MAS115 Calculus I

Week 2

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Revision: Inequalities

- Triangle inequality

$$|a+b| \leq |a| + |b|$$

- Arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \qquad \text{for } a, b \geq 0$$

- Cauchy-Schwarz inequality

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$$

including their proofs

Repeat: the Cauchy-Schwarz Inequality

- Start with the fact that

$$0 \leq (ax + c)^2 + (bx + d)^2$$

- Expand the RHS and collect equal powers of x

$$0 \leq (a^2 + b^2)x^2 + 2(ac + bd)x + (c^2 + d^2)$$

- The RHS is quadratic in x with discriminant

$$D = 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2)$$

- As $0 \leq (ax + c)^2 + (bx + d)^2$ for all x, $D \leq 0$
- $D \leq 0$ is equivalent to

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$$

This is the Cauchy-Schwarz inequality

Reading Assignment

Read Chapter 1.2

Lines, Circles, and Parabolas

Functions and Their Graphs

What do we mean when we say

"y is a function of x"?

Symbolically, we write

$$y = f(x)$$

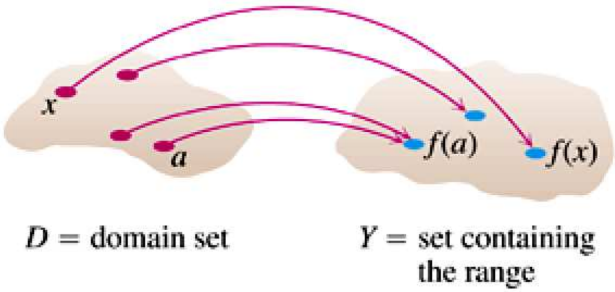
- x independent variable (input value)
- y dependent variable (output value)
- f function (rule that assigns)

- Important: rule is unique, only one value $f(x)$ for every x

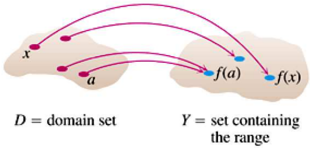
Definition of a function

Definition

A function from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.



Further definitions and notations



- The set D of all possible input values is called the *domain* of f
- The set R of all values of $f(x)$ as x varies throughout D is called the *range* of f (a subset of Y)
- We write "f maps D to Y" symbolically as

$$f : D \to Y$$

- We write "f maps x to $y = f(x)$" symbolically as

$$f : x \mapsto y = f(x)$$

Note the different arrow symbols used

Natural domain

The *natural domain* is the largest set of real x which the rule f can be applied to.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Note: a function is specified by the rule f and the domain D

$$f : x \mapsto x^2 , \quad D(f) = [0, \infty)$$

and

$$f : x \mapsto x^2 , \quad D(f) = (-\infty, \infty)$$

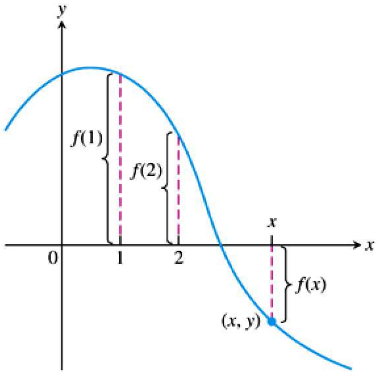
are *different* functions

Graphs of functions

Definition

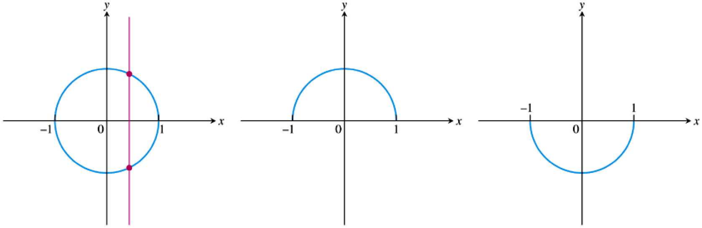
If f is a function with domain D, its graph consists of the points (x, y) whose coordinates are the the input-output pairs for f:

$$\{(x, f(x)) | x \in D\}$$



Curves that are graphs of functions

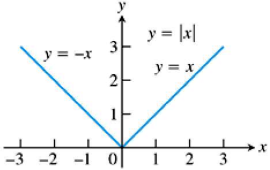
The vertical line test



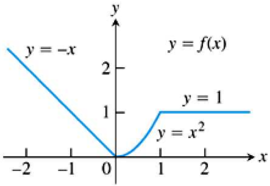
(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

Piecewise defined functions

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



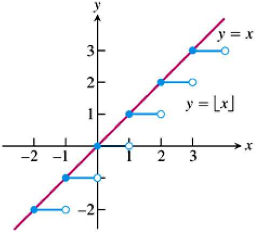
Floor and ceiling functions

The floor function

$$f(x) = \lfloor x \rfloor$$

the greatest integer less than or equal to x:

$$\lfloor 1.3 \rfloor = 1, \ \lfloor -2.7 \rfloor = -3$$

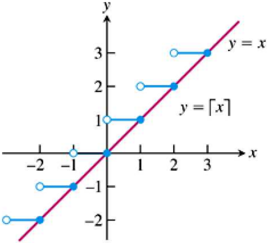


The ceiling function

$$f(x) = \lceil x \rceil$$

the smallest integer greater than or equal to x:

$$\lceil 3.5 \rceil = 4, \ \lceil -1.8 \rceil = -1$$



Revision: Functions and their Graphs

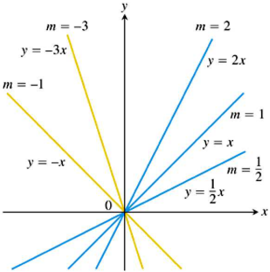
- Definition of a function
- Domain and range of a function
- Graph of a function
- Piecewise defined functions

Identifying functions

Linear function $f(x) = mx + b$

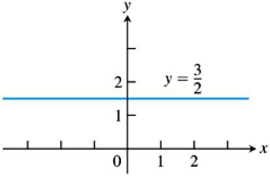
$b = 0$: lines passing through the origin

$$f(x) = mx$$



$m = 0$: constant function

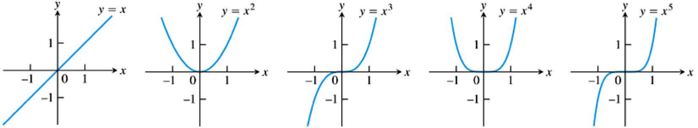
$$f(x) = b$$



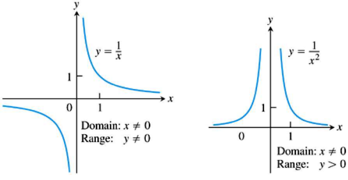
Identifying functions

Power function $f(x) = x^a$

- $a = n$, a positive integer



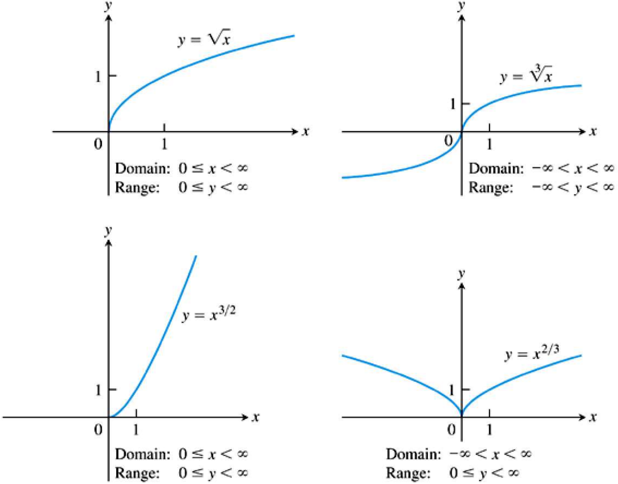
- $a = -1$ or $a = -2$



Identifying functions

Power function $f(x) = x^a$

- $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $\frac{2}{3}$



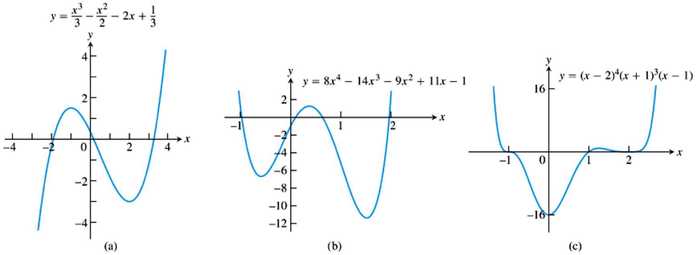
Identifying functions

- Polynomials

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

with $a_n \neq 0$ and $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$

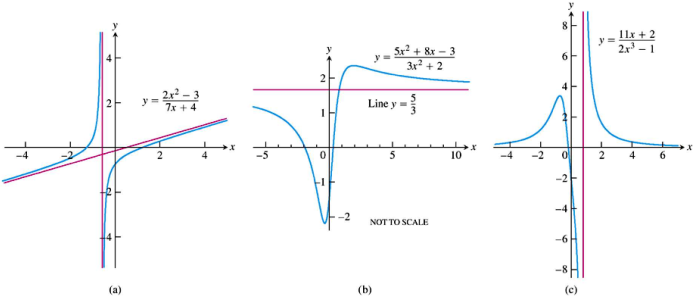
- n *degree* of the polynomial



- Domain: $\mathbb{R}$

Identifying functions

- Rational functions
$$f(x) = \frac{p(x)}{q(x)}$$
with $p(x)$ and $q(x)$ polynomials



- Domain: $\mathbb{R} \setminus \{x | q(x) = 0\}$ (never divide by zero!)

Identifying functions

Other classes (to come later)

- Algebraic functions
- Trigonometric functions
- Exponential functions
- Logarithmic functions
- ...

Increasing/decreasing functions

Informally,

- f is called increasing, if the graph of f "climbs" or "rises" as you move *from left to right*.
- f is called decreasing, if the graph of f "descends" or "falls" as you move *from left to right*.

Function	**Where increasing**	**Where decreasing**
$y = x^2$	$0 \leq x < \infty$	$-\infty < x \leq 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \leq x < \infty$	Nowhere
$y = x^{2/3}$	$0 \leq x < \infty$	$-\infty < x \leq 0$

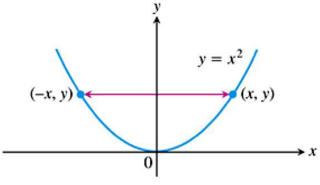
Even/odd functions

DEFINITIONS **Even Function, Odd Function**

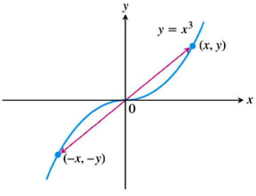
A function $y = f(x)$ is an

$$\textbf{even function of } x \quad \text{if } f(-x) = f(x),$$
$$\textbf{odd function of } x \quad \text{if } f(-x) = -f(x),$$

for every x in the function's domain.



(a)



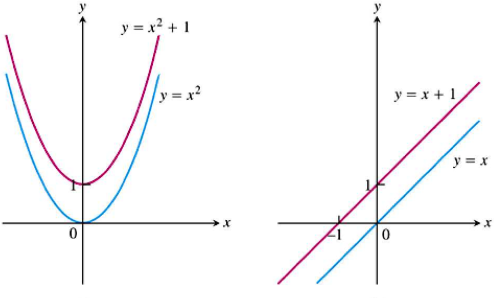
(b)

the graph of f is symmetric with respect to the y-axis

the graph of f is symmetric with respect to the origin

Examples

1. $f(x) = x^2$
2. $f(x) = x^2 + 1$
3. $f(x) = x$
4. $f(x) = x + 1$



(a) (b)

Sums, differences, products, quotients

If f and g are functions, then for every

$$x \in D(f) \cap D(g)$$

(that is, every x that belongs to the domains of f and g) we define

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\(f-g)(x) &= f(x)-g(x)\\(fg)(x) &= f(x)g(x)\\(f/g)(x) &= f(x)/g(x) \quad \text{if } g(x)\neq 0\end{aligned}$$

Special case: multiplication by constant:

$$(cf)(x) = c\,f(x)$$

(take $g(x) = c$ constant function)

Examples

$$f(x) = \sqrt{x} \qquad g(x) = \sqrt{1-x}$$

Domains:

$$D(f) = [0, \infty) \qquad D(g) = (-\infty, 1]$$

Intersection:

$$D(f) \cap D(g) = [0, \infty) \cap (-\infty, 1] = [0, 1]$$

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Revision: Functions and their Graphs

- power functions, polynomials, rational functions
- increasing/decreasing functions
- even/odd functions
- sums, differences, products, quotients of functions

Compositions of functions

Definition

If f and g are functions, the composite function $f \circ g$ ("f composed with g") is defined by

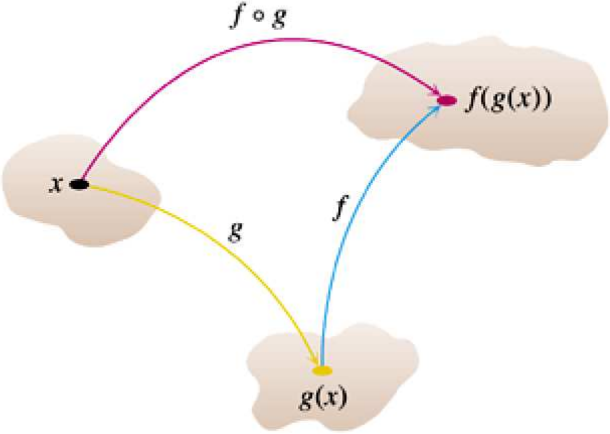
$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f, i.e.

$$D(f \circ g) = \{x | x \in D(g) \text{ and } g(x) \in D(f)\}$$

Compositions of functions

$$D(f \circ g) = \{x | x \in D(g) \text{ and } g(x) \in D(f)\}$$



Examples

$$f(x) = \sqrt{x} \text{ with } D(f) = [0, \infty)$$
$$g(x) = 1 + x \text{ with } D(g) = (-\infty, \infty)$$

- $(f \circ g)(x) = f(g(x)) = \sqrt{x+1}$, Domain $[-1, \infty)$
- $(g \circ f)(x) = g(f(x)) = \sqrt{x} + 1$, Domain $[0, \infty)$
- $(f \circ f)(x) = f(f(x)) = x^{1/4}$, Domain $[0, \infty)$
- $(g \circ g)(x) = g(g(x)) = x + 2$, Domain $(-\infty, \infty)$

Examples

$$f(x) = \sqrt{x} \text{ with } D(f) = [0, \infty)$$
$$g(x) = x^2 \text{ with } D(g) = (-\infty, \infty)$$

- $(f \circ g)(x) = f(g(x)) = |x|$, Domain $(-\infty, \infty)$
- $(g \circ f)(x) = g(f(x)) = x$, Domain $[0, \infty)$

$$f(x) = 1/x \text{ with } D(f) = (-\infty, 0) \cup (0, \infty)$$

- $(f \circ f)(x) = f(f(x)) = x$, Domain $(-\infty, 0) \cup (0, \infty)$

Shifting a graph of a function

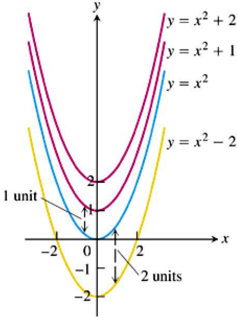
Shift Formulas

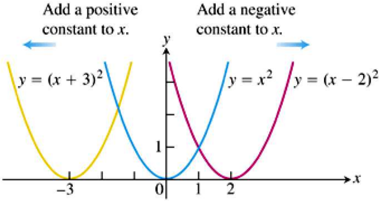
Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$ Shifts it *down* $\lvert k\rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$ Shifts it *right* $\lvert h\rvert$ units if $h < 0$





Scaling a graph of a function

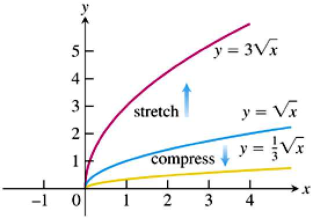
Vertical and Horizontal Scaling and Reflecting Formulas

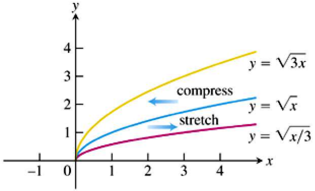
For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.





Reflecting a graph of a function

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
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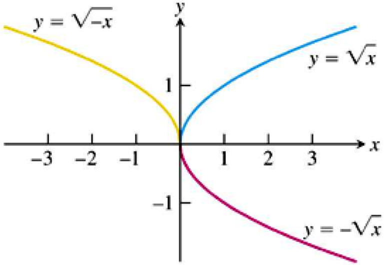
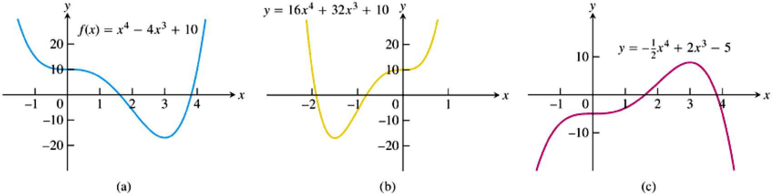


Combining scalings and reflections



(a) the original graph of $f(x)$: $y = f(x)$

(b) horizontal compression by a factor of 2, followed by a reflection across the y-axis: $y = f(-2x)$

(c) vertical compression by a factor 2, followed by a reflection across the x-axis: $y = -\frac{1}{2}f(x)$

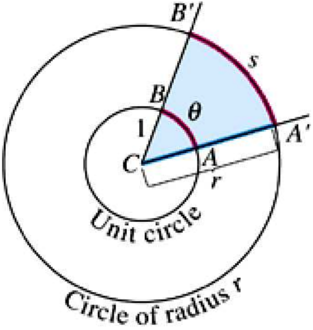
Reading Assignment

Read Chapter 1.6

Trigonometric Functions

especially Trigonometric Identities

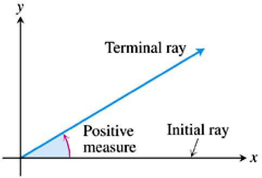
Radian measure

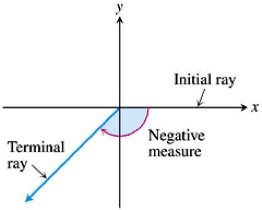


360° corresponds to 2π

$$\frac{\text{angle in radians}}{\text{angle in degrees}} = \frac{\pi}{180}$$

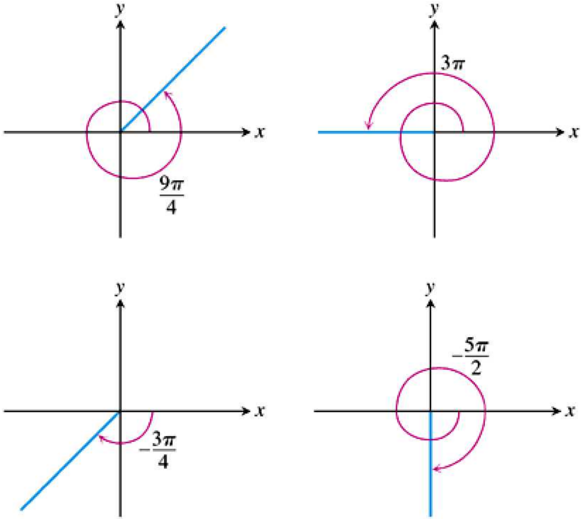
Signed angles



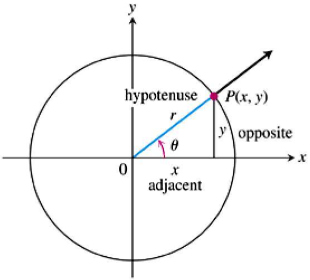


- angles are *oriented*
- positive angle: anti-clockwise
- negative angle: clockwise
- angles can be larger than 2π

Examples

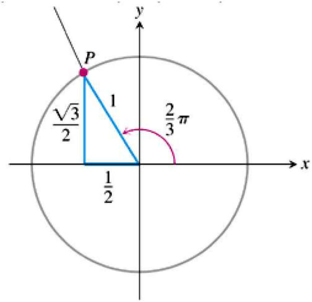


Trigonometric functions



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Example



$$\theta = \frac{2}{3}\pi$$

$$x = -\frac{1}{2}\,, \quad y = \frac{\sqrt{3}}{2}\,, \quad r = 1$$

(Can you answer why?)

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

$$\sin(\frac{2}{3}\pi) = \frac{\sqrt{3}}{2} \qquad \csc(\frac{2}{3}\pi) = \frac{2}{3}\sqrt{3}$$

The End