MTH5105 Differential and Integral Analysis 2008-2009

Exercises 5

Exercise 1: Let $f(x) = \exp(\sqrt{x})$, $g(x) = \sin(\pi x)$, and $P = \{0, 1, 4, 9\}$.

(a) Find the upper and lower sums $U(f, P)$ and $L(f, P)$ of f for the partition P. Use these sums to give bounds for $\int_0^9 f(x)\,dx$. [5 marks]

(b) Find the upper and lower sums $U(g, P)$ and $L(g, P)$ of g for the partition P. Use these sums to give bounds for $\int_0^9 g(x)\,dx$. [5 marks]

Solution: (a) Recall that $I_i = [x_i - x_{i-1}]$, $\Delta x_i = x_i - x_{i-1}$, $M_i = \sup_{x \in I_i} f(x)$, and $m_i = \inf_{x \in I_i} f(x)$. We have

$$\begin{aligned}
I_1 &= [0, 1]\ , & \Delta_1 &= 1\ , & M_1 &= \exp(1)\ , & m_1 &= \exp(0)\ ,\\
I_2 &= [1, 4]\ , & \Delta_2 &= 3\ , & M_2 &= \exp(2)\ , & m_2 &= \exp(1)\ ,\\
I_3 &= [4, 9]\ , & \Delta_3 &= 5\ , & M_3 &= \exp(3)\ , & m_3 &= \exp(2)\ .
\end{aligned}$$

[2 marks]

Therefore

$$U(f, P) = \sum_{i=1}^{3} M_i \Delta x_i = 1\exp(1) + 3\exp(2) + 5\exp(3)\ ,$$

$$L(f, P) = \sum_{i=1}^{3} m_i \Delta x_i = 1\exp(0) + 3\exp(1) + 5\exp(2)\ .$$

[2 marks]

Hence we have

$$1 + 3e + 5e^2 \le \int_0^9 f(x)\,dx \le e + 3e^2 + 5e^3\ .$$

[1 marks]

(In fact, the integral evaluates to $2 + 4e^3 \approx 82.3$, while the lower and upper sums are approximately 46.1 and 125.3.)

(b) We have now

$$M_1 = 1\ , \quad m_1 = 0\ , \quad M_2 = 1\ , \quad m_2 = -1\ , \quad M_3 = 1\ , \quad m_3 = -1\ .$$

[2 marks]

Therefore

$$U(g, P) = 1 \cdot 1 + 3 \cdot 1 + 5 \cdot 1\ , \quad L(g, P) = 1 \cdot 0 + 3 \cdot (-1) + 5 \cdot (-1)\ .$$

[2 marks]

Hence we have

$$-8 \le \int_0^9 g(x)\,dx \le 9\ .$$

[1 marks]

Exercise 2: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \,. \end{cases}$$

(a) Given a partition P of $[-1, 1]$, what is $L(f, P)$?
What is $\underline{\int_{-1}^{1}} f(x)\, dx$? [3 marks]

(b) For fixed $\epsilon > 0$, find a partition P of $[-1, 1]$ such that $U(f, P) < \epsilon$.
What is $\overline{\int_{-1}^{1}} f(x)\, dx$? [5 marks]

(c) Is f integrable on $[-1, 1]$? If so, what is its integral? [2 marks]

Solution: (a) Given a partition P of $[-1, 1]$, the function f has infimum 0 in any subinterval. Therefore $L(f, P) = 0$ for any partition P. [2 marks]
Hence $\underline{\int_{-1}^{1}} f(x)\, dx = 0$. [1 mark]

(b) For $0 < \delta < 1$, choose $P = \{-1, -\delta, \delta, 1\}$. On the intervals $[-1, -\delta]$ and $[\delta, 1]$ the function f has maximum value 0. On the interval $[-\delta, \delta]$ it has maximum value 1. Therefore

$$U(f, P) = ((-\delta) - (-1)) \cdot 0 + (\delta - (-\delta)) \cdot 1 + (1 - \delta) \cdot 0 = 2\delta \text{ ,`}$$

and if we choose $\delta < \epsilon/2$, we have $U(f, P) < \epsilon$. [3 marks]
Hence $\overline{\int_{-1}^{1}} f(x)\, dx \leq 0$. Using (a), we have

$$0 = \underline{\int_{-1}^{1}} f(x)\, dx \leq \overline{\int_{-1}^{1}} f(x)\, dx \leq 0 \,,$$

so that $\underline{\int_{-1}^{1}} f(x)\, dx = 0$. [2 marks]

(c) As

$$\underline{\int_{-1}^{1}} f(x)\, dx = \overline{\int_{-1}^{1}} f(x)\, dx = 0 \,,$$

f is integrable and $\int_{-1}^{1} f(x)\, dx = 0$. [2 marks]

Exercise 3: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the equidistant partitions P_n of $[0,1]$ into n subintervals.

(a) Find $U(f, P_n)$. What can you say about $\overline{\int_0^1} f(x)\,dx$? [4 marks]

(b) Find $L(f, P_n)$. What can you say about $\underline{\int_0^1} f(x)\,dx$? [4 marks]

(c) Is f integrable on $[0,1]$? If so, what is its integral? [2 marks]

[Hint: $\sum_{j=1}^n j^2 = \frac{1}{6}n(n+1)(2n+1)$.]

Solution: We have

$$P_n = \{0/n, 1/n, \ldots, n/n\}\ ,$$

or $x_i = i/n$ for $i = 0, \ldots, n$. Thus, $I_i = [(i-1)/n, i/n]$ and $\Delta x_i = 1/n$.

(a) We have $M_i = (i/n)^2$ and thus

$$\begin{aligned} U(f,P) &= \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3}\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\ . \end{aligned}$$

[3 marks]

Hence, $\overline{\int_0^1} f(x)\,dx \leq 1/3$. [1 mark]

(b) Similarly we have $m_i = ((i-1)/n)^2$ and thus

$$\begin{aligned} L(f,P) &= \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3}\sum_{i=1}^n (i-1)^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\ . \end{aligned}$$

[3 marks]

Hence, $\underline{\int_0^1} f(x)\,dx \geq 1/3$. [1 mark]

(c) Combining these we see that $\int_0^1 x^2\,dx$ exists and equals $1/3$. [2 marks]