Counting Defective Parking Functions

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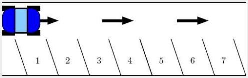
1. A parking problem (defective parking functions)
2. Enumeration (generating functions)
3. Asymptotics (limit distributions)
4. Conclusion and outlook

A parking problem

- Consider n parking spaces in a *one-way street*



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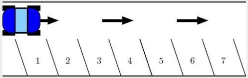

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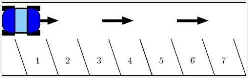

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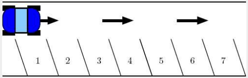

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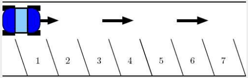

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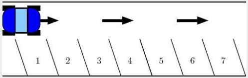

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- If k drivers fail to park, the sequence is called a

  defective parking function of degree k

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Theorem

Every permutation of a defective parking function of degree k is also a defective parking function of degree k.

The counting problem

Enumerate the number

$$\mathrm{cp}(n, m, k)$$

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The probabilistic question

What is the probability

$$p_{n,m}(k) = \frac{1}{n^m}\,\mathrm{cp}(n, m, k)$$

that for a randomly chosen assignment exactly k drivers leave? In particular, are there interesting limiting distributions?

Many equivalent or related formulations, for example

- Hashing with linear probing

  [Konheim and Weiss, 1966]

  [Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation

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Connections to other combinatorial objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, inversion in trees

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$\mathrm{cp}(n,n,k)$	$k=0$	1	2	3	4	5	6
$n=1$	1						
2	3	1					
3	16	10	1				
4	125	107	23	1			
5	1296	1346	436	46	1		
6	16807	19917	8402	1442	87	1	
7	262144	341986	173860	41070	4320	162	1

$\mathrm{cp}(n,n,k)$: the number of car parking assignments of n cars to n spaces such that k cars are not parked.

Theorem

For $m \leq n$*,*

$$\mathrm{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

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Proof (adapted from Pollak ($m = n$), 1974).

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- Space $n + 1$ is empty with probability $(n + 1 - m)/(n + 1)$.

$\square$

Definition

For $r, s, k \in \mathbb{N}_0$ let $a(r, s, k)$ denote the number of choices for which r spaces remain empty, s spaces are occupied in the end and k people drive home.

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- $n = s + r$ parking spaces, $m = s + k$ drivers
- $\mathrm{cp}(n, m, k) = a(n - m + k, m - k, k)$

Lemma (A recursion)

For $r, s, k \in \mathbb{N}_0$, the number of assignments of $s+k$ drivers to $s+r$ spaces such that r spaces remain empty, s spaces are occupied and k drivers leave is recursively defined by

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$$a(r,s,k) = \begin{cases} 1 & \text{if } r=s=k=0, \\ \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r,s-1,i) & \text{if } k>0, \\ a(r-1,s,0)+ & \text{if } k=0 \text{ and} \\ \sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r,s-1,i) & (r>0 \text{ or } s>0). \end{cases}$$

Lemma (A functional equation)

Let A be the formal power series in the three variables u, v, and t defined by

$$A(u,v,t) := \sum_{r,s,k\geq 0} a(r,s,k)\cdot u^r \frac{v^s t^k}{(s+k)!}\,.$$

Then $A(u,v,t)$ is the unique solution of

$$0 = (\tfrac{v}{t}e^t - 1)\cdot A(u,v,t) + (u - \tfrac{v}{t})\cdot A(u,v,0) + 1$$

in the ring of formal power series in u, v and t.

Solving

$$0 = \left(\frac{v}{t} e^{t} - 1\right) \cdot A(u, v, t) + \left(u - \frac{v}{t}\right) \cdot A(u, v, 0) + 1$$

with the Kernel Method:

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- This is solved by $t = T(v)$ where T is the *tree function*
$$T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i$$
- Now we can solve
$$0 = (u - \tfrac{v}{T(v)}) \cdot A(u,v,0) + 1$$
for $A(u,v,0)$.

Lemma (The explicit generating function)

The generating function for the car parking problem is given by

$$A(u,v,t) = \frac{1}{1-\frac{v}{t}e^t} + \frac{u-\frac{v}{t}}{1-\frac{v}{t}e^t} \cdot \frac{e^{T(v)}}{1-ue^{T(v)}},$$

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Now extract the coefficients $a(r,s,k)$ ...

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Theorem

The number of car parking assignments of m cars on n spaces such that at least *k cars do not find a parking space is given by*

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i} \, .$$

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- $S(n,m,k)=\sum_{j=k}^{m}\mathrm{cp}(n,m,j)$

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- $S(n,m,k) = \sum_{j=k}^{m} \mathrm{cp}(n,m,j)$
- The car parking numbers $\mathrm{cp}(n,m,k)$ are given by

$$\boxed{\mathrm{cp}(n,m,k) = S(n,m,k) - S(n,m,k+1)}$$

- Remembering that $cp(n,m,0)=(n+1-m)(n+1)^{m-1}$, we can rewrite

$$S(n,m,k)=\sum_{i=0}^{m-k}\binom{m}{i}\,\mathrm{cp}(n-m+k+i-1,i,0)\cdot(m-k-i)^{m-i}\ .$$

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- If at least k cars do not find a parking space, then there are at least $r = n + k - m$ empty parking spaces, and we rewrite

$$S(n, m, r+m-n) = \sum_{i=0}^{n-r} \binom{m}{i} \cdot \mathrm{cp}(r+i-1, i, 0) \cdot (n-r-i)^{m-i} \ .$$

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- This expression leads to a direct combinatorial proof.

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Proof.

- Assuming the r-th empty space occurs at position $r + i$, there are i cars successfully parked in the $r + i - 1$ spaces to the left of it, which is counted by $\mathrm{cp}(r + i - 1, i, 0)$.

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- The remaining $m-i$ cars are assigned to the $n-r-i$ rightmost spaces in $(n-r-i)^{m-i}$ different ways.

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- The remaining $m - i$ cars are assigned to the $n - r - i$ rightmost spaces in $(n - r - i)^{m-i}$ different ways.
- Sum over all possible values of i.

$\square$

Notice that

$$S(n,m,k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}$$

are partial sums occurring in

Lemma (Abel's Binomial Identity, 1826)

For all $a, b \in \mathbb{R}, m \in \mathbb{N}_0$,

$$\sum_{i=0}^{m} \binom{m}{i} \cdot a \cdot (a+i)^{i-1} \cdot (b-i)^{m-i} = (a+b)^m$$

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- In fact, $S(n, m, 0) = n^m$ gives a combinatorial proof for Abel's identity for $a = n - m$ and $b = m$.

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$$p_{n,m}(k) = \frac{1}{n^m}\,\mathrm{cp}(n,m,k)\ ,$$

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  for n (parking spaces) or m (cars) large.
- Different regimes:
  - $m \ll n$: cars park with probability 1
  - $m \sim \lambda n$, $\lambda < 1$: discrete limit law
  - ($\sqrt{m} \ll m - n \ll m$: exponential distribution)
  - $m - n \sim \lambda\sqrt{m}$: linear-exponential distribution
  - ($\sqrt{m} \ll n - m \ll m$: exponential distribution)
  - $m \sim \lambda n$, $\lambda > 1$: discrete limit law
  - $m \gg n$: cars leave with probability 1

Define

$$P_{n,m}(k) = \frac{1}{n^m} S(n,m,k) = \sum_{j=k}^{m} p_{n,m}(j)$$

- $m \ll n$: $P_{n,m}(k) \to 0$

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- $m \gg n$: $P_{n,m}(k) \to 1$

The interesting scaling regime $m - n \sim \lambda\sqrt{m}$

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Theorem

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$

The interesting scaling regime $m - n \sim \lambda\sqrt{m}$

Theorem

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y\sqrt{n} \rfloor$ drivers to n spaces at least $\lfloor x\sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore we have approximately

$$\frac{\mathrm{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\lim_{n\to\infty} P_{n,n+\lfloor y\sqrt{n}\rfloor}(\lfloor x\sqrt{n}\rfloor)$$
$$= \int_0^1 \frac{x-y}{\sqrt{2\pi\alpha^3(1-\alpha)}} \cdot \exp\left(-\frac{(x-(1-\alpha)y)^2}{2\alpha(1-\alpha)}\right) d\alpha\,.$$

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Under the substitution $\alpha=\frac{u(x-y)}{x+u(x-y)}$ this integral simplifies to

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}}\cdot\int_0^\infty\sqrt{\frac{x(x-y)}{u^3}}\cdot\exp\left(-x\cdot(x-y)\cdot\frac{(1+u)^2}{2u}\right)du \\ &= \exp(-2x(x-y))\,. \end{aligned}$$

$\square$

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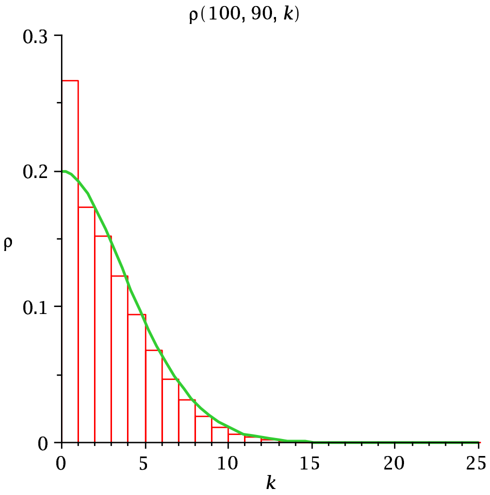
$$= \exp(-2x(x-y))\,.$$

$\square$

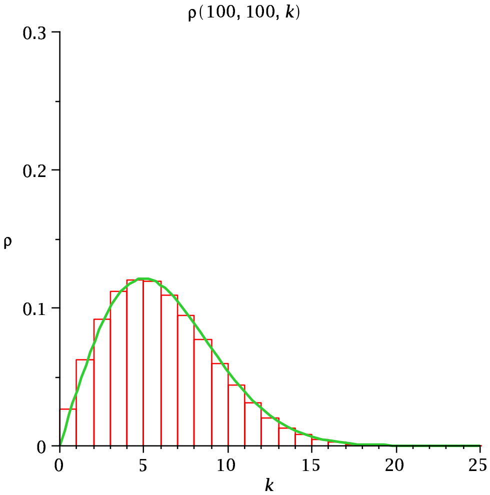
NB: neither Maple nor Mathematica can do the above integral!

Less cars than parking spaces ($n > m$):



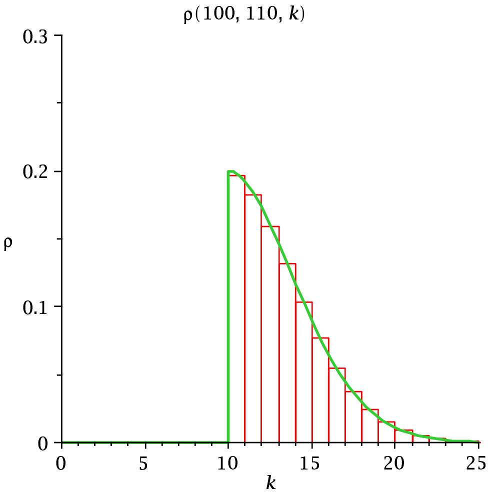
$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

Equal number of cars and parking spaces ($n = m$):



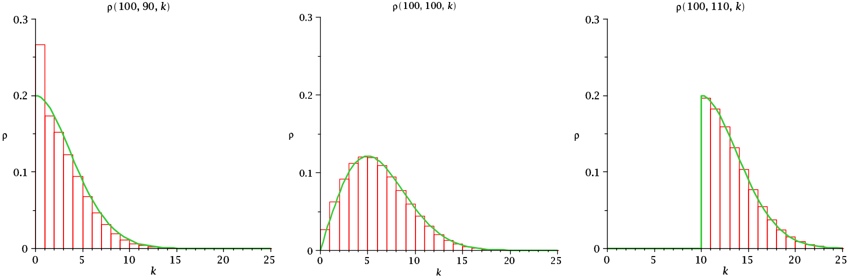
$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

More cars than parking spaces ($n < m$):



$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

The approximation is surprisingly accurate:



$$\frac{\mathrm{cp}(n,m,k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$

What is the probability that the car park is full?

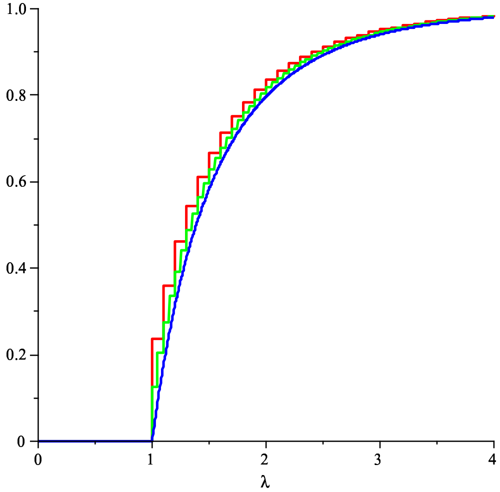
What is the probability that the car park is full?

Theorem

Let $\lambda \in \mathbb{R}^+$. Then the limiting probability that in a random assignment of $\lfloor \lambda n \rfloor$ drivers to n spaces all spaces are occupied is

$$\lim_{n\to\infty} \frac{\mathrm{cp}(n, \lfloor \lambda n \rfloor, \lfloor \lambda n \rfloor - n)}{n^{\lfloor \lambda n \rfloor}} = \begin{cases} 0 & \text{if } \lambda \leq 1, \\ 1 - \frac{1}{\lambda} \cdot T\left(\lambda e^{-\lambda}\right) & \text{if } \lambda > 1. \end{cases}$$

$n = 10, 20, \infty$:



$$\left.\frac{\mathrm{cp}(n, m, m-n)}{n^m}\right|_{m=\lfloor \lambda n \rfloor} \approx 1 - \frac{1}{\lambda} T(\lambda e^{-\lambda})$$

Reference for this work:

- P. J. Cameron, D. Johannsen, T. Prellberg, and P. Schweitzer, "Counting defective parking functions", Electron. J. Combinat. **15** (2008) R92 (arXiv:0803.0302v2) (submitted to EJC on March 3, 2008)

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Related work by Alois Panholzer, TU Wien:

- "Limiting distribution results for a discrete parking problem", GOCPS 2008 (talk presented on March 5, 2008)
- "On a discrete parking problem", AofA 2008 (talk presented on April 17, 2008)

Summary:

- Introduced interesting and apparently new parking problem
- Solved the counting problem (Kernel method)
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Outlook:

- So far only "weak limit laws" - can refine analysis
- Extension to several generalised parking problems in the literature possible

The End