Tutte polynomials for counting and classifying orbits

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Abstract

Given a graph Γ and an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, we define some polynomials which count and classify the orbits of G on various structures on Γ, as counted by the Tutte polynomial, while also specialising to the Tutte polynomial.

1 Introduction

Given a graph Γ, the Tutte polynomial is defined by

$$T(\Gamma; x, y) = \sum_{D \subseteq E} (x-1)^{\rho\Gamma - \rho D}(y-1)^{|D| - \rho D}$$

(where ρ is the rank function of the graphic matroid associated with Γ).
This polynomial can be evaluated at various points to give, among other things, the numbers of nowhere zero flows, nowhere zero tensions, proper vertex colourings, acyclic orientations, and spanning trees of Γ.
We will first show that it makes sense to talk about a group $G \leq \mathrm{Aut}(\Gamma)$, then define several polynomials to count orbits on some of these structures. The first will be based on nowhere zero flows and nowhere zero tensions, but will also be capable of dealing with proper vertex colourings. This polynomial will then be refined to give more functionality, in particular to give the Tutte polynomial of the graph obtained when a graph is 'factored out' by an automorphism. A third polynomial will then be defined which counts the numbers of structures on Γ fixed by the entire group G; which will be used with Möbius inversion to find the sizes of the orbits of G acting on the various structures.

2 Tutte Structures and Group Actions

Definition If $\Gamma = (V, E)$ is a graph (with some arbitrary orientation) then define a *Tutte structure* on Γ as a pair (S_V, S_E) of functions

$$\begin{aligned} S_V : V &\longmapsto A_V \\ S_E : E &\longmapsto A_E \end{aligned}$$

(where A_V and A_E are some sets, depending on the type of structure with some conditions (also depending on the type of structure) on the functions such that for each type of structure, the number of possible structures is an evaluation of the Tutte polynomial $T(\Gamma; x, y)$, possibly with a normalising factor.

Note that, if the structure only requires a function of the vertices (or edges) of Γ, let A_E (or A_V be a singlet, since there is exactly one map from any set to a singlet.

Definition If $\mathcal{S}$ is a type of Tutte structure such that for any Tutte structure $S = (S_V, S_E)$ of type $\mathcal{S}$ on a graph Γ, and g is an automorphism of Γ; and g maps S to another Tutte structure of type $\mathcal{S}$ under the map

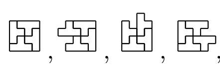
$$\begin{array}{rccc} g: & v & \longmapsto & v' \\ & e & \longmapsto & e' \end{array} \qquad \begin{array}{rccc} g: & S_V(v) & \longmapsto & S_V(v') \\ & S_E(e) & \longmapsto & S_E(e') \end{array}$$

$(v, v' \in V(\Gamma),\ e, e' \in E(\Gamma))$
then call every Tutte structure of type $\mathcal{S}$ a *symmetric Tutte structure.*

Note that there may be further conditions on the type of graph and the automorphism group in order for a (symmetric) Tutte structure to exist, as is the case with T-tetromino tilings.

Examples

- Proper k vertex colourings are a type of symmetric Tutte structure: A_V is the set of k vertex colours, $A_E = \{1\}$, the conditions on S_V are that no two adjacent vertices may have the same colour; and the number of nowhere zero A-flows is given by $(-1)^{\rho(\Gamma)}k^{c(\Gamma)}T(\Gamma; 1-k, 0)$, where $\rho(\Gamma)$ is the rank of the cycle matroid of Γ and $c(\Gamma)$ is the number of connected components of Γ.
- Nowhere zero A-flows (for some finite abelian group A with $|A| = k$) are a type of symmetric Tutte structure: $A_V = \{1\}$, $A_E = A \setminus \{0\}$, the conditions on S_E are the flow conditions; and the number of nowhere zero A-flows is given by $(-1)^{|E|-\rho(\Gamma)}T(\Gamma; 0, 1-k)$.
- Spanning trees are a type of symmetric Tutte structure (for a connected graph Γ): $A_V = \{1\}$, $A_E = \{0, 1\}$; the condition is that set $\{e\} \subseteq E$ of edges with $S_E(e) = 1$ must form a spanning tree. The number of spanning trees is given by $T(\Gamma; 1, 1)$.
- If Γ is the $m \times n$ grid graph and $G \leq \mathrm{Aut}(\Gamma)$ is (a subgroup of) the symmetry group of a $4m \times 4n$ rectangle; then the T-tetromino tilings of the $4m \times 4n$ rectangle are a symmetric Tutte structure of Γ: $A_E = \{1\}$, A_V is the set of admissible 'tilings' of a 4×4 square, that is, arrangements of 4 T-tetrominos such that the middle four of the 4×4 are covered, and at most one square overhangs the 4×4 square on each side. There are 32 such tilings:

  {  , etc }

  The condition on A_V is that adjacent vertices must have tessellating tilings, and the side and corner vertices must have the relevant tilings. The number of T-tetromino tilings of the $4m \times 4n$ rectangle is given by $2T(\Gamma; 3, 3)$ (see [4] for more about T-tetromino tilings).
- The acyclic orientations of Γ are a type of symmetric Tutte structure: $A_V = \{1\}$, $A_E = \{-1, 1\}$, S_E maps each edge of Γ onto A_E depending on whether or not the orientation assigned agrees with the arbitrary orientation of Γ. The condition is that no directed cycles of Γ are formed by S_E. The number of acyclic orientations of a graph Γ with n vertices, c connected components and rank ρ is given by $(-1)^{c+\rho+n}T(\Gamma; 2, 0)$.
- If Γ is a connected graph and v is a vertex of Γ then the number of acyclic orientations of Γ with a single source at v is a Tutte structure but not a symmetric Tutte structure (since v may be moved by an automorphism of Γ). The number of such orientations is given by $T(\Gamma; 1, 0)$. However the number of acyclic orientations of Γ with a single source at any vertex of Γ is a symmetric Tutte structure, and the number of such orientations is given by $|V|T(\Gamma; 1, 0)$. A_V, A_E and S_E are the same as for acyclic orientations.

Theorem 1 *If Γ is a graph, $G \leq \mathrm{Aut}(\Gamma)$ is an automorphism group, and $\mathcal{S}$ is a type of symmetric Tutte structure; then, under the maps below, if $I \leq G$ is the subgroup of G which fixes every structure on Γ of type $\mathcal{S}$, G/I acts as a permutation group on the set of structures on Γ of type $\mathcal{S}$.*

$$\begin{array}{rccc} g: & v & \longmapsto & v' \\ & e & \longmapsto & e' \end{array} \qquad \begin{array}{rccc} g: & S_V(v) & \longmapsto & S_V(v') \\ & S_E(e) & \longmapsto & S_E(e') \end{array}$$

$(v, v' \in V(\Gamma),\ e, e' \in E(\Gamma),\ (S_V, S_E)$ structure on Γ of type $\mathcal{S}$)

Proof The proof of this theorem is a straightforward exercise of checking the four group axioms and is omitted here.

The Orbit Counting Lemma Much of the study of the study of symmetric Tutte structures depends on the orbit counting lemma

$$\#\text{ orbits} = \frac{1}{|G|}\sum_{g\in G}\text{fix}(g)$$

which counts the number of orbits of a permutation group G acting on a set T (fix(g) is the number of elements of S fixed by g).
Ordinarily it will not be possible to determine what the group I, which fixes every structure of type $\mathcal{S}$, actually is. It is therefore not possible in general to find G/I, and so it is convenient to talk about the action of G on a set of structures (rather than the action of G/I, which is induced by G); and indeed the (number of) G-orbits on a set of structures.
Every element of a coset gI maps a structure S to the same structure S', so every element of gI fixes the same set of structures. Also, every coset gI has the same size $|I|$. Therefore the number of orbits of G/I on a set of structures is given by

$$\begin{aligned}\#\text{ orbits} &= \frac{1}{|G/I|}\sum_{g\in G}\text{fix}(g) \quad (\text{some } g\in gI)\\ &= \frac{|I|}{|G|}\sum_{gI\in G/I}\frac{1}{|I|}\sum_{g\in gI}\text{fix}(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\text{fix}(g)\end{aligned}$$

Therefore it makes sense (since we get the same result) to count the orbits of G on a set of structures.

3 The (First) Orbital Tutte Polynomial

We now begin to define a polynomial which counts orbits of $G \leq \text{Aut}(\Gamma)$. Given the orbit counting lemma

$$\#\text{ orbits} = \frac{1}{|G|}\sum_{g\in G}\text{fix}(g)$$

it only remains to find a way to calculate the number of structures on Γ fixed by the automorphism g.

3.1 Flows, tensions, and matrix equations

A function $f : E(\Gamma) \to A$ is an A-flow by definition if and only if

$$M\underline{f} = \underline{0}$$

where $\underline{f}$ is a column vector representing the elements of A assigned to each edge of $E(\Gamma)$, $\underline{0}$ is a column vector with each entry being the zero element of A, and M is the matrix whose columns are labelled by $\vec{E}$, whose row set corresponds directly to V and whose entries are

$$M(\vec{\Gamma})_{ij} = \begin{cases} 0 & \text{edge } e_j \text{ and vertex } v_i \text{ not incident}\\ 1 & \text{edge } e_j \text{ points out from vertex } v_i\\ -1 & \text{edge } e_j \text{ points into vertex } v_i\end{cases}$$

Furthermore, a function $f : E(\Gamma) \to A$ is fixed by an automorphism g if and only if

$$(g^E - I)\underline{f} = \underline{0}$$

where I is an identity matrix with the relevant size and g^E is an $|E| \times |E|$ matrix with ij entry 1 if g maps e_i to e_j in the same direction (with respect to some arbitrary orientation of E), -1 if g maps e_i to e_j in the opposite direction, and 0 otherwise.

Thus a function $f : E(\Gamma) \to A$ is a fixed A-flow if and only if

$$M_g \underline{f} = \underline{0}$$

where $M_g = \begin{bmatrix} M \\ g^E - I \end{bmatrix}$.
Hence the number of A-flows fixed by g is simply the number of solutions $\underline{f}$ over A to the above matrix equation. Putting the matrix M_g in Smith normal form, we can count these solutions by substituting $\alpha_i = \#\text{solutions } x \in A \text{ to } ix = 0$ for x_i in the monomial $x(M_g) = \prod x_i$ where the is are the invariant factors of M_g and the product is over all of the invariant factors of M_g (with repetitions, and including 0 as an invariant factor).
The question of a flow being nowhere zero or not is taken care of by the principle of inclusion and exclusion, and restricting the columns of matrices. We shall use $M_g[D]$ to denote the matrix M_g restricted to the columns in D where $D \subseteq E$. We can now use the orbit counting lemma to calculate the number of orbits of $\mathrm{Aut}(\Gamma)$ on nowhere zero A-flows of Γ as

$$\text{number of orbits} = \frac{1}{|G|} \sum_{g \in G} \sum_{D \subseteq E} (-1)^{|E|-|D|} x(M_g[D]) \Big|_{x_i = \alpha_i}$$

Having solved the problem for nowhere zero flows, it is straightforward to solve the problem for nowhere zero tensions by replacing the matrix M with the matrix M^* whose columns again correspond to the edges of Γ, and whose rows correspond to the cycles of Γ, each with a direction given, so that the entries of M^* are given by

$$M^*(\vec{\Gamma})_{ij} = \begin{cases} 0 & \text{edge } e_j \text{ does not appear on cycle } C_i \\ 1 & \text{edge } e_j \text{ appears on cycle } C_i \text{ in the same direction} \\ -1 & \text{edge } e_j \text{ appears on cycle } C_i \text{ in the opposite direction} \end{cases}$$

One then proceeds in exactly the same way as for flows, replacing $x(M_g)$ with $x^*(M_g^*)$ to get the number of orbits on nowhere zero A-tensions as

$$\text{number of orbits} = \frac{1}{|G|} \sum_{g \in G} \sum_{D \subseteq E} (-1)^{|E|-|D|} x^*(M_g^*[D]) \Big|_{x_i^* = \alpha_i}$$

3.2 The (first) orbital Tutte polynomial

We are now in a position to define the (first) orbital Tutte polynomial.

Definition If Γ is a graph and $G \leq \mathrm{Aut}(\Gamma)$ is an automorphism group, then define the *orbital Tutte polynomial* $\mathrm{OT}(\Gamma, G)$ by

$$\mathrm{OT}(\Gamma, G) = \frac{1}{|G|} \sum_{g \in G} \sum_{D \subseteq E} x(M_g[D]) x^*(M_g^*[E \setminus D])$$

At the moment, we have that the number of orbits of G on the nowhere zero A-flows of Γ is given by

$$\mathrm{OT}(\Gamma, G; x_0 \leftarrow |A|, x_i \leftarrow 1\ \forall i \geq 1, x_j^* \leftarrow -1\ \forall j)$$

and that the number of orbits of G on the nowhere zero A-tensions of Γ is given by

$$\mathrm{OT}(\Gamma, G; x_i \leftarrow -1\ \forall i, x_0^* \leftarrow |A|, x_j^* \leftarrow 1\ \forall j \geq 1)$$

In the next subsection we will see that this polynomial can be used to count orbits on proper vertex colourings, and then we will see that, when G is the trivial automorphism group, it can be specialised to give the Tutte polynomial of Γ. There are other evaluations of this polynomial, but all of the known evaluations are also evaluations of the more general second orbital Tutte polynomial.

3.3 Proper vertex colourings and the orbital Tutte polynomial

We will show that the orbital Tutte polynomial can be used to count the number of orbits of G on the proper vertex colourings of a connected graph Γ.
Before the main result we need to state and prove the following lemma.

Lemma 2

$$(g^V - I)M = M(g^E - I)$$

where g^V is the matrix whose rows and columns are indexed by $V(\Gamma)$ and whose i^{th} row has a 1 in the j^{th} column and zeros everywhere else whenever v_j is the image of v_i under g.

Proof If v and w are vertices, the matrix g^V has a (v, w) entry equal to 1 if w is the image of v and 0 otherwise. This means that the (v, e) entry of $g^V M$ is equal to $M_{v^g e}$.
Now, if e and f are edges, the matrix g^E has an (e, f) entry equal to 1 if e is the image of f under g, -1 if $-e$ is the image of f under g (e with its direction reversed is the image of f under g), and 0 otherwise. This means that the (v, e) entry of the matrix Mg^E is equal to $\pm M_{vf}$, where $f_g = \pm e$.
To see the equality of these two (v, e) entries; note that the fact that g is an automorphism means that if $v^g = w$ then $M_{v^g f^g} = \pm M_{vf}$; and that the sign change is the same as that given by the (f, e) entry of the matrix g^E.
Now we have $g^V M = Mg^E$. Subtracting M from both sides gives the required result. This completes the proof.

Theorem 3 *Given a connected graph Γ, we can use the orbital Tutte Polynomial to count the number of orbits of a subgroup G of the automorphism group $Aut(\Gamma)$ on proper k-vertex colourings of Γ via*

$$\#orbits = kOT(\Gamma, G; x_i \leftarrow -1\ \forall i, x_0^* \leftarrow k, x_j^* \leftarrow 1 \forall j \geq 1)$$

Proof We first deal with the relationship between the general (not necessarily proper) vertex colourings of Γ fixed by some $g \in G$ and the invariant factors of the corresponding M_g^*.
First note that

$$x^*(M_g^*)\Big|_{x_0^*=k, x_i^*=1,\ \forall i \neq 0} = k^{n(M_g^*)}$$

where $n(N)$ denotes the nullity of the (matroid with linear independence consisting of the columns of the) matrix N; and that the number of (not necessarily proper) vertex colourings from k colours fixed by the automorphism g is given by

$$k^{n(g^V - I)}$$

where g^V is the matrix whose rows and columns are indexed by V and whose i^{th} row has a 1 in the j^{th} column and zeros everywhere else whenever v_j is the image of v_i under g. This comes from the (easily checked) fact that $n(g^V - I)$ gives the number of vertex automorphism cycles of g, each of which must have the same colour on each vertex.
Therefore, if we can prove that

$$n(g^V - I) = n(M_g^*) + 1$$

then we will have proved that the number of (not necessarily proper) vertex colourings from k colours fixed by the automorphism g is given by

$$kx^*(M_g^*)\Big|_{x_0^*=k, x_i^*=1,\ \forall i \neq 0}$$

By lemma 2, for any $1 \times |V|$ row vector $\underline{x}$ we get

$$\underline{x}(g^V - I)M = \underline{x}M(g^E - I)$$

Now, the 'row' null space of $g^V - I$ is simply the set of vectors which are constant on the vertex automorphism cycles of g. Clearly this space has dimension equal to

$$\#\text{vertex automorphism cycles of } g = n(g^V - I)$$

Also, the 'row' null space of M is spanned by the $1 \times |V|$ row vector with entry equal to 1 (call it $\underline{1}$), since Γ is connected.

$$\therefore \underline{x}(g^V - I)M = \underline{0} \Longrightarrow \underline{x}(g^V - I) = \alpha\underline{1}$$

for some constant α.
However this means that $\alpha = 0$ because the coordinate sum of $\underline{x}g^V$ is equal to the coordinate sum of $\underline{x}$. (Since g^V is a reordering of the rows of I).
So the 'row' null space of $(g^V - I)M$ (call it W) is the same as the 'row' null space of $(g^V - I)$. And, by the Lemma, the 'row' null space of $M(g^E - I)$ is simply W, which is to say

$$W = \{\underline{x} : \underline{x}(g^V - I)M = \underline{0}\} = \{\underline{x} : \underline{x}(g^V - I) = \underline{0}\} = \{\underline{x} : \underline{x}M(g^E - I) = \underline{0}\}$$

We saw earlier that

$$\dim\{\underline{x} : \underline{x}(g^V - I) = \underline{0}\} = \dim(W) = n(g^V - I)$$

Now we calculate $\dim(W)$ using another method. Label the null space of M_g^* by V and note that $\dim(V) = n(M_g^*)$. Then

$$\begin{array}{rcll} \underline{y} \in V & \Leftrightarrow & M^*\underline{y} = \underline{0} & \text{and} \quad (g^E - I)\underline{y} = \underline{0} \\ & \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} \quad (g^E - I)\underline{y} = \underline{0} \end{array}$$

for some $|V| \times 1$ row vector $\underline{x}$, since the row space of M is equal to the null space of M^*.

$$\begin{array}{cll} \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} \quad (g^E - I)(\underline{x}M)^T = \underline{0} \\ \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} \quad \underline{x}M(g^E - I)^T = \underline{0} \\ \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} \quad \underline{x}M((g^{-1})^E - I) = \underline{0} \\ \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} \quad \underline{x}((g^{-1})^V - I)M = \underline{0} \end{array}$$

by the Lemma, so this

$$\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T \quad \text{and} \quad \underline{x} \text{ is constant on the vertex automorphism cycles of } g^{-1}$$

Clearly, any colouring fixed by g^{-1} is also fixed by g so this

$$\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T \quad \text{and} \quad \underline{x} \text{ is constant on the vertex automorphism cycles of } g$$

So we have

$$\underline{y} \in V \Leftrightarrow \underline{y} \in \text{image of } W \text{under } M$$

But the null space of M is spanned by $\underline{1}$ $(= \langle\underline{1}\rangle)$ and $\dim(\langle\underline{1}\rangle) = 1$, and $\langle\underline{1}\rangle \subset W$.

$$\therefore \dim(V) = \dim(W) - \dim(\langle\underline{1}\rangle)$$

$$\therefore n(M_g^*) = n(g^V - I) - 1$$

So the number of (not necessarily proper) vertex colourings from k colours fixed by the automorphism g is given by

$$kx^*(M_g^*)\Big|_{x_0^*=k, x_i^*=1,\ \forall i\neq 0}$$

To complete the proof we must now show that the inclusion/exclusion takes care of the somewhere improper vertex colourings in the same way that it takes care of the somewhere zero flows and tensions.

Suppose that $\underline{x}$ is a vertex colouring fixed by the automorphism g with adjacent vertices v and w (joined by an edge e) having the same colour. That is, suppose that $\underline{x} \in W$ and $\underline{x}_v = \underline{x}_w$

Then, since $W = \{\underline{x} : \underline{x}(g^V - I) = \underline{0}\} = \{\underline{x} : \underline{x}M(g^E - I) = \underline{0}\}$ we must have the entry e in $\underline{x}M$ equal to zero. Let $\underline{y} = (\underline{x}M)^T$. Note that the converse also holds:$(\underline{x}M)_e = 0 \Rightarrow \underline{x}_v = \underline{x}_w$.

Running a previous part of the proof backwards we get:

$$\begin{array}{llll}
\multicolumn{4}{l}{\underline{x} \in W \text{ and } \underline{x}_v = \underline{x}_w} \\
\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x} \text{ is constant on the vertex automorphim cycles of } g & \text{and} \quad \underline{y}_e = 0 \\
\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x} \text{ is constant on the vertex automorphim cycles of } g^{-1} & \text{and} \quad \underline{y}_e = 0 \\
\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x}((g^{-1})^V - I)M = \underline{0} & \text{and} \quad \underline{y}_e = 0 \\
\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x}M((g^{-1})^E - I) = \underline{0} & \text{and} \quad \underline{y}_e = 0 \\
\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x}M(g^E - I)^T = \underline{0} & \text{and} \quad \underline{y}_e = 0 \\
\Leftrightarrow \quad \underline{y} = (\underline{x}M)^T & \text{and} & (g^E - I)(\underline{x}M)^T = \underline{0} & \text{and} \quad \underline{y}_e = 0 \\
\Leftrightarrow \quad M_g^* \underline{y} = \underline{0} & \text{and} & (g^E - I)\underline{y} = \underline{0} & \text{and} \quad \underline{y}_e = 0 \\
\multicolumn{4}{l}{\Leftrightarrow \underline{y} \in V \text{ and } \underline{y}_e = 0}
\end{array}$$

So we have

$$\underline{x} \in W \text{ and } \underline{x}_v = \underline{x}_w \Leftrightarrow \underline{y} = (\underline{x}M)^T \in V \text{ and } \underline{y}_e = 0$$

Now

$$k^{\dim\{\underline{x}:\underline{x}\in W \text{ and } \underline{x}_v=\underline{x}_w\}} = \#\text{ fixed } k \text{ vertex colourings with } c^V(v) = c^V(w)$$

and by the previous arguments,

$$\dim\{\underline{x} : \underline{x} \in W \text{ and } \underline{x}_v = \underline{x}_w\} = \dim\{\underline{y} : M_g^* \underline{y} = \underline{0} \text{ and } \underline{y}_e = 0\} + 1$$

We saw in our analysis of flows and tensions that

$$\{\underline{y} : M_g^* \underline{y} = \underline{0} \text{ and } \underline{y}_e = 0\} = \{\underline{y} : (M_g^* \setminus \{e\})\underline{y} = \underline{0}\} = \{\underline{y} : M_g^*[E \setminus \{e\}]\underline{y} = \underline{0}\}$$

So the number of k-vertex colourings fixed by g with $c^V(v) = c^V(w)$ is equal to

$$\begin{aligned}
& k^{\dim\{\underline{y}:(M_g^*\setminus\{e\})\underline{y}=\underline{0}\}+1} \\
= \; & k^{n(M_g^*\setminus\{e\})+1} \\
= \; & kx^*(M_g^* \setminus \{e\})\Big|_{x_0^*=k, x_i^*=1, \; \forall i\neq 0}
\end{aligned}$$

Now suppose $\underline{x}$ is a vertex colouring fixed by g and we have a subset $F \subseteq E$ such that each edge $f_i \in F$ joins vertices v_i and w_i (note that we could have $v_i = v_j$ or $v_i = w_j$ $w_i = w_j$ for some i and j, although this makes no difference to the analysis.) and v_i and w_i have the same vertex colour for each i.

If we run the arguments above again, we get

$$\begin{array}{llll}
& \underline{x} \in W & \text{and} & \underline{x}_{v_i} = \underline{x}_{w_i} \; \forall f_i \in F \\
\Leftrightarrow & \underline{y} \in V & \text{and} & \underline{y}_{f_i} = 0 \; \forall f_i \in F
\end{array}$$

then

$$k^{\dim\{\underline{x}:\underline{x}\in W \text{ and } \underline{x}_{v_i}=\underline{x}_{w_i} \; \forall f_i\in F\}} = \#\text{ fixed } k\text{-vertex colourings with } c^V(v_i) = c^V(w_i) \; \forall f_i \in F$$

and as above

$$k^{\dim\{\underline{x}:\underline{x}\in W \text{ and } \underline{x}_{v_i}=\underline{x}_{w_i} \; \forall f_i\in F\}} = k^{\dim\{\underline{y}:\underline{y}\in V \text{ and } \underline{y}_{f_i}=0 \; \forall f_i\in F\}+1}$$

But

$$\dim\{\underline{y} : \underline{y} \in V \text{ and } \underline{y}_{f_i} \in F\} = \dim\{\underline{y} : M_g^*[D]\underline{y} = \underline{0}\} = n(M_g^*[D])$$

where $D = E \setminus F$, giving that the number of k-vertex colourings with the same colour at both ends of each edge in a set $F \subseteq E$ of edges is

$$k^{n\{\underline{y}:M_g^*[D]\underline{y}=\underline{0}\}+1} = kx^*(M_g^*[D])\Big|_{x_0^*=k, x_i^*=1, \; \forall i\neq 0}$$

where $D = E \setminus F$.
So, by the principal of inclusion and exclusion, the number of proper vertex colourings from k colours fixed by the automorphism g is given by

$$k \sum_{D \subseteq E} (-1)^{|E \setminus D|} x^*(M_g^*[D]) \Big|_{x_0^*=k, x_i^*=1,\ \forall i \neq 0}$$

Applying the orbit counting lemma completes the proof.

It is not possible to use the orbital Tutte polynomial in its current form to count the orbits of an automorphism group on the proper vertex colourings of a disconnected graph. However, as we shall see later, it is possible if some adjustments are made.

3.4 The ordinary Tutte polynomial and the orbital Tutte polynomial

We will show that the ordinary Tutte polynomial is a specialisation of the orbital Tutte polynomial. In order to do this, we need the following result concerning the matroids consisting of the column sets of M and M^* with linear independence.
Note that it is easily shown that the row space of the matrix M is the same as the null space of the matrix M^* and vice versa.

Lemma 4 *Let M and M^* be two matrices over some principle ideal domain with the same number of columns, and which are dual in the sense that the row space of one is the same as the null space of the other and vice versa.*
Then the vectorial matroids consisting of the columns of M and M^* respectively are dual. (The $1-1$ correspondence between the column sets being that the i^{th} column of M corresponds to the i^{th} column of M^*).

Proof Each column of $M^\top$ is a row of M, and so is in the row space of M, which is the null space of M^*. So we have

$$M^* M^\top = 0$$

where 0 is an all zero matrix of the relevant size.
Each entry of $M^* M^\top$ is of the form

$$(\text{Row of } M^*) \cdot (\text{Row of } M^\top)$$

where $\cdot$ is the ordinary scalar dot product of two vectors.
From here on, if A is a set of columns of M (or a set of entries of a row of M) we will denote the corresponding set of columns in M^* (or set of entries of a row of M^*) by A^*.
Partition the whole column set E of M into two sets B and C (and E^* into B^* and C^*), and denote some entry of $M^* M^\top$ by n. (We have shown that each $n = 0$.) Then for some row of M^* and some row of M we have

$$(B^*\ C^*) \cdot (B\ C) = B^* \cdot B + C^* \cdots C = n$$

Note that this dot product is commutative and will be written in the reverse order from here on.
Now consider an elementary column operation of B on C; ie for a column $c \in C$, perform

$$c \longrightarrow c + \sum b \in B a_{b,c} b$$

for some scalars $a_{b,c}$.
n can be retrieved by performing the following operations of C^* on B^*.

$$b^* \longrightarrow b^* + \sum b \in B a_{b,c} c^*$$

for the same scalars $a_{b,c}$.
For example, if there are just two columns;

$$\begin{aligned}(b,c)\cdot(b^*,c^*) &= bb^*+cc^*=n\\ (b,c+ab)\cdot(b^*-ac^*) &= bb^*-bac^*+abc^*+cc^*\\ &= bb^*+cc^*=n\end{aligned}$$

where a is some scalar.
More generally, we can denote this by

$$(B\ C)\cdot(B^*\ C^*)=(B\ C+f(B))\cdot(B^*-f^{-1}(C^*)\ C^*)$$

Now suppose that B is a base of M (the matroid consisting of the columns of M). Then there are columns operations of B on C that reduce C to zero. That is, $C+f(B)=0$. Then

$$\begin{aligned}(B\ C)\cdot(B^*\ C^*) &= (B\ C+f(B))\cdot(B^*-f^{-1}(C^*)\ C^*)\\ &= (B\ 0)\cdot(B^*-f^{-1}(C^*)\ C^*)\\ &= B\cdot B^*-B\cdot f^{-1}(C)=0\end{aligned}$$

Thus $B\cdot(B^*-f^{-1}(C))=0$, so either B is orthogonal to $B^*-f^{-1}(C)$ or $B^*-f^{-1}(C)=0$. But B could be anything in the row space of $M[B]$, (M restricted to the columns of B), which in general will not be orthogonal to $B^*-f^{-1}(C)$. Hence C^* is spanning in the matroid M^*.
It now remains to show that C^* is minimally spanning in M^*.
Suppose that $C^*=E^*\setminus B^*$ is spanning but not minimally spanning in M^*, and that $F^*\subset C^*$ is maximally independent in C^* (minimally spanning in M^*).
Since C^* is spanning there exists a set g of column operations such that $g(C^*)=B^*$. But g is only really a function of F^* since $C^*\setminus F^*$ depends on F^*.
By the previous arguments we can show that $E\setminus F$ generates F in M, but since g is a function of F^* in M^*, g^{-1} must be a function of $E\setminus F$ in M.
So F depends on the whole of $E\setminus F$ in M, which means that $C\supset F$ cannot be generated by $B\subset E\setminus F$ in M.
This contradicts our assumption, so C^* must be minimally spanning.
We have now shown that if B is a base in M then $E^*\setminus B^*$ is a base in M^*, (equivalent operations show the converse), so that M and M^* are dual as matroids.
This completes the proof.

We can now show the relationship between the ordinary and orbital Tutte polynomials.

Theorem 5 *Given a graph Γ with edge set E, the ordinary Tutte polynomial, defined by*

$$T(\Gamma;x,y)=\sum_{D\subseteq E}(x-1)^{\rho E-\rho D}(y-1)^{\rho^*E-\rho^*(E\setminus D)}$$

can be obtained from the orbital Tutte polynomial via

$$T(\Gamma;x,y)=\mathrm{OT}(\Gamma,\langle\mathrm{id}\rangle;x_0\leftarrow y-1,x_i\leftarrow 1\ \forall i\geq 1,x_0^*\leftarrow x-1,x_j^*\leftarrow 1\ \forall j\geq 1)$$

Proof Let

$$Q=\mathrm{OT}(\Gamma,\langle\mathrm{id}\rangle;x_0\leftarrow y-1,x_i\leftarrow 1\ \forall i\geq 1,x_0^*\leftarrow x-1,x_j^*\leftarrow 1\ \forall j\geq 1)$$

We want to verify that

$$Q=T(\Gamma;x,y)$$

Let $F\subseteq E$ The exponent of x_0 in $x(M[D])$ is equal to the nullity of $M[D]$, or $|F|$ minus the rank of $M[F]$ (within the matroid M). Similarly, within the matroid M^*, the exponent of x_0^* in $x^*(M[E\setminus F])$ is equal to the nullity of $E\setminus F$, or $|E\setminus F|$ minus the rank of $E\setminus F$. Hence

$$Q=\sum_{F\subseteq E}(x-1)^{|F|-\rho F}(y-1)^{|E\setminus F|-\rho^*(E\setminus F)}$$

Using the matroid duality of M and M^*, and the matroid identity

$$\rho^*(E \setminus F) = |E| - |F| - \rho E + \rho F$$

we see that

$$|E \setminus F| - \rho^*(E \setminus F) = \rho E - \rho F$$

and that

$$|F| - \rho F = \rho^* E - \rho^*(E \setminus F)$$

Now let $F = D$ and we have

$$Q = T(\Gamma; x, y)$$

as required. This completes the proof.

4 Collapsed Graphs and a Second Orbital Tutte Polynomial

4.1 Collapsed Graphs

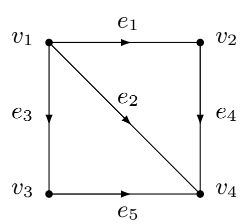
It seems that for some of the structures on a graph Γ that the Tutte polynomial counts, it is possible to count the number of these structures fixed by an automorphism g using the Tutte polynomial of the 'collapsed' graph Γ/g. Here we develop an orbital Tutte polynomial which exploits this fact, but can still count the number of some structures fixed by g which cannot be counted via the Tutte polynomial of Γ/g.

4.1.1 The 'collapsed' graph Γ/g

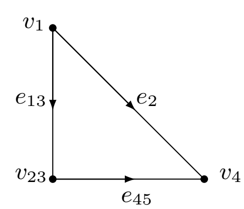
Definition Given a graph Γ and an automorphism g, let Γ/g denote the graph with vertex set the cycles of g on vertices of Γ and edge set the set of cycles of g on edges of Γ.
Obviously if g is the identity automorphism then Γ/g is just Γ.
Note that this graph has $n(g^V - I)$ vertices, and will have a loop whenever two adjacent vertices lie in the same cycle of g on vertices of Γ.
We use notation C_i^V and C_i^E for both the vertices and edges of Γ/g and the corresponding cycles of g in Γ.

Example

$\Gamma =$



$g = (v_2 v_3) = (e_1 e_3)(e_4 e_5)$ $\quad \Gamma/g =$



In this example Γ has an orientation which is fixed by g. The existence of such an orientation will become quite pertinent later on.

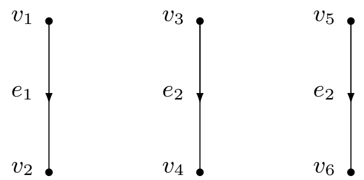
It is easy to see that this graph is well defined: Let C_1^V and C_2^V be vertex cycles of Γ, and suppose a vertex $v \in C_1^V$ is adjacent to a vertex $w \in C_2^V$. Then, since g is an automorphism, every vertex in C_1^V is adjacent to a vertex in C_2^V, so that the corresponding vertices in $g(\Gamma)$ must be adjacent via the edge corresponding to the edge cycle of g which contains the edge vw.

Loops in Γ/g The loops of the graph Γ/g fall into two categories: those which arise from an edge of Γ being reversed by some iteration of g, and those which do not.

We shall refer to those loops of Γ/g that arise from an edge of Γ being reversed by some iteration of g as loops from *eventually* reversed edges. Clearly in this case no orientation of the loop (or indeed Γ) can be fixed by g, so it does not make any sense to give the loop a direction in Γ.

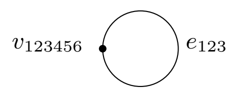
Example

$\Gamma =$



$g = (v_1 v_3 v_5 v_2 v_4 v_6)$

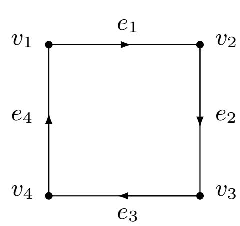
$\Gamma/g =$



The loops of Γ/g which do not come from eventually reversed edges of must have in their preimage in Γ at least one cycle, which has been collapsed to a single vertex and a single edge. This category includes cycles of g on loops of Γ. Suppose C is a cycle of Γ and n is the smallest non zero integer such that g^n fixes C setwise. Then if g^n maps C to itself after a rotation of order coprime to the length of C, all of the edges of C will be in the same cycle of g on edges of Γ and all of the vertices of C are in the same cycle of g on vertices of Γ, so that the image of C and all of its images under g in Γ/g will be a loop without g having reversed any edges. In this case the loop may be given a direction, and may be considered to have two possible directions, because there are two orientations of the edges in the preimage of the loop which can be fixed by g.

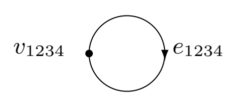
Example

$\Gamma =$



$g = (v_1 v_2 v_3 v_4)$

$\Gamma/g =$



4.1.2 Proper vertex colourings of Γ/g

The most natural structure that the graph Γ/g can be used to count is the number of proper vertex colourings of Γ fixed by g.

Theorem 6 *The number of proper vertex colourings of Γ fixed by g is given by the chromatic polynomial of Γ/g.*

Proof It is clear that the general colourings of Γ/g are in bijection with the general vertex colourings of Γ fixed by g since the vertices of Γ/g correspond to the cycles of g on vertices of Γ. It is also clear that the bijection preserves the property of a vertex colouring being proper, because two vertices of Γ/g will be adjacent if and only if each of the Γ vertices in the vertex-cycle of g corresponding to one of the Γ/g vertices will be adjacent to a Γ vertex in the vertex-cycle of g corresponding to the other Γ/g vertex.
If Γ/g has a loop, then it has no proper vertex colourings, and if Γ/g has a loop then two adjacent vertices of Γ, which cannot have the same colour if a vertex colouring is to be proper, will be in the same g cycle on vertices, and so must have the same colour if a vertex colouring is to be fixed. Thus if Γ/g has a loop, g fixes no proper vertex colourings of Γ.
This completes the proof.

4.2 Defining $\mathrm{OT_{II}}(\Gamma, G)$

From here on, unless otherwise stated, we will choose an orientation of Γ which is fixed by g if possible. If there are edges of Γ which are eventually reversed by g then we will choose an orientation of Γ in which cycles of g on edges of Γ which do not eventually reverse edges preserve edge direction, and an edge cycle of g which does eventually reverse an edge preserves all of the edge directions except one.

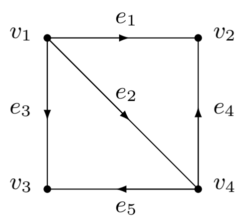
4.2.1 Definitions of M'_g and $M^{*\prime}_g$

If g reverses no edges of Γ then the sum of the columns of $g^E - I$ constituting a cycle of g on edges of Γ will be a zero vector. If C^E is a cycle of g on edges of Γ which eventually reverses an edge then the sum of the columns of $g^E - I$ which make up C^E will be a vector which has one entry -2 and the rest zeros.

Definition Given a graph Γ and an automorphism g of Γ, let M'_g (and $M^{*\prime}_g$) be the matrices obtained from the matrices $M_g = \begin{pmatrix} M \\ g^E - I \end{pmatrix}$ (and $M^*_g = \begin{pmatrix} M^* \\ g^E - I \end{pmatrix}$) by adding together and identifying the columns of M_g (and M^*_g) corresponding to each cycle of g on edges of Γ.

Example

$\Gamma =$



$g = (v_1 v_4) = (e_1 e_4)(e_3 e_5)(\overline{e_2})$

The $\overline{e_2}$ denotes that the direction of e_2 is reversed by g.

$$M_g = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \qquad M'_g = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M^*_g = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \qquad M^{*\prime}_g = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4.2.2 Fixed flows and tensions from M'_g and $M^{*\prime}_g$

Theorem 7 *If A is a finite abelian group, A-flows of Γ fixed by g are in bijection with column vectors $\underline{v}$ over A such that*

$$M'_g \underline{v} = \underline{0}$$

Simlarly, A-tensions of Γ fixed by g are in bijection with column vectors $\underline{v}$ over A such that

$$M^{*\prime}_g \underline{v} = \underline{0}$$

Further, these bijections preserve the property of a flow or tension being nowhere zero.

Proof We prove the theorem for flows. Equivalent arguments apply for tensions.
First recall that the A-flows of Γ fixed by g are in bijection with the column vectors $\underline{v}$ such that

$$M_g\underline{v} = \underline{0}$$

Each row of $g^E - I$ has at most 2 non zero entries. If a row of $g^E - I$ has a 1 and a -1 then the corresponding coordinates of $\underline{v}$ must be equal in order for $M_g\underline{v} = \underline{0}$ to hold. Hence the contributions of the columns containing the 1 and the -1 to the equation corresponding to the M part of M_g may be added together and identified without essentially changing the equation.
Now, for every row of $g^E - I$ containing a 1 and a -1 we can add together and identify the columns of M_g containing these entries without essentially changing the fixed flow matrix equation.
Having chosen an orientation of Γ which has a minimal number of edges eventually reversed, if we keep adding together and identifying columns until $g^E - I$ has no rows containing a 1 and a -1 left we will have added together and identified the columns forming each cycle of g on edges of Γ and formed the matrix M_g' without essentially changing the fixed flow matrix equation.
Any -2s remaining in $g^E - I$ having reduced M_g to M_g' will correspond to edge cycles of g requiring group elements of order 2 (or the identity) in order for $M_g\underline{v} = \underline{0}$ to hold. Unless g maps an edge onto itself in reverse, these -2s will each have resulted from a submatrix $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, so the algorithm as described is valid.
We now have a bijection between fixed A-flows and solutions $\underline{v}$ to $M_g'\underline{v} = \underline{0}$.
Since the columns of M_g' correspond to the cycles of g on edges of Γ, and each edge of a g cycle must receive the same group element for a flow to be fixed, it is easy to see that the bijection preserves the nowhere zero property.
It is also clear that the same arguments will apply to M_g^* and $M_g^{*\prime}$.
This completes the proof.

4.2.3 M_g', $M_g^{*\prime}$ and $T(\Gamma/g; x, y)$

Here we see that M_g' and $M_g^{*\prime}$ are essentially the graphic and co-graphic matroids of Γ/g, and use them to find the Tutte polynomial of Γ/g. Each of these results is generalised by a result concerning the fixed point Tutte polynomial in the next section, so the proofs are omitted for now.

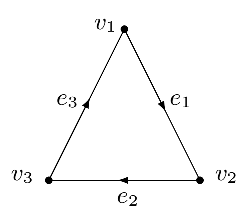
Theorem 8 *If g does not eventually reverse any edges of Γ then M_g' and $M(\Gamma/g)$ are isomorphic as vectorial matroids over the column sets of the matrices.*

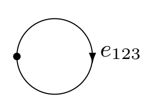
Proof This result is a specialisation of theorem 20.

Note that there is no problem, for loops in Γ/g which do not arise from eventually reversed edges, for example;

Example

$g = (v_1v_2v_3) = (e_1e_2e_3)$

$\Gamma =$ 

$\Gamma/g =$ v_{123} 

$$M_g = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \qquad M_g' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad M(\Gamma/g) = (0)$$

M_g' and $M(\Gamma/g)$ both have a single column of zeros, which is a loop in a vectorial matroid.

Theorem 9 *If g does not eventually reverse any edges of Γ, then ${M_g^*}'$ and $M^*(\Gamma/g)$ are isomorphic as matroids on their respective column sets.*

Proof This result is a specialisation of theorem 21.

We now deal with the possibility of edges of Γ being eventually reversed by g, and how a cycle of g on edges of Γ which eventually reverses an edge affects the ranks of subsets of M_g' and ${M_g^*}'$.

Theorem 10 *${M_g^*}'$ and $M^*(\Gamma/g))$ are isomorphic as matroids, whether or not g eventually reverses edges of Γ.*

Proof This result is a specialisation of theorem 21.

Unfortunately it is not so straightforward to deal with eventually reversed edges in the case of M_g' and $M(\Gamma/g)$.
If g reverses no edges of Γ we can obtain the Tutte polynomial of Γ/g via

$$\begin{aligned} T(\Gamma/g;x,y) &= \sum_{D\subseteq E(\Gamma/g)} x(M(\Gamma/g)[D])x^*(M^*(\Gamma/g)[E\setminus D])\Bigg|_{\substack{x_0=y-1\\ x_0^*=x-1\\ x_i=x_i^*=1\ \forall i\geq 1}} \\ &= \sum_{D\subseteq E(\Gamma/g)} x(M_g'[D])x^*({M_g^*}'[E\setminus D])\Bigg|_{\substack{x_0=y-1\\ x_0^*=x-1\\ x_i=x_i^*=1\ \forall i\geq 1}} \end{aligned}$$

because we have seen that if g reverses no edges of Γ, $M(\Gamma/g)$ and M_g', and $M^*(\Gamma/g)$ and ${M_g^*}'$ are isomorphic as matroids.
It is still possible though to obtain the Tutte polynomial of Γ/g from M_g' and ${M_g^*}'$ in the case where there are edges of Γ which are eventually reversed by g.

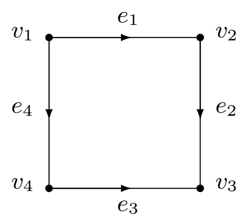
Theorem 11 *Whether or not g reverses edges of Γ, we can obtain the Tutte polynomial of Γ/g via*

$$T(\Gamma/g;x,y) = z(g^E-I)\sum_{D\subseteq E(\Gamma/g)} x(M_g'[D])x^*({M_g^*}'[E\setminus D])\Bigg|_{\substack{x_0=y-1\\ x_0^*=x-1\\ x_i=x_i^*=1\ \forall i\geq 1\\ z_2=y/2\ z_j=1\ \forall j\neq 2}}$$

Proof A loop in Γ corresponds to a loop (a dependent set of size 1) in the matroid $M(\Gamma/g)$. However, if C_l^E is a cycle of g on edges of Γ which eventually reverses an edge, then the corresponding column of M_g' will be a bridge of the matroid, since the lower part of M_g' will have a non zero entry in the column C_l^E in a row with now other non zero entries. Hence if g reverses a single edge of Γ, using M_g' and ${M_g^*}'$ will give double the Tutte polynomial of $(\Gamma/g)\setminus C_l^E$, because the deletion of C_l^E will not change the nullity of any subsets of M_g' or ${M_g^*}'$ containing C_l^E, and there are twice as many subsets giving these nullities in M_g' and ${M_g^*}'$ as there are in $M_g'\setminus C_l^E$ and ${M_g^*}'\setminus C_l^E$.
The number of edge cycles of g which eventually reverse edges of Γ is given by the number of invariant factors 2 of g^E-I, so we have proven the case where g reverses a single edge of Γ.
If g has m edge cycles which eventually reverse edges of Γ (say $\{C_l^E\}$ with $|\{C_l^E\}|=m$) then using M_g' and ${M_g^*}'$ will give 2^m times the Tutte polynomial of $(\Gamma/g)\{\setminus C_l^E\}$ because the deletion of any subset of $\{C_l^E\}$ will not change the nullity of any subsets of M_g' or ${M_g^*}'$ containing such a subset, and there are 2^m times as many subsets giving these nullities in M_g' and ${M_g^*}'$ as there are in $M_g'\setminus C_l^E$ and ${M_g^*}'\setminus C_l^E$.
In this case, g^E-I will have m invariant factors 2, and Γ/g will have m loops in addition to $(\Gamma/g)\setminus\{C_l^E\}$, so the proof is complete.

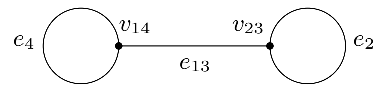
Example

$\Gamma =$



$g = (v_1v_4)(v_2v_3)$

$\Gamma/g =$



Here we have

$$M_g = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad M_g^* = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$M_g' = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad M_g^{*\prime} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

so that

$$\sum_{D \subseteq E(\Gamma/g)} x(M_g'[D])x^*(M_g^{*\prime}[E \backslash D]) = x_1^2x_2 + x_1^2x_1^* + x_1^2x_1^* + x_1^2x_0^* + x_1x_1^*x_2^* + x_1x_0^*x_1^* + x_1x_0^*x_1^* + x_0^*(x_1^*)^2$$

Substituting $x_0 = y - 1$, $x_0^* = x - 1$, $x_i = x_i^* = 1 \ \forall i \geq$ gives $4x$, which is 2^2 times the Tutte polynomial of Γ/g with the 2 loops deleted.

4.2.4 The Second Orbital Tutte Polynomial

We are now in a position to define the new orbital Tutte polynomial.

Definition Given a graph Γ and a group G of automorphisms of Γ, define the *second orbital Tutte polynomial* as

$$\mathrm{OT_{II}}(\Gamma, G) = \frac{1}{|G|} \sum_{g \in G} [\textit{prefactors}] z(g^E - I) \sum_{D \subseteq E(\Gamma/g)} x(M_g'[D])x^*(M_g^{*\prime}[E \setminus D])$$

where *prefactors* is left for the prefactors that are often needed in order to use the Tutte polynomial to count structures on graphs.

The most natural evaluation of $\mathrm{OT_{II}}(\Gamma, G)$ is the following.

$$\mathrm{OT_{II}}(\Gamma, G; x_0 \leftarrow y-1, x_0^* \leftarrow x-1, z_2 \leftarrow y/2,\ x_i, x_i^*, z_j \leftarrow 1 \ \forall i \geq 1, j \neq 2) = \frac{1}{|G|} \sum_{g \in G} T(\Gamma/g; x, y)$$

In the case where $G = \langle \mathrm{id} \rangle$ is the trivial automorphism group this evaluation simply gives the Tutte polynomial of Γ.

4.3 Evaluations of $\mathrm{OT}_{\mathrm{II}}$

$\mathrm{OT}_{\mathrm{II}}(\Gamma, G)$ can be used to count orbits of $G \leq \mathrm{Aut}(\Gamma)$ on the following structures on Γ. The proofs of these results can be found later in section 5.
Given a graph Γ and an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, the number of ... fixed by every $g \in G$ is given by

	prefactors	z_0	z_1	z_2	x_0	x_0^*	x_i $(i \geq 1)$	x_i^* $(i \geq 1)$
nowhere zero A-flows	1	1	1	1	$\lvert A\rvert$	-1	α_i	-1
nowhere zero A-tensions	1	1	1	1	-1	$\lvert A\rvert$	-1	α_i
proper k-vertex colourings	$k^{c(\Gamma/g)}$	1	1	0	-1	$-k$	-1	1
acyclic orientations	$(-1)^{n(g^V-I)+c(\Gamma/g)}$	1	1	0	-1	1	-1	1
SSAOs	$\mathrm{fix}_V(G)$	1	1	0	-1	0	1	1
orientations of Γ	1	2	1	0	1	0	1	0

Also, the ordinary Tutte polynomial of Γ can be obtained via

$$T(\Gamma; x, y) = \mathrm{OT}_{\mathrm{II}}(\Gamma, \langle \mathrm{id} \rangle; z_i \leftarrow 1 \forall i, \textit{prefactors} \leftarrow 1, x_0 \leftarrow y-1, x_j \leftarrow 1 \forall j \neq 0, x_0^* \leftarrow x-1, x_k^* \leftarrow 1 \forall k \neq 0)$$

5 The Fixed Point Tutte Polynomial

Given a graph Γ and a generating set $\{g_i\}$ of an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, we define a polynomial which counts the numbers of structures on Γ fixed by every $g \in G$ for as many structures on Γ counted by the Tutte polynomial as possible, while itself specialising to give the Tutte polynomial of Γ.

5.1 The Matrices M_G and M_G^*

First we define some matrices. For each $g_i \in \{g_i\}$, form the matrix g_i^E with respect to the same orientation of Γ, then let

$$\{g_i\}^E = \begin{pmatrix} g_1^E \\ \vdots \\ g_m^E \end{pmatrix} \qquad \text{and} \qquad \{g_i - I\}^E = \begin{pmatrix} g_1^E - I \\ \vdots \\ g_m^E - I \end{pmatrix}$$

(where $m = |\{g_i\}|$).
Also, let

$$M_G = \begin{pmatrix} M \\ \{g_i - I\}^E \end{pmatrix} \qquad \text{and} \qquad M_G^* = \begin{pmatrix} M^* \\ \{g_i - I\}^E \end{pmatrix}$$

where M and M^* are formed with respect to the same orientation of Γ as each of the g_i^E.

We now come to the fundamental result that allows us to use a generating set rather than having to check the whole of G.

Theorem 12 *A function $\underline{v}$ of $\vec{E}$ (the directed edge set of Γ) is fixed by every every $g \in G$ if and only if it is fixed by every $g_i \in \{g_i\}$*

Proof If $\underline{v}$ is not fixed by every $g_i \in \{g_i\}$ then it is clearly not fixed by every $g \in G$.
Suppose $g \in G$ is not in $\{g_i\}$ and that $g = g_1^{n_1} g_2^{n_2} \ldots g_m^{n_m}$, where $g_i \in \{g_i$, the n_i's are integers and that $\underline{v}$ is fixed by every $g_i \in \{g_i\}$. Then $\underline{v}$ is unchanged by the (repeated) action of each g_i, and so is fixed by g, and indeed every $g \in G$. This completes the proof.

Of course theorem 12 may be restated as follows:
A function $\underline{v}$ of $\vec{E}$ is fixed by every $g \in G$ if and only if

$$\{g_i - I\}^E \underline{v} = \underline{0}$$

so that the number of A-flows of Γ (A is a finite abelian group) fixed by every $g \in G$ is the number of solutions $\underline{f}$ over A to

$$M_G \underline{f} = \underline{0}$$

and the number of A-tensions fixed by every $g \in G$ is the number of solutions $\underline{t}$ to

$$M_G^* \underline{t} = \underline{0}$$

(Inclusion/Exclusion can be used in the usual way to count the number of nowhere zero flows/tensions). Note that, for any given G, the matrices M_G and M_G^* will be the same in terms of their invariant factors regardless of the choice of $\{g_i\}$; because by theorem 1, the solution sets of $\{g_i - I\}^E \underline{v} = \underline{0}$, $M_G \underline{f} = \underline{0}$ and $M_G^* \underline{t} = \underline{0}$ will not be changed. The same can be said for $M_G[D]$ and $M_G^*[D]$, for $D \in E(\Gamma)$.

5.2 "Everywhere fixed" vertex colourings and M_G^*

We now use the matrix M_G^* to count the proper k vertex colourings of Γ fixed by every $g \in G$. ("Everywhere fixed" vertex colourings).

Definitions Define the matrix $\{g_i - I\}^V$ by

$$\{g_i - I\}^V = \begin{pmatrix} g_1^V - I \\ \vdots \\ g_m^V - I \end{pmatrix}$$

and the matrix M_D^m by

$$M_D^m = \underbrace{\begin{pmatrix} \begin{matrix} M & 0 \\ 0 & M \end{matrix} & & 0 & \\ & & & \\ & 0 & \ddots & 0 \\ & & 0 & M \end{pmatrix}}_{m \text{ blocks}}$$

where $m = |\{g_i\}|$.
The following is a generalisation of lemma 2.
Lemma 13

$$(\{g_i^{-1}\}^V)^T M_D^m = M(g_i^{-1}\}^E)^T$$

Proof From the proof of the number of vertex colourings fixed by an automorphism g (lemma 2) we have

$$g^V M = M g^E$$

so clearly

$$((g_1^V - I)M \dots (g_m^V - I)M) = (M(g^E - I) \dots M(g^E - I))$$

Bearing in mind that $(g^{-1})^V = (g^V)^T$ and $(g^{-1})^E = (g^E)^T$ it is easy to see that the LHS= $(\{g_i^{-1}\}^V)^T M_D^m$ and the RHS= $M(g_i^{-1}\}^E)^T$. This completes the proof.

Theorem 14 *For a connected graph Γ and an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, the number of proper k vertex colourings of Γ fixed by every $g \in G$ is given by*

$$k \sum_{D \subseteq E} (-1)^{|E \setminus D|} x^*(M_G^*[D]) \Bigg|_{\substack{x_0^* = k \\ x_i^* = 1\ \forall l \neq 0}}$$

Proof This result can be proved by using the same arguments as for theorem 3 and making the appropriate generalisations.

We now come to the case when Γ might be disconnected.

Lemma 15 *Given a graph Γ and a group $G \leq \mathrm{Aut}(\Gamma)$ of automorphisms of Γ, (with some generating set $\{g_i\}$ of G),*

$$n(M_G^*) = n(\{g_i - I\}^V) - c_G(\Gamma)$$

where $c_G(\Gamma)$ is the number of orbits of G on the connected components of Γ.

Proof The row null space of $\{g_i - I\}^V$ is the set of vectors which are constant on the G-vertex orbits, that is, $\{\underline{x} : \underline{x}\{g_i - I\}^V = \underline{0}\}$, and this space has dimension $n(\{g_i - I\}^V$.
Now consider the null space of M_G^*, $\{\underline{y} : M_G^*\underline{y} = \underline{0}\}$.

$$\begin{array}{rcccc} M_G^*\underline{y} = \underline{0} & \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} & \{g_i - I\}^E(\underline{x}M)^T = \underline{0} \\ & \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x}M(\{g_i - I\}^E)^T = \underline{0} \\ & \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x}M(\{g_i^{-1} - I\}^E)^T = \underline{0} \\ & \Leftrightarrow & \underline{y} = (\underline{x}M)^T & \text{and} & \underline{x}(\{g_i^{-1} - I\}^V)^T M_D^m = \underline{0} \end{array}$$

by lemma 13.
Suppose then that $\underline{x}(\{g_i^{-1} - I\}^V)^T M_D^m = \underline{0}$. Then we have

$$\underline{x}(\{g_i^{-1}\}^V)M - \underline{x}M = \underline{0}$$

for each $g_i \in (\{g_i\}$. But $(\{g_i^{-1}\}^V)M$ reorders the rows of M since g_i is an automorphism, so

$$\underline{x} = \underline{0} \Leftrightarrow \underline{x}(g_i^{-1})^V = \underline{0} \Leftrightarrow \underline{x} \text{ constant on the vertices of } \Gamma \text{ making up connected components}$$

If $\underline{x}M \neq \underline{0}$ and $\underline{x}(\{g_i^{-1}\}^V)M = \underline{x}M$ then what can we say about $\underline{x}$?
Suppose u and v are adjacent vertices in Γ via an edge e directed from u to v. Then

$$\underline{x}M = \underline{x}(g_i^{-1})^V M = \underline{0} \Leftrightarrow \underline{x}_u M_{ue} + \underline{v}M_{ve} = \underline{x}_{u^{g_i^{-1}}}((g_i^{-1})^V M)_{u^{g_i^{-1}}e} + \underline{x}_{v^{g_i^{-1}}}((g_i^{-1})^V M)_{v^{g_i^{-1}}e}$$

but

$$M_{ue} = ((g_i^{-1})^V M)_{u^{g_i^{-1}}e} = 1 \text{ and } M_{ve} = ((g_i^{-1})^V M)_{v^{g_i^{-1}}e} = -1$$

so

$$\begin{array}{rcl} \underline{x}M = \underline{x}(g_i^{-1})^V M = \underline{0} & \Leftrightarrow & \underline{x}_u - \underline{x}_v = \underline{x}_{u^{g_i^{-1}}} - \underline{x}_{v^{g_i^{-1}}} \\ & \Leftrightarrow & (\underline{x}_{u^{g_i^{-1}}} - \underline{x}_{v^{g_i^{-1}}}) - (\underline{x}_u - \underline{x}_v) = \underline{0} \end{array}$$

This must hold for all adjacent vertices u, v of Γ. We have said that $\underline{x}M = \underline{0}$, do $\underline{x}$ is not constant on the vertices of each connected component of Γ, so in general $(\underline{x}_u - \underline{x}_v) \neq \underline{0}$ and $(\underline{x}_{u^{g_i^{-1}}} - \underline{x}_{v^{g_i^{-1}}}) \neq \underline{0}$. Hence

$$\begin{array}{rcl} (\underline{x}_{u^{g_i^{-1}}} - \underline{x}_{v^{g_i^{-1}}}) - (\underline{x}_u - \underline{x}_v) & = & \underline{0} \\ (\underline{x}_{u^{g_i^{-1}}} - \underline{x}_u) - (\underline{x}_{v^{g_i^{-1}}} - \underline{x}_v) & = & \underline{0} \end{array}$$

gives us $\underline{x}((g_i^{-1})^V - I) = \underline{0}$ for each $g_i \in \{g_i\}$, so $\underline{x}$ is constant on the vertex cycles of each g_i, and thereefore on the G-vertex orbits on Γ; so that

$$\underline{x}(\{g_i^{-1} - I\}^V)^T) = \underline{0}$$

Now we have

$$M_G^*\underline{y} = \underline{0} \Leftrightarrow \underline{y} = (\underline{x}M)^T \text{ and } (\underline{x}M = \underline{0} \text{ or } \underline{x}(\{g_i^{-1} - I\}^V)^T = \underline{0})$$

The space $\{\underline{y} : M_G^*\underline{y} = \underline{0}\}$ has dimension $n(M_G^*)$.
On the RHS, the space $\{\underline{y} : \underline{y} = (\underline{x}M)^T = \underline{0}\}$ has dimension zero, so the space $\{\underline{y} : \underline{y} = (\underline{x}M)^T, (\underline{x}M = \underline{0}$ or $\underline{x}(\{g_i^{-1} - I\}^V)^T = \underline{0})\}$ has dimension

$$\dim\{\underline{y} : \underline{y} = (\underline{x}M)^T \text{ and } \underline{x}(\{g_i^{-1} - I\}^V)^T = \underline{0}\} - \dim\{\underline{x} : \underline{x}M = \underline{0} \text{ and } \underline{x}(\{g_i^{-1} - I\}^V)^T = \underline{0}\}$$

Now $\underline{x}$ with $\underline{x}M = \underline{0}$ and $\underline{x}(\{g_i^{-1} - I\}^V)^T = \underline{0}$ must be constant on the vertices making up the connected components of Γ and on the G-vertex orbits of Γ, and so must be constant on the vertices making up the orbits of G on the connected components of Γ, so

$$\dim\{\underline{x} : \underline{x}M = \underline{0} \text{ and } \underline{x}(\{g_i^{-1} - I\}^V)^T = \underline{0}\} = c_G(\Gamma)$$

giving

$$n(M_G^*) = n(\{g_i - I\}^V) - c_G(\Gamma)$$

as required. This completes the proof.

Theorem 16 *The number of proper k-vertex colourings of a (not necessarily connected) graph Γ fixed by every $g \in G \leq \mathrm{Aut}(\Gamma)$ is given by*

$$k^{c_G(\Gamma)} \sum_{D \subseteq E} (-1)^{|E \setminus D|} x^*(M_G^*[D]) \Bigg|_{\substack{x_0^* = k \\ x_i^* = 1\ \forall l \neq 0}}$$

Proof The fact that the number of (not necessarily proper) k-vertex colourings of Γ is

$$k^{c_G} x^*(M_G^*) \Bigg|_{\substack{x_0^* = k \\ x_i^* = 1\ \forall l \neq 0}}$$

is a direct consequence of lemma 15, since the required number is clearly $n(\{g_i - I\}^V)$.
The same arguments as for theorem 13 (that is, the arguments for theorem 3) can be used to apply inclusion/exclusion and give the rest of the result. This completes the proof.

5.3 The 'collapsed' graph Γ/G

It seems (as we will see when looking at evaluations of the fixed point Tutte polynomial) that for many of the structures on Γ counted by the Tutte polynomial, in order to count the number of structures fixed by every $g \in G$, it is necessary to sum over subsets of the G-edge orbits, and the important relationships between the G-edge orbits are given by the graph Γ/G.

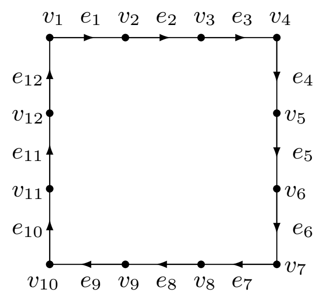
5.3.1 Definition

Given a graph Γ and a generating set $\{g_i\}$ of an automorphism group $G \leq \mathrm{Aut}(\Gamma)$ of Γ, let Γ/Gbe the graph with vertex set theset of G-vertex orbits of Γ, edge set the set of G-edge orbits of Γ, and where incidence and adjacency relationships of Γ are preserved.

We use notation O_i^V and O_i^E for both the vertices and edges of Γ/G and the corresponding G-vertex orbits and G-edge orbits in Γ.

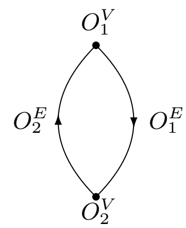
Example

$\Gamma =$



$g_1 = (e_1e_5e_9)(e_2e_6e_{10})(e_3e_7e_{11})(e_4e_8e_{12})$ $g_2 = (e_1e_7)(e_2e_8)(e_3e_9)(e_4e_{10})(e_5e_{11})(e_6e_{12})$
$G = \langle g_1, g_2 \rangle$

$\Gamma/G =$



G-vertex orbits in Γ: $O_1^V = \{v_1, v_3, v_5, v_7, v_9, v_{11}\}$, $O_2^V = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}$
G-edge orbits in Γ: $O_1^E = \{e_1, e_3, e_5, e_7, e_9, e_{11}\}$, $O_2^E = \{e_2, e_4, e_6, e_8, e_{10}, e_{12}\}$
Note that the orientation of Γ shown is fixed by every $g \in G$, so it makes sense to orient Γ/G in the corresponding way.

It is easy to see that Γ/G is well defined: Let O_1^V and O_2^V be G-vertex orbits in Γ, and suppose a vertex $v \in O_1^V$ is adjacent to a vertex $w \in O_2^V$. Then, since every $g \in G$ is an automorphism, every vertex in O_1^V is adjacent to a vertex in O_2^V, so that the corresponding vertices in Γ/G will be adjacent by the edge corresponding to the G-edge orbit containing the edge vw in Γ.

5.3.2 Loops in Γ/G

Like the graph Γ/g, (which is the same graph as Γ/G in the case when $G = \langle g \rangle$ is a cyclic group), the graph Γ/G will have a loop whenever two adjacent vertices of Γ lie in the same G-vertex orbit. These loops fall into two categories: those which arise from an edge of Γ being reversed by some $g \in G$ (*loops from reversed edges*), and those which do not.

5.4 Proper vertex colourings of Γ/G

The most natural structures to use Γ/G to count are the proper vertex colourings of Γ fixed by every $g \in G$.

Theorem 17 *The number of proper k-vertex colourings of Γ fixed by every $g \in G$ is given by the number of proper k-vertex colourings of Γ/G.*

Proof This result is a generalisation of theorem 6, and it is straightforward to prove it by making simple adjustments to the proof of theorem 6.

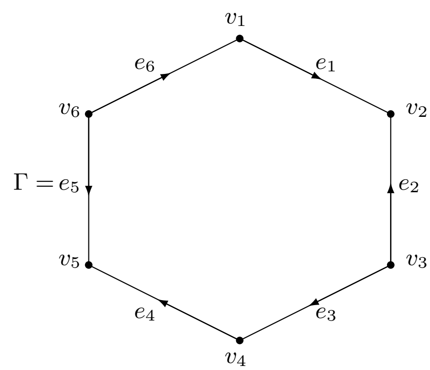
5.5 The matrices M_G' and $M_G^{*\prime}$

From here on, unless otherwise stated, matrices will be generated with respect to an orientation of Γ fixed by every $g \in G$ where possible. If this is not possible, an orientation will be used such

that the orientations of as many G-edge orbits as possible are fixed by every $g \in G$, and for those G-edge orbits which have an edge e which is reversed by some $g \in G$ (in which case every edge e of the G-edge orbit is reversed by some $g \in G$), unless the G-edge orbit has size 1, e is mapped to its image, or its preimage is mapped to e, in the same direction by at least one $g_i \in \{g_i\}$. Here, for every such edge e, there is a row of $\{g_i - I\}^E$ in which the entry in the column e is ± 1 and there is a ∓ 1 elsewhere in that row, in another column from the same G-edge orbit.
If O^E is G-edge orbit not containing an edge reversed by some $g \in G$ then the sum of the columns of $\{g_i - I\}^E$ making up O^E will be a zero vector. If some $e \in O^E$ is reversed by some $g \in G$ then the sum of the columns of $\{g_i - I\}^E$ making up O^E will contain one or more -2 and the rest zeros.

Definition Given a graph Γ and a generating set $\{g_i\}$ of an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, let M'_G and $M^{*}_G{}'$ be the matrices obtained from M_G and M^*_G by adding together and identifying the columns making up the G-edge orbits in Γ.

Example



$g_1 = (v_1 v_3)(v_4 v_6) = (e_1 e_2)(e_3 \overline{e_6})(e_4 e_5)$, $g_1 = (v_1 v_4)(v_2 v_5)(v_3 v_6) = (e_1 e_4)(e_2 e_5)(e_3 e_6)$
$G = \langle g_1, g_2 \rangle$
$O_1^E = \{e_1, e_2, e_4, e_5\}, O_2^E = \{e_3, e_6\}$

Having constructed the relevant M_G and M^*_G we get

$$M'_G = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 1 \\ 1 & -1 \\ -2 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad M^{*}_G{}' = \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

5.5.1 Flows and tensions with M_G' and ${M_G^*}'$

Theorem 18 *If A is a finite abelian group, the A-flows of Γ fixed by every $g \in G$ are in bijection with the column vectors $\underline{f}$ over A such that*

$$M_G' \underline{f} = \underline{0}$$

Similarly, the A-tensions of Γ fixed by every $g \in G$ are in bijection with the column vectors $\underline{t}$ over A such that

$${M_G^*}' \underline{t} = \underline{0}$$

Further, these bijections preserve the property of a flow or tension being nowhere zero.

Proof Keeping theorem 12 in mind, this theorem is a generalisation of theorem 7 and can be proved by making the appropriate adjustments to the arguments that proved theorem 7.

Corollary 19 *For a subset D of the G-edge orbits of Γ,*

$$n(M_G'[D]) = n(M_G[D])$$

and

$$n({M_G^*}'[D]) = n(M_G^*[D])$$

where $M_G^{(*)}[D]$ is $M_G^{(*)}$ restricted to the columns corresponding to the edges of Γ making up the G-edge orbits of $D \subset E(\Gamma/G)$, and $n(N)$ is the nullity of the vectorial matroid made up of the columns of the matrix N.

Proof Suppose we are counting A-flows of Γ fixed by every $g \in G$ where A is a finite abelian group with no solutions x to $ix = 0$ for i matching any $x_i^{(*)}$ in $x_i^{(*)}(M_G^{(*)}[D])$.
Then, by the bijection between the solutions $\underline{f}$ to $M_G^{(*)}\underline{f} = \underline{0}$ and ${M_G^{(*)}}'\underline{f} = \underline{0}$, we must have the same number of free choices, that is, the degree of $x_0^{(*)}$ in $x_0^{(*)}(M_G^{(*)}[D])$ and $x_0^{(*)}({M_G^{(*)}}'[D])$.
When forming ${M_G^{(*)}}'$ from $M_G^{(*)}$, no column of D will have been added to a column in $E \setminus D$, so that ${M_G^{(*)}}'[D] = (M_G^{(*)}[D])'$. The nullity of a matrix is simply the number of zero invariant factors, so the result follows.
This completes the proof.

5.5.2 The Tutte Polynomial $T(\Gamma/G; x, y)$

For many of the structures on Γ counted by $T(\Gamma; x, y)$, it is necessary to have the Tutte polynomial of Γ/G in order to calculate the number of structures fixed by every $g \in G$.

Theorem 20 *If there exists an orientation of Γ fixed by every $g \in G$, then M_G' and $M(\Gamma/G)$ are isomorphic as vectorial matroids over their column sets.*

Proof Choose an orientation of Γ fixed by every $g \in G$.
Form the matrix N below, and add together and identify rows and columns until the $\{g_i - I\}^V$ and $\{g_i - I\}^E$ are reduced to zero matrices. Then delete the all zero rows and columns.

$$N = \begin{pmatrix} M & (\{g_i^{-1} - I\}^V)^T \\ \{g_i - I\}^E & 0 \end{pmatrix}$$

The rows of the resulting matrix N' will correspond to the G-vertex-orbits and the columns to the G-edge orbits.
Suppose O_1^V and O_2^V are G-vertex orbits and O^E is a G-edge orbit.
If a vertex $v_1 \in O_1^V$ of Γ is adjacent to a vertex $v_2 \in O_2^V$ of Γ via an edge $e \in O^E$ of Γ, and e points from v_1 to v_2; then every vertex $v_1' \in O_1^V$ is adjacent to a vertex $v_2' \in O_2^V$ via an edge $e' \in O^E$ which points from v_1' to v_2', and all edges in O^E point from a vertex in O_1^V to a vertex in O_2^V.
So in the column of N' corresponding to C^E, the row corresponding to O_1^V will have a positive entry, the row corresponding to O_2^V will have a negative entry, and all other entries will be zero.
But this is how Γ/G is defined, so replacing positive entries of N' with 1 and negative entries with

-1 would give the matrix $M(\Gamma/G)$.
Now, given a G-vertex orbit O^V, each vertex $v \in O^V$ of Γ will be incident to the same number of edges from each G-edge orbit O^E, and in the same direction. Hence each row $v \in O^V$ of M'_G will be the same, and the only difference between M'_G and N' is that these rows are added together and identified in N'. Thus replacing positive and negative entries with 1 and -1 respectively would make M'_G and N', and therefore M'_G and $M(\Gamma/G)$ isomorphic as matroids. However, it is not necessary to replace the positive and negative entries with 1 and -1 because the rank of a subset is given by the number of non zero invariant factors, which will be the same whether or not the entries are replaced.
This completes the proof.

Note that, if Γ/G has a loop that does not arise from some $g \in G$ reversing some edge of Γ, M'_G and $M(\Gamma/G)$ are still isomorphic as matroids over their respective column sets, provided that this is the only type of loop in Γ/G because then M'_G will have a column of zeros, which is a loop in the vectorial.

Theorem 21 *The matrices $M^{*\prime}_G$ and $M^*(\Gamma/G)$ form isomorphic vectorial matroids over their respective column sets.*

Proof We first deal with the case where there is an orientation of Γ fixed by every $g \in G$, that is, there is no $g \in G$ which reverses an edge of Γ.

The columns of $M^{*\prime}_G$ and $M^*(\Gamma/G)$ both correspond to the set of G-edge orbits of Γ, that is, $E(\Gamma/G)$.
We know how to calculate the chromatic polynomial of Γ/G using M_G and M^*_G (the number of proper vertex colourings fixed by every $g \in G$), and of course using $M(\Gamma/G)$ and $M^*(\Gamma/G)$. We prove the theorem by comparing the two polynomials, using the fact that $n(M^{*\prime}_G[D]) = n(M^*_g[D])$ for $D \subseteq E(\Gamma/G)$, where $M^*_G[D]$ is M^*_G restricted to the columns corresponding to the edges of the G-edge orbits in D.

$$\begin{aligned}
\chi(\Gamma/G;k) &= k^{c(\Gamma/G)} \sum_{D\subseteq E(\Gamma/G)} x(M_G[D])x^*(M^*_G[E\setminus D])\Bigg|_{\substack{x_i=-1\ \forall i\\ x^*_0=k\\ x^*_j=1\ \forall j\neq 0}} \\
&= k^{c(\Gamma/G)} \sum_{D\subseteq E(\Gamma/G)} (-1)^{|D|} k^{n(M^*(\Gamma/G)[E\setminus D])}
\end{aligned}$$

We also have that

$$\begin{aligned}
& \# \text{ proper } k \text{ vertex colourings of } \Gamma \text{ fixed by every } g \in G \\
= \chi(\Gamma/G;k) \quad &= k^{c(\Gamma/G)} \sum_{F\subseteq E(\Gamma)} (-1)^{|F|} k^{n(M^*_G[E\setminus F])}
\end{aligned}$$

where $c(\Gamma/G)$ is the number of connected components of the graph Γ/G.

First note that

$$n(M^*_G[E\setminus F]) = n(M^*_G[\text{complete } G\text{-edge orbits in } E\setminus F])$$

For suppose that $F \setminus D$ contains an incomplete edge orbit. Then $(\{g_i - I\}^E)[E\setminus F]$ will have a row with a single non zero entry, which can be used via elementary row and column operations to transform the submatrix of $(\{g_i - I\}^E)[E\setminus F]$ corresponding to the columns of the incomplete edge orbit into an identity matrix, which clearly makes the columns of $M^*_G[E\setminus F]$ corresponding to the incomplete edge orbit independent, and so adding nothing to the nullity of $M^*_G[E\setminus F]$.
The major obstacle in comparing the above two polynomials is that the summations are over the subsets of different sets. However we can use the complete edge cycle argument to show that the

extra terms in the polynomial from M_G and M_G^* essentially cancel each other out.
If $D = F = \emptyset$, then $E \setminus D$ and $E \setminus F$ are $E(\Gamma/G)$ and $E(\Gamma)$ respectively, so the polynomials both have (equal) positive terms $k^{c(\Gamma/G)}k^{n(M^*(\Gamma/G))}$ and $k^{c(\Gamma/G)}k^{n(M_G^*)}$.
If $E \setminus F$ has all but one complete edge orbit, say $O_1^E \in \{O^E\}$, where $\{O^E\}$ is the set of orbits of G on edges of Γ, then the polynomial from M_G and M_G^* must deliver a term in $k^{n(M_G^*[\{O^E\}\setminus O_1^E])}$ with coefficient -1 to match the term in $k^{n(M^*(\Gamma/G)[E\setminus O_1^E])}$ from $M(\Gamma/G)$ and $M^*(\Gamma/G)$.

$$
\begin{aligned}
\sum_{F\subseteq O_1^E} k^{n(M_G^*[E\setminus F])}(-1)^{|F|} &= k^{n(M_G^*)}(-1)^0 + \sum_{\substack{F\subseteq O_1^E\\ F\neq\emptyset}} k^{n(M^*(\Gamma/G)[E\setminus G_1^E])}(-1)^{|F|}\\
&= k^{n(M_G^*)} + k^{n(M^*(\Gamma/G)[E\setminus O_1^E])}\sum_{i=1}^{|O_1^E|}\binom{|O_1^E|}{i}(-1)^i\\
&= k^{n(M_G^*)} - k^{n(M^*(\Gamma/G)[E\setminus O_1^E])}
\end{aligned}
$$

as required.

Suppose $E \setminus F$ contains all but m complete edge orbits (say $O_i^E,\ 1 \leq i \leq m$). The sum over all such Fs must give a term in the polynomial from M_G and M_G^* of

$$(-1)^m k^{n(M_G^*[\{O^E\}\setminus\bigcup_{i=1}^m O_i^E])}$$

Each F contains at least one element from each O_i^E, so let each $F = \bigcup_{i=1}^m F_i$, where no F_i is empty.
Then in the polynomial from M_G and M_G^*, the sum over all such Fs is (omitting the prefactor $k^{c(\Gamma/G)}$)

$$
\begin{aligned}
& k^{n(M_G^*[\{O^E\}\setminus\bigcup_{i=1}^m O_i^E])}\sum_{\substack{F_1\subseteq O_1^E\\ F_1\neq\emptyset}}(-1)^{|F_1|}\sum_{\substack{F_2\subseteq O_2^E\\ F_2\neq\emptyset}}(-1)^{|F_2|}\cdots\sum_{\substack{F_m\subseteq O_m^E\\ F_m\neq\emptyset}}(-1)^{|F_m|}\\
= {} & k^{n(M_G^*[\{O^E\}\setminus\bigcup_{i=1}^m O_i^E])}\sum_{l=1}^{|O_1^E|}\binom{|O_1^E|}{l}(-1)^l\sum_{\substack{F_2\subseteq O_2^E\\ F_2\neq\emptyset}}(-1)^{|F_2|}\cdots\sum_{\substack{F_m\subseteq O_m^E\\ F_m\neq\emptyset}}(-1)^{|F_m|}\\
= {} & k^{n(M_G^*[\{O^E\}\setminus\bigcup_{i=1}^m O_i^E])}(-1)\sum_{l=1}^{|O_2^E|}\binom{|O_2^E|}{l}(-1)^l\sum_{\substack{F_3\subseteq O_3^E\\ F_3\neq\emptyset}}(-1)^{|F_3|}\cdots\sum_{\substack{F_m\subseteq O_m^E\\ F_m\neq\emptyset}}(-1)^{|F_m|}\\
& \vdots\\
= {} & (-1)^m k^{n(M_G^*[\{C^E\}\setminus\bigcup_{i=1}^m O_i^E])} \quad \text{as required.}
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\chi(\Gamma/G;k) &= k^{c(\Gamma/G)}\sum_{D\subseteq E(\Gamma/G)}(-1)^{|D|}k^{n(M^*(\Gamma/G)[E\setminus D])}\\
&= k^{c(\Gamma/G)}\sum_{F\subseteq\{O^E\}}(-1)^{|F|}k^{n(M^*(\Gamma/G)[\{O^E\}\setminus F])}
\end{aligned}
$$

so that $n(M^*(\Gamma/G)[D]) = n(M_G^*[D]) = n(M_G^{*\prime}[D])$ where D is a subset of the set of cycles of G-edge orbits of Γ and $M_G^*[D]$ is M_G^* restricted to the columns corresponding to the edges of the

G-edge orbits in D.
Hence $M^*(\Gamma/G)$ and ${M_G^*}'$ are isomorphic matroids and the proof is complete, provided that no $g \in G$ reverses an edge of Γ.
We no deal with the possibility that some $g \in G$ reverses an edge of Γ.
A loop in Γ/G corresponds to a bridge in the matroid $M^*(\Gamma/G)$. If O_l^E is a G-edge orbit of Γ which reverses an edge, then the set of columns O_l of $\{g_i - I\}^E$ will be independent. Consequently in ${M_G^*}'$ the column O_l^E will have a non zero entry in the lower part, and this non zero entry will be the only non zero entry in that row, so that O_l^E is a bridge of the matroid ${M_G^*}'$.
Let $\Gamma \setminus \{O_l^E\}$ be the graph Γ with all edge orbits which reverse an edge deleted. Then G is an automorphism group of $\Gamma \setminus \{O_l^E\}$ and $(\Gamma \setminus \{O_l^E\})/G = (\Gamma/G) \setminus \{O_l^E\}$, where $(\Gamma/G) \setminus \{O_l^E\}$ is the graph Γ/G with the loops from $\{O_l^E\}$ deleted.
We know that ${M_G^*}' \setminus \{O_l^E\}$ is isomorphic to $M^*((\Gamma/G) \setminus \{O_l^E\})$ (since G reverses no edges of $\Gamma \setminus \{O_l^E\}$), and that $\{O_l^E\}$ are bridges of ${M_G^*}'$ and $M^*(\Gamma/G)$.
Hence ${M_G^*}'$ and $M_G^*(\Gamma/G)$ are isomorphic matroids even if there are edges which (some elements of) G reverses. This completes the proof.

We are now in a position to find the Tutte polynomial of the graph Γ/G. From theorems 9 and 10 we have that, in the case where there exists an orientation of Γ fixed by every $g \in G$, the Tutte polynomial of Γ/G is given by

$$T(\Gamma/G; x, y) = \sum_{D \subseteq E(\Gamma/G)} x(M'_G[D]) x^*({M_G^*}'[E \setminus D]) \Bigg|_{\substack{x_0 = y-1 \\ x_0^* = x-1 \\ x_i = x_i^* = 1 \; \forall i \geq 1}}$$

On the other hand, if some edge of Γ is reversed by some $g \in G$ we have the following:

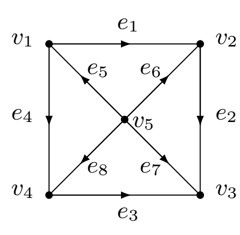
Theorem 22 *Whether or not some $g \in G$ reverses some edge of Γ, we can obtain the Tutte polynomial of Γ/G via*

$$T(\Gamma/G; x, y) = z(\{g_i - I\}^E) \sum_{D \subseteq E(\Gamma/G)} x(M'_G[D]) x^*({M_G^*}'[E \setminus D]) \Bigg|_{\substack{x_0 = y-1 \\ x_0^* = x-1 \\ x_i = x_i^* = 1 \; \forall i \geq 1 \\ z_2 = y/2 \\ z_j = 1 \; \forall j \neq 2}}$$

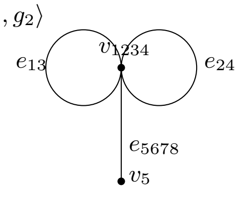
Proof This result is easily proven as it is a generalisation of theorem 11, and can be proven by making the appropriate adjustments to the proof of theorem 11.

Example

$\Gamma =$



$g_1 = (v_1v_3)(v_2v_4)$ $g_2 = (v_1v_4)(v_2v_3)$
$G = \langle g_1, g_2 \rangle$
$\Gamma/G =$



In this case, having found M'_G and ${M_G^*}'$, we have that

$$\sum_{D \subseteq E(\Gamma/G)} x(M'_G[D]) x^*({M_G^*}'[E \setminus D]) = x_1x_2x_4 + x_1x_2x_0^* + 2x_1x_4x_1^* + 2x_1x_0^*x_1^* + x_4x_1^*x_3^* + x_0^*x_1^*x_3^*$$

Substituting $x_0 = y - 1$, $x_0^* = x - 1$, and $x_i = x_i^* = 1 \; \forall i \neq 0$ gives $4x$, which is 2^2 times the Tutte polynomial of Γ/G with the 2 loops deleted.

5.6 The Fixed Point Tutte Polynomial

We can now define the fixed point Tutte polynomial.

Definition Given a graph Γ and a generating set $\{g_i\}$ of an automorphism group $G \leq \mathrm{Aut}(\Gamma)$ of Γ, define the *fixed point Tutte polynomial* as

$$\mathrm{FPT}(\Gamma, G) = [prefactors] z(\{g_i - I\}^E) \sum_{D \subseteq E(\Gamma/G)} x(M'_G[D]) x^*({M^*_G}'[E \setminus D])$$

It is easy to see that if $G \leq \mathrm{Aut}(\Gamma)$ is a cyclic group generated by the automorphism g, then, at least as far as the invariant factors of the matrices formed by subsets of the column set of the original matrix, we have

$$\begin{aligned} g^E - I &= \{g_i - I\}^E \\ M_g &= M_G \\ M^*_g &= M^*_G \\ M'_g &= M'_G \\ {M^*_g}' &= {M^*_G}' \end{aligned}$$

This means that the second orbital Tutte polynomial can be expressed in terms of the fixed point Tutte polynomial via

$$\mathrm{OT}_{\mathrm{II}}(\Gamma, G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{FPT}(\Gamma, \langle g \rangle)$$

Also, theorem 5 could be restated to say that the ordinary Tutte polynomial can be obtained via

$$T(\Gamma; x, y) = \mathrm{FPT}(\Gamma, \langle \mathrm{id} \rangle; x_0 \leftarrow y - 1, x_i \leftarrow 1\ \forall i \geq 1, x^*_0 \leftarrow x - 1, x^*_j \leftarrow 1\ \forall j \geq 1)$$

6 Evaluations of $\mathrm{FPT}(\Gamma, G)$

We have already seen that $\mathrm{FPT}(\Gamma, G)$ can be used to count the nowhere zero flows, nowhere zero tensions and proper vertex colourings of Γ fixed by every $g \in G$. We also have an evaluation that counts the number of orientations of Γ fixed by every $g \in G$ (shown in the table below, the evaluation is easily checked). We now see that there are a few other structures on Γ fixed by every $g \in G$ that the fixed point Tutte polynomial can be used to count. Each of these is also an evaluation of the second orbital Tutte polynomial, since it can be expressed in terms of the fixed point Tutte polynomial.

6.1 Acyclic orientations and SSAO's

Theorem 23 *The number of acyclic orientations of Γ fixed by every $g \in G$ is equal to the number of acyclic orientations of Γ/G.*

Proof Suppose that Γ/G has a loop. Then Γ/G has no acyclic orientations.
If g reverses an edge of Γ for some $g \in G$ then there are no orientations of Γ fixed by every $g \in G$, acyclic or otherwise, as required.
If no $g \in G$ reverses an edge of Γ but Γ/G has a loop then the preimage in Γ of the loop together with its incident vertex must contain a cycle. Thus in any orientation of Γ there must be edges in the preimage of the loop pointing both in and out of the vertex incident to the loop; so that in order for an orientation of Γ to be fixed a cycle in the preimage of the loop with its incident vertex must be directed.
Thus if Γ/G has a loop then there are no acyclic orientations of Γ fixed by every $g \in G$.

Now assume that Γ/G has no loops. In this case the orientations of Γ/G are in bijection with the orientations of Γ fixed by every $g \in G$. (Since no $g \in G$ reverses an edge of Γ). We will show that this bijection preserves the property of an orientation being acyclic.

We first show that if Γ/G has a directed cycle then the preimage of this cycle in Γ has a directed cycle.

Suppose $C = O_1^V O_1^E O_2^V O_2^E \dots O_n^V O_n^E O_1^V$ is a cycle in Γ/G, and that some orientation of Γ fixed by every $g \in G$ causes C to be directed in the direction $O_i^V \to O_i^E \to O_{i+1}^V$.
Now choose a vertex $v_1 \in O_1^V$ in Γ.

v_1	must be incident to an edge	$e_1 \in O_1^E$ in Γ
e_1	must be incident to a vertex	$e_2 \in O_1^V$ in Γ
$\vdots$	$\vdots$	$\vdots$
$e_n \in O_n^E$	must be incident to a vertex	$v_1' \in O_1^V$ in Γ

v_1' may or may not be v_1, but if one continues in this way one will eventually hit a vertex in some O_i^V for a second time, and so find a cycle in Γ, whose image in Γ/G is C. Since C is directed in Γ/G, this cycle of Γ must also be directed.

(We can also say that cycles are mapped to cycles by every $g \in G$, and as a set of vertices and edges of Γ, C is closed under the action of G, so C must be a G-orbit on cycles of Γ.)

We now show that if Γ has a directed cycle then a fixed orientation causes Γ/G to have a directed cycle.

If Γ/G is loopless then the image of a cycle C of Γ in Γ/G is either a path or it contains a cycle. If the image is a path then only a non directed cycle can be fixed. Suppose then that the image of C in Γ/G contains a cycle. Let $C = v_1 e_1 v_2 e_2 \dots v_n e_n v_1$, and let the images of v_i and e_i in Γ/g be O_i^V and O_i^E respectively. (Not all O_i^Vs and O_i^Es will necessarily be distinct). Suppose further that C is directed in the direction $v_i \to e_i \to v_{i+1}$
In Γ/G, $O_1^V \sim O_1^E$, $O_1^E \sim O_2^V$, $O_2^V \sim O_2^E$, etc, ($\sim$ denotes incidence), and since the image of C in Γ/G contains a cycle we must hit one of the O_i^Vs for a second time, so that a cycle in the image of C will be directed if an orientation is to be fixed.

Hence if Γ/G has no loops and has a cycle C, then an orientation of Γ/G in which C is directed will correspond to an orientation of Γ fixed by every $g \in G$ with directed cycles. So non acyclic orientations of Γ/G correspond to fixed non acyclic orientations of Γ.

We now have a bijection between the acyclic orientations of Γ/G and those of Γ fixed by every $g \in G$, and so the theorem is proven.

Thus, by a theorem of Stanley, the number of acyclic orientations of a graph Γ fixed by every $g \in G$ is given by

$$(-1)^{n(\{g_i - I\}^V)} \chi(\Gamma/G, -1)$$

that is,

$$\mathrm{FPT}(\Gamma, G; \textit{prefactors} \leftarrow (-1)^{n(\{g_i - I\}^V)}, z_i \leftarrow 1 \forall i, x_j \leftarrow -1 \forall j, x_0^* \leftarrow -1, x_k \leftarrow 1 \forall k \neq 0)$$

We now turn to *single sourced acyclic orientations* (SSAO's). The following theorem comes from [2].

Theorem For every connected graph Γ and every vertex $v \in V(\Gamma)$, the number of acyclic orientations of Γ with a single source v is given by $a_v(\Gamma) = T(\Gamma; 1, 0)$.

Obviously then the total number of single sourced acyclic orientations (SSAOs) of Γ is given by $|V(\Gamma)|T(\Gamma; 1, 0)$.

Theorem 24 *The number of SSAOs fixed by every $g \in G$ is given by*

$$\mathit{fix}_V(G)T(\Gamma/G; 1, 0)$$

where $\mathit{fix}_V(G)$ is the number of vertices of Γ fixed by every $g \in G$.

Proof If an orientation of Γ/G (loopless) has a source O^V, then every $v \in O^V$ in Γ is a source in the orientation of Γ fixed by every $g \in G$ giving rise to the orientation of Γ/G. If O^V has only one vertex v in Γ (v is a fixed vertex) then every $g \in G$ fixes an SSAO. This completes the proof. (There does not appear to be a natural way to calculate $\mathrm{fix}_V(G)$).)

6.2 Evaluations of $\mathrm{FPT}(\Gamma, G)$

To date, we have the following evaluations of the fixed point Tutte polynomial.
Given a graph Γ and an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, the number of ... fixed by every $g \in G$ is given by

	prefactors	z_0	z_1	z_2	x_0	x_0^*	x_i $(i \geq 1)$	x_i^* $(i \geq 1)$
nowhere zero A-flows	1	1	1	1	$\lvert A\rvert$	-1	α_i	-1
nowhere zero A-tensions	1	1	1	1	-1	$\lvert A\rvert$	-1	α_i
proper k-vertex colourings	$k^{c(\Gamma/g)}$	1	1	0	-1	$-k$	-1	1
acyclic orientations	$(-1)^{n(g^V - I)+c(\Gamma/g)}$	1	1	0	-1	1	-1	1
SSAOs	$\mathrm{fix}_V(G)$	1	1	0	-1	0	1	1
orientations of Γ	1	2	1	0	1	0	1	0

Also, the ordinary Tutte polynomial of Γ can be obtained via

$$T(\Gamma; x, y) = \mathrm{FPT}(\Gamma, \langle \mathrm{id} \rangle; z_i \leftarrow 1 \forall i, \mathit{prefactors} \leftarrow 1, x_0 \leftarrow y-1, x_j \leftarrow 1 \forall j \neq 0, x_0^* \leftarrow x-1, x_k^* \leftarrow 1 \forall k \neq 0)$$

Note: There may be substitutions into the fixed point Tutte polynomial to count T-tetromino tilings of a $4m \times 4n$ rectangle fixed by a subgroup G of the symmetry group of the rectangle, but this would involve variable *prefactors*, depending on which G is used, and would therefore, it seems, be "cheating".

7 Sizes and Types of G-Orbits on Symmetric Tutte Structures, and the Third Orbital Tutte Polynomial

In this section we aim to calulate the sizes of the G-orbits on the various structures, and to classify these orbits in terms of the action of G on each orbit.

7.1 Some definitions

Definition (well known) If G is a permutation group acting on a set Ω, for $\alpha \in \Omega$, the stabiliser G_α in G of α is the maximal subgroup $H \leq G$ such that every $h \in H$ fixes α.
(For a subset $A \subseteq \Omega$, the pointwise stabiliser is the maximal subgroup $H \subseteq G$ which fixes $\alpha \in A$)

Definition For a subgroup $H \leq G$, let $\{\alpha_i\}_H \subseteq \Omega$ be the maximal subset of Ω such that H is the stabiliser of every $\alpha_i \in \{\alpha_i\}_H$.
Note that $\{\alpha_i\}_H$ is not the necessarily the full fixed point set of H: If $J \geq H$, there may be a β which is fixed by every $j \in J$, (and thus for every $h \in H$), so that H is not the stabiliser of β. For the same reason $\{\alpha_i\}_H$ is not the largest set for which H is the pointwise stabiliser.

7.2 Some results

Lemma 25 *For $H_1, H_2 \leq G$ ($H_1 \neq H_2$), the sets $\{\alpha_i\}_{H_1}$ and $\{\alpha_i\}_{H_2}$ are disjoint.*

Proof If not, then $\exists \beta \in \{\alpha_i\}_{H_1} \cap \{\alpha_i\}_{H_2}$ in which case β is fixed by every element of the group $\langle H_1, H_2 \rangle$. Thus H_1 is not maximal with respect to fixing β, and so is not the stabiliser of β, which is a contradiction: $\{\alpha_i\}_{H_1}$ and $\{\alpha_i\}_{H_2}$ must be disjoint. This completes the proof.

We now calculate $|\{\alpha_i\}_H|$ for each $H \leq G$.

Theorem 26 *If G is a permutation group acting on a set Ω, and $H \leq G$; the number of elements of Ω for which H is the stabiliser is given by*

$$|\{\alpha_i\}_H| = \sum_{(G \geq) J \geq H} \mu(H, J) \mathrm{fix}(J)$$

where μ is the Möbius function on the poset of subgroups of G, ordered by inclusion, defined recursively by

$$\mu(H, J) = \begin{cases} 1 & H = J \\ 0 & H \not\leq J \\ -\displaystyle\sum_{J \geq K > H} \mu(K, J) & H \leq J \end{cases}$$

and $\mathrm{fix}(J)$ *is the number of elements of Ω fixed by every $j \in J$.*

Proof From the lemma we have that

$$|\{\alpha_i\}_H| = \mathrm{fix}(H) - \sum_{(G \geq) J > H} |\{\alpha_i\}_J|$$

where the sum does not include $J = H$. This comes from the fact that if $\alpha \in \Omega$ is fixed by every $j \in J$, then it is fixed by every $h \in H$, where $H < J$; and the particular case when J is the stabiliser of α.
This gives us that

$$\mathrm{fix}(H) = \sum_{(G \geq) J \geq H} |\{\alpha_i\}_J|$$

The (generalised) Möbius inversion formula is that if $(P, \leq)$ is a poset and $f : P \to \mathbb{R}$ is a function with

$$g(x) = \sum_{y \leq x} f(y)$$

then

$$f(x) = \sum_{y \leq x} \mu(y, x) g(y)$$

where

$$\mu(y, x) = \begin{cases} 1 & y = x \\ 0 & y \not\leq x \\ -\displaystyle\sum_{y \leq z < x} \mu(y, z) & y \leq x \end{cases}$$

Now replace x by H, $g(x)$ by $\mathrm{fix}(H)$, y by J, $f(y)$ by $|\{\alpha_i\}_J|$, $\leq$ by $\geq$, and μ by ν. (P is replaced by the poset of subgroups of G, ordered by inclusion). This gives us

$$\mathrm{fix}(H) = \sum_{(G \geq) J \geq H} |\{\alpha_i\}_J| \Longrightarrow |\{\alpha_i\}_H| = \sum_{(G \geq) J \geq H} \mathrm{fix}(J) \nu(J, H)$$

where

$$\nu(J,H) = \begin{cases} 1 & J = H \\ 0 & J \ngeq H \\ -\displaystyle\sum_{J \geq K > H} \mu(J,K) & J \geq H \end{cases}$$

Then letting $\nu(a,b) = \mu(b,a)$ gives the result.
This completes the proof.

7.3 Sizes and types of orbits

The following is a well known result of permutation group theory [1].

Lemma

(a) Let Ω be a transitive G-space (a G-orbit). Then Ω is isomorphic to the (right) coset space $H\backslash G$, where $H = G_\alpha$ for $\alpha \in \Omega$. In particular, the index $|G|/|H|$ of H in G is equal to the size of Ω.

(b) Two coset spaces $H\backslash G$ and $K\backslash G$ are isomorphic if and only if H and K are conjugate subgroups if G.

We omit the proof.

Now we can calculate the numbers of elements of Ω with stabilisers of each index, and hence the sizes of the orbits. Clearly

$$\#\text{ orbits of size } n = \frac{\left[\begin{array}{l} \#\text{ elements lying in orbits of size } n \\ = \#\text{ elements with stabilisers of index } n \end{array}\right]}{n}$$

Thus the ordinary generating function for the size of the orbits is

$$f_G(t) = \frac{1}{|G|} \sum_{H \leq G} |H| t^{|G|/|H|} \sum_{(G \geq) J \geq H} \mu(H,J) \mathrm{fix}(J)$$

(The second sum calculates the number of elements with H as their stabiliser). If G is an automorphism group of a graph Γ and Ω is a set of structures (eg proper k-vertex colourings) on Γ, then $fix(J) = \mathrm{FPT}(\Gamma, J)$ with the relevant substitutions.

By the lemma, we can not only calculate the sizes of the orbits, but also classify the orbits in terms of the action of G on the orbits, by the conjugacy classes $\mathcal{C}_i$ of subgroups of G. With each $\mathcal{C}_i$, associate a variable u_i, and for each subgroup H of G, let $\mathcal{C}_{i(H)}$ be the conjugacy class containing H.

7.4 The third orbital Tutte polynomial $\mathrm{OT}_{\mathrm{III}}(\Gamma, G)$

Definition Given a graph Γ and an automorphism group $G \leq \mathrm{Aut}(\Gamma)$, define the *third orbital Tutte polynomial* by

$$\begin{aligned} \mathrm{OT}_{\mathrm{III}}(\Gamma, G) &= \frac{1}{|G|} \sum_{H \leq G} |H| u_{i(H)} \sum_{(G \geq) J \geq H} \mu(H,J) [\mathit{prefactors}] z(\{j_i - I\}^E) \sum_{D \subseteq E(\Gamma/G)} x(M'_J[D]) x^*(M^{*\prime}_J[E \setminus D]) \\ &= \frac{1}{|G|} \sum_{H \leq G} |H| u_{i(H)} \sum_{(G \geq) J \geq H} \mu(H,J) \mathrm{FPT}(\Gamma, J) \end{aligned}$$

Then the number of orbits of G on structures on Γ of type j is found by putting $u_j = 1$ and $u_i = 0$ for $i \neq j$; and the ordinary generating function for the size of the orbits is given by putting $u_{i(H)} \leftarrow t^{|G|/|H|}$. Further, the number of orbits on various structures can be found by putting $u_j = 1\ \forall j$ and using the relevant substitutions for the fixed point Tutte polynomial, which gives rise to the following theorem.

Theorem 27

$$\mathrm{OT_{III}}(\Gamma, G; u_i \leftarrow 1 \forall i) \equiv \mathrm{OT_{II}}(\Gamma, G)$$

Proof The two polynomials to be compared are

$$\mathrm{OT_{III}}(\Gamma, G; u_i \leftarrow 1 \forall i) = \frac{1}{|G|} \sum_{H \leq G} |H| \sum_{(G\geq) J \geq H} \mu(H, J) \mathrm{FPT}(\Gamma, J)$$

and

$$\mathrm{OT_{II}}(\Gamma, G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{FPT}(\Gamma, \langle g \rangle)$$

We can immediately disregard the factor $\frac{1}{|G|}$ because it occurs in both polynomials.
If $h \in G \leq \mathrm{Aut}(\Gamma)$, and h generates the automorphism group $H \leq G$, then it is clear that $M_h = M_H$, $M_h^* = M_H^*$, $M_h' = M_H'$, and $M_h^{*\prime} = M_H^{*\prime}$; at least as far as the invariant factors of column restrictions of the matrices. This means that every summand

$$\mathrm{FPT}(\Gamma, \langle g \rangle) = \sum_{D \subseteq E(\Gamma/g)} x(M_g'[D]) x^*(M_g^{*\prime}[E \setminus D])$$

of $\mathrm{OT_{II}}(\Gamma, G)$ is contained in $\mathrm{OT_{III}}(\Gamma, G; u_i \leftarrow 1 \forall i)$, so we need to show that the coefficient in $\mathrm{OT_{III}}(\Gamma, G; u_i \leftarrow 1 \forall i)$ of $\mathrm{FPT}(\Gamma, J)$, which is clearly $\sum_{H \leq J} \mu(H, J)|H|$, satisfies

$$\sum_{H \leq J} \mu(H, J)|H| = \#\text{ single element generators of } J = \begin{cases} 0 & J \text{ is not cyclic} \\ \#\text{ single element generators of } J & \text{otherwise} \end{cases}$$

since the number of single element generators of a cyclic group $J \leq G$ is clearly the coefficient of $\mathrm{FPT}(\Gamma, \langle g \rangle)$ in $\mathrm{OT_{II}}(\Gamma, G)$.

If $H \leq J \leq G \leq \mathrm{Aut}(\Gamma)$, let $f(H)$ denote the number of single element generators of the group H. Note that $f(H) = 0$ if and only if H is non cyclic. Then it is easy to see that

$$\sum_{H \leq J} f(H) = |J|$$

because every element $j \in J$ generates a unique cyclic subgroup of J.
Using (generalised) Möbius inversion we find that

$$f(J) = \sum_{H \leq J} \mu(H, J)|H|$$

as required. This completes the proof.

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