Enumeration of area-weighted Dyck paths with restricted height

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Abstract

We derive explicit expressions for q-orthogonal polynomials arising in the enumeration of area-weighted Dyck paths with restricted height.

1 Introduction and Statement of Results

Dyck paths are directed walks on $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the line $y=0$, which have no vertices with negative y-coordinates, and which have steps in the $(1,1)$ and $(1,-1)$ directions. We impose the additional geometrical constraint that the paths have height at most h, *i.e.*, they lie between lines $y=0$ and $y=h$. Given a Dyck path π, we define the length $n(\pi)$ to be half the number of its steps, and the area $m(\pi)$ to be the sum of the starting heights of all steps in the $(1,1)$ direction in the path. We denote by $u(\pi)$ and $v(\pi)$ the number of vertices in the line $y=0$ (excluding the vertex $(0,0)$) and the number of vertices in the line $y=h$, respectively. Let $\mathcal{D}_h$ be the set of Dyck paths with height at most h, and define the generating function

$$D_h(a,b;q,t) = \sum_{\pi\in\mathcal{D}_h} a^{u(\pi)} b^{v(\pi)} q^{m(\pi)} t^{n(\pi)} \ .$$

The purpose of this note is to prove the following identity for $D_h(a,b;q,t)$.

Theorem 1. *For $h \geq 0$,*

$$D_h(a,b;q,t) = \frac{\displaystyle\sum_{m=0}^{\infty} (-t)^m q^{m(m-1)} \left((1-b) {\textstyle\begin{bmatrix} h-m \\ m \end{bmatrix}_q} + b {\textstyle\begin{bmatrix} h+1-m \\ m \end{bmatrix}_q} - (1-b) {\textstyle\begin{bmatrix} h+1-m \\ m-1 \end{bmatrix}_q} - b {\textstyle\begin{bmatrix} h-m \\ m-1 \end{bmatrix}_q} \right)}{\displaystyle\sum_{m=0}^{\infty} (-t)^m q^{m(m-1)} \left((1-b) {\textstyle\begin{bmatrix} h-m \\ m \end{bmatrix}_q} + b {\textstyle\begin{bmatrix} h+1-m \\ m \end{bmatrix}_q} - (1-a)(1-b) {\textstyle\begin{bmatrix} h+1-m \\ m-1 \end{bmatrix}_q} - (1-a) b {\textstyle\begin{bmatrix} h-m \\ m-1 \end{bmatrix}_q} \right)} \ . \tag{1}$$

Here, we have used the standard notation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} , \quad \text{where} \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) .$$

For $a = b = 1$, this identity simplifies considerably.

Corollary 2. *For $h \geq 0$,*

$$D_h(1,1;q,t) = \frac{\displaystyle\sum_{m=0}^{\infty} (-t)^m q^{m^2} \begin{bmatrix} h-m \\ m \end{bmatrix}_q}{\displaystyle\sum_{m=0}^{\infty} (-t)^m q^{m(m-1)} \begin{bmatrix} h+1-m \\ m \end{bmatrix}_q} .$$

Taking the limit $h \to \infty$, we recover the well-known result [3] that the area-weighted generating function $D(q,t)$ for Dyck paths without height restriction is given by

$$D(q,t) = \frac{\displaystyle\sum_{m=0}^{\infty} \frac{(-t)^m q^{m^2}}{(q;q)_m}}{\displaystyle\sum_{m=0}^{\infty} \frac{(-t)^m q^{m(m-1)}}{(q;q)_m}} .$$

2 Proofs

We use as the starting point of our derivation a well-established connection between lattice path enumeration and continued fractions [2].

Proposition 3. *$D_0(a,b,q,t) = b$, $D_1(a,b,q,t) = 1/(1-abt)$, and for $h \geq 2$,*

$$D_h(a,b;q,t) = \cfrac{1}{1 - \cfrac{at}{1 - \cfrac{qt}{1 - \cfrac{q^2 t}{1 - \cfrac{q^3 t}{\ddots \, \cfrac{q^{h-2} t}{1 - b q^{h-1} t}}}}}} . \tag{2}$$

While this can easily be proved by specialising the general theory in [2] to the case at hand, we shall give a direct combinatorial proof.

Proof. The only Dyck path of height zero is the zero step Dyck path. If $h = 0$ then it has weight b, whence $D_0(a,b;q,t) = b$. Let now $h \geq 1$. Except for the zero-step Dyck path with weight 1, every Dyck path of height at most h can be decomposed uniquely into a Dyck path of height at most $(h-1)$ bracketed by a pair of steps into the $(1,1)$ and $(1,-1)$ directions, followed by another Dyck path of height h. The associated generating functions are $atD_{h-1}(1,b;q,qt)$ and $D_h(a,b;q,t)$, respectively. This decomposition leads to the functional-recurrence

$$D_h(a,b;q,t) = 1 + atD_{h-1}(1,b;q,qt) D_h(a,b;q,t) .$$

Iterating $D_h(a,b;q,t) = 1/(1 - atD_{h-1}(1,b;q,qt))$ gives (2). $\square$

It is clear that the generating function can also be written as a rational function, and from Section 3 in [2] we obtain the following three-term recurrence.

Proposition 4. *For $h \geq 1$,*

$$D_h(a,b;q,t) = \frac{Q_h(0,b;q,t)}{Q_h(a,b;q,t)}$$

where

$$Q_h(a,b;q,t) = \begin{cases} 1-abt\ , & h=1\ , \\ 1-at-bqt\ , & h=2\ , \\ Q_{h-1}(a,1;q,t) - bq^{h-1}tQ_{h-2}(a,1;q,t) & h \geq 3\ . \end{cases} \tag{3}$$

Proof. The initial conditions follow from $D_1(a,b;q,t) = 1/(1-abt)$ and $D_2(a,b;q,t) = (1-bqt)/(1-at-bqt)$, and the factor $bq^{h-1}t$ in the three-term recurrence is just the final term in the continued fraction (2). □

We proceed by considering the generating function of the denominators $Q_h(a,b;q,t)$,

$$W(z;a,b;q,t) = \sum_{h=0}^{\infty} Q_h(a,b;q,t) z^w\ .$$

The next proposition expresses $W(z;a,b;q,t)$ in terms of the basic hypergeometric series $\phi(z,q,t) = {}_1\phi_2(q;0,z;q,t)$ [4], *i.e.*,

$$\phi(z,q,t) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}t^n}{(z;q)_n}\ .$$

Proposition 5.

$$\begin{aligned} W(z;a,b;q,t) = &\frac{abt^2z^3 - at^2z^3 + abtz - abt - atz - btz - bz + b + z}{tz} \\ &+ \frac{(bz-b-z)(1-at)}{zt}\phi(z,q,-tz^2) - (bz-b-z)\phi(z,q,-qtz^2)\ . \end{aligned} \tag{4}$$

Proof. The recurrence (3) implies that $W(x;a,b;q,t)$ satisfies a functional equation,

$$W(z;a,b;q,t) = z(1-z)(1-abt) + z^2(1-at-bqt) + zW(z;a,1;q,t) - z^2bqtW(qz;a,1;q,t)\ .$$

Solving this functional equation for $W(z;a,1;q,t)$ by iteration gives

$$\begin{aligned} W(z;a,1;q,t) = &\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{n^2+n} t^n \left(1-at-zq^{n+1}t\right)}{(z;q)_{n+1}} \\ = &\frac{1-at}{tz} - 1 - \frac{1-at}{tz}\phi(z,q,-tz^2) + \phi(z,q,-qtz^2)\ . \end{aligned}$$

Inserting this expression into the functional equation gives (4). □

Proposition 6.

$$\begin{aligned} Q_h(a,b;q,t) = &\sum_{m=0}^{\infty} (-t)^m q^{m(m-1)} \times \\ &\left((1-b)\begin{bmatrix} h-m \\ m \end{bmatrix}_q + b\begin{bmatrix} h+1-m \\ m \end{bmatrix}_q - (1-a)(1-b)\begin{bmatrix} h+1-m \\ m-1 \end{bmatrix}_q - (1-a)b\begin{bmatrix} h-m \\ m-1 \end{bmatrix}_q \right)\ . \end{aligned} \tag{5}$$

Proof. We obtain $Q_h(a,b;q,t)$ by extracting the coefficient of z^h in $W(z;a,b;q,t)$. We expand the q-product in the function ϕ with the help of the q-binomial theorem [4] to obtain

$$\phi(z,q,tz^2) = 1 + \sum_{m=0}^{\infty} z^m \sum_{n=1}^{\infty} q^{n(n-1)} \begin{bmatrix} m-n-1 \\ n-1 \end{bmatrix}_q t^n \; .$$

Inserting this expansion into (4) and collecting terms with equal powers in z gives (5). ☐

Theorem 1 now follows from Propositions 4 and 6.

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