MAS115 Calculus I

Week 1

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What is Calculus I

- Study of functions of real variables
  - one real variable
  - many variables (Calculus II)
- Fundamental: real numbers
- Geometric view: graph of a function
  - slope $\leftrightarrow$ derivative
  - area $\leftrightarrow$ integral
- many techniques
- many applications

Real numbers and the real line

- Properties of real numbers $\mathbb{R}$
    - algebraic (rules of calculation)
        - formalisation of rules of calculation such as

$$\begin{aligned} 2(3+5) =& 2 \cdot 3 + 2 \cdot 5 \\ =& 6 + 10 = 16 \end{aligned}$$

    - order (geometric picture: the real number line)
        - inequalities such as

$$a < b \quad \Rightarrow -b < -a$$

    - completeness
        - There are "no gaps" on the real number line

Algebraic properties

- algebraic properties $a, b, c \in \mathbb{R}$

(A1) $a + (b + c) = (a + b) + c$
(A2) $a + b = b + a$
(A3) there is a 0 such that $a + 0 = a$
(A4) there is an x such that $a + x = 0$

(M1) $a(bc) = (ab)c$
(M2) $ab = ba$
(M3) there is a 1 such that $a\,1 = a$
(M4) there is an x such that $a\,x = 1$ (for $a \neq 0$)

(D) $a(b + c) = ab + ac$

Order: the real number line

- Order properties
  - (O1) for any $a, b \in \mathbb{R}$, $a \leq b$ or $b \leq a$
  - (O2) if $a \leq b$ and $b \leq a$ then $a = b$
  - (O3) if $a \leq b$ and $b \leq c$ then $a \leq c$
  - (O4) if $a \leq b$ then $a + c \leq b + c$
  - (O5) if $a \leq b$ and $0 \leq c$ then $a\,c \leq b\,c$

Rules for inequalities

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Subsets of the real numbers $\mathbb{R}$

- Start with "counting numbers" $1, 2, 3, \ldots$
  - $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ natural numbers

    can we solve $a + x = b$ for x?
  - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ integers

    can we solve $ax = b$ for x?
  - $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z},\ q \neq 0\}$ rational numbers

    can we solve $x^2 = 2$ for x?
  - $\mathbb{R}$ real numbers
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- $\mathbb{N}$ and $\mathbb{Z}$ clearly have gaps but $\mathbb{Q}$ is "dense"
  - "dense" $\Leftrightarrow$ between any two rationals there is another one
- Is $\mathbb{R}$ really bigger than $\mathbb{Q}$? Are there "holes" in $\mathbb{Q}$?

$\mathbb{Q}$ has "holes"

- Yes, there are "holes": Irrational numbers such as $\sqrt{2} = 1.414\ldots$ or $\pi = 3.141\ldots$
- $\sqrt{2}$ is the positive solution to the equation $x^2 = 2$.

Theorem

$x^2 = 2$ has no solution $x \in \mathbb{Q}$

- Completeness
  - the real numbers $\mathbb{R}$ correspond to all points on the line, here are no "holes" or "gaps" (proof covered in MAS111 Convergence and Continuity, 2nd year module)

Revision: Real numbers

- Properties of real numbers $\mathbb{R}$
  - algebraic
    - rules of calculation
  - order
    - order, inequalities
  - completeness
    - “no gaps”

$\sqrt{2}$ is irrational

Theorem

$x^2 = 2$ has no solution $x \in \mathbb{Q}$

Proof.

Assume there is an $x \in \mathbb{Q}$ with $x^2 = 2$. This must be of the form $x = \frac{p}{q}$ and we can assume that p and q have no common factors.

$x^2 = 2$ implies then $(\frac{p}{q})^2 = 2$, or $\boxed{p^2 = 2q^2}$ so that p is even.

Write $p = 2p_1$, so that $p^2 = (2p_1)^2$, or $4p_1^2 = 2q^2$, or $\boxed{2p_1^2 = q^2}$ so that q is also even.

We have now shown that both p and q must be even, so they share a common factor 2.

This is a contradiction, therefore the assumption is wrong. $\square$

Definitions, Theorems, Proofs, ...

You have just seen a "theorem with proof".

- University mathematics is built upon
  - Basic properties (Axioms, Definitions)
  - Statements deduced from these (Lemma, Proposition, Theorem, Corollary, ...) and their proofs!

    (The technique in the previous proof is called
    - Proof by Contradiction

    There will be many different ones to come!)
- Of course there will also be
  - examples, exercises, applications, ...

Intervals

Definition

A subset of the real line is called an interval if it contains at least two numbers and all the real numbers between any two of its elements.

Types of Intervals

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Examples

(a)

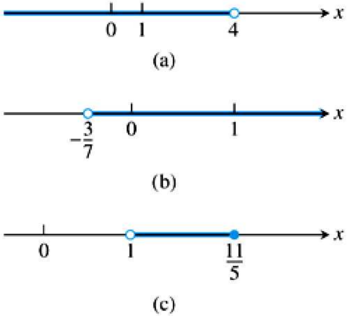
$$2x - 1 < x + 3$$

(b)

$$-\frac{x}{3} < 2x + 1$$

(c)

$$\frac{6}{x-1} \geq 5$$

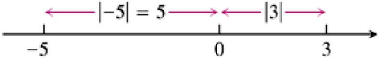


Absolute Value

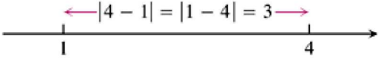
- The absolute value of a real number x is

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

- Geometrically: $|x|$ is distance between x and 0



- $|x - y|$ is distance between x and y



- Alternatively

$$|x| = \sqrt{x^2}$$

Taking the square root always gives a non-negative result!

Revision:

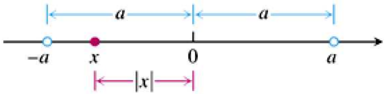
Definitions, Theorems, Proofs

- Theorem and proof
  - Irrationality of $\sqrt{2}$
- Definitions
  - Intervals (a, b), $[a, b]$, $[a, \infty)$, etc.
- Absolute value $|x|$ and distances

Inequalities with $|x|$

$$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

(need $a > 0$, otherwise no solution)



Properties:

1. $|-a| = |a|$
2. $|ab| = |a|\,|b|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$
4. $|a+b| \leq |a| + |b|$, the *Triangle Inequality*

Some simple proofs

1. Proof of $|-a| = |a|$: use $|x| = \sqrt{x^2}$:

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$$

   Note: we have used a direct proof: we started on the left hand side (LHS) of the equation and transformed it step by step until we have arrived at the right hand side (RHS)

2. Proof of $|ab| = |a|\,|b|$:

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|$$

3. Proof of $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$:

$$|a/b| = \sqrt{(a/b)^2} = \sqrt{a^2/b^2} = \sqrt{a^2}/\sqrt{b^2} = |a|/|b|$$

Proof of the Triangle Inequality

4. Proof of $|a+b| \leq |a| + |b|$: use a little trick and prove

$$|a+b|^2 \leq (|a|+|b|)^2$$

instead:

$$\begin{aligned} |a+b|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &\leq a^2 + 2|a|\,|b| + b^2 \\ &= |a|^2 + 2|a|\,|b| + |b|^2 \\ &= (|a|+|b|)^2 \end{aligned}$$

Further properties

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

Examples

(a)

$$|2x - 3| \leq 1$$

(a)

(b)

$$|2x - 3| \geq 1$$

(b)

Important Inequalities

Triangle inequality

$$|a+b| \leq |a| + |b|$$

- arithmetic mean $\frac{1}{2}(a+b)$
- geometric mean $\sqrt{ab}$

Arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \qquad \text{for } a, b \geq 0$$

Cauchy-Schwarz inequality

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$$

Proof of $\sqrt{ab} \leq \frac{1}{2}(a+b)$

- multiply inequality by 2 and square

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \quad \Leftrightarrow 4ab \leq (a+b)^2$$

(why is this equivalent? can you justify this?)

- Use direct proof: start on one side until the other side is obtained

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ &= 4ab + a^2 - 2ab + b^2 \\ &= 4ab + (a-b)^2 \end{aligned}$$

$(a-b)^2 \geq 0$ and therefore

$$\geq 4ab$$

Proof of $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$

- Use direct proof: start on one side until the other side is obtained
- Decide which side:

$$
\begin{aligned}
(ac+bd)^2 &= a^2c^2 + 2abcd + b^2d^2 \\
(a^2+b^2)(c^2+d^2) &= a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2
\end{aligned}
$$

- Start on RHS and work it out

$$
\begin{aligned}
(a^2+b^2)(c^2+d^2) =& a^2c^2 + 2abcd + b^2d^2 \\
& + b^2c^2 - 2abcd + a^2d^2 \\
=& (ac+bd)^2 + (bc-ad)^2 \\
\geq& (ac+bd)^2
\end{aligned}
$$

This concludes the proof.

Proof of $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$

A second proof, which uses a "trick":

- Consider
$$0 \leq (ax+c)^2 + (bx+d)^2$$
(the RHS is non-negative, as it is the sum of squares)
- Expand the RHS and collect equal powers of x
$$0 \leq (a^2+b^2)x^2 + 2(ac+bd)x + (c^2+d^2)$$
- The RHS is quadratic in x. Now remember quadratic equations ...

Aside: quadratic equations and parabolas

- Equation of a parabola

$$y = \alpha x^2 + \beta x + \gamma$$

  - When is $y = 0$?

$$x_{1,2} = \frac{1}{2\alpha}\left(-\beta \mp \sqrt{D}\right) \quad \text{with } D = \beta^2 - 4\alpha\gamma$$

  - When are there two distinct solutions x_1 and x_2? If $D > 0$
  - When is there just one solution (i.e. $x_1 = x_2$)? If $D = 0$
  - When is there no solution? If $D < 0$

- When does the parabola *not* cross the x-axis?

If $\boxed{D \leq 0}$

Proof of $(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$

- Equate

$$(a^2 + b^2)x^2 + 2(ac + bd)x + (c^2 + d^2) = \alpha x^2 + \beta x + \gamma$$

- Read off $\alpha = a^2 + b^2$, $\beta = 2(ac + bd)$, $\gamma = c^2 + d^2$
- Compute

$$D = \beta^2 - 4\alpha\gamma = 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2)$$

The proof now follows from two observations

1. As $0 \leq (ax + c)^2 + (bx + d)^2$ is always true, it follows that

$$D \leq 0$$

2. The Cauchy-Schwarz Inequality is equivalent to $D \leq 0$

This concludes the proof.

The End