MAS115 Calculus I

Week 6

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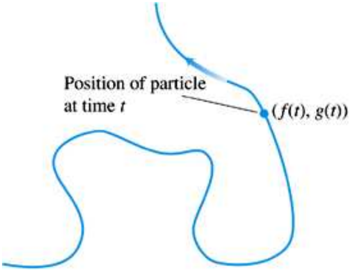
Revision

- Rules of Differentiation
- Higher Derivatives
- Derivatives of Trigonometric Functions
- The Chain Rule

Parametric equations



Describe a point moving in the xy-plane as a function of a parameter t ("time") by two functions

$$x = x(t)\ , \quad y = y(t)$$

This may be the graph of a function, but it need not be.

Parametric Curve

DEFINITION Parametric Curve
If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

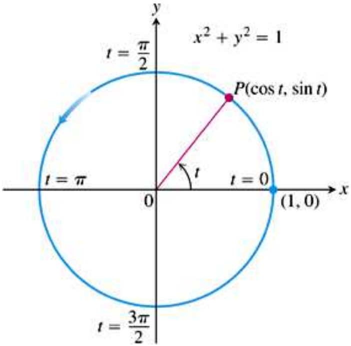
- t is called a parameter for the curve.
- If $t \in [a, b]$, then

$(f(a), g(a))$ is the initial point

$(f(b), g(b))$ is the terminal point

Example: Motion on a Circle

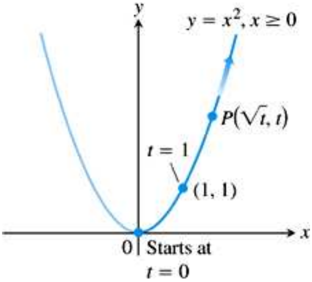
$$x = \cos t\ , \quad y = \sin t\ , \quad 0 \leq t \leq 2\pi$$



describes motion on a circle with radius 1:
The motion starts at initial point $(1, 0)$ and traverses the circle $x^2 + y^2 = 1$ anti-clockwise once, ending at the terminal point $(1, 0)$.

Example: Moving along a parabola

$$x = \sqrt{t}\,, \quad y = t\,, \quad t \geq 0$$



solve this as $y = f(x)$:

$$y = t = (\sqrt{t})^2 = x^2$$

Note that the domain of f is $[0, \infty)$

Parametrising a Line Segment

Find a parametrisation for the line segment from $(-2, 1)$ to $(3, 5)$:

- start at $(-2, 1)$ for $t = 0$

$$x = -2 + at\ , \quad y = 1 + bt$$

- end at $(3, 5)$ for $t = 1$

$$3 = -2 + a\ , \quad 5 = 1 + b$$

- we conclude $a = 5$ and $b = 4$

Solution: $x = -2 + 5t$, $y = 1 + 4t$, $0 \leq t \leq 1$

Slopes of Parametrised Curves

A parametrised curve $x = f(t)$, $y = g(t)$ is differentiable at t if f and g are differentiable at t.
If y is a differentiable function of x, say $y = h(x)$, then $y = h(x(t))$ and by the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Therefore, if $\frac{dx}{dt} \neq 0$

Parametric Formula for dy/dx
If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Example: Moving along an Ellipse

Compute the slope at a point (x, y) of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Parametrisation (use $\cos^2 t + \sin^2 t = 1$):

$$x = a\cos t\ , \quad y = b\sin t\ , \quad 0 \leq t \leq 2\pi$$

- Differentiate $\frac{dx}{dt} = -a\sin t\ , \quad \frac{dy}{dt} = b\cos t$ and therefore

$$\frac{dy}{dx} = \frac{b\cos t}{-a\sin t}$$

- Eliminating t, we obtain

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

Higher derivatives

$$y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\,, \qquad y'' = \,?$$

Remember $y'' = (y')'$:

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Example continued: $y' = -\frac{b}{a}\frac{\cos t}{\sin t}$ gives

$$y'' = \frac{\frac{d}{dt}\left[-\frac{b}{a}\frac{\cos t}{\sin t}\right]}{-a\sin t} = -\frac{b}{a^2}\frac{1}{\sin^3 t} = -\frac{b^4}{a^2}\frac{1}{y^3}$$

Summary of Parametrisation

Standard Parametrizations and Derivative Rules

CIRCLE $x^2 + y^2 = a^2$:

$$x = a\cos t \qquad y = a\sin t \qquad 0 \le t \le 2\pi$$

ELLIPSE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$:

$$x = a\cos t \qquad y = b\sin t \qquad 0 \le t \le 2\pi$$

FUNCTION $y = f(x)$:

$$x = t \qquad y = f(t)$$

DERIVATIVES

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Implicit differentiation

We want to compute y' but don't have $y = f(x)$ but rather

$$F(x, y) = 0 \,,$$

an implicit relation between x and y.

- One way is parametrisation, for example $x = \cos t$, $y = \sin t$ gives

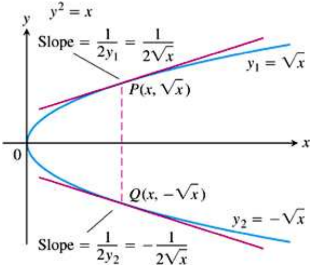
  $$F(x, y) = x^2 + y^2 - 1 = 0 \,.$$

  We've just done this.
- Another way, if no obvious parametrisation of $F(x, y) = 0$ is possible: $\boxed{\text{Differentiate the relation directly!}}$

Example

Given $y^2 = x$, compute y':

- Of course we know already that we have two solutions $y_{1,2} = \pm\sqrt{x}$ with derivatives $y'_{1,2} = \pm\frac{1}{2\sqrt{x}}$.



New method:

- differentiating $y^2 = x$ directly gives

$$2yy' = 1$$

- solve for y' to get

$$y' = \frac{1}{2y}$$

- Substituting $y = y_{1,2} = \pm\sqrt{x}$ gives the above result.

General recipe

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Return to ellipse: differentiate

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

directly

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

Solve for y':

$$y' = -\frac{b^2}{a^2}\frac{x}{y}$$

Higher Derivatives

Implicit differentiation also works for higher derivatives:
We had

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

Differentiate again:

$$\frac{2}{a^2} + \frac{2(y'^2 + yy'')}{b^2} = 0$$

Now insert $y' = -\frac{b^2}{a^2}\frac{x}{y}$ and simplify (this takes a few steps)

$$y'' = -\frac{b^4}{a^2}\frac{1}{y^3}$$

Notice that y' and y'' are identical to the result obtained using parametric equations.

Revision

- Parametric Equations
- Parametric Differentiation
- Implicit Relation
- Implicit Differentiation

Power rule for rational powers

Differentiate $y = x^{\frac{p}{q}}$ using implicit differentiation:

- write
$$y^q = x^p$$
- differentiate
$$qy^{q-1}y' = px^{p-1}$$
- solve for y':

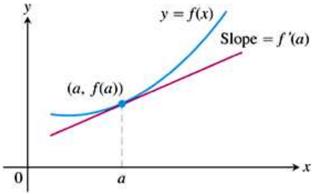
$$y' = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^p}{y^q}\frac{y}{x} = \frac{p}{q}\frac{y}{x} = \frac{p}{q}\frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}x^{\frac{p}{q}-1}$$

THEOREM 4 Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Linearisation



"Close to" the point $(a, f(a))$, the tangent

$$y = f(a) + f'(a)(x - a)$$

is a "good" approximation for $y = f(x)$

DEFINITIONS Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

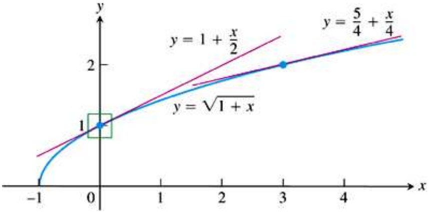
$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Example

Compute the linearisation for

$$f(x) = \sqrt{1+x}\,, \quad a = 0$$



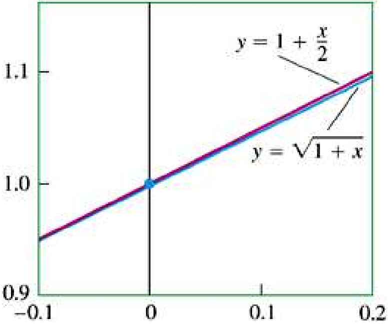
$$f(0) = 1\,, \quad f'(0) = \tfrac{1}{2}$$

$$L(x) = 1 + \frac{1}{2}x$$

So “near” $x = 0$ we have

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

Example



Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Differentials

- The derivative $y' = \frac{dy}{dx}$ is not a ratio!
- Introduce two new variables dx and dy with the property that if their ratio exists, it will be equal to the derivative:

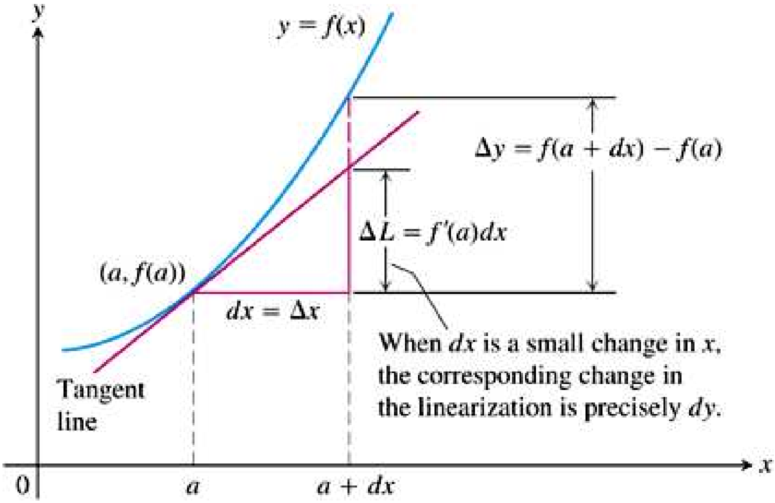
DEFINITION Differential

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Differentials

Geometrically, dy is the change in the linearisation of f if x changes by dx



Estimating with Differentials

- True value:

$$f(a+\Delta x) = f(a) + \Delta f$$

- Approximation:

$$\begin{aligned} f(a+\Delta x) &\approx f(a) + \Delta y \\ &= f(a) + f'(a)\Delta x \end{aligned}$$

- The approximation error is

$$\begin{aligned} \Delta f - f'(a)\Delta x &= f(a+\Delta x) - f(a) - f'(a)\Delta x \\ &= \underbrace{\left[\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right]}_{\epsilon} \Delta x \end{aligned}$$

- As $\Delta x \to 0$, we find that $\epsilon \to 0$.

Proof of the Chain Rule

Theorem

If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x then $(f \circ g)'(x) = f'(g(x))g'(x)$.

Proof.

We have $\Delta u = (g'(x) + \epsilon_1)\Delta x$ with $\epsilon_1 \to 0$ as $\Delta x \to 0$.
Similarly, $\Delta y = (f'(u) + \epsilon_2)\Delta u$ with $\epsilon_2 \to 0$ as $\Delta u \to 0$.
Together,

$$\frac{\Delta y}{\Delta x} = (f'(u) + \epsilon_2)(g'(x) + \epsilon_1) \ .$$

Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u)g'(x) \ .$$

$\square$

Extreme Values of Functions

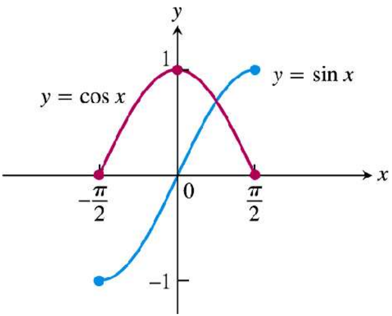
DEFINITIONS **Absolute Maximum, Absolute Minimum**

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

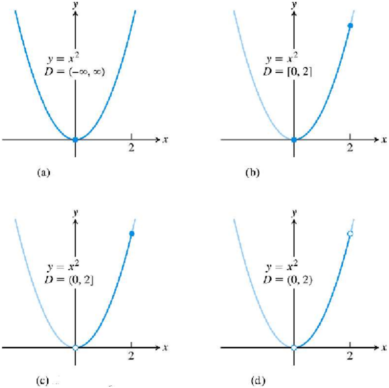
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$



Example

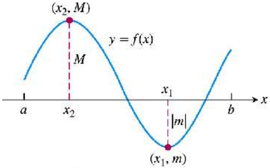


	Domain	abs. max.	abs. min.
(a)	$(-\infty, \infty)$	none	0, at 0
(b)	$[0, 2]$	4, at 2	0, at 0
(c)	$(0, 2]$	4, at 2	none
(d)	$(0, 2)$	none	none

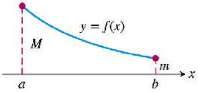
Existence of a Global Maximum/Minimum

THEOREM 1 The Extreme Value Theorem

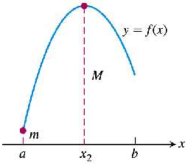
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).



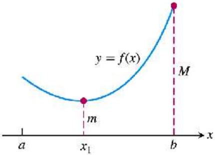
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

Local (Relative) Extreme Values

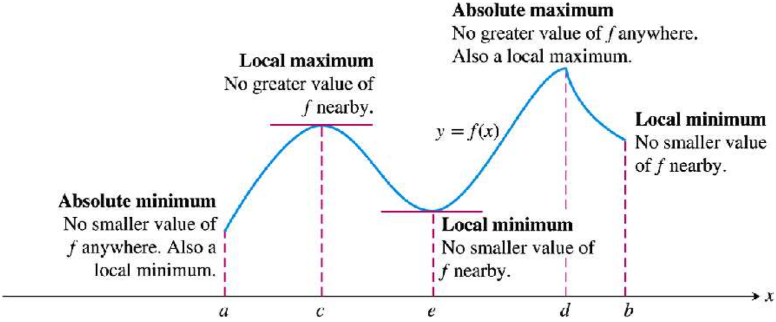
DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

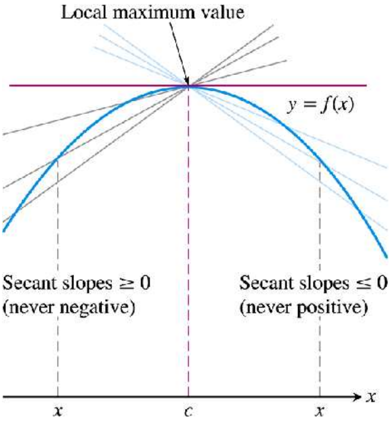
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$



Finding Extreme Values

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.



Finding Extreme Values

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof.

If at a local *maximum* c the derivative

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then

$$f'(c) = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

and

$$f'(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

so that $f'(c) = 0$. (Similarly for *minimum*.) $\square$

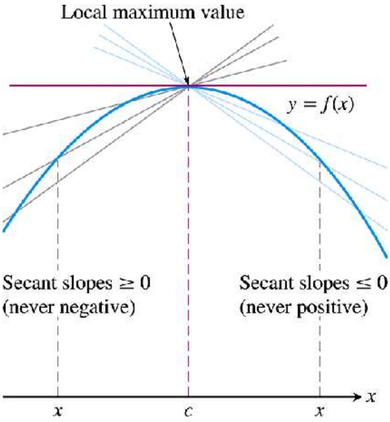
Revision

- Linearisation, Differentials
- Proof of the Chain Rule
- Local and Global Extrema

Finding Extreme Values

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.



Finding Extreme Values

Where can a function f possibly have an extreme value?

- at interior points where $f' = 0$
- at interior points where f' is not defined
- at endpoints of the domain of f.

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example

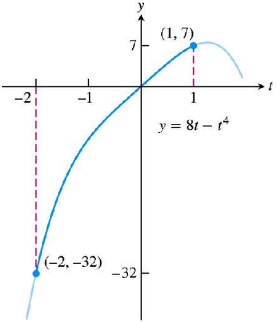
Find the absolute extrema of $f(x) = x^2$ on $[-2, 1]$:

- f is differentiable on $[-2, 1]$ with $f'(x) = 2x$
- critical point: $f'(x) = 0 \quad \Rightarrow \quad x = 0$
- endpoints $x = -2$, $x = 1$
- $f(0) = 0$, $f(-2) = 4$, $f(1) = 1$

Therefore f has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Example

Find the absolute extrema of $g(t) = 8t - t^4$ on $[-2, 1]$:

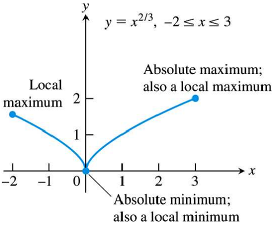


- g is differentiable on $[-2, 1]$ with $g'(t) = 8 - 4t^3$
- critical point: $g'(t) = 0 \quad \Rightarrow \quad t = \sqrt[3]{2} > 1$
- endpoints $t = -2$, $t = 1$
- $g(-2) = -32$, $g(1) = 7$

Therefore g has an absolute maximum value of 7 at $t = 1$ and an absolute minimum value of -32 at $t = -2$.

Example

Find the absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$:



- f is differentiable on $[-2, 0) \cup (0, 3]$ with $f'(x) = \frac{2}{3}x^{-1/3}$
- critical point: $f'(x) = 0$ or $f'(x)$ undefined $\Rightarrow \quad x = 0$
- endpoints $x = -2$, $x = 3$
- $f(-2) = \sqrt[3]{4}$, $f(0) = 0$, $f(3) = \sqrt[3]{9}$

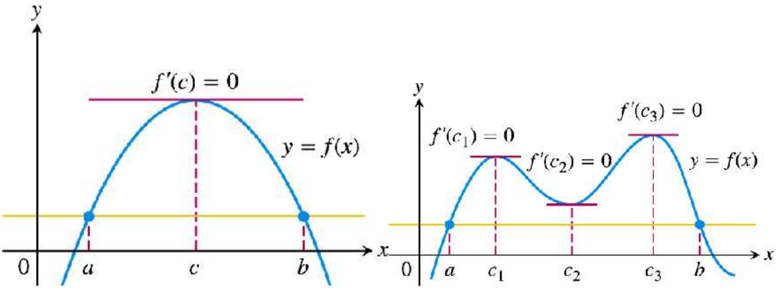
Therefore f has an absolute maximum value of $\sqrt[3]{9}$ at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Rolle's Theorem

Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

$$f'(c) = 0\ .$$



Rolle's Theorem

Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

$$f'(c) = 0 \; .$$

Proof.

- f is continuous on $[a, b]$, so it has absolute maximum and minimum.
- these occur only if $f'(x) = 0$ on (a, b), or else at a or b.
- if one of them occurs at $c \in (a, b)$, then $f'(c) = 0$ (and we're done).
- if not, both must occur at the endpoints. But as $f(a) = f(b)$, $f(x)$ must then be constant and therefore $f'(x) = 0$ on $[a, b]$.

$\square$

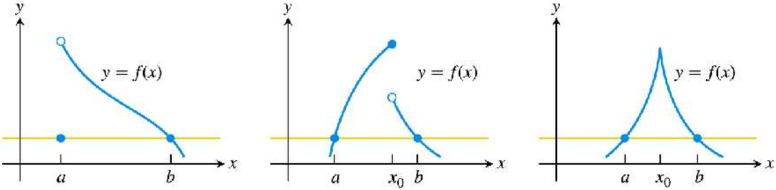
Rolle's Theorem

Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

$$f'(c) = 0 \ .$$

All assumptions are necessary:



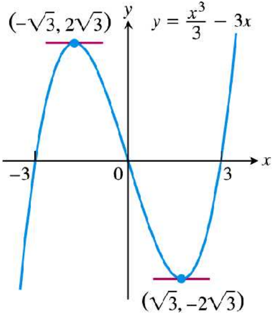
(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Example

Horizontal tangents of $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$:

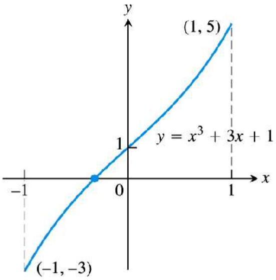


- $f(-3) = 0$, $f(3) = 0$
- by Rolle's Theorem there exists a $c \in [-3, 3]$ with $f'(c) = 0$

We find indeed from $f'(x) = x^2 - 3 = 0$ that $x = \pm\sqrt{3}$.

Example

Show that $x^3 + 3x + 1 = 0$ has exactly one real solution:



- Consider $f(x) = x^3 + 3x + 1$
- $f'(x) = 3x^2 + 3 > 0$ for all $x \in (-\infty, \infty)$
- If there were two solutions with $f(x) = 0$, then by Rolle's Theorem for some c we have $f'(c) = 0$. Therefore $f(x) = 0$ can have at most one solution.

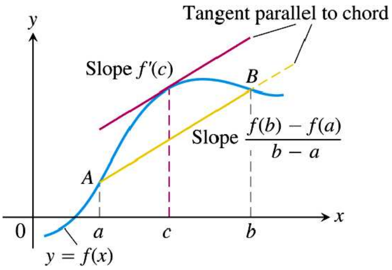
Furthermore, as $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem implies that there is a solution to $f(x) = 0$ in $(-1, 0)$.

The Mean Value Theorem

Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a} \, .$$



The Mean Value Theorem

Theorem

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}\,.$$

Proof.

- straight line through $(a, f(a))$ and $(b, f(b))$ given by

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

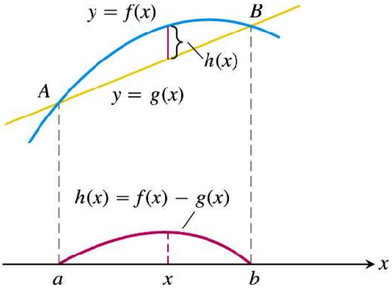
- for $h(x) = f(x) - g(x)$, both $h(a) = 0$ and $h(b) = 0$
- apply Rolle's Theorem: there is a $c \in (a,b)$ with $h'(c) = 0$
- as $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, this implies $f'(c) = \frac{f(b)-f(a)}{b-a}$ $\square$

The Mean Value Theorem

Theorem

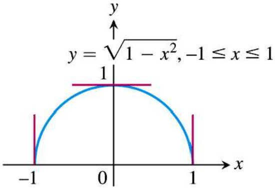
Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$



Example

$f(x) = \sqrt{1-x^2}$ on $[-1,1]$:



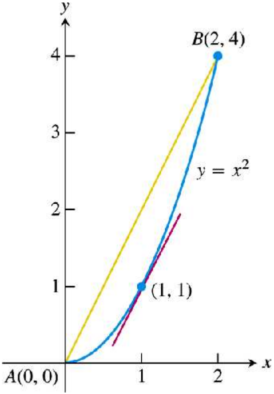
- $f(x)$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$ (but not at ± 1)
- therefore there is a $c \in (-1,1)$ with

$$f'(c) = \frac{f(-1) - f(1)}{1-(-1)} = 0$$

(We compute easily that $c = 0$)

Example

$f(x) = x^2$ on $[0, 2]$:



- $f(x)$ is continuous and differentiable on $[0, 2]$
- therefore there is a $c \in (0, 2)$ with

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 2$$

(We compute easily that $c = 1$)

Consequences of the Mean Value Theorem

Corollary

If $f'(x) = 0$ on (a, b) then $f(x) = C$ for all $x \in (a, b)$.

Proof.

For any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ there is a $c \in (x_1, x_2)$ with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

but as $f'(c) = 0$ by assumption, it follows that

$$f(x_2) = f(x_1) \ .$$

As x_1 and x_2 are chosen arbitrarily in (a, b), $f(x)$ is constant for all $x \in (a, b)$. $\square$

Consequences of the Mean Value Theorem

Corollary

If $f'(x) = g'(x)$ on (a, b) then

$$f(x) = g(x) + C$$

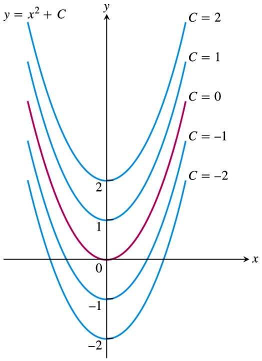
for all $x \in (a, b)$.

Proof.

Consider $h(x) = f(x) - g(x)$. As

$$h'(x) = f'(x) - g'(x) = 0$$

on (a, b), $h(x) = C$ by the previous Corollary and so $f(x) = g(x) + C$. $\square$



Examples

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through $(0, 2)$:

- $g(x) = -\cos x$ satisfies

$$g'(x) = \sin x = f'(x)$$

- Therefore $f(x) = g(x) + C$, i.e.

$$f(x) = -\cos x + C$$

- $f(0) = 2$ gives

$$2 = -\cos 0 + C$$

  so that $C = 3$

$$f(x) = 3 - \cos x$$

Reading Assignment

Sections 4.3 and 4.4
(needed for coursework 7)

The End