MTH5105 Differential and Integral Analysis 2009-2010

Exercises 6

There are two sections. Questions in Section 1 will be marked and will form your coursework mark. Questions in Section 2 are voluntary but highly recommended.

1 Exercise for Feedback/Assessment

1) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Show that f is uniformly continuous. [4 marks]
[Hint: Use that if $|f'(x)| \le M$ for all $x \in \mathbb{R}$ then $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$ (from Exercise sheet 2).]

(b) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ and $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$. Prove or disprove:

(i) f is uniformly continuous. [3 marks]

(ii) g is uniformly continuous. [3 marks]

(iii) fg is uniformly continuous. [5 marks]

(iv) $x \mapsto \begin{cases} g(x)/f(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$ is uniformly continuous. [5 marks]

2 Extra Exercises

2) Let $f : (0, 1) \to \mathbb{R}$ be continuous. Show that f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

[Note: the converse is also true, but much harder to show.]

3) Let $\alpha \in \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha & x \in \{1/k; k \in \mathbb{N}\} , \\ 0 & \text{else.} \end{cases}$$

For which values of α is f Riemann-integrable? If f is Riemann-integrable, what is the value of $\int_0^1 f(x)\, dx$?

4) Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable and $c \in \mathbb{R}$.

(a) Given a partition P of $[a, b]$, show that

$$U(cf, P) - L(cf, P) \le |c|(U(f, P) - L(f, P)) .$$

(b) Deduce from (a) that cf is integrable and

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx .$$

[This completes the proof of Theorem 7.4.]

The deadline is 5.00pm (strict) on Monday 15th March. Please hand in your coursework to the red coursework box on the ground floor.

Thomas Prellberg, March 2010