MTH5105 Differential and Integral Analysis 2009-2010

Exercises 3

There are two sections. Questions in Section 1 will be marked and will form your coursework mark. Questions in Section 2 are voluntary but highly recommended.

1 Exercise for Feedback/Assessment

1) The function $\tanh : \mathbb{R} \to \mathbb{R}$ is given by

$$x \mapsto \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)} .$$

   (a) Prove that tanh is differentiable and that $\tanh' = 1 - \tanh^2$. [4 marks]

   (b) Prove that tanh is strictly increasing, and hence invertible. [4 marks]

   (c) Prove that $\lim_{x\to\pm\infty} \tanh(x) = \pm 1$ and hence that $\tanh(\mathbb{R}) = (-1, 1)$. [4 marks]

   (d) Prove that $\operatorname{artanh} = \tanh^{-1} : (-1, 1) \to \mathbb{R}$ is differentiable, and that

$$\operatorname{artanh}'(x) = \frac{1}{1 - x^2} .$$

   [4 marks]

   (e) Prove the identity $\operatorname{artanh}(a) + \operatorname{artanh}(b) = \operatorname{artanh}\left(\frac{a+b}{1+ab}\right)$ for $a, b \in (-1, 1)$ by considering the derivative of the function

$$f(x) = \operatorname{artanh}(x) + \operatorname{artanh}(b) - \operatorname{artanh}\left(\frac{x+b}{1+xb}\right)$$

   for fixed $b \in (-1, 1)$. [4 marks]

2 Extra Exercises

2) (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continous inverse.

   (b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove: If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} f'(x) = 0$.

3) Using the Intermediate Value Theorem, prove that a continuous function maps intervals to intervals.

The deadline is 5.00pm (strict) on Monday 8th February. Please hand in your coursework to the red coursework box on the ground floor.

Thomas Prellberg, January 2010