A number theoretical approach for faster square root computation

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1 Algorithm

1.1 step 1

Initialize an integer type variable N and store the input integer in it. Initialize two more integer type variables $temp$ and $unitsize$ with values N and 1 respectively.

1.2 step 2

Iteratively divide $temp$ by 9 while $temp >= 9$ and for each iteration multiply $unitsize$ by 3.

1.3 step 3

Initialize an integer type variable $startTerm$ valued 1. Initialize an integer type variable $sum1$ storing the arithmetic sum of first $unitSize$ terms starting from $startTerm$ and having a common difference of 2 calculated as,

$$sum1 = (unitSize * (2 * startTerm + (unitSize - 1) * 2))/2 \tag{1}$$

also initialize another integer type variable $sum2$ storing the arithmetic sum of first $2 * unitSize$ terms starting from $startTerm$ and having a common difference of 2 calculated as,

$$sum2 = (2 * unitSize * (2 * startTerm + (2 * unitSize - 1) * 2))/2 \tag{2}$$

and finally initialize another integer type variable $squareRoot$ valued 0.

1.4 step 4

Now there are two cases,

1.4.1 $temp >= 4$

In this case, Reassign $temp$ as $temp = N - sum2$ and $squareRoot$ as $squareRoot = 2 * unitSize$.

1.4.2 $1 <= temp < 4$

In this case, Reassign $temp$ as $temp = N - sum1$ and $squareRoot$ as $squareRoot = unitSize$.

1.5 step 5

Reassign $startTerm$ as $startTerm = 1 + 2 * squareRoot$ and $unitSize = unitSize$

1.6 step 6

Reassign $sum1$ again as,

$$sum1 = (unitSize * (2 * startTerm + (unitSize - 1) * 2))/2 \tag{3}$$

and also $sum2$ as,

$$sum2 = (2 * unitSize * (2 * startTerm + (2 * unitSize - 1) * 2))/2 \tag{4}$$

1.7 step 7

Now following cases are possible,

1.7.1 $temp >= sum2$

In this case reassign $temp$ as $temp = temp - sum2$ and $squareRoot$ as $squareRoot = squareRoot + 2 * unitSize$.

1.7.2 $sum1 <= temp < sum2$

In this case reassign $temp$ as $temp = temp - sum1$ and $squareRoot$ as $squareRoot = squareRoot + unitSize$.

1.8 step 8

Reassign $startTerm = 1 + 2 * squareRoot$ and $unitSize = unitSize/3$.

1.9 step 9

Iteratively repeat the steps from step 4 to step 8 until either $temp$ or $unitSize$ becomes smaller than 1.

2 Mathematical proofs

The algorithm uses inherent mathematical structures and insights to compute integer value of square roots of integers with high convergence rate while maintaining the correctness.

The mathematical insights forming the basis of algorithm are as follows,

2.1 result 1

For every N belonging to the set of all positive integers the value of N^2 is the sum of the first N consecutive odd integers starting from 1. Formally,

$$N^2 = sum_{n=1}^{n=N}(2 * n + 1) \tag{5}$$

where n belongs to the set of all natural numbers. Also,

$$N^2 = (N * (2 * 1 + (N - 1) * 2))/2 \tag{6}$$

that is N^2 equals to the arithmetic sum of N terms starting from 1 and having a common difference 2.

2.1.1 proof

Given N belonging to the set of all positive integers.

$$N^2 = (1 + N - 1)^2 \tag{7}$$

$$N^2 = 1^2 + (N - 1)^2 + 2(N - 1) \tag{8}$$

$$N^2 = 1^2 + 2 * N - 2 + (N - 1)^2 \tag{9}$$

$$N^2 = 2 * N - 1 + (N - 1)^2 \tag{10}$$

The equation 10 proves that every N belonging to the set of all positive integers can be expressed as a sum the Nth odd integer and the square of the integer $N - 1$. Also for every k belonging to the set of all positive integers less than N that is $N - k >= 1$ and $k >= 1$.

$$(N - k)^2 = (1 + ((N - k) - 1)^2 \tag{11}$$

$$(N - k)^2 = 1^2 + 2 * ((N - k) - 1) + ((N - k) - 1)^2 \tag{12}$$

$$(N - k)^2 = 1 + 2 * (N - k) - 2 + ((N - k) - 1)^2 \tag{13}$$

$$(N - k)^2 = 2 * (N - k) - 1 + ((N - k) - 1)^2 \tag{14}$$

Equation 14 can be recursively solved for all values of k starting from 1 to $N - 1$ as (for the sake of illustration only N is assumed to be a large integer which does not become less than 1 for the values of k used in the illustration).

$$N^2 = 2 * N - 1 + (N - 1)^2 \tag{15}$$

$$N^2 = 2 * N - 1 + 2 * (N - 1) - 1 + (N - 2)^2 \tag{16}$$

$$N^2 = 2 * N - 1 + 2 * (N - 1) - 1 + 2 * (N - 2) - 1 + (N - 3)^2 \tag{17}$$

$$\vdots N^2 = 2 * N - 1 + 2 * (N - 1) - 1 + 2 * (N - 2) - 1 + \cdots + 1 \tag{18}$$

The equation 18 proves the consistency of result 1.

2.2 result 2

The sum of first 3^k odd integers is equal to 9^k for all k belonging to the set of all natural numbers. Formally if,

$$S = sum_{n=1}^{n=3^k}(2n - 1) \tag{19}$$

where n and k belong to the set of all natural numbers then it implies

$$S = 9^k \tag{20}$$

2.2.1 proof

Given k and n belonging to the set of all natural numbers. Assume the sum of the first 3^k is given by S then,

$$S = sum_{n=1}^{n=3^k}(2*n-1) \tag{21}$$

$$S = sum_{n=1}^{n=3^(k-1)}(2*a-5+2*a-3+2*a-1) \tag{22}$$

$$\tag{23}$$

where,

$$a = 3*n \tag{24}$$

$$S = sum_{n=1}^{n=3^(k-1)}(6*n-5+6*n-3+6*n-1) \tag{25}$$

$$S = sum_{n=1}^{n=3^(k-1)}(18*n-9) \tag{26}$$

$$S = sum_{n=1}^{n=3^(k-1)}9(2*n-1) \tag{27}$$

$$S = sum_{n=1}^{n=3^(k-2)}9^2(2*n-1) \tag{28}$$

$$\vdots S = sum_{n=1}^{n=3^(k-k)}9^k(2*n-1) \tag{29}$$

$$S = sum_{n=1}^{1}9^k(2*1-1) \tag{30}$$

$$S = 9^k \tag{31}$$

Equation 31 proves the consistency of result 2.