In De Bruijn's book, page 108, equation (6.2.7) it is stated that

$$\begin{aligned}\frac{\log B_n}{n} &= \log n - \log\log n - 1 + \frac{\log\log n}{\log n} \\ &\quad + \frac{1}{\log n} + \frac{1}{2}\left(\frac{\log\log n}{\log n}\right)^2 + O\left(\frac{\log\log n}{(\log n)^2}\right)\end{aligned}$$

and that the $O(\cdot)$ term can be replaced by an asymptotic series with terms of the form $(\log\log n)^k(\log n)^{-m}$. From earlier remarks it seems that this series converges absolutely but De Bruijn does not say this explicitly.

I would like to show something like

$$\frac{B_{n-r}}{B_n} = \exp\left(-r\log n + r\log\log n + \frac{r}{\log n} + c_n r + d_n r^2 + O\left(n^{-1-\epsilon}\right)\right)$$

for $r = O\left((\log n)^{3/2}\right)$ (say) where $c_n$ and $d_n$ are sequences of order $c_n = o\left((\log n)^{-1}\right)$ and $d_n = O\left(n^{-1}\right)$ and $\epsilon > 0$. If the formal asymptotic series replacing the $O(\cdot)$ term in $B_{n-r}$ is expanded using Taylor's theorem then the conjecture follows, but I'm worried about convergence.

If the conjecture is true that we can show the number of 2-covers on $n$ elements $t_n$ is asymptotically

$$t_n \sim \frac{1}{\sqrt{2en}} B_{2n}.$$

**Reply:**

It got later than I had expected, so I'll be brief. I seem to get

$$\log B_{n-r} - \log B_n = -rw + \frac{rw}{2n}\left(\frac{r}{w+1} + \frac{1}{(w+1)^2}\right) + O\left(\frac{r^3 w}{n^2}\right)$$

which should be transformable to your desired result, if you invert $n = we^w$ and insert the resulting double series in $\log n$ and $\log\log n$.

Regarding your comment of convergence: It is true that the double series expansion of $w$ is convergent. The expansion of the Bell numbers however is only asymptotic.

Maple gives:

```
soln := -r*w+1/2*(r*w+1+r)*r*w/n/(1+w)^2+1/24*w^2*
r*(-9*w^3-18*w^2-16*w+10-2*w^4+18*r+24*r*w+6*r*w^2
+8*r^2+20*r^2*w+16*r^2*w^2+4*r^2*w^3)/(1+w)^5/n^2+
O(1/(n^3))
```

Here, the O-term suppresses powers of $w$ and $r$.

Regarding the validity of the calculations:

1. For $\log B_n$ you have a complete asymptotic expansion in $1/n$ where the coefficients are rational functions in $w$, with controlled degree growth of the denominator and enumerator polynomials.

2. Due to the structure of the asymptotic scale ($w$ and $n$), an expansion of $\log B_{n-r}$ in $r$ produces additional terms on an asymptotic scale $r/n$, again with coefficients being rational functions in $w$.

3. The resulting truncated asymptotic formula is correct up to $O(r^{k+1}w^{\gamma_k}/n^k)$.