MTH5105 Differential and Integral Analysis 2010-2011

Solutions 4

1 Exercises for Feedback

1) Let the function $f\colon (0,\pi) \to \mathbb{R}$ be given by $x \mapsto \cos(x)$. Show that f is invertible and that the inverse $g(y) = f^{-1}(y)$ is differentiable. Find the derivative g'.

Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g. What is the remainder term in the Lagrange form?

Hence show that for $|y| \leq 1/2$

$$|g(y) - \pi/2 + y| \leq \sqrt{3}/18 \approx 0.096 \ .$$

Solution:

As $f'(x) = -\sin(x) < 0$ on $(0,\pi)$, f is strictly decreasing and therefore invertible, with differentiable inverse $g : (-1,1) \mapsto \mathbb{R}$. (Of course we recognize from Calculus that $g = \arccos$.)

To compute the inverse, note that $f'(x) = -\sqrt{1-\cos^2(x)}$, and thus $g'(y) = 1/f'(x) = -1/\sqrt{1-y^2}$.

We have $g(0) = \pi/2$ and $g'(0) = -1$, so that $T_{1,0}(y) = \pi/2 - y$.

From $g''(y) = -y(1-y^2)^{-3/2}$, the remainder term in the Lagrange form is given by

$$R = \frac{1}{2}g''(c)y^2 = -\frac{cy^2}{2(1-c^2)^{3/2}} \ .$$

There exists a $|c| < |y|$ such that $g(y) - T_{1,0}(y) = R$. For $|y| \leq 1/2$ we can get an explicit bound on $|R|$ by estimating

$$|R| \leq \frac{|y|^3}{2(1-y^2)^{3/2}} \leq \frac{(1/2)^3}{2(1-1/4)^{3/2}} = \frac{1}{18}\sqrt{3} \ .$$

2 Extra Exercises

2) Let $f : (-1,\infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0 \ .$$

Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0 \ .$$

Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$.
Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h, $(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)}h^{(k)}$.

Solution:

We find

$$
\begin{aligned}
f(x) &= \sin(\pi\sqrt{1+x}) \\
f'(x) &= \frac{\cos(\pi\sqrt{1+x})\pi}{2\sqrt{1+x}} \\
f''(x) &= -\frac{\sin(\pi\sqrt{1+x})\pi^2}{4(1+x)} - \frac{\cos(\pi\sqrt{1+x})\pi}{4(1+x)^{3/2}}
\end{aligned}
$$

From this, the identity immediately follows.

Differentiating $g(x) = 4(1+x)f''(x)$ n times, we find

$$
g^{(n)}(x) = 4(1+x)f^{(n+2)}(x) + 4nf^{(n+1)}(x)
$$

(this is where the Leibniz formula might be useful, otherwise you might need to use induction). Thus, differentiating the identity gives

$$
4(1+x)f^{(n+2)}(x) + 2(2n+1)f^{(n+1)}(x) + \pi^2 f^{(n)}(x) = 0
$$

which for $x = 0$ simplifies to the needed formula.

We compute now $f(0) = 0$, $f'(0) = -\pi/2$, and recursively

$$
\begin{aligned}
f''(0) &= -\frac{1}{4}\left(2f'(0) + \pi^2 f(0)\right) = \frac{\pi}{4} \\
f'''(0) &= -\frac{1}{4}\left(6f''(0) + \pi^2 f'(0)\right) = \frac{\pi}{8}(\pi^2 - 3) \\
f''''(0) &= -\frac{1}{4}\left(10f''(0) + \pi^2 f''(0)\right) = \frac{3\pi}{16}(5 - 2\pi^2)
\end{aligned}
$$

from whence

$$
T_{4,0}(x) = -\frac{\pi}{2}x + \frac{\pi}{8}x^2 + \frac{\pi}{48}(\pi^3 - 3)x^3 + \frac{\pi}{128}(5 - 2\pi^2)x^4
$$

follows.

3) In the lecture we have shown that the number e can be expressed as

$$
e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!} \, .
$$

Show that remainder term r_n in

$$
n!e = n! \sum_{k=0}^{n} \frac{1}{k!} + r_n \, .
$$

cannot be an integer. Hence deduce that e is irrational.

<u>Solution:</u>

We find

$$
r_n = \sum_{k=n+1}^{\infty} \frac{n!}{k!} \, .
$$

We estimate each term as

$$
0 < \frac{n!}{k!} = \frac{1}{(n+1)(n+2)\dots k} < \frac{1}{(n+1)^{k-n}}
$$

so that

$$
0 < r_n < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \sum_{l=1}^{\infty} \frac{1}{(n+1)^l} = \frac{1}{n} \, .
$$

Hence

$$0 < r_n < \frac{1}{n} < 1 ,$$

and r_n therefore cannot be an integer.

Now $\sum_{k=0}^{n} \frac{n!}{k!}$ is an integer. Therefore we have shown that for all $n \in \mathbb{N}$, $n!e$ cannot be an integer. It follows that e cannot be a rational number. (If $e = p/q$ was a rational number, then $n!q/qp$ would have to be an integer for n sufficiently large.)

4*) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, and let $M_i = \sup\limits_{x\in\mathbb{R}} |f^{(i)}(x)|$ for $i = 0, 1, 2$. Show that

$$M_1^2 \leq 4M_0M_2 .$$

Hint: apply Taylor's theorem to $f(a + h)$.

<u>Solution:</u>

By Taylor's theorem $f(a + h) = f(a) + f'(a)h + f''(c)h^2/2$ for some $c \in (a, a + h)$. Hence

$$f'(a) = \frac{1}{h}(f(a + h) - f(a)) - \frac{h}{2}f''(c)$$

and thus

$$|f'(a)| \leq \frac{2}{h}M_0 + \frac{h}{2}M_2 .$$

The right hand side is minimal for $h = 2\sqrt{M_0/M_2}$, and inserting this gives

$$M_1 \leq 2\sqrt{M_0M_2} .$$

Thomas Prellberg, February 2011