column or row has a label between 1 and n. If a column, or row, is labelled by i and does not contain a rook, it is a fixed point of $\alpha(R)$. And if there is a rook at the intersection of column i and row j, then $\alpha(R)$ sends i to j (and j to i).

Given a rook placement R of half-perimeter n, we also define a Young diagram $\beta(R)$ by the following construction: if we remove all rows and columns of R containing a rook, the remaining cells form a Young diagram, which we denote by $\beta(R)$. We also define $\phi(R) = (\alpha(R), \beta(R))$. See Figure 1 for an example.

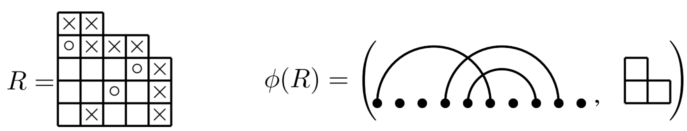

Fig. 1: Example of a rook placement and its image by the map ϕ.

Proposition 7 *The map ϕ is a bijection between rooks placements in Young diagrams of half-perimeter n, and couples (I, λ) where I is an involution on $\{1, \ldots, n\}$ and λ a Young diagram of half-perimeter $|Fix(I)|$. If $\phi(R) = (I, \lambda)$, the number of crosses in R is the sum of $|\lambda|$ and some parameter $\mu(I)$.*

Proof: This kind of bijection rather classical, see for instance [13, 4]. Note that these couples (I, λ) may be seen as involutions on $\{1, \ldots, n\}$ with a weight 2 on each fixed point. For the second part of the proposition, we just have to distinguish different kinds of crosses in the rook placements R. The crosses with no rook in the same line and column are enumerated by $|\lambda|$ for example. $\square$

Corollary 2 *Let $T_{j,k,n}$ be the sum of weights of rook placements of half perimeter n, with k lines and j lines without rooks. Then for any j,k,n, we have:*

$$T_{j,k,n} = \begin{bmatrix} n-2k+2j \\ j \end{bmatrix}_q y^j T_{0,k-j,n}. \tag{7}$$

Proof: The previous proposition means that the number of crosses is an additive parameter with respect to the decomposition $R \mapsto (I, \lambda)$. This naturally lead to a factorisation of the generating function. □

Corollary 3 *We have the following recurrence relation:*

$$T_{0,k,n} = T_{0,k,n-1} + py[n+1-2k]_q T_{0,k-1,n-1}. \tag{8}$$

Proof: We have the relation:

$$T_{0,k,n} = T_{0,k,n-1} + pT_{1,k,n-1}. \tag{9}$$

Indeed, we can distinguish two cases, wether a rook placement enumerated by $T_{0,k,n}$ has a rook in its first column or not. These two cases give respectively the two terms of the previous identity. To end the proof we can apply the previous corollary to the second term. □

3.2 The recurrence

Proposition 8 *The recurrence (9) is solved by:*

$$T_{0,k,n} = q^{-2k} \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left(\binom{n}{k-i} - \binom{n}{k-i-1} \right). \tag{10}$$

From this and the factorisation (7) we derive a formula for $T_{j,k,n}$.

Proposition 9 *Thanks to a q-binomial identity, we have the simplification:*

$$\sum_{j=0}^{k} T_{j,k,n} = \sum_{j=0}^{k} \left(\tbinom{n}{j} - \tbinom{n}{j-1} \right) \left(\tfrac{q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}}{(1-q)q^n} \right).$$

Proposition 10 *Summing the previous equality over k gives :*

$$\langle W|(y\hat{D} + \hat{E})|V\rangle = (1+y)G(n) - G(n+1), \tag{11}$$

$$\textit{where} \quad G(n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{j} - \binom{n}{j-1} \right) \sum_{i=0}^{n-2j} y^{i+j-1} q^{i(n+1-2j-i)}.$$

This formula is a linear combination of the polynomials $P_k = \sum_{i=0}^{k} y^i q^{i(k+1-i)}$, the coefficients being polynomials in y just as in Theorem 1. With the previous proposition and the inversion formula (5), we can prove Theorem 1: the last step is a simple binomial simplification.

4 Applications

Among the several objects of the list in Proposition 1, the most studied are probably permutations and the pattern 13-2, see for example [5, 7, 17, 18]. In particular in [5, 18] we can find methods for obtaining, as a function of n for a given k, the number of permutations of size n with exactly k occurrences of pattern 13-2. By taking the Taylor series of (1), we obtain direct and quick proofs for these previous results. As an illustration we give the formulas for $k \leq 3$ in the following proposition.

Proposition 11 *The order 3 Taylor series of $\langle W|(D+E)^{n-1}|V\rangle$ is:*

$$\langle W|(D+E)^{n-1}|V\rangle = C_n + \binom{2n}{n-3}q + \frac{n}{2}\binom{2n}{n-4}q^2 + \frac{(n+1)(n+2)}{6}\binom{2n}{n-5}q^3 + O(q^4),$$

where C_n is the nth Catalan number.

More generally, a computer algebra system can provide higher order terms, for example it takes no more than a few seconds to obtain the following closed formula for $[q^{10}]\langle W|(D+E)^{n-1}|V\rangle$:

$$\tfrac{(2n)!}{10!(n+12)!(n-8)!}\Big(n^{13}+70\,n^{12}+2093\,n^{11}+32354\,n^{10}+228543\,n^9-318990\,n^8-17493961\,n^7-104051458\,n^6$$

$$-6828164\,n^5+2022876520\,n^4+6310831968\,n^3+5832578304\,n^2+14397419520\,n+5748019200\Big),$$

which is quite an improvement compared to the methods of [18]. Besides these exact formulas, the following proposition gives the asymptotic for permutations with a given number of occurrences of pattern 13-2.

Theorem 3 *for any $m \geq 0$ we have the following asymptotic when n goes to infinity:*

$$[q^m]\langle W|(D+E)^{n-1}|V\rangle \sim \frac{4^n n^{m-\frac{3}{2}}}{\sqrt{\pi}m!}.$$

Proof: When n goes to infinity, the numbers $\binom{2n}{n-k} - \binom{2n}{n-k-2}$ are dominated by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It implies that in $(1-q)^n\langle W|(D+E)^{n-1}|V\rangle$, each higher order term grows at most as fast as the constant term C_n. On the other side, the coefficient of q^m in $(1-q)^{-n}$ is equivalent to $n^m/m!$. □

Since any occurrence of the pattern 13-2 in a permutation is also an occurrence of the pattern 1-3-2, a permutation with k occurrences of the pattern 1-3-2 has at most k occurrences of the pattern 13-2. So we get the following corollary.

Corollary 4 *Let $\psi_k(n)$ be the number of permutations in $\mathfrak{S}_n$ with at most k occurrences of the pattern 1-3-2. For any constant $C > 1$ and $k \geq 0$, we have:*

$$\psi_k(n) \leq C\frac{4^n n^{k-\frac{3}{2}}}{\sqrt{\pi}k!}$$

when n is sufficiently large.

So far we have only used Corollary 1. Now we illustrate what we can do with the refined formula given in Theorem 1. First, when $q = 0$ the coefficient of y^m is $\sum_{k=0}^{n}(-1)^k\left(\binom{n}{m}\binom{n}{m+k} - \binom{n}{m-1}\binom{n}{m+k+1}\right)$. This is equal to the Narayana number $N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$. See [26] for a combiantorial proof.

We can also get the coefficients of higher degree in q. For example it is conjectured in [26] that the coefficient of qy^m in $\langle W|y(yD+E)^{n-1}|V\rangle$ is equal to $\binom{n}{m+1}\binom{n}{m-2}$. In view of our results we can prove:

Proposition 12 *The coefficients of qy^m and q^2y^m in $\langle W|y(yD+E)^{n-1}|V\rangle$ are respectively:*

$$\binom{n}{m+1}\binom{n}{m-2} \qquad \textit{and} \qquad \binom{n+1}{m-2}\binom{n+1}{m+2}\frac{nm+m-m^2-4}{2(n+1)}.$$

Proof: A naive expansion of the Taylor series in q gives a lengthy formula, which is simplified easily after noticing it is the product of $\binom{n}{m}^2$ and a rational fraction of n and m. □

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