Time series in epidemiology

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Objectives of this module

- To understand the limitations of standard linear regressions models for time series data.
- To model temporal trends (both seasonal and non-seaonal) through the use of explanatory variables.
- To understand and apply basic models for time series analysis.

1. Review of linear regression

Linear regression model

$$Y_i = \beta_0 + \beta_1 x_{1i} + ... + \beta_k x_{ki} + Z_i : i = 1, ..., n$$

- explanatory variables/factors $x_{1i}, ..., x_{ki}$
- regression parameters $\beta_0, ..., \beta_k$
- residuals Z_i
- responses Y_i

```
fit1<-lm(y x); summary(fit1)
xsq<-x*x; fit2<-lm(y x+xsq); summary(fit2)
names(fit2)
xx<-0.1*(0:200); beta<-fit2$coef;
ff<-beta[1]+beta[2]*xx+beta[3]*xx*xx
par(mfrow=c(1,1)); plot(x,y,pch=19,xlim=c(0,20));
lines(xx,ff,col="blue",lwd=3)
```

Parameter estimation: simple linear regression

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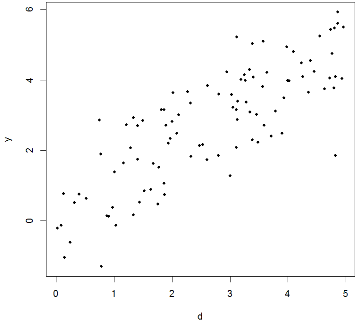

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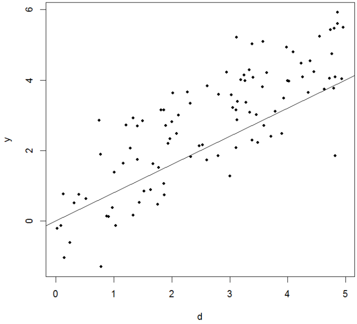

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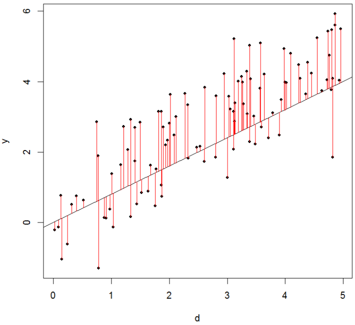

$$RSS = \sum_{i=1}^{n} (y_i - \beta_1 - \beta_2 x_i)^2$$

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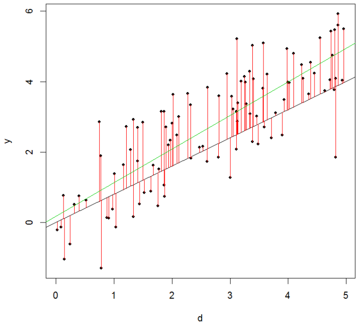

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General linear model: things to remember

- Transformation of x and/or Y widens the scope of the model

  Examples:
  - $Y_i = \beta_1 + \beta_2 \log x_i + Z_i$
  - $\log Y_i = \beta_1 + \beta_2 x_i + Z_i$
- Normality of the Z_i is less important than independence and constant variance
- But if the Z_i are iid $\mathrm{N}(0, \sigma^2)$, likelihood-based inference is straightforward (and least squares estimates are maximum likelihood estimates)
- Diagnostic checking of residuals is an important part of model-building, but requires subjective judgement.

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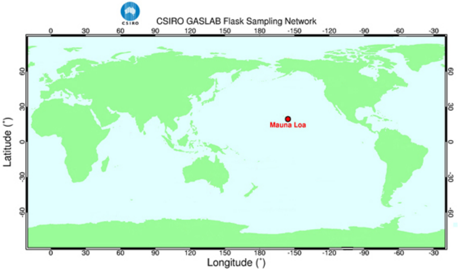
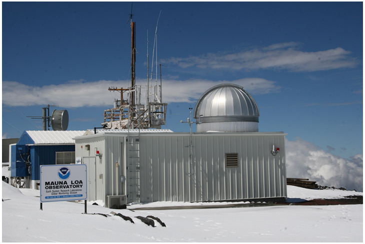
- Our model for nature,

$$[\mathrm{Y}] = [Y_1] \times [Y_2|Y_1] \times \ldots \times [Y_t|Y_{t-1}, Y_{t-2}, \ldots, Y_1].$$

Example: CO_2 time series

http://yearsoflivingdangerously.com/watch/season-1/





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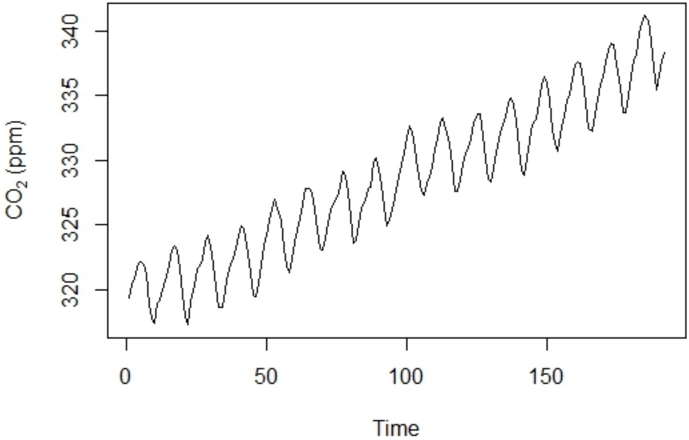
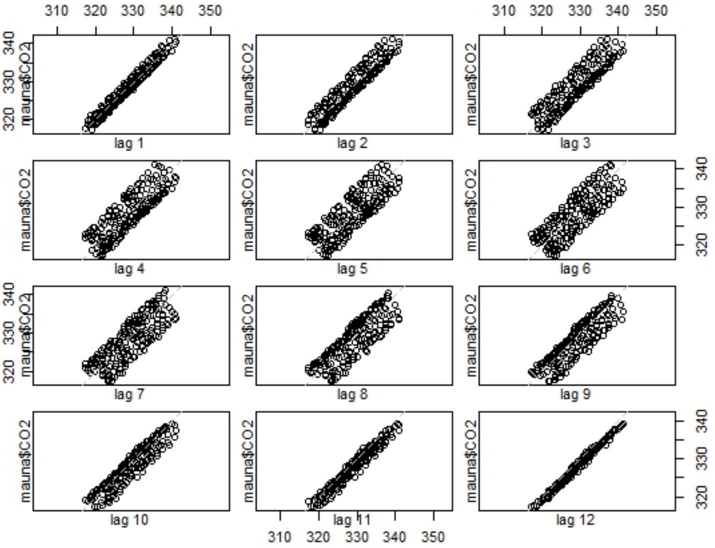


Lag plots 



310 320 330 340 350
mauna$CO2
320 330 340
lag 1
lag 2
lag 3
lag 4
lag 5
lag 6
lag 7
lag 8
lag 9
lag 10
lag 11
lag 12

Harmonic regression

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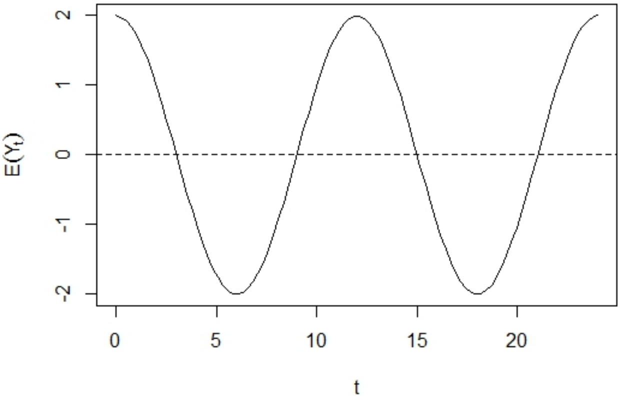

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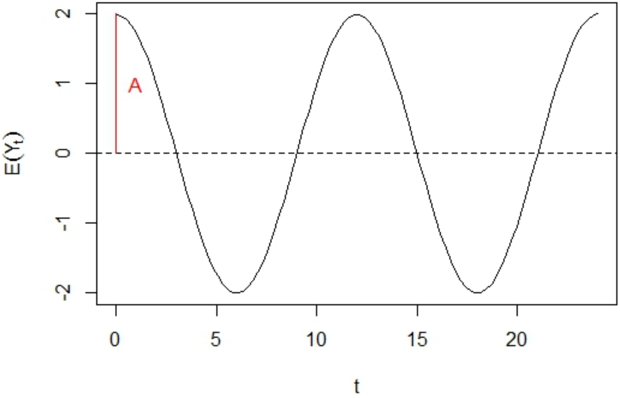

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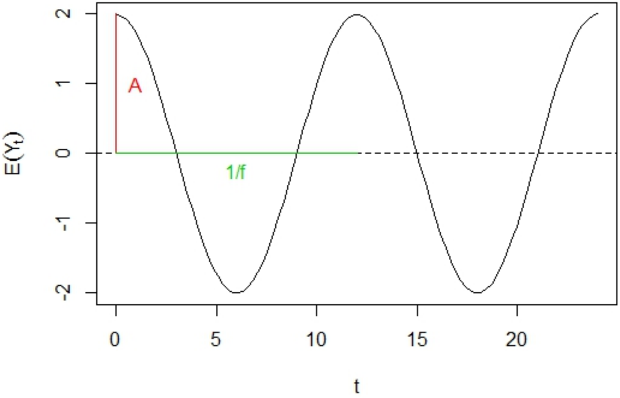
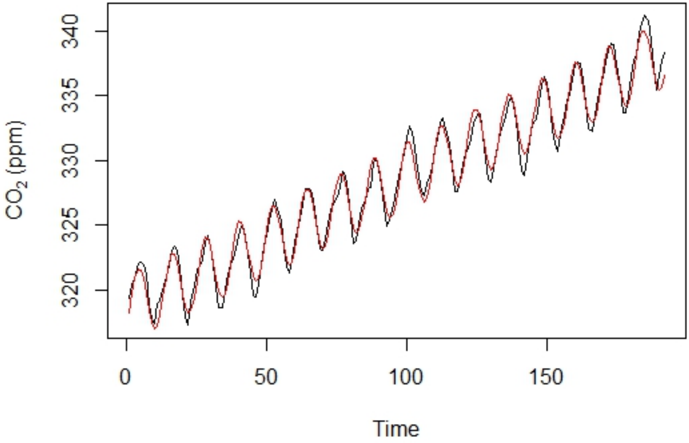


Do what you know from the linear model 



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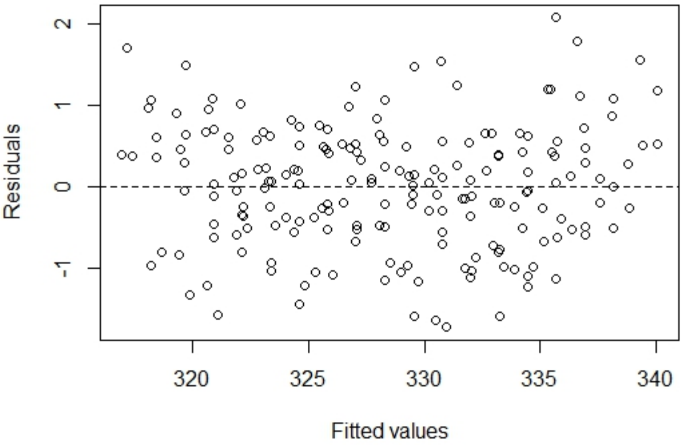

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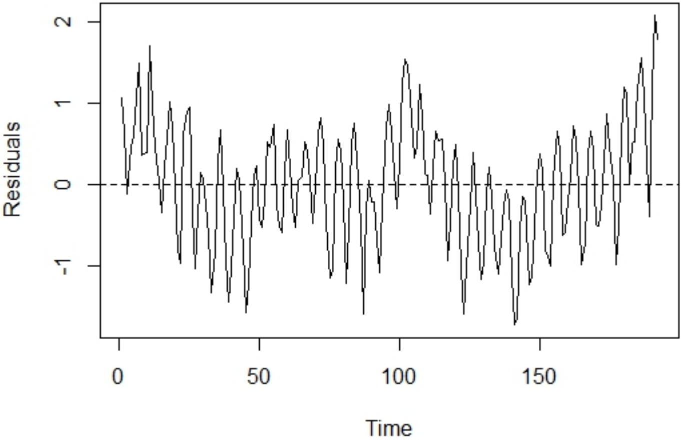

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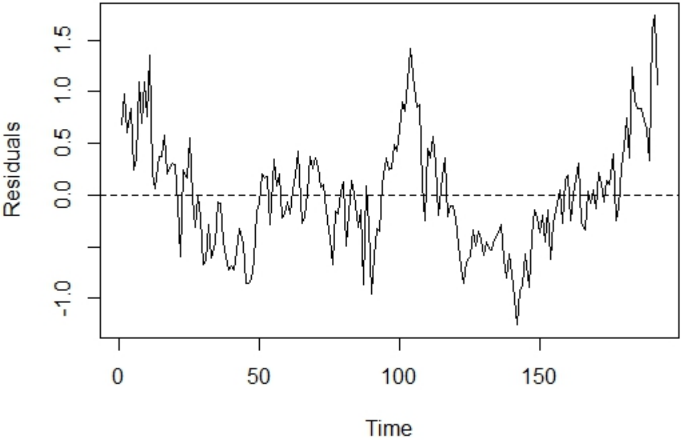

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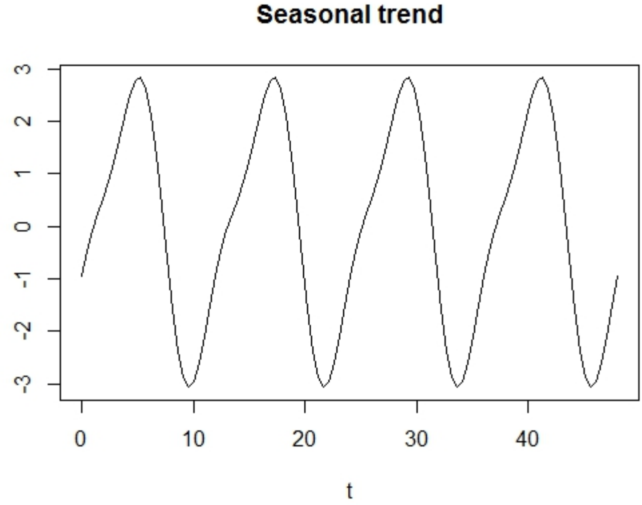

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- Second order stationarity

$$V[Y_t] = \sigma^2, \text{ for all } t$$

and

$$\mathrm{COV}(Y_t, Y_{t-k}) = \gamma_k = \sigma^2 \rho_k.$$

- What value does γ_0 take?

Correlogram 

$$\hat{\gamma}_k = \frac{1}{n}\sum_{t=1}^{n-k}(y_t - \bar{y})(y_{t-k} - \bar{y})$$

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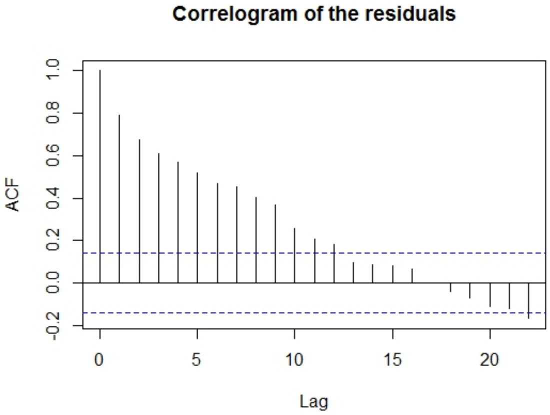

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Stationarity

An AR(p) process is stationary if and only if the roots of the equation

$$1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$$

are in absolute value smaller than 1.

Example: AR(1)

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- $1 - \phi x = 0 \iff x = \phi^{-1}$, hence $|\phi| < 1$.
- $V[Y_t] = \sigma^2/(1-\phi^2)$, $\gamma_k = \sigma^2\phi^k/(1-\phi^2)$.

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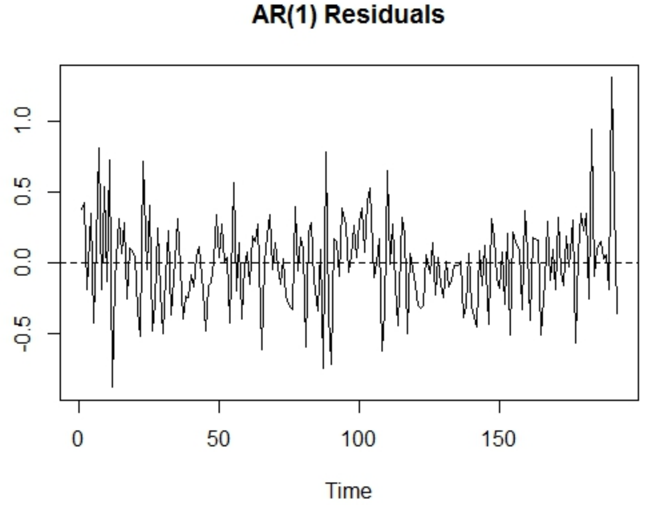
$$\begin{aligned}\log L(\boldsymbol{\theta}) &= -\frac{1}{2}\Bigg[n \log\{\sigma^2\} - \log(1-\phi^2) + (1-\phi^2)\frac{y_1^2}{\sigma^2} \\ &\qquad + \sum_{t=2}^{n} \frac{(y_t - \phi y_{t-1})^2}{\sigma^2}\Bigg]\end{aligned}$$

CO_2 time series (continue) 



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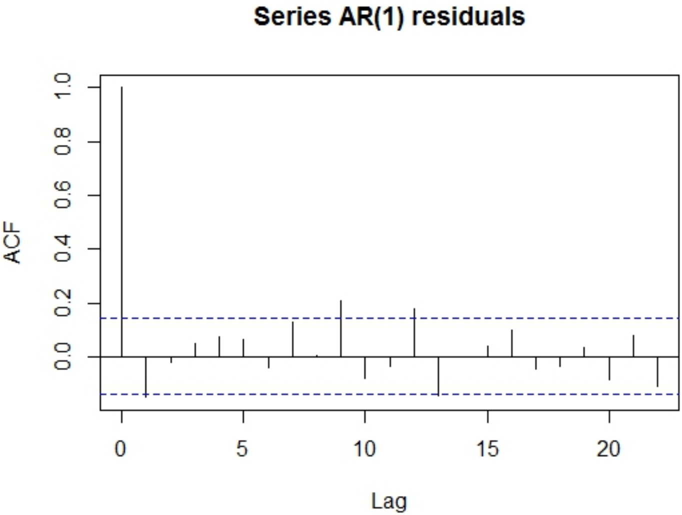

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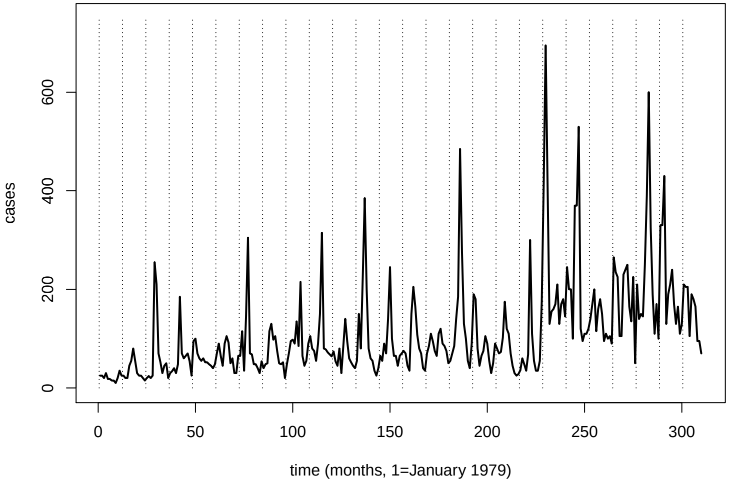
- A model for the data

$$Y(t_i) = d(t_i)^\top \beta + W(t_i) + Z(t_i).$$

Malaria cases in Kericho, Kenya



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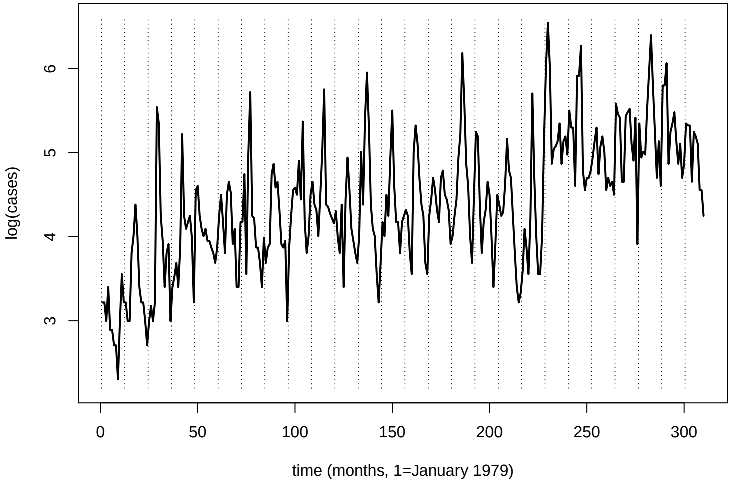

Standard linear regression analysis

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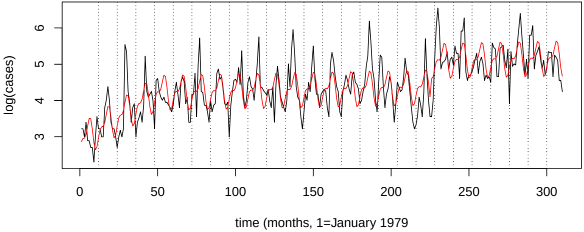
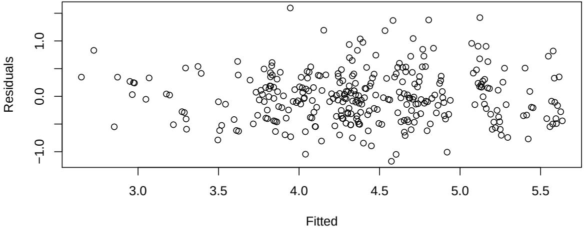
$$
\begin{aligned}
Y(t) \quad = \quad & \beta_0 + \beta_1 t + \beta_2 \max\{t - 50, 0\} + \\
& \beta_3 I(t > 225) + \\
& \beta_2 \sin(2\pi t/12) + \beta_3 \cos(2\pi t/12) + \\
& \beta_4 \sin(2\pi t/6) + \beta_5 \cos(2\pi t/6) + \\
& Z(t) \qquad (1)
\end{aligned}
$$

Standard linear regression analysis





The theoretical variogram

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- Assumption: $W(t)$ is a stationary zero-mean stochastic process with covariance function

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- Theoretical varioram (definition)

$$\begin{aligned} v(h) &= \frac{1}{2} E[(W(t) + Z(t) - W(t') - Z(t'))^2] \\ &= \tau^2 + \sigma^2 (1 - \rho(h)) \end{aligned}$$

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- The empirical variogram (definition)

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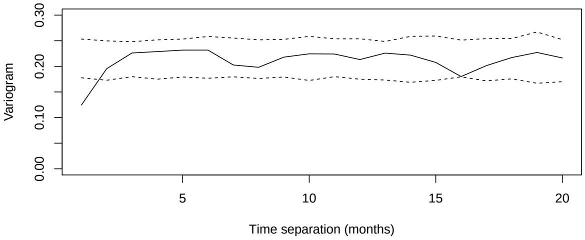
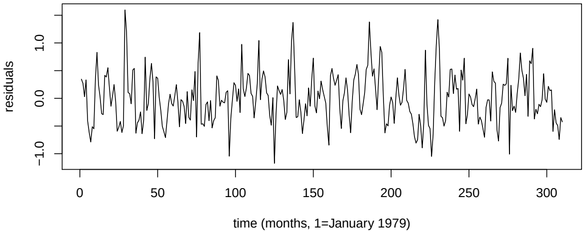
The empirical variogram





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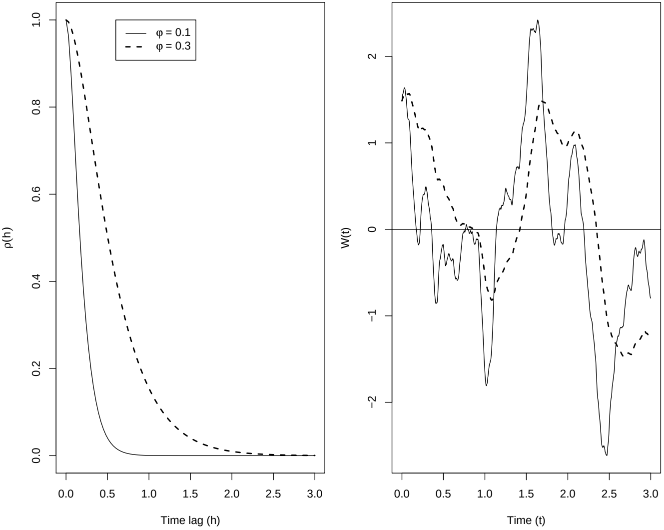
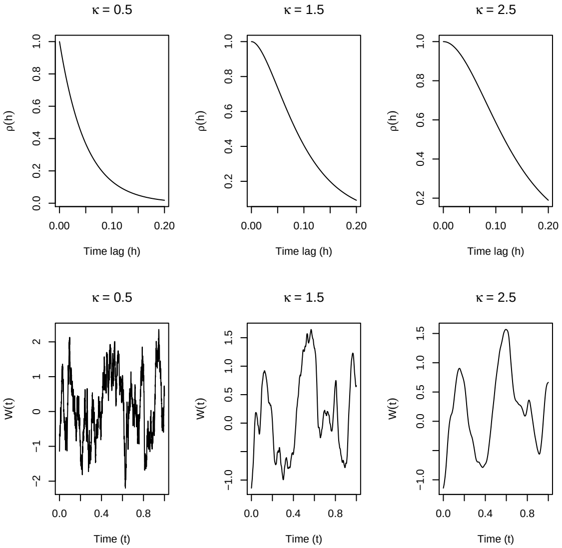
- Special case ($\kappa = 1/2$)

$$\rho(h) = \exp\{-h/\phi\}. \tag{5}$$

The scale parameter ϕ



The smoothness parameter κ



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- The vector $(Y(t_1), \ldots, Y(t_n))$ follows a multivariate Gaussian distribution with mean $D\beta$ and covariance matrix $\Omega = \sigma^2 R + \tau^2 I$ where

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Fitting a Matern process

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- The likelihood function

$$l(\theta) = -\frac{1}{2}\left\{n \log \sigma^2 + \log|\Omega| + (y - D\beta)^\top \Omega^{-1}(y - D\beta)/\sigma^2\right\}, \tag{6}$$

Kericho data: parameter estimation

	$\kappa = 0.5$	
Parameter	Point estimate	95% CI
β_0	2.892	(2.535, 3.248)
β_1	0.137	(0.038, 0.235)
β_2	-0.345	(-0.444, -0.247)
β_3	0.162	(0.086, 0.238)
β_4	0.126	(0.050, 0.202)
β_5	0.026	(0.018, 0.035)
β_6	-0.025	(-0.035, -0.016)
β_7	0.700	(0.397, 1.004)
σ^2	0.214	(0.177, 0.258)
ϕ	1.149	(0.870, 1.516)
τ^2	-	-