Indeterminate Forms and L'Hôpital's Rule

If $f(a)=0$ and $g(a)=0$, how can we compute

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad ?$$

Idea: consider linearisation

$$f(a+\Delta x) \approx \cancel{f(a)} + f'(a)\Delta x = f'(a)\Delta x$$

$$g(a+\Delta x) \approx \cancel{g(a)} + g'(a)\Delta x = g'(a)\Delta x$$

therefore

$$\frac{f(a+\Delta x)}{g(a+\Delta x)} \approx \frac{f'(a)\cancel{\Delta x}}{g'(a)\cancel{\Delta x}} = \frac{f'(a)}{g'(a)}$$

Can we prove this?

Theorem : [4-68]

Proof:

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{\cancel{x-a}}}{\frac{g(x)-g(a)}{\cancel{x-a}}} = \lim_{x\to a} \frac{f(x)-\overbrace{f(a)}^{=0}}{g(x)-\underbrace{g(a)}_{=0}}$$

$$= \lim_{x\to a} \frac{f(x)}{g(x)} \qquad \square$$

Caution: do NOT compute

$$\left(\frac{f}{g}\right)'(x), \quad \text{this is not} \quad \frac{f'(x)}{g'(x)}$$

(I've seen this with 2nd year students!)

<u>Examples</u>

"0/0" ✓ $\displaystyle \lim_{x\to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}\Bigg|_{x=0} = \frac{3-1}{1} = 2$

"0/0" ✓ $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1}\Bigg|_{x=0} = \frac{1}{2}$

not "0/0" $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{1-x} = \cancel{\frac{\sin x}{1}\Big|_{x=0} = 1} = 0 \quad \frac{0}{1}$

"0/0" ✓ $\displaystyle \lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1-\cos x}{3x^2}\Bigg|_{x=0}$ whoops!

2, 1/2

Theorem: [4.69]

Example:

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \qquad \text{"}0/0\text{"}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \qquad \text{"}0/0\text{"}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \qquad \text{"}0/0\text{"}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

To prove the stronger form of l'Hôpital's rule, we need (without proof)

Cauchy's Mean Value Theorem [4-70]

Let f and g be continuous on $[a,b]$ and differentiable on (a,b), with $g'(x) \neq 0$ on (a,b). Then there is a $c \in (a,b)$ with

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

MVT get from $g(x) = x$

Proof: let $x > a$. Then for some $c \in (a,x)$,

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$$ and

$$\lim_{x \to a^+} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{c \to a} \frac{f'(c)}{g'(c)}$$

Case $x<a$ similarly. $\square$

So far, indeterminate form "$0/0$".

What about "∞/∞", "$\infty \cdot 0$", "$\infty - \infty$"?

"$\frac{\infty}{\infty}$": use

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{1/g(x)}{1/f(x)} \qquad \text{"}0/0\text{"}$$

"$\infty \cdot 0$": use

$$\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} \frac{g(x)}{1/f(x)} \qquad \text{"}0/0\text{"}$$

"$\infty - \infty$": simplify (later: use exponentiation)

$$\lim_{x \to a} \big(f(x) - g(x)\big) = \lim_{x \to a} \left(\frac{1}{1/f(x)} - \frac{1}{1/g(x)} \right)$$

$$= \lim_{x \to a} \frac{1}{\frac{1}{f(x)} \frac{1}{g(x)}} \left(\frac{1}{g(x)} - \frac{1}{f(x)} \right)$$

looks cumbersome ...

Examples and Tricks:

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin 1/x}{1/x} = \begin{cases} \frac{\left(\cos\frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = \\ \lim_{h\to 0} \frac{\sin h}{h} = \end{cases}$$

"$\infty \cdot 0$"　　"$0/0$"

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \quad \text{"}0/0\text{"}$$

$x>0$　"$\infty - \infty$"

$x<0$　"$-\infty + \infty$"

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \quad \text{"}0/0\text{"}$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$$

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+x}\right) = \lim_{x\to\infty} \frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}$$

"$\infty - \infty$"

$$= \lim_{x\to\infty} \frac{-x}{x+\sqrt{x^2+x}} = \ldots$$

"∞/∞"

Antiderivatives

goal: given that $f = F'$, find F

Definition [4-88]: If $F'(x) = f(x)$ on an interval I, then F is called **antiderivative** of f on I

A Corollary of the Mean Value Theorem was:

> Any two antiderivatives of a function differ by a constant

Consequently: [4-89]

Finding antiderivatives:

- Table of formulas, e.g. [4-90]

- List of rules:

  basic rules [4-91]

  (later, more advanced techniques)

Example:

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x$$

$$= 3\,g(x) + h(x)$$

$$g(x) = \frac{1}{\sqrt{x}}, \quad G(x) = 2\sqrt{x} + C_1$$

$$h(x) = \sin 2x, \quad H(x) = -\frac{1}{2}\cos 2x + C_2$$

Therefore $F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C$

$$C = C_1 + C_2$$

Initial value problems and Differential equations

$$\frac{dy}{dx} = f(x)$$ differential equation for unknown $y(x)$

$$y(x_0) = y_0$$ initial condition

Example Find the curve whose slope at (x,y) is $3x^2$ if $(1,-1)$ lies on the curve:

$$\frac{dy}{dx} = 3x^2$$

$$y(1) = -1$$

Solution: $\left.\begin{aligned} y(x) &= x^3 + C_1 \\ y(1) &= -1 \end{aligned}\right\}$ $\underline{y(x) = x^3 - 2}$

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of f [4.93]

$$\int (2x)\,dx = x^2 + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int (2x + \cos x)\,dx = x^2 + \sin x + C$$

Using linearity rules, we also have

$$\int (2x + \cos x)\,dx = 2\int x\,dx + \int \cos x\,dx$$

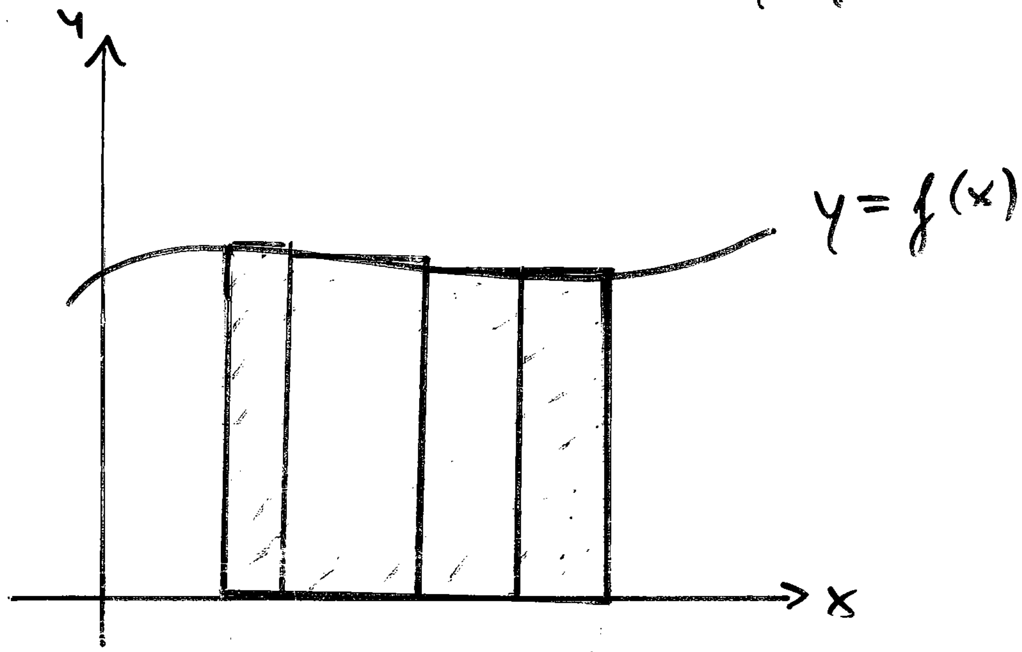
$$= x^2 + C_1 + \sin x + C_2$$

$$= x^2 + \sin x + C$$

these are different

Integration

Estimating an area between the x-axis and the curve $y=f(x)$:



Idea: approximate by
"lots of small rectangles"

"more rectangles" $\rightsquigarrow$ "better approximation"

[5-4, 5, 6, 7] [flash animation]

Summary:

- Subdivide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$

- Choose points c_k in the k-th subinterval

- Form the sum

$$f(c_1)\Delta x + f(c_2)\Delta x + \ldots f(c_n)\Delta x$$

- upper sum: choose c_k such that $f(c_k)$ is maximal

- lower sum: choose c_k such that $f(c_k)$ is minimal

- midpoint rule: choose c_k in the middle of the interval

Sigma-notation

To handle sums with many terms,

we need a better notation [5-14]:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

So, instead of writing

$$f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x$$

we can write

$$\sum_{k=1}^{n} f(c_k)\,\Delta x$$

This needs practice [5-15]

and rules [5-16]

Example

Express the sum

$$1 + 3 + 5 + 7 + 9$$

in sigma-notation:

$$\sum_{k=0}^{4} (2k+1) \quad , \text{ or}$$

$$\sum_{k=1}^{5} (2k-1) \quad , \text{ or}$$

$$\sum_{r=-3}^{1} (2r+7)$$

these sums are all equal!

<u>Example</u>: The sum of the first n integers

$$S = 1 + 2 + 3 + \dots + (n-1) + n$$

$$+ \quad +$$

$$S = n + (n-1) + (n-2) + \dots + 2 + 1$$

n terms $\quad n+1 \quad n+1 \quad n+1 \quad \dots \quad n+1$

- $2s = (n+1)\,n$, or $s = \dfrac{n(n+1)}{2}$

[Carl-Friedrich Gauß, ≈ 1784, 7 years old]

This shows $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

- other simple sums see [5-17]

Such formulae can be proved by <u>mathematical induction</u>.

Limits of finite sums

Example: compute the area below the graph of $y = 1 - x^2$ and above the interval $[0,1]$.

- subdivide the interval into n subintervals of width $\Delta x = \frac{1}{n}$:

$$\left[0, \tfrac{1}{n}\right], \left[\tfrac{1}{n}, \tfrac{2}{n}\right], \left[\tfrac{2}{n}, \tfrac{3}{n}\right], \ldots, \left[\tfrac{n-1}{n}, \tfrac{n}{n}\right]$$

- choose lower sum:

  c_k is rightmost point, $c_k = \frac{k}{n}$

$$\left(\text{i.e. } c_1 = \tfrac{1}{n},\ c_2 = \tfrac{2}{n},\ c_3 = \tfrac{3}{n},\ \ldots,\ c_n = 1\right)$$

- do the summation:

$$\sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}$$

$$= \sum_{k=1}^{n} \left(1-\left(\frac{k}{n}\right)^2\right)\frac{1}{n}$$

$$= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n}\, n - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \ldots = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

- lower sum $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$

- upper sum $\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$

  [not done here]

- as $n \to \infty$, both sums tend to $\frac{2}{3}$

- any other choice of c_k would give the same result, as the result would be between the lower and upper sums

- Therefore, the area is equal to $\frac{2}{3}$

Riemann sums

[5-18]

- allow f to be +ve or −ve
- partition the interval $[a,b]$ by [5-19]

  choosing $n-1$ points between a and b:

  $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

  (i.e. Δx may vary!)

- choose finer and finer partitions [5-21,22]
- The resulting sums are called

  <u>Riemann sums for f on $[a,b]$</u>

- Take the limit such that the width of the largest subinterval goes to zero.
- notation: partition $P = \{x_0, x_1, \dots, x_n\}$

  width of the largest interval: $\|P\|$

The definite integral

Definition : [5-24]

Notation : [5-25]

Shorthand:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx$$

Note: as in $\sum$-notation,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \ldots$$

is independent of the letter chosen for the "dummy variable"

Question: when does the integral exist?

Answer: when f is continuous on $[a,b]$

[5-26]

Question: when does the integral fail to exist?

Answer: when f is "sufficiently" discontinuous

<u>Example</u>

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$\int_0^1 f(x)\,dx$ does not exist, as

upper sum is always $\sum_{k=1}^{n} 1\,\Delta x_k = 1$

lower sum is always $\sum_{k=1}^{n} 0\,\Delta x_k = 0$

Properties of definite integrals

- extend $\int_a^b f(x)\,dx$ to $b<a$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

[reason: when changing a & b, each Δx_k changes sign, leading to an overall change of sign]

- define $\int_a^a f(x)\,dx = 0$ (makes sense)

- Table of properties: [5-28]

  graphical interpretation: [5-29]

Area under the graph of a non-negative function $f(x)$ over a closed interval $[a,b]$ is now <u>defined</u> as

$$A = \int_a^b f(x)\,dx$$

<u>Example:</u> $f(x) = x$, $a = 0$, $b > 0$ [5-31]

(a) graphically, $A = \frac{1}{2} b^2$

(b)

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \underset{\substack{\uparrow \\ f(c_n)}}{\frac{kb}{n}} \; \underset{\substack{\uparrow \\ \Delta x}}{\frac{b}{n}}$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2} \frac{n(n+1)}{2} = \frac{b^2}{2}$$

Average value of a continuous function

The average value of $f(x)$ on $[a,b]$ is given by [5-34]: $\Delta x = \frac{b-a}{n}$

$$\lim_{n\to\infty} \frac{1}{n}\left(f(c_1)+f(c_2)+\dots+f(c_n)\right)$$

$$= \lim_{n\to\infty} \frac{\Delta x}{b-a}\left(f(c_1)+f(c_2)+\dots+f(c_n)\right)$$

$$= \frac{1}{b-a} \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x}_{\int_a^b f(x)\,dx}$$

Definition: [5-35]