MTH5105 Differential and Integral Analysis 2010-2011

Solutions 1

1 Exercises for Feedback

1) Using the definition of the derivative of a function, investigate for which values of x each of the following two functions is differentiable, and find the derivatives, if they exist.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto (x+1)|x|$,

   (b) $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto (x-1)|x-1|$.

Solution:

(a) We need to distinguish three cases: (1) $a > 0$, (2) $a < 0$, and (3) $a = 0$:

(1) For $a > 0$, we find

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(x+1)|x|-(a+1)|a|}{x-a} \\ &= \lim_{x\to a} \frac{(x+1)x-(a+1)a}{x-a} = \lim_{x\to a}(x+a+1) = 2a+1\ . \end{aligned}$$

Some argument is needed as to why we can replace $|x|$ by x when calculating the limit. It suffices to say that x becomes positive as $x \to a$ when $a > 0$.

(More formally, in the definition of the limit one can replace δ by $\delta' = \min\{\delta, a\}$ as then $|x-a| < \delta'$ implies $|x-a| < a$ and thus $x > 0$. However, I don't require this degree of formality.)

(2) For $a < 0$, we find

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{(x+1)|x|-(a+1)|a|}{x-a} \\ &= \lim_{x\to a} \frac{(x+1)(-x)-(a+1)(-a)}{x-a} = \lim_{x\to a}(-x-a-1) = -2a-1\ . \end{aligned}$$

Some argument is needed as to why we can replace $|x|$ by $-x$ when calculating the limit. It suffices to say that x becomes negative as $x \to a$ when $a < 0$.

(3) For $a = 0$, we find

$$\frac{f(x)-f(0)}{x-0} = \frac{(x+1)|x|-0}{x} = \begin{cases} x+1 & x > 0 \\ -x-1 & x < 0 \end{cases}$$

so that the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x}$ does not exist.

Taken together, this shows that f is differentiable for all $x \in \mathbb{R} \setminus \{0\}$ and that

$$f'(x) = \begin{cases} 2x+1 & x > 0 \\ \text{undefined} & x = 0 \\ -2x-1 & x < 0\ . \end{cases}$$

(b) We need to distinguish three cases: (1) $a > 1$, (2) $a < 1$, and (3) $a = 1$:

(1) For $a > 1$, we find

$$\begin{aligned} g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{(x-1)|x-1|-(a-1)|a-1|}{x-a} \\ &= \lim_{x\to a} \frac{(x-1)^2-(a-1)^2}{x-a} = \lim_{x\to a}(x+a-2) = 2(a-1)\ . \end{aligned}$$

Some argument is needed as to why we can replace $|x-1|$ by $x-1$ when calculating the limit. It suffices to say that $x-1$ becomes positive as $x \to a$ when $a > 1$.

(2) For $a < 1$, we find

$$\begin{aligned} g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{(x-1)|x-1|-(a-1)|a-1|}{x-a} \\ &= \lim_{x\to a} \frac{-(x-1)^2+(a-1)^2}{x-a} = \lim_{x\to a}(-x-a-2) = -2(a-1)\ . \end{aligned}$$

Some argument is needed as to why we can replace $|x-1|$ by $-(x-1)$ when calculating the limit. It suffices to say that $x-1$ becomes negative as $x \to a$ when $a < 1$.

(3) For $a = 1$, we find

$$g'(1) = \lim_{x\to 1} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1} \frac{(x-1)|x-1|-0}{x-1} = \lim_{x\to 1}|x-1| = 0\ .$$

Taken together, this shows that g is differentiable for all $x \in \mathbb{R}$ and that

$$g'(x) = \begin{cases} 2(x-1) & x > 1 \\ 0 & x = 1 \\ -2(x-1) & x < 1 \end{cases}$$

or, simply, $g'(x) = 2|x-1|$.

2 Extra Exercises

2) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at zero and find $f'(0)$.

Find $f'(x)$ for $x \neq 0$ assuming that $\sin' = \cos$.

Give a rough sketch of the curve $f'(x)$ for small x and mark $f'(0)$ clearly on your sketch.

Solution:

Consider the difference quotient

$$\frac{f(x)-f(0)}{x-0} = \frac{x^2\sin(1/x^2)-0}{x} = x\sin(1/x^2)\ .$$

Since $|\sin(1/x^2)| \leq 1$,

$$\frac{f(x)-f(0)}{x-0} \to 0$$

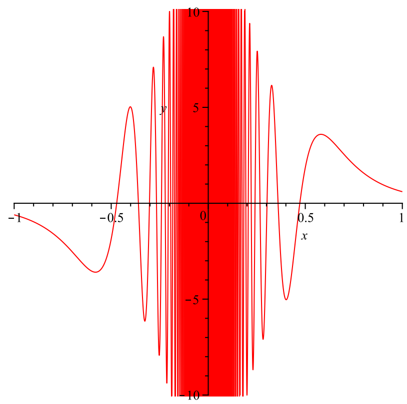
as $x \to 0$. Therefore f is differentiable at zero with $f'(0) = 0$.

For $x \neq 0$ differentiation gives

$$f'(x) = 2x\sin(1/x^2) - \frac{2}{x}\cos(1/x^2)\ .$$

Graph of $f'(x)$:

For $x \to 0$, $2x\sin(1/x^2) \to 0$ and the second term dominates. The graph of f' oscillates rapidly with increasing amplitude as $x \to 0$. At zero, the derivative is zero.



3) Let $f : [-1,1] \to \mathbb{R}$ be continuous on $[-1,1]$, differentiable at zero and $f(0) = 0$. Show that the function

$$g(x) = \begin{cases} f(x)/x & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

is continuous at zero.

Is g continuous for $x \neq 0$?

Deduce that there is some number M such that

$$f(x)/x \leq M \quad \text{for all} \quad x \in [-1,1] \setminus \{0\}\ .$$

Solution:

A function g is continuous at a if $\lim_{x\to a} g(x) = g(a)$.

With $a = 0$, this gives

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0)$$

so g is continuous at 0.

For $x \neq 0$, g is continuous since it is a quotient of continuous functions.

By the boundedness principle, a continuous function on a closed interval attains it maximum and minimum.

Therefore there exists a number M such that $g(x) \leq M$ for all $x \in [-1,1]$.

4) Give an example of a function that is differentiable on (a, b) but that cannot be made differentiable on $[a, b]$ by any definition of $f(a)$ or $f(b)$. Can you give an example where f is bounded?

Solution:

There are many possible examples, for example if we define $f(x) = \dfrac{1}{(x-a)(b-x)}$ on (a, b) then f is differentiable on (a, b) but cannot be made to be continuous at a or b by any definition of $f(a)$ or $f(b)$.

We get a bounded function if we compose this with sin, i.e. if we define

$$f(x) = \sin\left(\frac{1}{(x-a)(b-x)}\right)$$

on (a, b), then again f is differentiable on (a, b) but cannot be made to be continuous at a or b by any definition of $f(a)$ or $f(b)$. However, once $f(a)$ and $f(b)$ are defined, the resulting function is clearly bounded on $[a, b]$.

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