MAS115 Calculus I

Week 9

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Revision

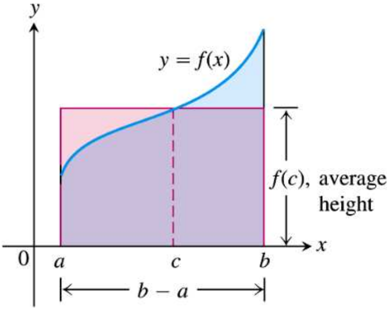
- Limits of Finite Sums
- Riemann Sums
- The Definite Integral
- Area under a Graph
- Average Value of a Function

The Mean Value Theorem for Definite Integrals

Theorem

If f is continuous on $[a, b]$, then there is a $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \ .$$



f assumes its average value somewhere on $[a, b]$.

The Mean Value Theorem for Definite Integrals

Theorem

If f is continuous on $[a, b]$, then there is a $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx \; .$$

Proof.

Write

$$\mathrm{av}(f) = \frac{1}{b-a} \int_a^b f(x)dx$$

for the average value of f on $[a, b]$. The Max-Min-Inequality says that

$$\min(f) \leq \mathrm{av}(f) \leq \max(f) \; .$$

By the intermediate value theorem for continuous functions, there is a $c \in [a, b]$ such that

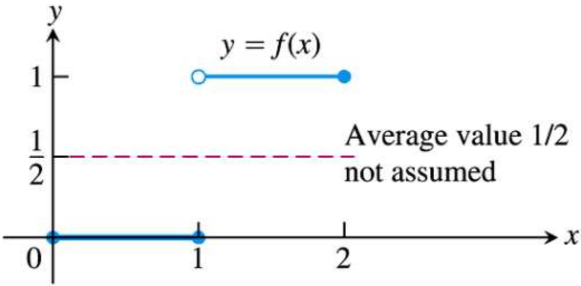
$$f(c) = \mathrm{av}(f) \; . \qquad \square$$

The Mean Value Theorem for Definite Integrals

Theorem

If f is continuous on $[a, b]$, then there is a $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx\ .$$



Continuity of f is necessary.

Example

Let f be continuous on $[a, b]$ with $a \neq b$. If

$$\int_a^b f(x)dx = 0$$

then there is a $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx = 0$$

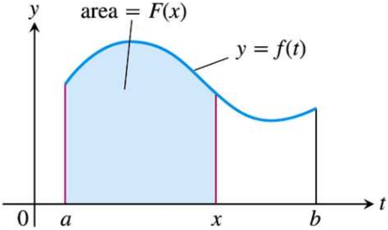
so that $f(x) = 0$ at least once in $[a, b]$.

The Fundamental Theorem of Calculus

For a continuous function f, define

$$F(x) = \int_a^x f(t)dt \; .$$

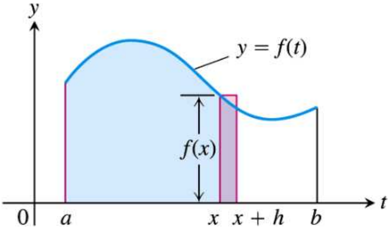
Geometric interpretation:



Now compute $F'(x)$ ...

The Fundamental Theorem of Calculus

Computation of $F'(x)$:



$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)$$
$$= \frac{1}{h}\int_x^{x+h} f(t)dt = f(c)$$

for some c with $x \le c \le x+h$ (why?). As $h \to 0$, $f(c) \to f(x)$ and therefore

$$F'(x) = f(x)$$

The Fundamental Theorem of Calculus

We have just proved

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Examples:

$$\frac{d}{dx}\int_a^x \cos t\,dt = cos(x)$$

$$\frac{d}{dx}\int_a^x \frac{1}{1+t^2}dt = \frac{1}{1+x^2}$$

More Examples

$$\begin{aligned}\frac{d}{dx}\int_x^5 3t\sin t\,dt &= -\frac{d}{dx}\int_5^x 3t\sin t\,dt\\ &= -3x\sin x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\int_1^{x^2}\cos t\,dt &= \left(\frac{d}{du}\int_1^u \cos t\,dt\Big|_{u=x^2}\right)\left(\frac{d}{dx}x^2\right)\\ &= \cos u\big|_{u=x^2}2x = 2x\cos(x^2)\end{aligned}$$

The Fundamental Theorem of Calculus

- So far we know that

$$\int_a^x f(t)dt = G(x)$$

is an antiderivative of f (as $G'(x) = f(x)$).

- The most general antiderivative is $F(x) = G(x) + C$.

- We have

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt \,. \end{aligned}$$

The Fundamental Theorem of Calculus

We have just proved

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Recipe to calculate

$$\int_a^b f(x)dx \quad :$$

1. Find any antiderivative F of f
2. Calculate $F(b) - F(a)$

Notation:

$$F(b) - F(a) = F(x)\big|_a^b$$

Examples

$$\int_0^\pi \cos x \, dx = \sin x|_0^\pi = \sin \pi - \sin 0 = 0$$

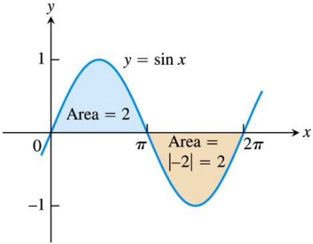
$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left(x^{3/2} + \frac{4}{x}\right)\Big|_1^4 \\ &= \left(4^{3/2} + \frac{4}{4}\right) - \left(1^{3/2} + \frac{4}{1}\right) = 4\end{aligned}$$

Finding Total Area

Note: $f(c_k) > 0 \quad \Rightarrow \quad f(c_k)\Delta x_k$ is area

$f(c_k) < 0 \quad \Rightarrow \quad f(c_k)\Delta x_k$ is negative area

Example:



compute the area between the x-axis and $y = \sin x$ over $[0, 2\pi]$:

$$\begin{aligned} A &= \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} (-\sin x) dx \\ &= -\cos x|_0^{\pi} + \cos x|_{\pi}^{2\pi} \\ &= (-cos\pi + \cos 0) + (\cos 2\pi - \cos \pi) = 4 \end{aligned}$$

Note that $\int_0^{2\pi} \sin x\, dx = 0$ does not give the total area.

Finding Total Area

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

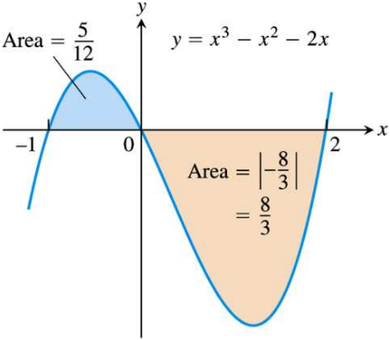
1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate over each subinterval.
3. Add the *absolute* value of the integrals.

Example: $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$

1. $f(x) = x(x+1)(x-2)$: zeros are -1, 0, 2
2. $$\int_{-1}^{0} (x^3 - x^2 - 2)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} = \frac{5}{12}$$
   $$\int_{0}^{2} (x^3 - x^2 - 2)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Bigg|_{0}^{2} = -\frac{8}{3}$$
3. $A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$

Finding Total Area

Example: $f(x) = x^3 - x^2 - 2,\ -1 \leq x \leq 2$



$$A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

Revision

- The Mean Value Theorem for Definite Integrals
- The Fundamental Theorem of Calculus
- Finding Total Area

Substitution Rule for Indefinite Integrals

- Recall the chain rule for $F(g(x))$:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

- If F is an antiderivative of f, then

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x)$$

- Now compute

$$\int f(g(x))g'(x)dx = \int \left(\frac{d}{dx}F(g(x))\right) dx$$

$$= F(g(x)) + C = F(u) + C = \int F'(u)du = \int f(u)du$$

where $u = g(x)$

Substitution Rule for Indefinite Integrals

We have just proved

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Method for

$$\int f(g(x))g'(x)dx \quad :$$

1. Substitute $u = g(x)$, $du = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace $u = g(x)$.

Example

Evaluate

$$\int \cos(7\theta + 5)d\theta \ :$$

1. Substitute $u = 7\theta + 5$, $du = 7d\theta$ to obtain

$$\int \cos(7\theta + 5)d\theta = \int \cos u \, \frac{1}{7} du$$

2. Integrate with respect to u:

$$\int \cos u \, \frac{1}{7} du = \frac{1}{7} \sin u + C$$

3. Replace $u = 7\theta + 5$:

$$\int \cos(7\theta + 5)d\theta = \frac{1}{7} \sin(7\theta + 5) + C$$

Example – Different Substitutions

Evaluate

$$\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz \ :$$

- Substitute $u = z^2 + 1$, $du = 2z\,dz$:

$$\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int u^{-1/3}\,du$$
$$= \frac{3}{2}u^{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C$$

- Substitute $u = \sqrt[3]{z^2+1}$ or $u^3 = z^2 + 1$, so that $3u^2du = 2z\,dz$:

$$\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int \frac{3u^2}{u}\,du = 3\int u\,du$$
$$= \frac{3}{2}u^2 + C = \frac{3}{2}(z^2+1)^{2/3} + C$$

Integration Using Identities

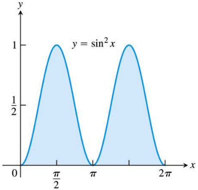
Evaluate
$$\int \sin^2 x \, dx \quad :$$

Use $2\sin^2 x = 1 - \cos 2x$ to write

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) dx$$
$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Compute the area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$:

$$\int_0^{2\pi} \sin^2 x \, dx = \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{2\pi} = \pi$$



Substitution in Definite Integrals

Theorem

If g is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \,.$$

Proof.

For F with $F' = f$ we have

$$\begin{aligned}\int_a^b f(g(x))g'(x)dx &= F(g(x))\big|_{x=a}^{x=b} \\ &= F(u)\big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)du\end{aligned}$$

$\square$

Example

Evaluate

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx \;\; :$$

- substitute $u = x^3 + 1$, $du = 3x^2 dx$

$$\begin{array}{lll} x = -1 & \text{gives} & u = (-1)^3 + 1 = 0 \\ x = 1 & \text{gives} & u = 1^3 + 1 = 2 \end{array}$$

- we obtain

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx = \int_0^2 \sqrt{u}du$$

$$= \frac{2}{3}u^{3/2}\Big|_0^2 = \frac{2}{3}2^{3/2} - 0 = \frac{4\sqrt{2}}{3}$$

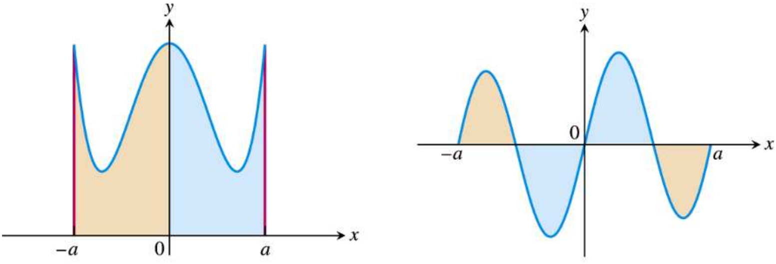
Integrals of Symmetric Functions

Theorem

Let f be continuous on the symmetric interval $[-a, a]$.

(a) *If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.*

(b) *If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.*



(a) (b)

Integrals of Symmetric Functions

Theorem

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(a) *If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.*

(b) *If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.*

Proof.

(a) Split $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$. and compute

$$\begin{aligned}\int_{-a}^{0} f(x)dx &= -\int_0^{-a} f(x)dx = -\int_0^a f(-u)(-du) \\ &= \int_0^a f(-u)du = \int_0^a f(u)du\end{aligned}$$

(b) similarly. $\square$

Integrals of Symmetric Functions

Theorem

Let f be continuous on the symmetric interval $[-a, a]$.

(a) *If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.*

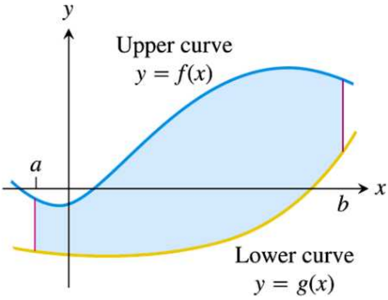
(b) *If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.*

This is very useful for simplifying calculations. For example,

$$\int_{-123456}^{123456} \left(\sin x + x^3 + \frac{1}{2} - \frac{x}{1+x^2}\right) dx$$

$$= \quad 2\int_{0}^{123456} \frac{1}{2}\, dx = 123456 \ .$$

Area Between Curves



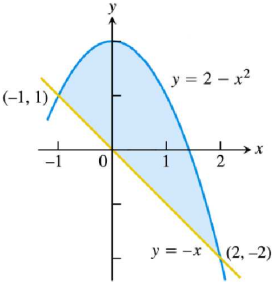
DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example

Find the area enclosed by $y = 2 - x^2$ and $y = -x$:



- solve $2 - x^2 = -x$:

$$x = -1\,, \quad x = 2$$

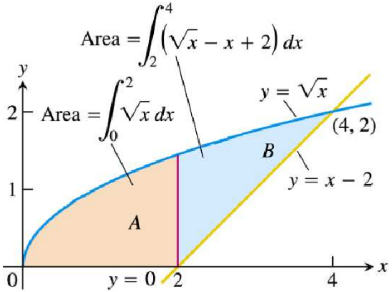
- On $[-1, 2]$, $2 - x^2 \geq -x$ and therefore

$$A = \int_{-1}^{2} \left[(2 - x^2) - (-x)\right] dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left(2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{-1}^{2} = \frac{9}{2}\,.$$

Example

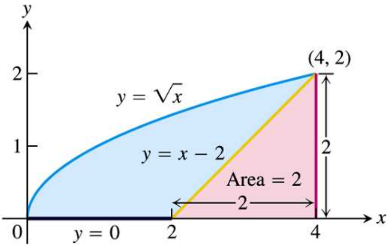
Find the area that is enclosed above by $y = \sqrt{x}$ and below by $y = 0$ and $y = x - 2$:



$$\begin{aligned} A &= \int_0^2 \sqrt{x} dx + \int_2^4 \sqrt{x} - (x-2) dx \\ &= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right)\Big|_2^4 = \frac{10}{3}\,. \end{aligned}$$

Example

Find the area that is enclosed above by $y = \sqrt{x}$ and below by $y = 0$ and $y = x - 2$: use some geometry!



The area below the parabola is $A_1 = \int_0^4 \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}$.
The area of the triangle is $A_2 = 2$, so that

$$A = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3} \ .$$

Revision

- Substitution rule for indefinite and definite integrals
- Area between curves
- Integration tricks: identities, symmetric functions, geometry

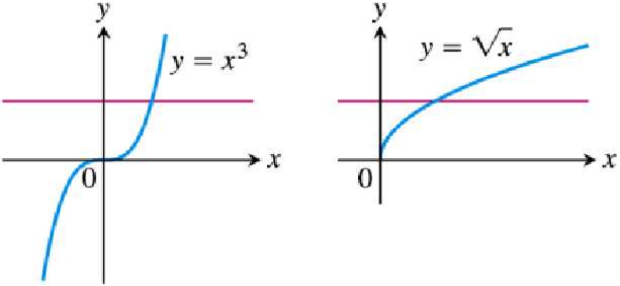
One-to-One Functions

DEFINITION One-to-One Function
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

These functions take on any value in their range exactly once.

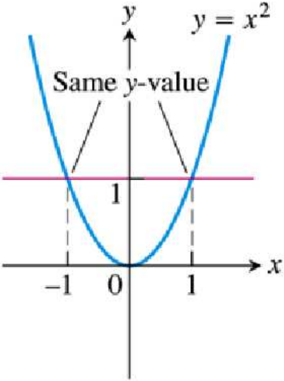
The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

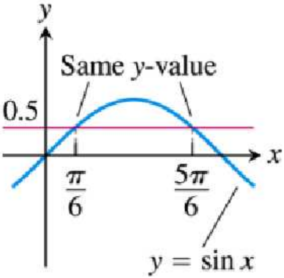
Examples



- $y = x^3$ one-to-one on $\mathbb{R}$
- $y = \sqrt{x}$ one-to-one on $\mathbb{R}_0^+$

Examples





- $y = x^2$ one-to-one on e.g. $\mathbb{R}_0^+$, but not $\mathbb{R}$
- $y = \sin x$ one-to-one on e.g. $[0, \pi/2]$, but not $\mathbb{R}$

Inverse Function

DEFINITION **Inverse Function**

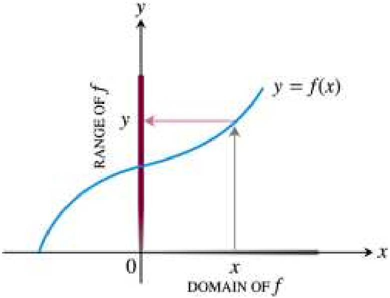
Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

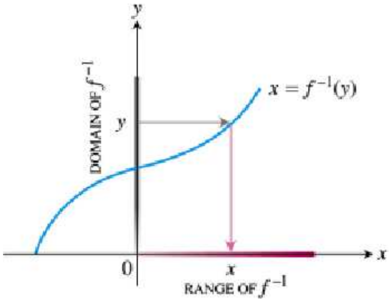
- f^{-1} read "f inverse"
- $f^{-1}(x)$ is not $(f(x))^{-1} = 1/f(x)$ (not an exponent)!
- $(f^{-1} \circ f)(x) = x$ for all $x \in D(f)$
- $(f \circ f^{-1})(x) = x$ for all $x \in R(f)$

Finding Inverses Graphically



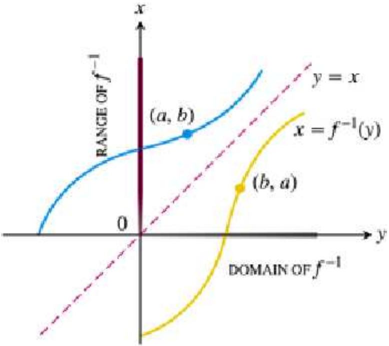
Change from $y = f(x)$ to $x = f^{-1}(y)$ ...

Finding Inverses Graphically



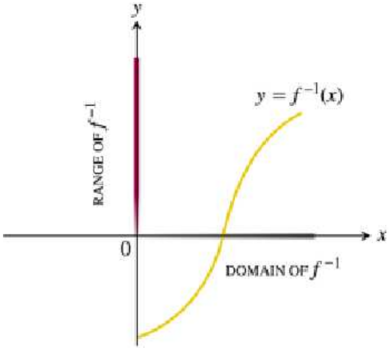
Note that $D(f) = R(f^{-1})$ and $R(f) = D(f^{-1})$.
Now reflect along $y = x$ ...

Finding Inverses Graphically



After reflection, x and y have changed places.
Swap x and y ...

Finding Inverses Graphically



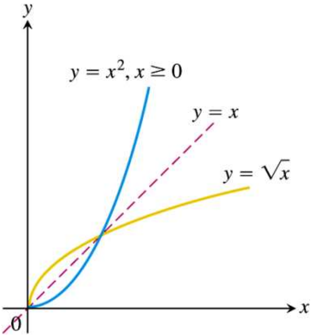
… and we arrive at $y = f^{-1}(x)$

Finding Inverses Algebraically

1. Solve $y = f(x)$ for x: $x = f^{-1}(y)$
2. Interchange x and y: $y = f^{-1}(x)$

Example: find the inverse of $y = x^2$, $x \geq 0$

1. Solve $y = f(x)$ for x: $\sqrt{y} = \sqrt{x^2} = |x| = x$ as $x \geq 0$
2. Interchange x and y: $y = \sqrt{x}$



Derivatives of Inverse Functions

Use implicit differentiation for $y = f^{-1}(x)$:

$$x = f(y) \qquad \Big| \frac{d}{dx}$$

$$1 = f'(y)\frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

Now $x = f(y)$ means $y = f^{-1}(x)$ so that

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$

Derivatives for Inverse Functions

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

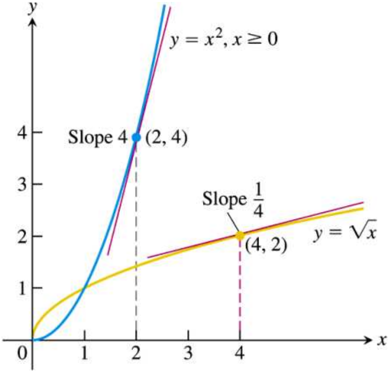
or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Example

Return to $f(x) = x^2$, $x \geq 0$:
$f^{-1}(x) = \sqrt{x}$ and $f'(x) = 2x$, so that

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2f^{-1}(x)} = \frac{1}{2\sqrt{x}}$$



Natural Logarithms

- For $a \in \mathbb{Q} \setminus \{-1\}$, we know that

$$\int_1^x t^a dt = \frac{1}{a+1}\left(x^{a+1} - 1\right) \ .$$

- What happens if $a = -1$?

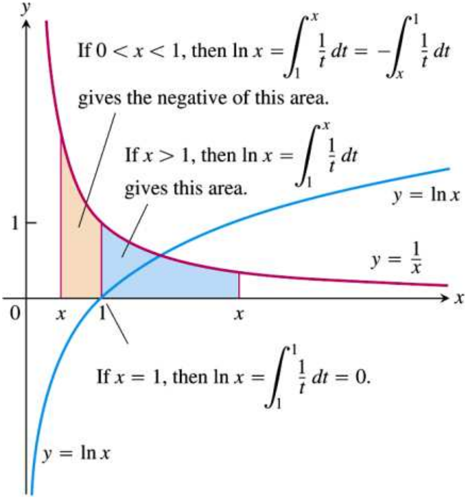
$$\int_1^x \frac{1}{t} dt \quad \text{is well defined for } x > 0.$$

But what is it?

DEFINITION **The Natural Logarithm Function**

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0$$

The Graph of $y = \ln x$



A special value: the number $e = 2.718281828459\ldots$:

$$\ln e = 1$$

Properties of $\ln x$

- Differentiating $\ln x$ is easy:

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}\ .$$

- If $u(x) > 0$, by the chain rule $\frac{d}{dx}\ln u = \frac{1}{u}u'$.
- In particular, if $u(x) = ax$ with $a > 0$,

$$\frac{d}{dx}\ln ax = \frac{1}{ax}a = \frac{1}{x}\ .$$

- $\ln ax$ and $\ln x$ have the same derivative, so that

$$\ln ax = \ln x + C\ .$$

  Substituting $x = 1$ we see that $C = \ln a1 - \ln 1 = \ln a$ and therefore

$$\ln ax = \ln a + \ln x$$

Properties of Logarithms

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

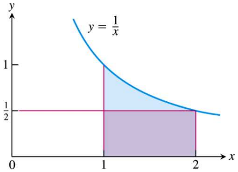
1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ r rational

Examples

$$\begin{aligned}
\ln 6 &= \ln(2\cdot 3) = \ln 2 + \ln 3\\
\ln 4 - \ln 5 &= \ln\frac{4}{5} = \ln 0.8\\
\ln\frac{1}{8} &= -\ln 8 = -\ln 2^3 = -3\ln 2
\end{aligned}$$

$$\begin{aligned}
\ln 4 + \ln\sin x &= \ln(4\sin x)\\
\ln\frac{x+1}{2x+3} &= \ln(x+1) - \ln(2x+3)\\
\ln\sqrt[3]{x+1} &= \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)\\
\ln\cot x &= \ln\frac{1}{\tan x} = -\ln\tan x
\end{aligned}$$

The Range of $\ln x$



- $\ln 2 > \frac{1}{2}$
- $\ln 2^n = n \ln 2$ and $\ln 2^{-n} = -n \ln 2$
- it follows that
$$\lim_{x\to\infty} \ln x = \infty$$
and
$$\lim_{x\to 0^+} \ln x = -\infty$$

Therefore, the range of ln is $\mathbb{R}$.

The Indefinite Integral $\int \frac{1}{x} dx$

- For $t > 0$, we already know

$$\int \frac{1}{t} dt = \ln t + C$$

- For $t < 0$, $(-t)$ is positive and we find

$$\int \frac{1}{t} dt = \int \frac{1}{(-t)} d(-t) = \ln(-t) + C$$

Together, this gives

$$\int \frac{1}{t} dt = \ln |t| + C$$

Substituting $t = f(x)$, $dt = f'(x)dx$ leads to a very useful formula:

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

Applications

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Substitute $t = \cos x$, $dt = -\sin x \, dx$:

$$\int \tan x \, dx = -\int \frac{1}{t} dt = -\log|t| + C = -\log|\cos x| + C$$

$$\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C = -\ln|\csc x| + C$$

The End