THE EULER-MACLAURIN SUMMATION FORMULA

We derive an expression for the *difference* between a divergent series and an infinite integral:

$$\triangle_0^\infty(f) := \left(\sum_{n=0}^{\infty} f(n)\right) - \int_0^\infty f(x)\,dx.$$

We proceed as Euler [4 , p326 et sui]:
Let $N \in \mathrm{Z}^+$. Then

$$\triangle_0^N(f) := \left(\sum_{n=0}^{N} f(n)\right) - \int_0^N f(x)\,dx =$$

$$1/2[f(N)+f(0)] + \sum_{n=0}^{N-1} 1/2[f(n+1)+f(n)] - \sum_{n=0}^{N-1}\int_n^{n+1} f(x)\,dx =$$

$$1/2[f(N+f(0)] + \sum_{n=0}^{N-1}\{1/2[f(n+1)+f(n)] - \int_n^{n+1} f(x)\,dx\}.$$

Now the $\{\cdots\}$ term above can be re-written by an integration by parts (IBP), so that

$$\{\cdots\} = 1/2[f(n+1)+f(n)] - [xf(x)]_n^{n+1} + \int_n^{n+1} xf'(x)\,dx =$$

$$-(n+1/2)[f(n+1)-f(n)] + \int_n^{n+1} xf'(x)\,dx =$$

$$-(n+1/2)\int_n^{n+1} f'(x)\,dx + \int_n^{n+1} xf'(x)\,dx =$$

$$\int_n^{n+1}(x-n-1/2)f'(x)\,dx.$$

Hence

$$\triangle_0^N(f) = 1/2[f(N)+f(0)] + \sum_{n=0}^{N-1}\int_n^{n+1}(x-[x]-1/2)f'(x)\,dx \qquad (1)$$

where $[x]$ is the greatest integer less than x , so that

$$[x] = n \quad \forall\, x \in (n, n+1).$$

Now $S_1(x) := x - [x] - 1/2$ is a *sawtooth* function [5, p280] , and we define the sequence of functions

$$S_0(x),\ S_1(x),\ S_2(x),\ \ldots$$

as follows: we let $S_0(x) := 1$, we let

$$S_k(x) = \int S_{k-1}(x)\,dx \qquad k \in \mathrm{Z}^+ \quad , \qquad\qquad (2)$$

and we choose the arbitrary constant of integration in (2) to be such that

$$\int_0^1 S_k(x)\,dx = 0 \quad \forall\, k \geq 1.$$

We defined the functions S_k as above in order to be able to integrate (1) K times by parts, where $K \in \mathrm{Z}^+$, (which assumes f to be K times differentiable),and we chose the constants of integration above so that (S_k) is a DECREASING sequence of functions, in order for our final expression for $\triangle_0^N$ - and therefore for $\triangle_0^\infty$ - to be summable.

We now proceed to integrate (1) once, twice, and by induction go to the integration K times, arriving at a preliminary expression for $\triangle_0^N(f)$ in terms of the functions S_k. The integral in (1) is expanded thus [5]:

$$\int_n^{n+1} S_1(x)f'(x)\,dx = [S_2(x)f'(x)]_n^{n+1} - \int_n^{n+1} S_2(x)f''(x)\,dx$$

where, as in (2),

$$S_2(x) = \int S_1(x)\,dx.$$

S_2 is the integral of 1-periodic S_1 (which has a jump discontinuity at each integer) , and

$$\int S_1(x)\,dx = \int x - [x] - 1/2\,dx = x^2/2 - x/2 + c$$

on the interval $[0, 1)$ and hence is as above on *each* period of length 1.Hence the behaviour on each period is identical to that on the interval $[0, 1]$. So $S_2(0) = S_2(1) = c$ and

$$S_2(n+1) = S_2(n) = S_2(0) = c \qquad \forall\; n \in \mathrm{Z}^+, \qquad (3)$$

ie $S_2(x)$ is *continuous and 1-periodic.* See graphs below (Figure 1, not to scale).

Further integration by parts then gives

$$\int_n^{n+1} S_1(x)f'(x)\,dx = S_2(0)(f'(n+1)-f'(n)) - [S_3(x)f''(x)]_n^{n+1} + \int_n^{n+1} S_3 f'''(x)\,dx,$$

due to (3), and where

$$S_3(x) = \int S_2(x)\,dx,$$

so

$$\int_n^{n+1} S_1(x)f'(x)\,dx = S_2(0)(f'(n{+}1){-}f'(n)){-}S_3(0)(f''(n{+}1){-}f''(n)){+}\int_n^{n+1} S_3(x)f^{(3)}(x)\,dx,$$

since

$$S_3(n+1) = S_3(n) = S_3(0)\ \forall\ n \in \mathrm{Z}^+, \qquad (4)$$

by a similar argument as for S_2.

FIGURE 1: $S_0(x)$, $S_1(x)$, $S_2(x)$ and $S_3(x)$ of the function sequence $S_k(x), k \in \mathrm{Z}^+$.(Not to scale)

Continuing in this way, we obtain, by induction, the following expression for the K-fold integration by parts of (1):

$$\triangle_0^N(f) = 1/2[f(N){+}f(0)]{+}\sum_{n=0}^{N-1}\left(\sum_{k=2}^{K-1}(-1)^k S_k(x)f^{(k)}(x)|_n^{n+1} + (-1)^{K+1}\int_n^{n+1} S_K(x)f^{(K)}(x)\,dx\right),$$

ie

$$\triangle_0^N(f) = 1/2[f(N){+}f(0)]{+}\sum_{k=2}^{K-1}(-1)^k S_k(x)f^{(k)}(x)|_0^N{+}(-1)^{K+1}\int_0^N S_K(x)f^{(K)}(x)\,dx. \quad (5)$$

(see [4] , p327).

This is our preliminary expression in terms of the functions S_k,as advertised above.We must now ask what explicit form the functions S_k should or could take.

Suppose ([5] ,p281)

$$\sum_{k=0}^{\infty} S_k(x)t^k = G(x,t) \qquad (0 \le x < 1) \quad (6)$$

Then, since $S'_k(x) = S_{k-1}$ (from (2)), we have

$$\frac{\partial\, G(x,t)}{\partial x} = tG(x,t),$$

since

$$\frac{\partial\, G(x,t)}{\partial x} = \sum_{k=0}^{\infty} S'_k(x)t^k = \sum_{k=0}^{\infty} S_{k-1}(x)t^{k-1}t = tG(x,t).$$

This suggests $G(x,t)$ to be of the form $g(t)e^{xt}$, since

$$\frac{\partial\,(g(t)e^{xt})}{\partial x} = tg(t)e^{xt}.$$

Recalling condition (2) (page 2), we have

$$\int_0^1 S_k(x)\,dx = 0 \;\; (k \ge 1) \Rightarrow \int_0^1 G(x,t)\,dx = \sum_{k=0}^{\infty}\left(t^k \int_0^1 S_k(x)\,dx\right) =$$

$$t^0 \int_0^1 S_0(x)\,dx = \int_0^1 1\,dx = 1.$$

So

$$\int_0^1 g(t)e^{xt} = 1 \Rightarrow \frac{g(t)e^{xt}}{t}|_0^1 = 1$$

$$\Rightarrow g(t) = \frac{t}{e^t - 1}.$$

Hence

$$\sum_{k=0}^{\infty} S_k(x)t^k = \frac{te^{xt}}{e^t - 1}.$$

Now, the *BERNOULLI POLYNOMIALS* are defined by the following expansion

$$\frac{te^{xt}}{e^t - 1} =: \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \qquad \forall x \in \Re, |t| < 2\pi. \;\; (7)$$

Hence

$$S_k(x) = \frac{B_k(x-[x])}{k!}.$$

We have $B_k(x-[x])$ on the rhs since our expansion (7) is then defined for the interval $[0,1)$ as our construction implies the S_k are polynomials of degree k in the interval [0,1) ([5] , p281).

So let ([4],p327)

$$S_k(x) = \frac{B_k(x)}{k!} \quad , k \in \mathrm{Z}^+ \qquad (8)$$

on $(0,1)$ *and 1-periodic thereafter.*

We have now defined our functions S_k explicitly,in terms of the standard Bernoulli polynomials,whose useful and relevant properties we now explore.

First, we prove the following

PROPOSITION 1

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!} \ \forall \ k \geq 1.$$

Proof

We have

$$\frac{te^{xt}}{e^t-1} =: \sum_{k=0}^{\infty} \frac{B_k(x)t^k}{k!}$$

Differentiating wrt x gives

$$t\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k'(x)t^k}{k!},$$

so

$$t\sum_{k=0}^{\infty} \frac{B_k(x)t^k}{k!} = \sum_{k=0}^{\infty} \frac{B_k'(x)t^k}{k!},$$

so

$$\sum_{k=0}^{\infty} \frac{B_k(x)t^k}{k!} = \sum_{k=0}^{\infty} \frac{B_k'(x)t^{k-1}}{k!}.$$

But

$$\sum_{k=1}^{\infty} \frac{B_{k-1}(x)t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{B_k(x)t^k}{k!} = \sum_{k=0}^{\infty} \frac{B_k'(x)t^{k-1}}{k!}.$$

Hence equating coefficients of t in the above expression gives us

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!} \ \forall \ k \geq 1 \quad QED.$$

Next we introduce the *BERNOULLI NUMBERS* $B_k := B_k(0)$. We prove that

PROPOSITION 2

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}\, x^j. \ \forall \ k \geq 0.$$

Proof

We have

$$\frac{te^{0t}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(0)t^k}{k!} \Rightarrow \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!},$$

so

$$e^{xt} \sum_{k=0}^{\infty} \frac{B_k t^k}{k!} = \frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)t^k}{k!}.$$

Equating coefficients of powers of t in the above expression, we have ([5],p282)

$$e^{xt} B_0 = B_0(x),$$

$$e^{xt} B_1 t = B_1(x)t,$$

and for positive integer k we have

$$e^{xt} B_k t = B_k(x)t.$$

So $B_k = B_k(0)$, as above!

Now, using Proposition 1, $B_k'(x) = kB_{k-1}(x)$,and by (8), we have

$$B_0(x) = 1,$$

so

$$B_0 = B_0(0) = 1$$

so

$$B_1(x) = \int_0^1 1 B_0 \, dx = B_0 x|_0^1 = B_0$$

and

$$B_1(x) = \int B_0\,dx = B_0x + c$$

so

$$B_1(0) = B_1 = c$$

so

$$B_1(x) = B_0x + B_1$$

ie

$$B_1(x) = \binom{1}{0}B_{1-0}x^0 + \binom{1}{1}B_{1-1}x^1 = \sum_{j=0}^{1}\binom{1}{j}B_{1-j}x^j.$$

Similarly,

$$B_2(x) = B_0x^2+2B_1x+B_2 = \binom{2}{0}B_{2-0}x^0+\binom{2}{1}B_{2-1}x^1+\binom{2}{2}B_{2-2}x^2 = \sum_{j=0}^{2}\binom{2}{j}B_{2-j}x^j.$$

So, by induction, we have the required result:

$$B_k(x) = \sum_{j=0}^{k}\binom{k}{j}B_{k-j}x^j. \quad QED. \qquad (9)$$

So far so good, but (9) is only useful if we know what the Bernoulli numbers are! We now derive an algorithm for recursively extracting the Bernoulli numbers:

Now, ([5],282), (7) is unchanged if we simultaneously replace x and t by $1-x$ and $-t$ respectively, hence

$$B_k(1-x) = (-1)^k B_k(x). \quad (10)$$

Moreover, apart from B_1 , all the Bernoulli numbers with an odd suffix are equal to zero. This is because

$$\frac{t}{e^t-1} + t/2 = \left(\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}\right) + t/2$$

is an *even* function of t , so odd powers of $-t$ must vanish from the series on the rhs of the above expression, (as the function on lhs above is ≥ 0 for positive t) beyond a certain value of k.(10) becomes:

$$B_k(1-0) = (-1)^k B_k(0)$$

ie

$$B_k(1) = B_k \ \ (k \geq 2). \quad (11)$$

So let $x = 1$ in (9) and we have

$$B_k(1) = B_k = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} \Rightarrow B_k = \sum_{j=0}^{k} \binom{k}{k-j} B_{k-j} = \sum_{j=0}^{k} \binom{k}{j} B_j,$$

by first recalling that $\binom{k}{j} = \binom{k}{k-j}$, and secondly just re-labelling $k-j$ as j .

So

$$B_k = \binom{k}{0} B_0 + \binom{k}{1} B_1 + \cdots + \binom{k}{k} B_k$$

$$\Rightarrow B_k - \binom{k}{k} B_k = 0 = \binom{k}{0} B_0 + \cdots + \binom{k}{k-1} B_{k-1},$$

$$\Rightarrow -\binom{k}{k-1} B_{k-1} = \sum_{j=0}^{k-2} \binom{k}{j} B_j,$$

which gives us the promised algorithm for recursively calculating the B_k as

$$B_{k-1} = -1/k \sum_{j=0}^{k-2} \binom{k}{j} B_j \qquad (k \geq 2).$$

Hence,recalling that $B_0 = 1$, we have ([5] p282)

$$B_1 = -1/2, \;\; B_2 = 1/6, \;\; B_3 = 0, \;\; B_4 = -1/30, \cdots$$

So,by (11), we have that the Bernoulli polynomials are 1-periodic,as $B_k(0) = B_k = B_k(1)$*.They are continuous as they are polynomials,and we will now see that successive Bernoulli polynomials have increasing powers of* x*,and so, on* $[0,1)$*,this means that they also fulfil our requirement that the* S_k *be a* DECREASING *sequence of functions:*

So, from Proposition 2, the first seven Bernoulli polynomials are :

$$B_0(x) = 1, \;\; B_1(x) = x-1/2, \;\; B_2(x) = x^2-x+1/6, \;\; B_3(x) = x^3-(3/2)x^2+(1/2)x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - 1/30, \;\; B_5(x) = x^5 - (5/2)x^4 + (5/3)x^3 - (1/6)x,$$

$$B_6(x) = x^6 - 3x^5 + (5/2)x^4 - (1/2)x^2 + 1/42$$

Hence, from all the above discussion of Bernoulli polynomials, (5) now becomes:

$$\triangle_0^N(f) = 1/2[f(N)+f(0)] + \sum_{k=2}^{K-1} \frac{(-1)^k B_k(x) f^{(k)}(x)|_0^N}{k!} + (-1)^{K+1} \int_0^N \frac{B_K(x) f^{(K)}(x)}{K!}\, dx.$$

From our remarks about Bernoulli numbers of odd suffix vanishing, and remembering that the second term above is an integration between *integer* values, so (11) applies and we can take the Bernoulli term out as a Bernoulli number, we have

$$\triangle_0^N(f) = 1/2[f(N)+f(0)] + \sum_{m=1}^{M(K)} \frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(x)\right]_0^N + (-1)^{K+1}\int_0^N \frac{B_K(x)f^{(K)}(x)}{K!}\,dx.$$

Where

$$M(K) := K/2\,,\ \ K\ \ even,\ \ and\ \ M(K) := (K-1)/2\,,\ \ K\ \ odd.$$

It will be noticed that we have kept only the odd derivatives of f.This is because ,in the above summation, we would lose either all even or all odd derivatives of f , since all odd Bernoulli numbers vanish above B_1. So we choose to keep the odd derivatives. This condition will preserve the first derivative of f.