Queen Mary University of London

M. Sci. Examination by course unit 2013

MTH744U Dynamical Systems

Duration: 3 hours

Date and time: 16 May 2013, 2.30–5.30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

This is a OPEN BOOK exam. Permitted are printed material, hand-written notes, and photocopies of any kind. Sharing material with other students is not permitted.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important note: the Academic Regulations state that possession of unauthorized material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, then please raise your hand and give them to an invigilator immediately. Please be aware that if you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. Disruption caused by mobile phones is also an examination offence.

Exam papers must not be removed from the examination room.

Examiner(s): T. Prellberg

Question 1 [26 marks]

For $r \in \mathbb{R}$, consider the differential equation

$$\dot{x} = x^3 + (2-r)x^2 - rx - r(2-r) \tag{1}$$

on the line.

(a) Show that $x^* = r - 2$ is a fixed point for any value of the parameter r, and determine its stability. Hence identify two bifurcation points r_1 and r_2. [5 marks]

(b) Show that for certain values of the parameter r there are additional fixed points x^*. For which values of r do these fixed points exist? Determine their stability and identify a further bifurcation point r_3. [7 marks]

(c) Using a Taylor expansion of (1), determine the normal forms of the bifurcation at r_1 and r_2. What type of bifurcations take place? [5 marks]

(d) Similarly, determine the normal form of the bifurcation at r_3. What type of bifurcation takes place? [5 marks]

(e) Sketch the bifurcation diagram for all values of r and x^*. (Use a full line to denote a curve of stable fixed points, and a dashed line for a curve of unstable fixed points!) [4 marks]

Question 2 [28 marks]

Consider the dynamical system

$$\begin{aligned} \dot{x} =& x(3 - 2x - y) \\ \dot{y} =& y(4 - x - 3y) \end{aligned} \tag{2}$$

in the (x, y)-plane.

(a) Determine the nullclines of the system (2) and find the fixed points. [8 marks]

(b) Give the Jacobian matrix. Hence, determine the linear stability of all fixed points. [12 marks]

(c) Draw the nullclines of the system (2). Hence, sketch the phase portrait and the flow for the system (2). Indicate the stable and unstable manifolds for any saddles. [8 marks]

Question 3 [26 marks] Given $\epsilon > 0$, consider the second-order differential equation

$$\ddot{x} + \epsilon(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0 \tag{3}$$

on the line.

(a) Using $y = \dot{x}$, state the system of first-order differential equations in the (x, y)-plane equivalent to (3). [4 marks]

(b) Find all the fixed points of this system. [4 marks]

(c) Derive an equation for $\dot{r}$ in terms of r and ϕ, where $x = r\cos\phi$ and $y = r\sin\phi$. *[Hint: differentiate $r^2 = x^2 + y^2$.]* [6 marks]

(d) By considering annuli $r_0 \le r \le r_1$ for $0 < r_0 < r_1$, construct a trapping region such that all solutions starting from initial conditions within this region stay inside. [6 marks]

(e) Deduce the existence of a limit cycle, and state its equation. Is the limit cycle stable or unstable? [6 marks]

Question 4 [20 marks] Consider the dynamical system

$$\dot{x} = f(x, y)\ , \quad \dot{y} = g(x, y) \tag{4}$$

with continuously differentiable functions f and g in the (x, y)-plane.

(a) Let C be a closed orbit of (4) and let A be the region inside C. Using Green's Theorem, show that

$$\iint_A \nabla \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} dxdy = 0\ .$$

[6 marks]

(b) Deduce that the dynamical system (4) has no closed orbits if

$$\frac{\partial f}{\partial x}(x, y) + \frac{\partial g}{\partial y}(x, y) > 0\ .$$

[6 marks]

Now consider the non-linear oscillator equation

$$\ddot{x} + \beta(x)\dot{x} + \kappa(x) = 0 \tag{5}$$

with continuously differentiable functions β and κ on the line.

(c) Using $y = \dot{x}$, state the functions f and g for which the system (4) is equivalent to equation (5). [4 marks]

(d) Show that if $\beta(x) < 0$ for all $x \in \mathbb{R}$ then (5) does not have any periodic solutions. [4 marks]

End of Paper