Counting knotted curves and surfaces in lattices

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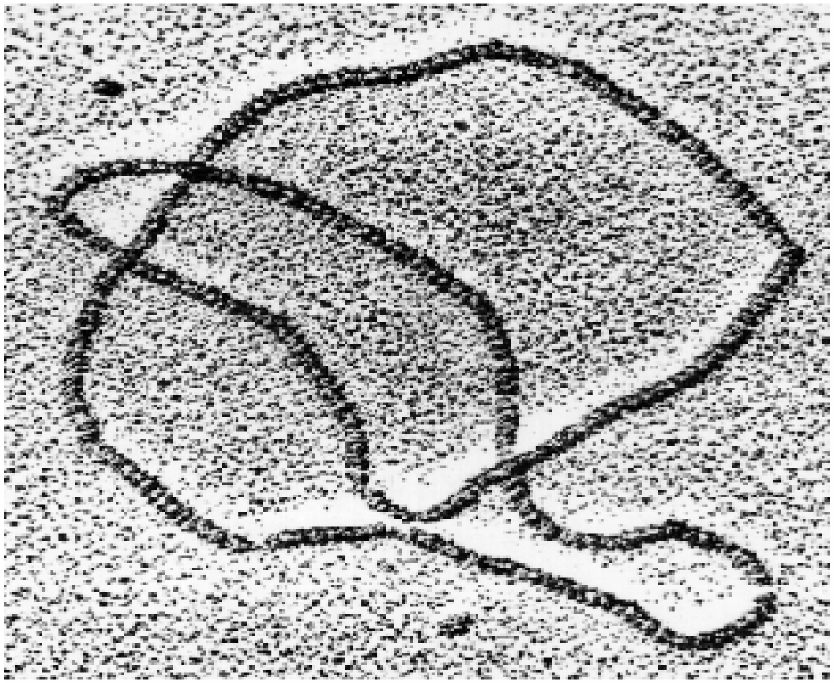
Long flexible objects are often highly self-entangled

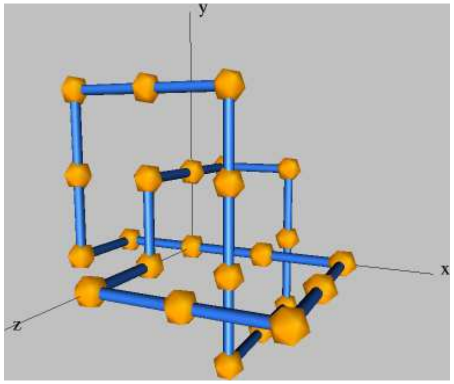


Macroscopic objects also get entangled

Knots in ring polymers:
The Frisch-Wasserman-Delbruck conjecture

Almost all sufficiently long ring polymers are knotted

Modelling ring polymers on a lattice



Counting polygons on Z^3

We can count polygons with n edges up to translation.

$$p_4 = 3$$

$$p_6 = 22$$

$$p_8 = 207$$

$$p_{20} = 1768560270$$

$$p_{32} = 53424552150523386 = 5.3\ldots \times 10^{16}$$

Large n behaviour?

Classic result due to John Hammersley:

$$\log 3 \leq \lim_{n \to \infty} n^{-1} \log p_n = \kappa \leq \log 5$$

Counting unknotted polygons on Z^3

If we write p_n^o for the number of *unknotted* polygons with n edges then

$$p_4^o = 3$$

$$p_6^o = 22$$

and in fact $p_n^o = p_n$ if $n < 24$ (Diao).

Unknotted polygons and pattern theorems

$$\lim_{n \to \infty} n^{-1} \log p_n^o = \kappa_o$$

and

$$\kappa_o < \kappa$$

which establishes the FWD conjecture for this model.

Unknotted polygons and pattern theorems

$$\lim_{n\to\infty} n^{-1} \log p_n^o = \kappa_o$$

and

$$\kappa_o < \kappa$$

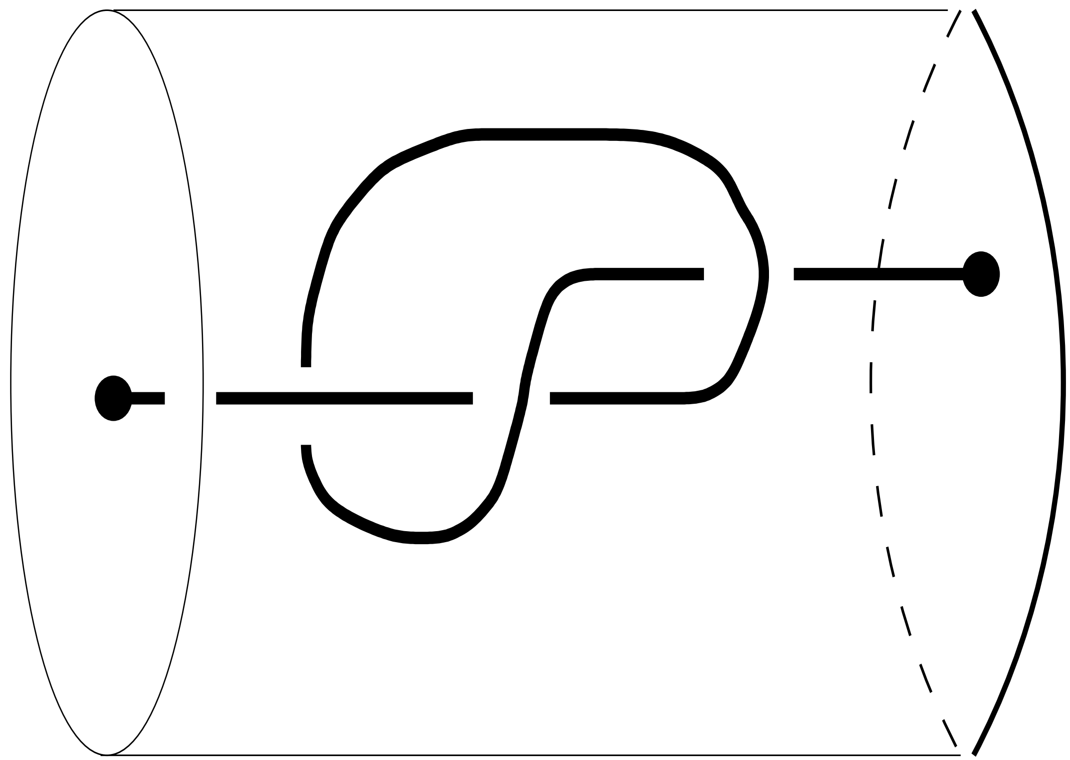
Idea of proof:

1. no antiknots

2. knotted ball pairs

3. Kesten's pattern theorem

Knotted ball pairs



Kesten's pattern theorem for polygons

- A *Kesten pattern* is any self-avoiding walk P for which there is a self-avoiding walk on which P occurs 3 times.

- Suppose that $p_n(\bar{P})$ is the number of n-edge polygons on which P never occurs. Then

$$\lim_{n\to\infty} n^{-1} \log p_n(\bar{P}) = \kappa(\bar{P}),$$

and

$$\kappa(\bar{P}) < \kappa$$

More details

$$p_n^o \leq p_n(\overline{3_1}) \leq p_n(\overline{P_{3_1}}) = e^{\kappa(\overline{P_{3_1}})n+o(n)}$$

Positive density results

- Polygons have a positive density of trefoils and, indeed, of every other (fixed) knot type.

- Hence they have lots of prime knots (a positive density) in their knot decomposition.

- Quantities which add for the prime knots in a composite knot will grow at least linearly with n.

- The take-home message is that polygons are very badly knotted.

Soteros, Sumners and Whittington, Entanglement complexity of graphs in Z^3, Math. Proc. Camb. Phil. Soc. **111** 75-91 (1992)

Some open questions

- How many trefoils are there?

- Is it true that the limit

$$\lim_{n\to\infty} n^{-1} \log p_n(3_1) \equiv \kappa(3_1)$$

  exists?

- Is it true that $\kappa(3_1) = \kappa_o$?

A partial answer

We do know that

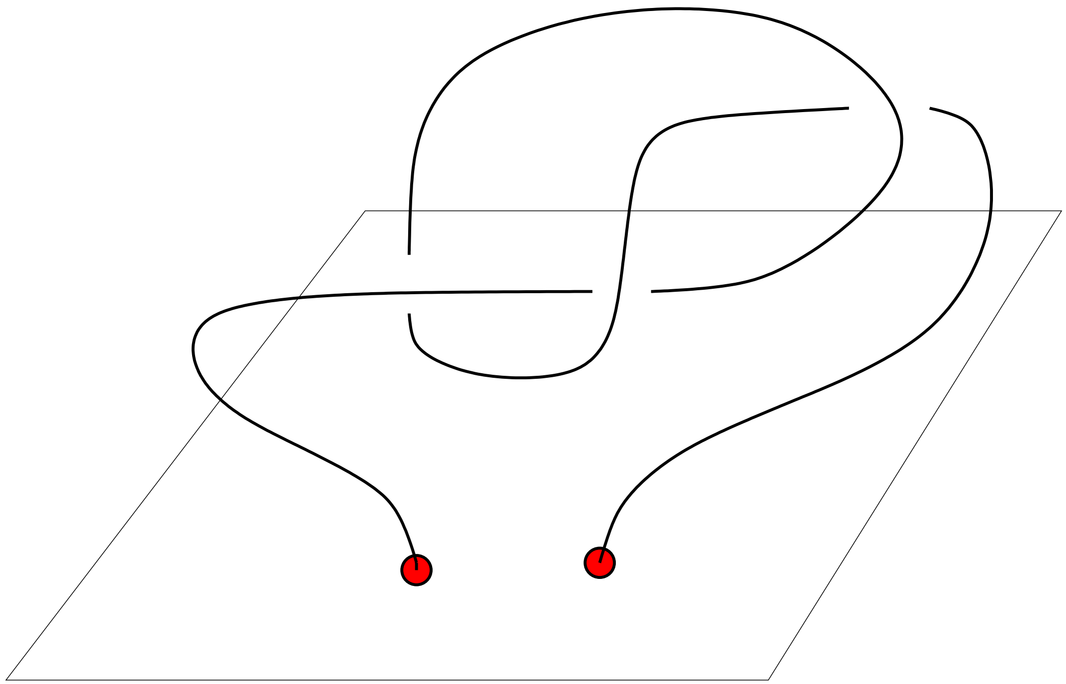
$$\kappa_o \le \liminf_{n\to\infty} n^{-1} \log p_n(3_1) \le \limsup_{n\to\infty} n^{-1} \log p_n(3_1) < \kappa$$

Can we prove a higher dimensional analogue?

- Higher dimensional analogue – we don't have a pattern theorem for 2-spheres in Z^4. If we had a pattern theorem for 2-spheres in Z^4 we would be able to prove that all except exponentially few 2-spheres are knotted.

- Why is it more difficult to prove a pattern theorem for 2-spheres?

What does a knotted 2-sphere look like?
Spinning a trefoil



Embedding a spun trefoil in Z^4

- Explicit construction

- Appeal to a general result by Boege, Hinojosa and Verjovsky, Rev Mat Complut (2010)

2-spheres in Z^4

If s_n is the number (mod translation) of 2-spheres in Z^4 with n plaquettes, and if s_n^0 is the number (mod translation) of unknotted 2-spheres in Z^4 with n plaquettes, then

$$\lim_{n\to\infty} n^{-1} \log s_n \equiv \lambda$$

$$\lim_{n\to\infty} n^{-1} \log s_n^0 \equiv \lambda_0$$

We would like to prove that $\lambda_0 < \lambda$

Tubes in Z^4

An L-tube, $T(L)$, in Z^4 is the set of vertices

$$\{(x_1, x_2, x_3, x_4) | 0 \leq x_1 \leq L, 0 \leq x_2 \leq L, 0 \leq x_3 \leq L, 0 \leq x_4\}$$

2-spheres in $T(L)$

Existence of limits

$$\lim_{n\to\infty} n^{-1} \log s_n(L) \equiv \lambda(L) \qquad \lim_{n\to\infty} n^{-1} \log s_n^0(L) \equiv \lambda_0(L)$$

2-spheres in $T(L)$

- Existence of limits

$$\lim_{n\to\infty} n^{-1} \log s_n(L) \equiv \lambda(L) \qquad \lim_{n\to\infty} n^{-1} \log s_n^0(L) \equiv \lambda_0(L)$$

- $\lambda(L) < \lambda(L+1) \ldots < \lambda$

2-spheres in $T(L)$

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2-spheres in $T(L)$

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2-spheres in $T(L)$

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- $\lim_{L\to\infty} \lambda_0(L) = \lambda_0$

- $\lambda_0(L) < \lambda(L)$

Take-home message

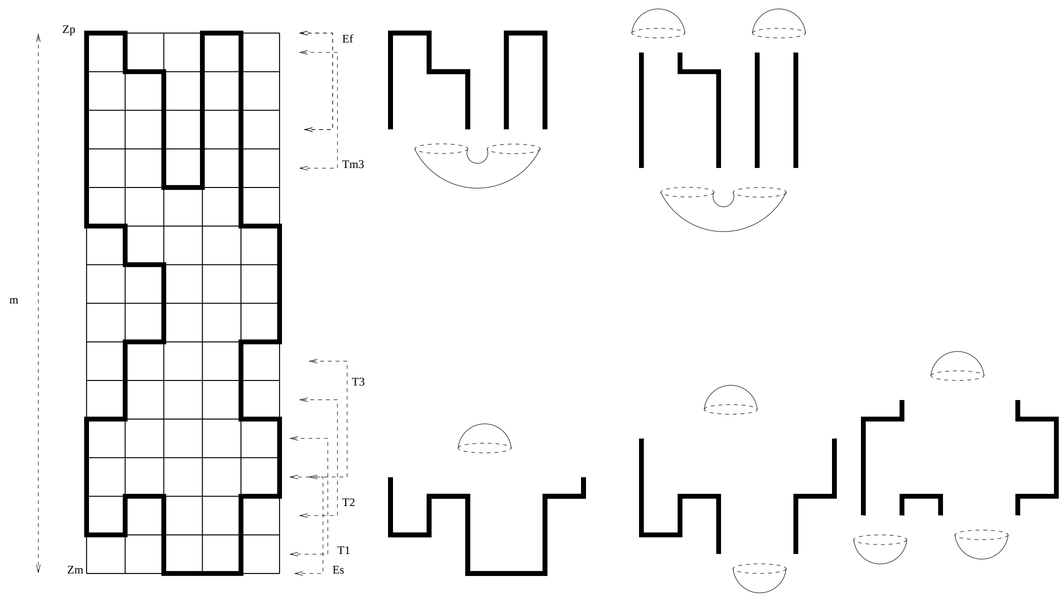
All except exponentially few sufficiently large 2-spheres in tubes in Z^4 are knotted.

Technical details

- Why are tubes easier?

- The quasi-one dimensional nature of the tube means that we can use transfer matrix techniques to prove a pattern theorem.

The idea behind transfer matrices



Topological input

- Since polynomial invariants multiply under connect sum, if the sphere has the spun trefoil as a summand then it is knotted.

- Think of the sphere in Z^4 as being made up of slices. These slices are closed curves or collections of closed curves. If one of these is the knot 6_1 (which is slice but not doubly-null-cobordant) then the sphere is knotted.

Topological entanglement complexity

In fact the spun trefoil occurs a positive density of times on (all but exponentially few sufficiently large) 2-spheres in a tube in Z^4. Since quantities like the span of the Alexander polynomial add under connect sum such measures of entanglement complexity increase (at least) linearly with the size of the 2-sphere in a tube.

Extensions?

- Dimensions larger than 4
- Linking in higher dimensions
- Almost unknotted surfaces