# The Farey Fraction Spin Chain in an External Field

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# Outline

- Definition of the spin chain model(s)
- Phase transition in zero field
- Coupling to an external field
    - Renormalization group analysis
    - Dynamical systems analysis
- Result: full phase diagram

# Farey Fraction Spin Chain - Definition

- Chain of $N$ spins $\vec{\sigma} = \{\sigma_i\}_{i=1}^N$ with $\sigma_i \in \{\uparrow, \downarrow\}$
- Associate with each spin $\sigma_i \in \{\uparrow, \downarrow\}$ a matrix

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- Energy of a configuration $\vec{\sigma}$

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- Thermodynamic limit $-\beta f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$

# Farey Fraction Spin Chain - Which Energy?

- Write

$$M_N = \prod_i A_{\sigma_i} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

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$\boxed{\text{Thermodynamic limit is the same}}$

# The Transfer Operator

- Using the notation

$$f(x)\Bigg|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{(cx+d)^{2\beta}}\, f\left(\frac{ax+b}{cx+d}\right)$$

we find

$$Z_N(\beta;x) = 1(x)\,\Big|(A_\uparrow + A_\downarrow)^N$$

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- Equivalently, using the transfer operator

$$\mathcal{L}_\beta = \mathcal{L}_\beta^\uparrow + \mathcal{L}_\beta^\downarrow$$

where

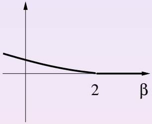
$$\mathcal{L}_\beta^\uparrow f(x) = f(x)\,|A_\uparrow \quad \text{and} \quad \mathcal{L}_\beta^\downarrow f(x) = f(x)\,|A_\downarrow$$

we obtain

$$Z_N(\beta;x) = \mathcal{L}_\beta^N 1(x)$$

# Phase Transition

- One-dimensional spin chain with phase transition at $\beta_c = 1$
- For $-\beta f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$ we have
  - $-\beta f(\beta)$ analytic in $\beta < \beta_c$
  - $-\beta f(\beta) \sim \frac{\beta_c - \beta}{-\log(\beta_c - \beta)}$ as $\beta \to \beta_c^-$
  - $-\beta f(\beta) = 0 \qquad \forall \beta \geq \beta_c$

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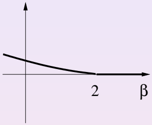


Fiala et al (2003) using results from Prellberg and Slawny (1992)

- Necessarily long-range interactions
- High temperature state is paramagnetic
- Low temperature state is completely ordered, no thermal effects
- The phase transition is second-order, but the magnetization jumps at $\beta_c$ from saturation to zero (first-order like)

# Farey Fraction Spin Chain with Field

- Natural generalisation: coupling to external magnetic field $h$

$$E_N(\vec{\sigma}, h) = E_N(\vec{\sigma}) + h \sum_i \left( \chi_\uparrow(\sigma_i) - \chi_\downarrow(\sigma_i) \right)$$

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- This leads directly to

$$Z_N(\beta, h; x) = 1(x) \Big| (e^{-\beta h} A_\uparrow + e^{\beta h} A_\downarrow)^N$$

respectively

$$Z_N(\beta, h; x) = \mathcal{L}_{\beta,h}^N 1(x)$$

where

$$\mathcal{L}_{\beta,h} = e^{-\beta h} \mathcal{L}_\beta^\uparrow + e^{\beta h} \mathcal{L}_\beta^\downarrow$$

# Renormalization Group Analysis

Fiala and Kleban (2004)

- Mean field expansion $f_{MF} = a + btM^2 + uM^4 - ghM + \dots$
- Two relevant fields $t = 1 - \beta/\beta_c$ and $h$, one marginal field $u$
- RG transformation for singular part $f_s(t, h, u)$
- Result for high-temperature phase

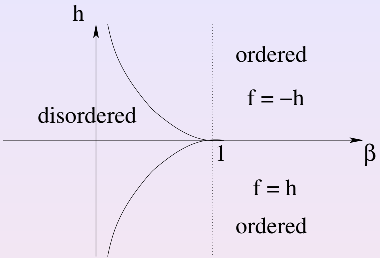
$$f_s(t,h,u) \sim \left|\frac{t}{t_0}\right| \left(\frac{x}{y_t} u \log\frac{t_0}{t}\right)^{-1} a - \frac{h^2}{t}\left(\frac{x}{y_t} u \log\frac{t_0}{t}\right)\frac{3g^2}{16b}$$

($x$, $y_t$ are scaling exponents)

- Combine with low-temperature result to get phase boundary

$$\boxed{-|h| \sim t/\log t}$$

# Phase Diagram from RG




- Disordered phase, small field: $t = 1 - \beta/\beta_c$

$$f(\beta, h) \sim a\frac{t}{\log t} - b\frac{h^2 \log t}{t}$$

- Phase boundary, $h_c = |h| = -f$:

$$h_c(\beta) \sim -a\frac{t}{\log t}$$

# The Associated Dynamical System

- The operator

$$\mathcal{L}_{\beta,h} = e^{-\beta h}\mathcal{L}_\beta^{\uparrow} + e^{\beta h}\mathcal{L}_\beta^{\downarrow}$$

is a (weighted) Ruelle-Perron-Frobenius operator of the map

$$T : x \mapsto \begin{cases} x/(1-x)\,, & 0 \le x < 1 \\ x-1\,, & x \ge 1 \end{cases}$$

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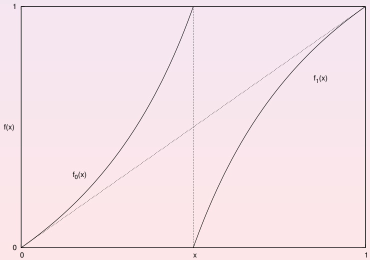
- Conjugating with $C(x) = x/(1-x)$ gives a symmetric map on $[0,1]$



# Identities and Spectral Relations I

- Consider the generating function

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$$\begin{aligned} G(\beta, h, z; x) &= 1(x) \Big| [1 - z(e^{-\beta h} A_{\uparrow} + e^{\beta h} A_{\downarrow})]^{-1} \\ &= [1 - z\mathcal{L}_{\beta,h}]^{-1} 1(x) \end{aligned}$$

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- We find

$$\beta f(\beta,h)=\log z_c(\beta,h)=-\log r(\beta,h)$$

$z_c(\beta,h)$ is the smallest singularity of $G(\beta,h,z;x)$
$r(\beta,h)$ is the spectral radius of $\mathcal{L}_{\beta,h}$

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This is hard to handle. But there is a trick we can use to overcome this problem.

# Identities and Spectral Relations II

## Lemma

$$[1+\tilde{\mathcal{M}}^{\downarrow}_{\beta,ze^{\beta h}}][1-z\mathcal{L}_{\beta,h}][1+\tilde{\mathcal{M}}^{\uparrow}_{\beta,ze^{-\beta h}}]=[1-\tilde{\mathcal{M}}^{\downarrow}_{\beta,ze^{\beta h}}\tilde{\mathcal{M}}^{\uparrow}_{\beta,ze^{-\beta h}}]$$

*where*

$$\tilde{\mathcal{M}}^{\uparrow}_{\beta,\tau}=\tau\mathcal{L}^{\uparrow}_{\beta}[1-\tau\mathcal{L}^{\uparrow}_{\beta}]^{-1} \quad \textit{and} \quad \tilde{\mathcal{M}}^{\downarrow}_{\beta,\tau}=\tau\mathcal{L}^{\downarrow}_{\beta}[1-\tau\mathcal{L}^{\downarrow}_{\beta}]^{-1}$$

*Moreover,* $1+\tilde{\mathcal{M}}^{\uparrow}_{\beta,\tau}=[1-\tau\mathcal{L}^{\uparrow}_{\beta}]^{-1}$ *and* $1+\tilde{\mathcal{M}}^{\downarrow}_{\beta,\tau}=[1-\tau\mathcal{L}^{\downarrow}_{\beta}]^{-1}$

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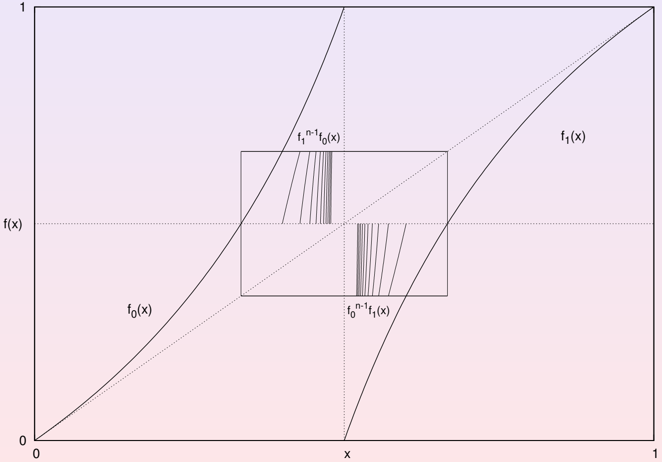
- A formal expansion for the associated matrices gives

$$\tilde{M}_{\uparrow}(\tau)=\sum_{n=1}^{\infty}\tau^n {A_{\uparrow}}^n \quad \text{and} \quad \tilde{M}_{\downarrow}(\tau)=\sum_{n=1}^{\infty}\tau^n {A_{\downarrow}}^n$$

- The operators $\tilde{\mathcal{M}}^{\uparrow}_{\beta,\tau}$ and $\tilde{\mathcal{M}}^{\downarrow}_{\beta,\tau}$ can be associated with a first-return map



# … Is the Gauss Map (well, nearly)

- Introduce the weighted transfer operator for the Gauss map $x \mapsto 1/x \mod 1$

$$\mathcal{M}_{\beta,\tau} f(x) = \sum_{n=1}^{\infty} \frac{\tau^n}{(n+x)^{2\beta}} f\left(\frac{1}{n+x}\right)$$

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**Lemma**

*Let $z \notin \left[0, e^{|\beta h|}\right]$. If $f$ is an eigenfunction of $\mathcal{M}_{\beta,ze^{\beta h}} \mathcal{M}_{\beta,ze^{-\beta h}}$ with eigenvalue 1, then $[1 + \tilde{\mathcal{M}}^{\uparrow}_{\beta,ze^{-\beta h}}]f$ is an eigenfunction of $\mathcal{L}_{\beta,h}$ with eigenvalue $\lambda = 1/z$.*

# Not-so-standard Perturbation Theory

- Consider normalised Eigenfunctions $g_{\beta,h,z}$ and Eigenmeasures $\mu_{\beta,h,z}$ associated with the Eigenvalue $\lambda_{\beta,h,z}$ of

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  on a suitable space of functions on $[1,\infty]$
- At $\beta = 1$, $z = 1$, and $h = 0$ we have

$$g_{1,1,0}(x) = \frac{1}{\log(2)x(1+x)} \quad \text{and} \quad \mu_{1,1,0} = \mu_L$$

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- Solve

$$1=\lambda_{\beta,h,z}=\mu_{\beta,h,z}\left(\mathcal{M}_{\beta,ze^{\beta h}}\mathcal{M}_{\beta,ze^{-\beta h}}g_{\beta,h,z}\right)$$

perturbatively to get $z(\beta,h)$ ...

# Perturbation Results

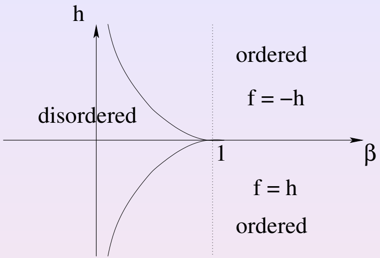
- Recall $z = e^{\beta f}$, so that $f(\beta, h) = \frac{1}{\beta} \log z(\beta, h)$
- To leading order

$$2\log(2)\lambda_G(1-\beta) \sim (f+h)\log(-(f+h)) + (f-h)\log(-(f-h))$$

where $\lambda_G$ is the Lyapunov exponent of the Gauss map $G(x) = 1/x \mod 1$

$$\lambda_G = \int_0^1 \log|G'(x)| \frac{1}{\log 2} \frac{dx}{1+x} = \frac{\zeta(2)}{\log 2}$$

# Phase Diagram revisited




- Disordered phase, small field: $\qquad t = 1 - \beta$

$$f(\beta, h) \sim \zeta(2)\frac{t}{\log t} - \frac{1}{2\zeta(2)}\frac{h^2}{t} \quad \text{where } |h| \ll |t/\log t|$$

- Phase boundary, $h_c = |h| = -f$:

$$h_c(\beta) \sim -\zeta(2)\frac{t}{\log t}$$

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- The application of RG, while slightly problematic ($h^2 \log t/t$ vs $h^2/t$), is surprisingly accurate
- The amplitude of the free energy expansion (pressure) at $\beta = 1$ is related to the Lyapunov exponent of the induced map