STATISTICS – PART II

SOME EXERCISES

EXERCISE 1

A box contains six marbles, four red and two green. A child selects two of them at random. What is the probability that exactly one is red?

EXAMPLE

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M & Ms.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green M & M.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M & M.

4×2 =8 ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

- A deck of cards consists of 52 cards, 13 "kinds" each of four suits (spades, hearts, diamonds, and clubs). The 13 kinds are Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), King (K). In many poker games, each player is dealt five cards from a well shuffled deck.

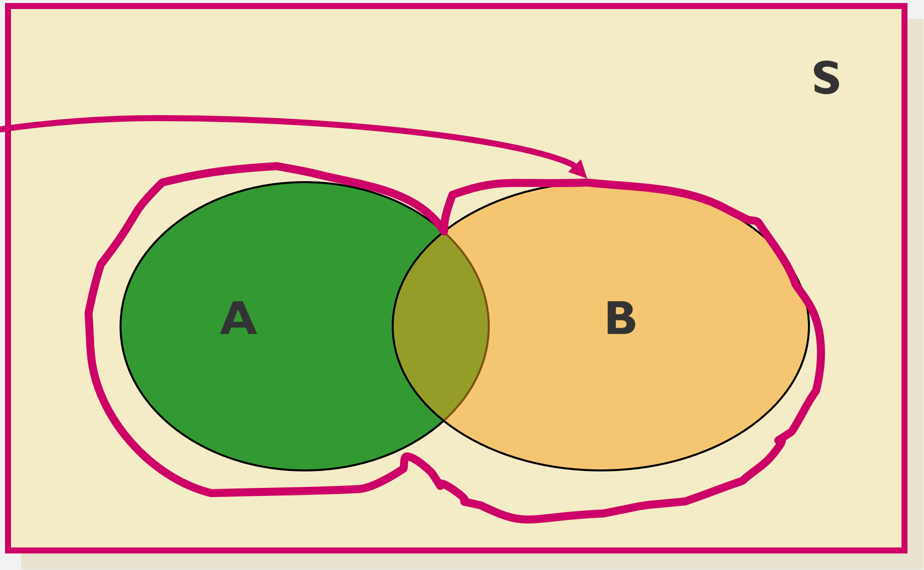
- Four of a kind: 4 of the 5 cards are the same “kind”. What is the probability of getting four of a kind in a five card hand?

EVENT RELATIONS

The beauty of using events, rather than simple events, is that we can **combine** events to make other events using logical operations: and, or and not.

The **union** of two events, **A** and **B**, is the event that either **A or B or both** occur when the experiment is performed. We write **A ∪ B**

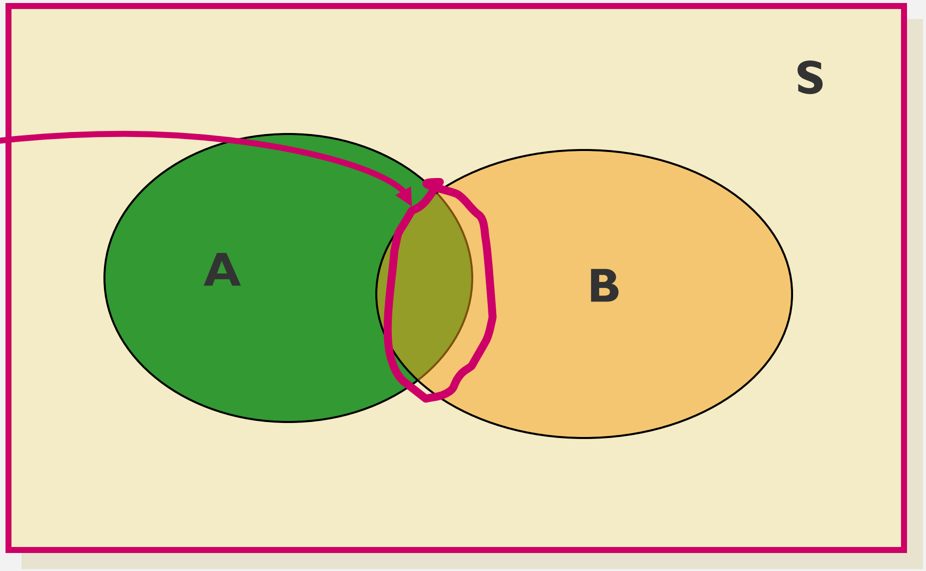
$A \cup B$



EVENT RELATIONS

The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B**.

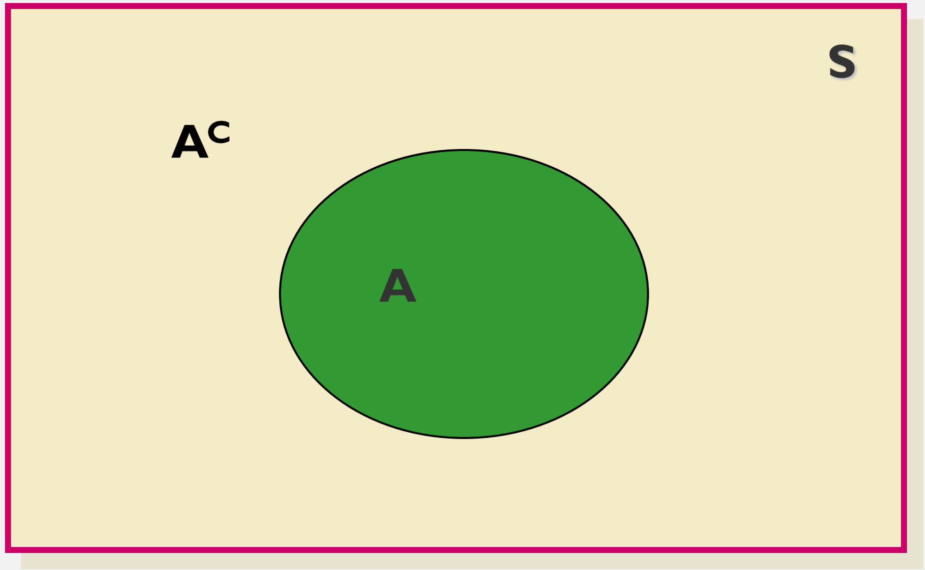
$A \cap B$



- If two events A and B are **mutually exclusive**, then $P(A \cap B) = 0$.

EVENT RELATIONS

The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event **A**. We write A^C.



EXAMPLE

Select a student from the classroom and record his/her **hair color** and **gender.**

- **A:** student has brown hair
- **B:** student is female
- **C:** student is male

Mutually exclusive; $B = C^C$

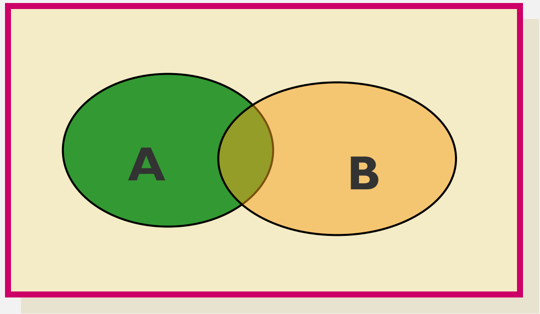
What is the relationship between events **B** and **C**?

- **A^C:** Student does not have brown hair
- **$B \cap C$:** Student is both male and female = $\varnothing$
- **$B \cup C$:** Student is either male and female = all students = S

CALCULATING PROBABILITIES FOR UNIONS AND COMPLEMENTS

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



EXAMPLE: ADDITIVE RULE

Example: Suppose that there were 120 students in the classroom, and that they could be classified as follows:

A: brown hair

P(A) = 50/120

B: female

P(B) = 60/120

	Brown	Not Brown
Male	20	40
Female	30	30

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 50/120 + 60/120 - 30/120$

$= 80/120 = 2/3$

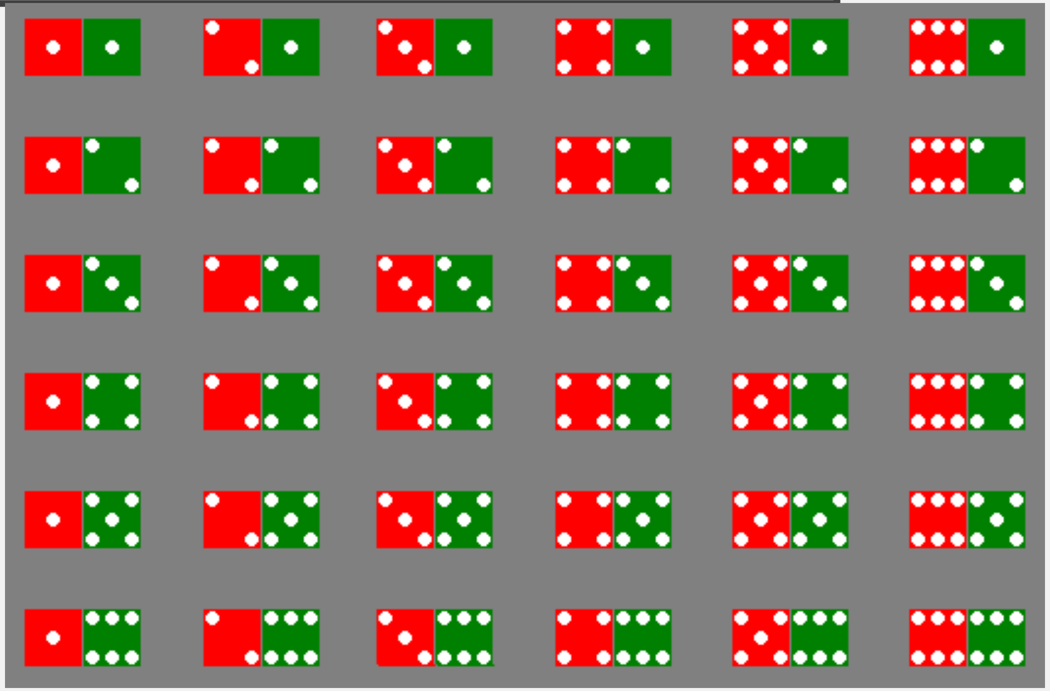
Check: $P(A \cup B)$

$= (20 + 30 + 30)/120$

EXAMPLE: TWO DICE

A: red die show 1

B: green die show 1



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 6/36 + 6/36 - 1/36$

$= 11/36$

A SPECIAL CASE

When two events A and B are **mutually exclusive**, $P(A \cap B) = 0$ and $\mathbf{P(A \cup B) = P(A) + P(B)}$.

A: male with brown hair

$P(A) = 20/120$

B: female with brown hair

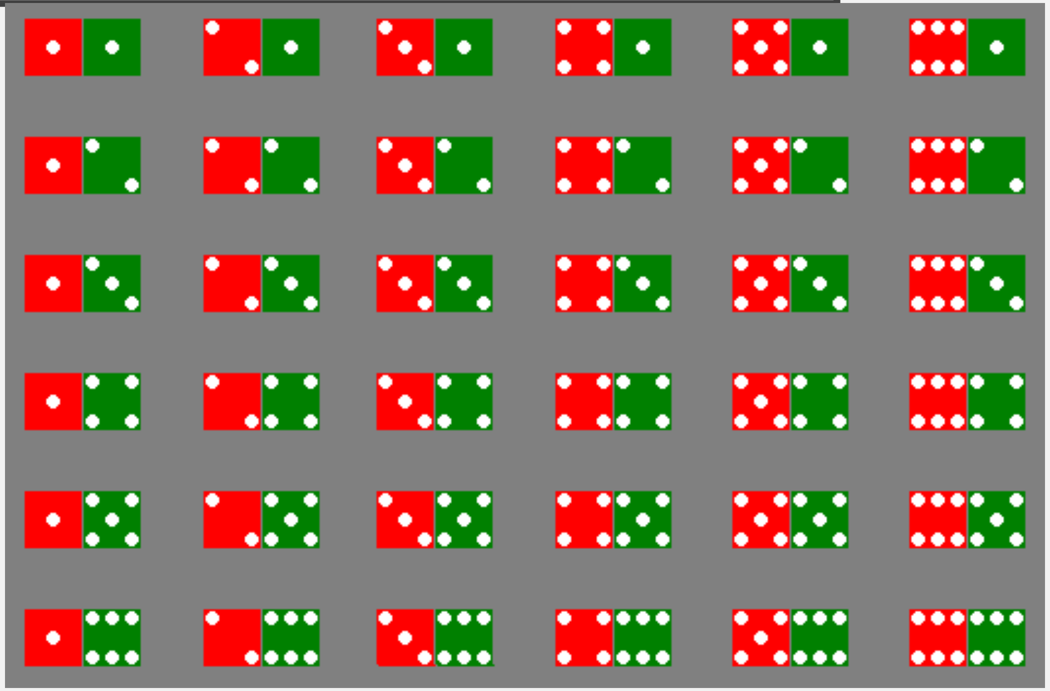
$P(B) = 30/120$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are mutually exclusive, so that

$$P(A \cup B) = P(A) + P(B)$$
$$= 20/120 + 30/120$$
$$= 50/120$$

EXAMPLE: TWO DICE



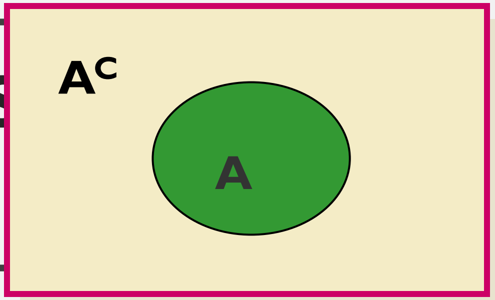
A: dice add to 3

B: dice add to 6

A and B are mutually exclusive, so that

$P(A \cup B) = P(A) + P(B)$
$= 2/36 + 5/36$
$= 7/36$

CALCULATING PROBABILITIES FOR COMPLEMENTS



- We know that for any event **A:**
  - $P(A \cap A^C) = 0$
- Since either **A** or **A^C** must occur,

  $P(A \cup A^C) = 1$
- so that $P(A \cup A^C) = P(A) + P(A^C) = 1$

$$\boxed{P(A^C) = 1 - P(A)}$$

EXAMPLE

Select a student at random from the classroom. Define:

A: male

$P(A) = 60/120$

B: female

$P(B) = ?$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are complementary, so that

$P(B) = 1 - P(A)$

$= 1 - 60/120 = 60/120$

CALCULATING PROBABILITIES FOR INTERSECTIONS

In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events.**

Two events, **A** and **B**, are said to be **independent** if the occurrence or nonoccurrence of one of the events does not change the probability of the occurrence of the other event.

CONDITIONAL PROBABILITIES

The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

"given"

EXAMPLE 1

Toss a fair coin twice. Define

- A: head on second toss
- B: head on first toss

HH	1/4
HT	1/4
TH	1/4
TT	1/4

$P(A|B) = ½$

$P(A|\text{not } B) = ½$

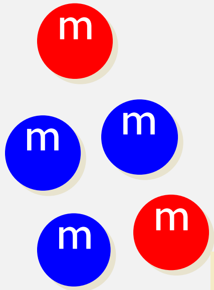
P(A) does not change, whether B happens or not…

A and B are independent!

EXAMPLE 2

A bowl contains five M&Ms®, two red and three blue. Randomly select two candies, and define

- A: second candy is red.
- B: first candy is blue.



$P(A|B) = P(2^{nd} \text{ red} | 1^{st} \text{ blue}) = 2/4 = 1/2$

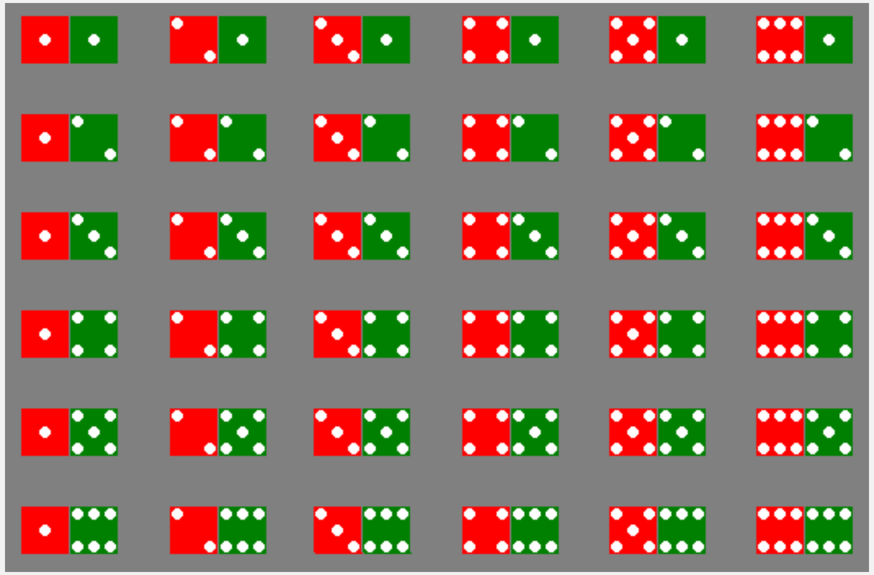
$P(A|\text{not } B) = P(2^{nd} \text{ red} | 1^{st} \text{ red}) = 1/4$

P(A) does change, depending on whether B happens or not… → A and B are dependent!

EXAMPLE 3: TWO DICE

Toss a pair of fair dice. Define

- A: red die show 1
- B: green die show 1



$P(A|B) = P(A \text{ and } B)/P(B)$

$=1/36/1/6=1/6=P(A)$

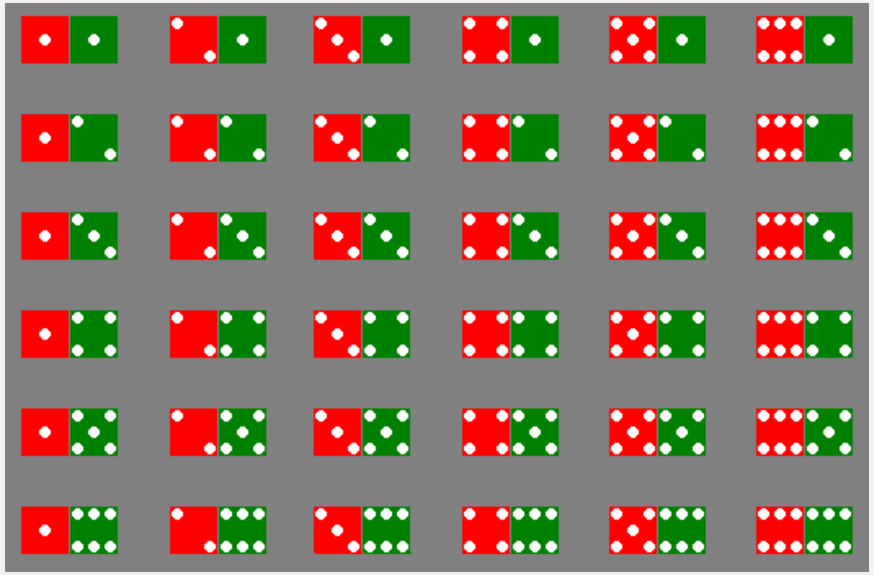
P(A) does not change, whether B happens or not…

A and B are independent!

EXAMPLE 3: TWO DICE

Toss a pair of fair dice. Define

- A: add to 3
- B: add to 6



P(A|B) = P(A and B)/P(B)

=0/36/5/6=0

P(A) does change when B happens

A and B are dependent! In fact, when B happens, A can't

DEFINING INDEPENDENCE

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

- Once you've decided whether or not two events are independent, you can use the following rule to calculate their intersection.

THE MULTIPLICATIVE RULE FOR INTERSECTIONS

- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)\ P(B \text{ given that } A \text{ occurred}) = P(A)P(B|A)$$

- If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)\ P(B)$$

EXAMPLE 1

In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk?

Define H: high risk N: not high risk

P(exactly one high risk) = P(HNN) + P(NHN) + P(NNH)

= P(H)P(N)P(N) + P(N)P(H)P(N) + P(N)P(N)P(H)

$= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$

EXAMPLE 2

Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?

Define H: high risk F: female

From the example, $P(F) = .49$ and $P(H|F) = .08$.
Use the Multiplicative Rule:

$$P(\text{high risk female}) = P(H \cap F)$$

$$= P(F)P(H|F) = .49(.08) = .0392$$