MTH5105 Differential and Integral Analysis 2008-2009

Exercises 7

Exercise 1: Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that if

$$\int_a^b f(x)\, dx = 0$$

then there exists a $c \in (a, b)$ such that $f(c) = 0$.

[Hint: use an antiderivative of f.]

[10 marks]

Solution: We use

$$F(t) = \int_a^t f(x)\, dx \ .$$

[2 marks]

Then $F(a) = 0$ and $F(b) = \int_a^b f(x)\, dx = 0$.

[2 marks]

This should remind you of Rolle's Theorem. We need to check whether we can apply it:

As f is continuous, F is an antiderivative of f: it is differentiable on $[a, b]$ and its derivative $F' = f$ is continuous on $[a, b]$.

[3 marks]

Thus the assumptions of Rolle's Theorem are satisfied, and we conclude that there is a $c \in (a, b)$ such that

$$0 = F'(c) = f(c) \ .$$

[3 marks]

Exercise 2: Evaluate

$$\lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin(nx)}{nx}\, dx \ .$$

[Hint: Choose an $\epsilon > 0$ and consider the intervals $[0, \epsilon]$ and $[\epsilon, \pi/2]$ separately.]

[8 marks]

Solution: Using that $|\sin(t)| \le |t|$ (compare with midterm test), we estimate

$$\left| \int_0^\epsilon \frac{\sin(nx)}{nx}\, dx \right| \le \int_0^\epsilon \left| \frac{\sin(nx)}{nx} \right| dx \le \int_0^\epsilon dx = \epsilon \ .$$

[3 marks]

Using that $|\sin(t)| \leq 1$, we estimate

$$\left|\int_{\epsilon}^{\pi/2} \frac{\sin(nx)}{nx}\, dx\right| \leq \int_{\epsilon}^{\pi/2} \left|\frac{\sin(nx)}{nx}\right| dx \leq \frac{1}{n}\int_{\epsilon}^{\pi/2} \frac{dx}{x} = \frac{1}{n}(\log(\pi/2) - \log\epsilon) \ .$$

[3 marks]

Hence

$$\left|\int_{0}^{\pi/2} \frac{\sin(nx)}{nx}\, dx\right| \leq \epsilon + \frac{1}{n}(\log(\pi/2) - \log\epsilon) \ ,$$

and choosing $\epsilon = 1/n$, we find

$$\left|\int_{0}^{\pi/2} \frac{\sin(nx)}{nx}\, dx\right| \leq \frac{1}{n}(1 + \log(\pi/2) + \log n) \to 0$$

as $n \to \infty$. [2 marks]

Exercise 3: Compute $\lim_{n\to\infty} f_n(x)$ and $\lim_{n\to\infty} f_n'(x)$ for the following functions:

(a) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{\sin(nx)}{\sqrt{n}} \ .$$

(b) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{n}(\sqrt{1 + n^2x^2} - 1) \ ,$$

(c) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{1 + nx^2} \ .$$

If the limit doesn't exist, please indicate clearly for which values of x this is the case and give a brief indication why (no complete proof necessary).

[12 marks]

Solution: (a) $|f_n(x)| \leq \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, hence

$$\lim_{n\to\infty} f_n(x) = 0 \ .$$

[2 marks]

$f_n'(x) = \sqrt{n}\cos(nx)$. With increasing n, this function oscillates with strictly increasing amplitude and frequency, so

$$\lim_{n\to\infty} f_n'(x) \text{ does not exist.}$$

[2 marks]

[A proof (not asked for) could be as follows. If $|\cos(nx)| \leq 1/2$ then $|\cos(2nx)| \geq 1/2$. Thus, for all x there exists an increasing subsequence n_k such that $|\cos(n_kx)| \geq 1/2$. This implies $|f_{n_k}'(x)| \geq \sqrt{n_k}/2$, so $f_n'(x)$ cannot converge.]

(b) $f_n(x) = \sqrt{x^2 + 1/n^2} - 1/n$, hence

$$\lim_{n\to\infty} f_n(x) = |x| \ .$$

[2 marks]

$f_n'(x) = nx/\sqrt{1 + n^2x^2} = x/\sqrt{x^2 + 1/n^2}$, hence

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 1 & x > 0 \ , \\ 0 & x = 0 \ , \\ -1 & x < 0 \ . \end{cases}$$

[2 marks]

(c) $f_n(x) = 1/(1 + nx^2)$ so that $f_n(0) = 1$, and for $x \neq 0$ we have $|f_n(x)| < 1/(nx^2)$, hence

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 0 \ , \\ 0 & x \neq 0 \ . \end{cases}$$

[2 marks]

$f_n'(x) = -2nx/(1 + nx^2)^2$, so that $f_n'(0) = 0$, and for $x \neq 0$ we have $|f_n(x)| < 2/(n|x|^3)$, hence

$$\lim_{n\to\infty} f_n'(x) = 0 \ .$$

[2 marks]

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