Fuzzy Relations

Fuzzy Relations

Cartesian Product

- ***Let A_1, A_2, ….., A_n be fuzzy sets in U_1, U_2, …U_n, respectively.***

  The Cartesian product of A_1, A_2, ….., A_n is a fuzzy set in the space U_1 x U_2 x…x U_n with the membership function as:

  $\mu_{A1 \times A2 \times \ldots \times An}(x_1, x_2, \ldots, x_n) = \min[\mu_{A1}(x_1), \mu_{A2}(x_2), \ldots. \mu_{An}(x_n)]$

- ***So, the Cartesian product of A_1, A_2, ….., A_n are donated by A_1 x A_2 x….. x A_n***

Fuzzy Relations

Cartesian Product: Example

- ***Let A =*** **{(3, 0.5), (5, 1), (7, 0.6)}**
- ***Let B =*** **{(3, 1), (5, 0.6)}**
- **Find the product**
- ***The product is all set of pairs from A and B with the minimum associated memberships***
- ***Ax B =*** **{[(3, 3), min (0.5, 1)], [(5, 3), min(1, 1)], [(7, 3), min(0.6, 1)], [(3, 5), min(0.5, 0.6)], [(5, 5), min(1, 0.6)], [(7, 5), min(0.6, 0.6)]}**

  = {[(3, 3), 0.5], [(5, 3), 1], [(7, 3), 0.6], [(3, 5), 0.5], [(5, 5), 0.6], [(7, 5), 0.6]}

Fuzzy Relations

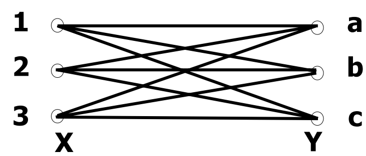
Crisp Relations

- ***The relation between any two sets is the Cartesian product of the elements of*** $A_1 \times A_2 \times \ldots \times A_n$
- ***For X and Y universes*** $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
- $\mu_{x \times y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}$
- ***This relation can be represented in a matrix format***

Fuzzy Relations

Crisp Relations: Example

- *Universe X =* {1, 2, 3}
- *Universe Y =* {a, b, c}



$$
\begin{array}{c c}
 & \\
\begin{array}{r} 1 \\ R = 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\
 & \begin{array}{ccc} a & \quad b & \quad c \end{array}
\end{array}
$$

Fuzzy Relations

Fuzzy Relations

- *Fuzzy relations are mapping elements of one universe, to those of another universe, Y, through the Cartesian product of two universes. X, Universe X =* {1, 2, 3}
- $R(X, Y) = \{[(x, y), \mu_R(x, y)] \mid (x, y) \in (X \times Y)\}$
- *Where the fuzzy relation R has membership function*
- $\mu_R(x, y) = \mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Fuzzy Relations

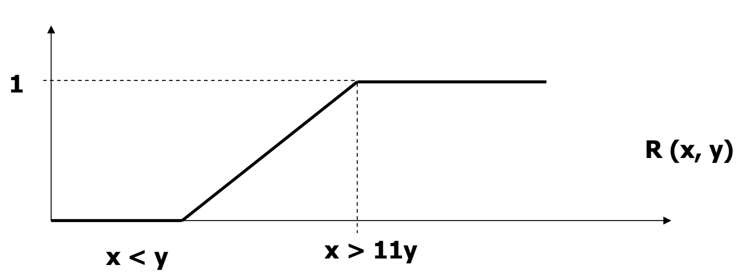
Fuzzy Relations

- ***It represents the strength of association between elements of the two sets***
- ***Ex: R = "x is considerably larger than y"***
- ***R (X, Y)= Relation between sets X and Y***
- ***R (x, y) = memebership function for the relation R (X, Y)***
- ***R (X, Y) = {R (x, y) / (x, y) | (x, y) Є (X × Y)}***

Fuzzy Relations

Fuzzy Relations

- $R(x,y) = \begin{cases} 0 & \text{for } x \leq y \\ \{x - y)/(10\text{-}y), & \text{for } y < x \leq 11y \\ 1 & \text{fro } x > 11y \end{cases}$



Fuzzy Relations

Operations on Fuzzy Relations

- *Since the fuzzy relation from X to Y is a fuzzy set in X × Y, then the operations on fuzzy sets can be extended to fuzzy relations. Let R and S be fuzzy relations on the Cartesian space* **X × Y** *then*:
- *Union:* $\mu_{R \cup S}(x, y) = \max[\mu_R(x, y), \mu_S(x, y)]$
- *Intersection:* $\mu_{R \cap S}(x, y) = \min[\mu_R(x, y), \mu_S(x, y)]$
- *Complement:* $\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$

Fuzzy Relations

Fuzzy Relations: Example

- *Assume two Universes*: **A = {3, 4, 5} and B = {3, 4, 5, 6, 7}**
- $\mu_R(x, y) = \begin{cases} (y-x)/(y+x+2) & \text{if } y > x \\ 0, & \text{if } y \le x \end{cases}$
- *This can be expressed as follow:*

$$R = \begin{array}{c} 3 \\ 4 \\ 5 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0 & 0.11 & 0.2 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{pmatrix} \\ \begin{matrix} 3 & \;\;4 & \;\;5 & \;\;6 & \;\;7 \end{matrix} \end{array}$$

Fuzzy Relations

Fuzzy Relations: Example

$$
\begin{array}{c}
 \\
R =
\end{array}
\begin{array}{c}
3 \\ 4 \\ 5 \\ \\
\end{array}
\begin{array}{c}
\left(\begin{array}{ccccc}
0 & 0.11 & 0.2 & 0.27 & 0.33 \\
0 & 0 & 0.09 & 0.17 & 0.23 \\
0 & 0 & 0 & 0.08 & 0.14
\end{array}\right) \\
\begin{array}{ccccc}
3 & 4 & 5 & 6 & 7
\end{array}
\end{array}
$$

- ***This matrix represents the membership grades between elements in X and Y***
- $\mu_R(x, y)$ = {[*0/(3, 3)*], [*0.11/(3, 4)*], [*0.2/(3, 5)*], *................*, [*0.14/(5, 7)*]}

Fuzzy Relations

Fuzzy Relations: Example

- ***Assume two fuzzy sets:*** $A = \{0.2/x_1 + 0.5/x_2 + 1/x_3\}$

  $B = \{0.3/y_1 + 0.9/y_2\}$

- ***Find the fuzzy relation (the Cartesian product)***

$$
A \times B = R = \begin{array}{cc}
\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} &
\left(\begin{array}{cc}
0.2 & 0.2 \\
0.3 & 0.5 \\
0.3 & 0.9
\end{array}\right) \\
 & \begin{array}{cc} y_1 & y_2 \end{array}
\end{array}
$$

Fuzzy Relations

Fuzzy Relations: Example

- *Consider the fuzzy relation. Express R using the resolution principle*

$$R = \begin{pmatrix} 0.4 & 0.5 & 0 \\ 0.9 & 0.5 & 0 \\ 0 & 0 & 0.3 \\ 0.3 & 0.9 & 0.4 \end{pmatrix}$$

- $R = 0.3\,R_{0.3} + 0.4\,R_{0.4} + 0.5\,R_{0.5} + 0.9\,R_{0.9}$

$$= 0.3 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} + 0.4 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + 0.5 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 0.9 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Fuzzy Relations

Composition of Fuzzy Relations

- *Composition of fuzzy relations used to combine fuzzy relations on different product spaces*
- *Having a fuzzy relation; R (X ×Y) and S (Y ×Z), then Composition is used to determine a relation T (X × Z),*

Fuzzy Relations

Composition of Fuzzy Relations

- ***Consider two fuzzy relation; R (X × Y) and S (Y × Z), then a relation T (X × Z), can be expressed as (max-min composition)***

  $T = R \circ S$

  $\mu_T(x, z) = \text{max-min}\ [\mu_R(x, y), \mu_S(y, z)]$

  $= \vee\ [\mu_R(x, y) \wedge \mu_S(y, z)]$

- ***If algebraic product is adopted, then max-product composition is adopted:***

  $T = R \circ S$

  $\mu_T(x, z) = \max\ [\mu_R(x, y) \cdot \mu_S(y, z)]$

  $= \vee\ [\mu_R(x, y) \cdot \mu_S(y, z)]$

Fuzzy Relations

Composition of Fuzzy Relations

- ***The max-min composition can be interpreted as indicating the strength of the existence of relation between the elements of X and Z***
- ***Calculations of*** **(R o S) is almost similar to matrix multiplication**
- ***Fuzzy relations composition have the same properties of:***

  Distributivity: **R o (S U T) = (R o S) U (R o T)**

  Associativity: **R o (S o T) = (R o S) o T**

Fuzzy Relations

Composition of Fuzzy Relations: Example

- Assume the following universes: $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$, with the following fuzzy relations.

$$R = \begin{array}{c} x_1 \\ x_2 \\ \end{array} \begin{array}{c} \begin{pmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{pmatrix} \\ \begin{array}{cc} y_1 & y_2 \end{array} \end{array} \quad \text{and} \quad S = \begin{array}{c} y_1 \\ y_2 \\ \end{array} \begin{array}{c} \begin{pmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{pmatrix} \\ \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \end{array}$$

- Find the fuzzy relation between X and Z using the max-min and max-product composition

Fuzzy Relations

Composition of Fuzzy Relations: Example

- By max-min composition

$$\mu_T(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

$$T = \begin{array}{c} x_1 \\ x_2 \\ \end{array} \begin{array}{c} \begin{pmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{pmatrix} \\ \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \end{array}$$

- By max-product composition

$$\mu_T(x_2, z_2) = \max[(0.8, 0.6), (0.4, 0.7)] = 0.48$$

$$T = \begin{array}{c} x_1 \\ x_2 \\ \end{array} \begin{array}{c} \begin{pmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{pmatrix} \\ \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \end{array}$$