Discrete Simulation of Flexible Structures

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***Abstract*— This report summarizes the results and insights drawn from the discrete simulations of (i) rigid spheres and elastic beam falling in viscous flow, (ii) a generalized case of elastic beam falling in viscous flow, and (iii) an elastic beam bending when subjected to an applied external force.**

I. Introduction

Simulation means computing the configuration of the system as a function of time. Simulation of the motion of flexible structures involve accounting for the different forces acting on the structure, defining the environmental constraints in terms of boundary conditions and then solving the force equation by computing the degrees of freedom at discretized time steps. Here, planar motion of structures is simulated for three different cases where conservative forces (elastic forces), viscous forces and other external forces are considered. The three cases include: (i) simulation of the motion of three connected spheres falling inside viscous fluid, (ii) simulation of the motion of N connected spheres falling inside viscous fluid, and (iii) simulation of the deformation of elastic beam and comparison with the Euler-Bernoulli beam theory. For implicit method of solving the force equations, Newton-Raphson method for non-linear system of equations is used to iteratively calculate the degrees of freedom at each time step. Whereas, for the explicit method, it is assumed that the change over a very small time step is negligible. Since only a planar motion is being simulated, the elastic beams represent stretching and bending only. The source code of the three simulations is written in Python and is stored in a GitHub repository.

II. Simulation of the Motion of Three Connected Spheres Falling Inside Viscous Fluid

Problem1.py in the GitHub repository contains the Python code for a solver that simulates the position and velocity of the spheres as a function of time both implicitly and explicitly. The case simulates the motion of three connected spheres of density 7000 kg/m^3, for a total time of 10 s, where the radius of the second sphere is 0.025 m and the radius of the other two spheres being 0.005 m. The spheres are connected by elastic beams and the system is dropped initially at a linear position, into a viscous fluid of viscosity 1000 Pa-s and density 1000 kg/m^3. The time step Δt, is taken to be 10^{-2} s for the implicit method and 10^{-5} s for the explicit method.

A. Shape of the Structure

The shape of the structure as a function of time is shown in Fig. 1 at time t = 0, 0.01, 0.05, 0.1, 1.0 and 10 s as calculated by the implicit method.

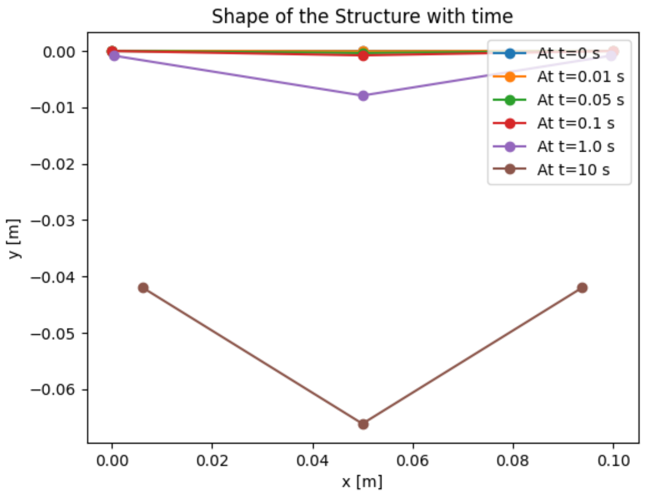

Figure 1

Fig. 2 and 3 show the position and velocity of the second sphere (R_2) respectively along *y-axis* as a function of time.

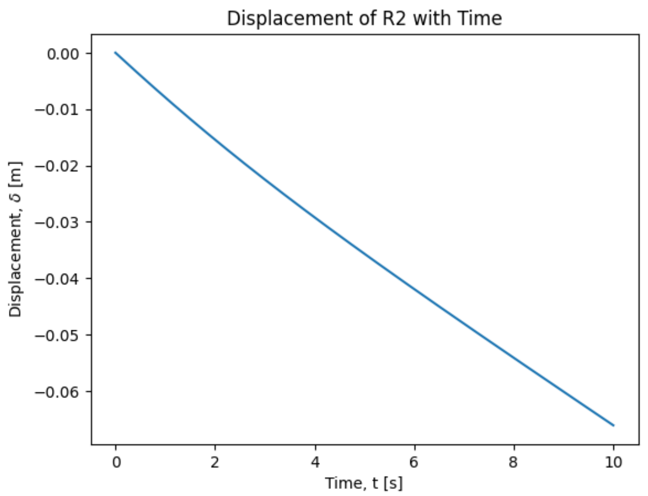

Figure 2

B. Terminal Velocity of the System

As shown in Fig. 3, based on the simulation, when approximated to a precision of 4 decimal places, the system achieves a terminal velocity of:

$$v_t = -0.0060 \text{ m/s}$$

The negative sign denotes the downward motion of the sphere.

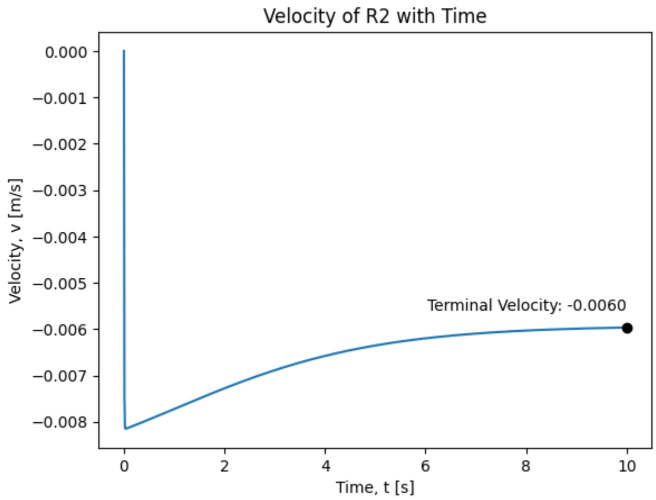

Figure 3

C. Special Case

In the earlier case, the middle sphere had a radius different from that of the other two spheres. That would mean, the force due to gravity and the viscous damping will be different for the middle sphere. This led the middle sphere to have larger displacement relatively as seen in Fig. 1. Here, a special case is discussed where the radii of all the three spheres are made equal. In such a case, it is observed that the shape of the structure does not change with time. Fig. 4 shows the shape of the structure with time when the radii of the three spheres are equal.

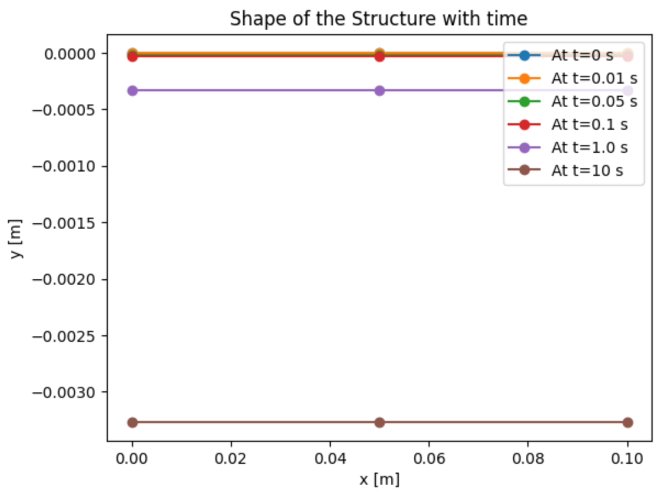

Figure 4

This result matches with our intuition. Since the radii of all the spheres are the same, the viscous forces and force due to gravity acting on them will be the same. Therefore, there is no moment generated on the elastic rod edges between the nodes. Hence, the shape of the structure does not change. The same can be observed in Fig. 5 which shows the turning angle at the middle node with time when the radii of the spheres are equal as compared to the turning angle shown in Fig. 6 when the radii of the spheres were different.

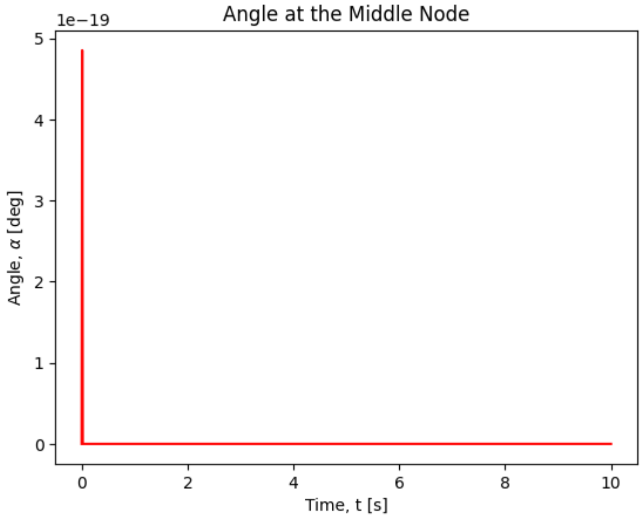

Figure 5

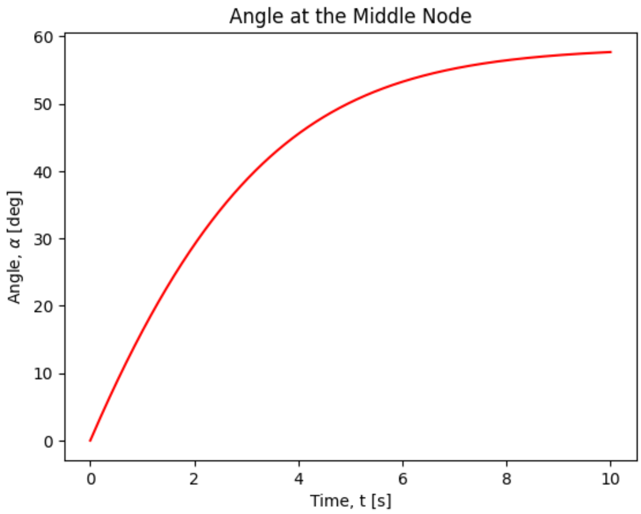

Figure 6

D. Comparison of Implicit and Explicit Method

The time step size (Δt) is varied for both implicit and explicit methods to observe the changes. It is unrealistic for the explicit approach to remain stable, since it requires a very small time step size (Δt) for the simulation, making it computationally expensive and time-consuming. A smaller Δt ensures that numerical instabilities are avoided, but this increases the total number of iterations needed, leading to a significantly longer simulation time. On the other hand, the implicit method is more stable and can handle larger time steps without diverging, even for stiff systems. However, the implicit approach involves solving a system of nonlinear equations at each time step, which requires computational effort and may be complex to implement. Thus, while explicit methods are simpler and straightforward for implementation, they suffer from stability issues with larger time steps, whereas implicit methods are computationally more intensive per iteration but allow larger time steps and are generally more stable for systems like this elastic beam problem.

III. SIMULATION OF THE MOTION OF N-CONNECTED SPHERES FALLING INSIDE VISCOUS FLUID

Now we extend the solution obtained in the earlier scenario to a general case of *N* spheres connected by *N-1* elastic rods. Problem2.py in the GitHub repository contains Python code for the solver that simulates the motion of N spheres connected by elastic beams falling in a viscous fluid. 21 spheres are considered for the simulation with a time step size (Δt) of 10^{-2} s and a total simulation time of 50 s. Due to longer simulation time only the implicit method is used for this case. The radius of the middle node is taken to be 0.025 m and the radius of the others nodes as $\Delta l/10$, where Δl is the length of the edge between the nodes.

A. Position and Velocity of the Middle Node

Fig. 7 shows the position of the middle node with time and Fig. 8 shows the velocity of the middle node with time along the *y-axis*.

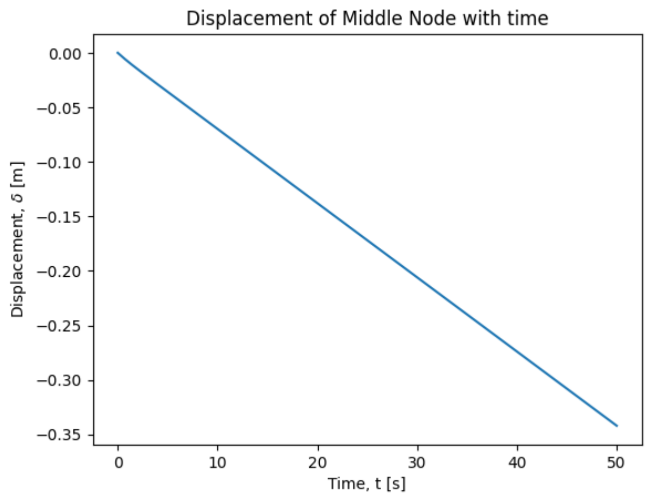

Figure 7

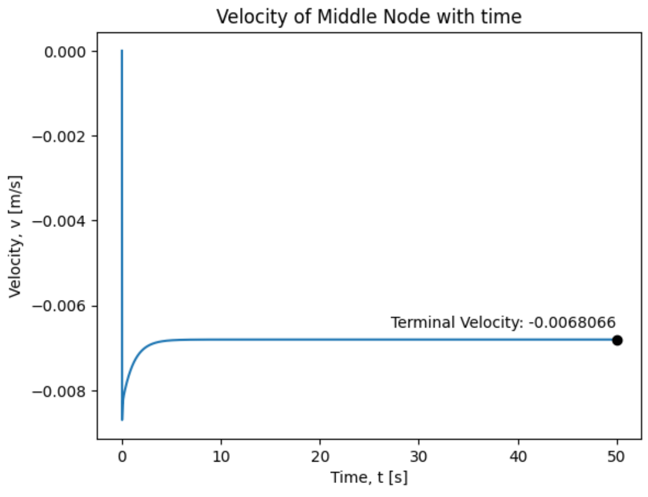

Figure 8

It can be seen that the velocity of the middle node approaches a steady state value. As per the simulation, when approximated to a precision of 7 decimal places, the middle node achieves a terminal velocity of:

$$v_t = -0.0068066 \text{ m/s}$$

B. Shape of the Structure

Fig. 9 shows the final shape of the structure after a simulation time of 50 s.

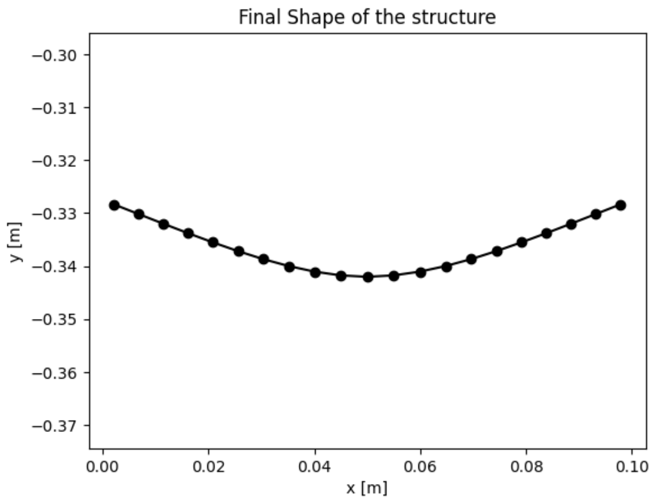

Figure 9

C. Spatial Discretization and Temporal Discretization

Any simulation must be sufficiently discretized such that the quantifiable metrics like terminal velocity do not vary much if *N* is increased and Δt is decreased. To discuss the importance of spatial and temporal discretization the number of nodes *N* and the time step size Δt was varied to analyze the sensitivity of terminal velocity to these parameters. Fig. 10 shows the terminal velocity sensitivity to Δt.

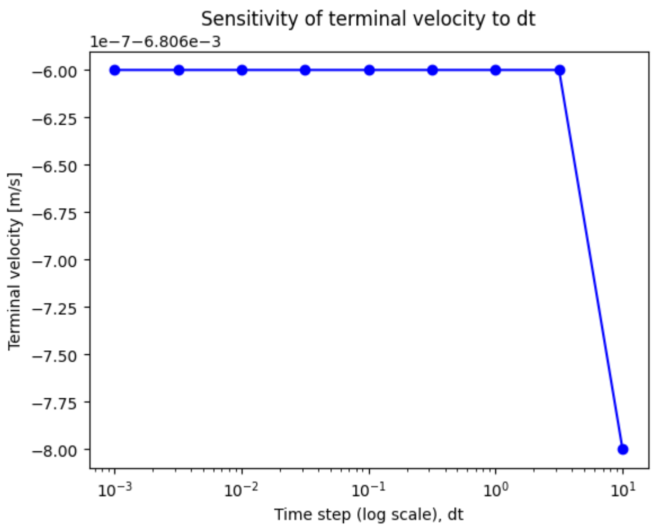

Figure 10

As can be seen, the terminal velocity does not change much with the changes in the time step size Δt. This implies that the implicit method is robust enough to yield accurate results even when the time step Δt is not small enough. This conclusion ties back to the inference from the earlier case on how the implicit method is better for larger time steps than explicit method. Fig. 11 shows the sensitivity of terminal velocity to number of nodes *N*.

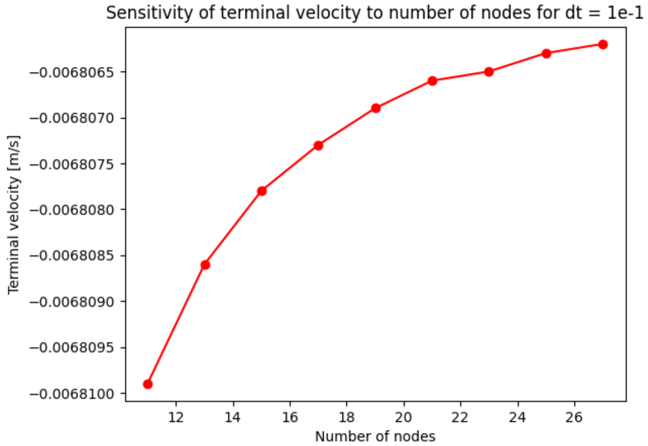

Figure 11

As seen in the plot, the terminal velocity is considerably affected by the change in number of nodes. For lower values of *N* the rate of change is higher compared to higher values of *N*. This demonstrates the significance of spatial discretization. Hence, the simulation must be spatially discretized, i.e., set *N* sufficiently large enough such that the simulation yields accurate results.

IV. Simulation of the Deformation of Elastic Beams and Comparison with Euler-Bernoulli Beam Theory

Using the discrete simulation approach from the generalized case, the deformation of a simply supported aluminum beam subjected to a single point load can be simulated. The beam is represented as a system of *N* nodes connected by elastic edges with a mass *m* located at each node given as:

$$m = \pi(R^2 - r^2) l \rho \,/\, (N-1)$$

where *R* (outer radius) = 0.013 m; *r* (inner radius) = 0.011 m; *l* (rod length) = 1 m; ρ (density of aluminum) = 2700 kg/m^3. The point load *P* of 2000 N is applied at 0.75 m away from the first node and is the only external force considered for this case disregarding the force due to gravity. The first node is constrained along both *x* and *y*-axes and the last node is constrained along y-axis which give the following boundary conditions:

$$x_1(t_{k+1}) = 0,$$

$$y_1(t_{k+1}) = 0,$$

$$y_N(t_{k+1}) = 0.$$

The solver considers 50 nodes to simulate the beam as a function of time for a simulation time of 1 s, with time step size Δt as 10^{-2} s for the implicit simulation.

A. Maximum Vertical Displacement

Fig. 12 shows the maximum deflection y_{max} of the beam as a function of time. It is observed that the value of y_{max} reaches a steady state. The steady state value is achieved because an artificial damping is implicitly part of the solver. The theoretical prediction of maximum deflection from Euler-Bernoulli beam theory is given by:

$$y_{\max} = \frac{Pc\,(L^2 - c^2)^{1.5}}{9\sqrt{3}EI\,l} \qquad \text{where} \qquad c = \min(d, l - d)$$

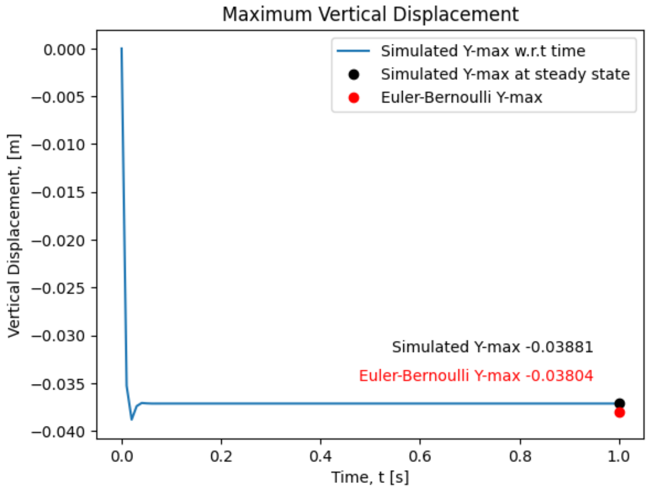

Figure 12

The maximum deflection of the beam based on the simulation is -0.03881 m at steady state. While the maximum deflection as predicted by the Beam theory is -0.03804 m which is comparable.

B. Comparison of Simulation and Beam Theory

To analyze the difference between the results from simulation and that from the beam theory, maximum vertical displacement was calculated for varying loads *P*. Fig. 13 compares the simulated result against the prediction from beam theory.

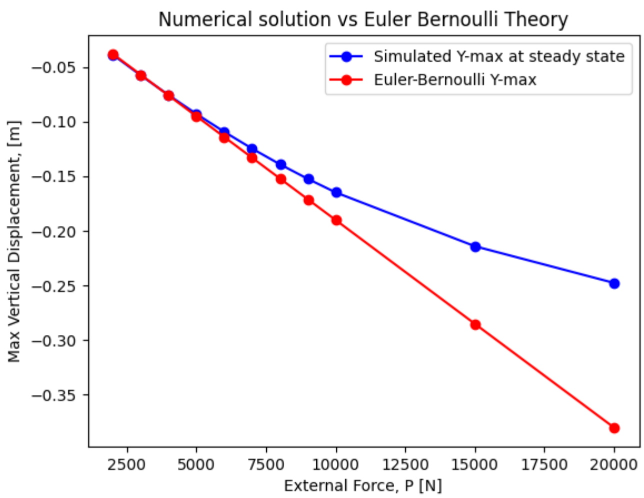

Figure 13

From a load of 5000 N the two solutions begin to diverge. It can be inferred that Euler beam theory is only valid for small deformation whereas the simulation is able to handle large deformations. This is because, as the deformations increase, the bending force becomes larger, which resists the said deformations. In the beam theory, this bending force is ignored but, in the simulation, it is considered. Hence, the simulation gives more accurate results for large loads.

REFERENCES

[1] Professor M. Khalid Jawed, MechAE 263F Course modules Link: https://bruinlearn.ucla.edu/courses/193842/modules Portion of this code and the helper functions or files have been referenced from the class resources on Bruinlearn (Link mentioned above).