MTH5105 Differential and Integral Analysis 2010-2011

Exercises 5

There are two sections. Questions in Section 1 will be used for feedback. Questions in Section 2 are voluntary but highly recommended. Starred questions are more difficult than unstarred ones.

1 Exercises for Feedback

1) Let $f(x) = \exp(\sqrt{x})$, $g(x) = \sin(\pi x)$, and $P = \{0, 1, 4, 9\}$.
   (a) Find the upper and lower sums $U(f, P)$ and $L(f, P)$ of f for the partition P. Use these sums to give bounds for $\int_0^9 f(x)\,dx$.
   (b) Find the upper and lower sums $U(g, P)$ and $L(g, P)$ of g for the partition P. Use these sums to give bounds for $\int_0^9 g(x)\,dx$.

2 Extra Exercises

2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0\,. \end{cases}$$

   (a) Given a partition P of $[-1, 1]$, what is $L(f, P)$? What is $\int_{*-1}^{1} f(x)\,dx$?
   (b) For fixed $\epsilon > 0$, find a partition P of $[-1, 1]$ such that $U(f, P) < \epsilon$. What is $\int_{-1}^{*1} f(x)\,dx$?
   (c) Is f integrable on $[-1, 1]$? If so, what is its integral?

3) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the equidistant partitions P_n of $[0, 1]$ into n subintervals.
   (a) Find $U(f, P_n)$. What can you say about $\int_0^{*1} f(x)\,dx$?
   (b) Find $L(f, P_n)$. What can you say about $\int_{*0}^{1} f(x)\,dx$?
   (c) Is f integrable on $[0, 1]$? If so, what is its integral?

   [Hint: $\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1)$*.]*

4*) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1/q & x = p/q \in \mathbb{Q} \text{ with } p, q \text{ coprime and } q > 0, \text{ and} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

   Prove that f is Riemann-integrable on $[0, 1]$ and that $\int_0^1 f(x)dx = 0$. If you want to practice old material, show also that f is discontinuous at $x \in \mathbb{Q}$ and continuous at $x \notin \mathbb{Q}$ (easy), and that it is nowhere differentiable (hard).

The deadline is 5.00pm (strict) on Monday 7th March. Please hand in your coursework to the orange coursework box on the second floor. Coursework will be returned during the exercise class immediately following the deadline.

Thomas Prellberg, February 2011