CPC Aptitude Quant Guide©

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Preface

"Aptitude is more of understanding English than doing Math."

- Iktomi Follower

"Motherflerken."

- Nick Fury

Tips for attending the aptitude paper :

Assuming the paper is 1 hour long, and you have 45 to 50 questions, first set a target for yourselves, say 20 or 25 questions. You should be accurate in those questions, and get them right. So, take a maximum of 50 minutes for these questions, which gives you around 2 to 2.5 minutes per question.

The next task is to choose the right questions to answer. Once you read a question ***properly***, you will understand its difficulty, and know if you can solve it in time or not. If yes, go for it, with maximum of 30 seconds buffer time (extra time), else put peace and move on.

Make sure you attain your target questions at 100% accuracy. The last 10 minutes is for extras. You can answer the 50-50 questions, which you feel you can answer with reasonable accuracy. The most important takeaway is that ***solving fewer questions with full accuracy is better*** than haphazard guessing.

All the best.

"In aptitude, some people move on.
But not us.
Not us."

- Nobody

Chapter 1

Clocks

1.1 Intro

A 12-hour clock contains a minute hand and a hour hand. The minute hand completes one revolution across the clock in one hour, which is 60 minutes. And since one revolution = 360°, one minute equals $\frac{360°}{60} = 6°$ rotation. In other words, a minute hand rotates 6° in one minute.

A hour hand completes one revolution in 12 hours. One revolution = 360°, so the hour hand rotates $\frac{360°}{12} = 30°$ in one hour. 1 hour = 60 minutes, so the hour hand rotates 0.5° in one minute.

1.2 How to approach a clock problem

Question : At what time between 4 and 5 o' clock will the hands of the clock be at right angle?

First, let us take 4 o' clock as the reference point. At that time, the minute hand is pointing at 12, and the hour hand is pointing at 4. They make an angle of 120° with each other.

If x minutes have passed, then the minute hand rotates $6x$ degrees, and the hour hand rotates $0.5x$ degrees. The starting angle between the hands is 120°.

The relative speed between minute and hour hand is $6x - 0.5x = 5.5x$.

Since we need them to be at right angles, we need to solve

$$\begin{aligned}|120 - 5.5x| &= 90\\ 120 - 5.5x &= \pm 90\\ 5.5x &= 120 \pm 90\\ x &= \frac{120 \pm 90}{5.5} = \frac{2 \times (120 \pm 90)}{11}\end{aligned}$$

So, $x = \frac{420}{11}$ minutes, or $x = \frac{60}{11}$ minutes. Since both solutions are smaller than 60 minutes, they occur between 4 o' clock and 5 o' clock. Hence, these are valid solutions.

We can generalise this problem to any degree from 0° to 180°. Let us denote the degree by d. If the initial angle between the hour and the minute hand is i, then we can find the minute x at which they are d degrees apart, using the formula

$$|i - 5.5x| = d$$

which reduces to

$$x = \frac{i \pm d}{5.5}$$

Check if x falls under the given time interval, and use a bit of intuition to find the number of possible solutions.

1.3 Gaining and losing time in clocks

This is a specific case of the ratio problem. We should approach the problem by finding the effective time gained or lost per hour or day, as per the question.

Question : A clock is set right at 8 am. It gains 10 minutes in a day. What will be the true time when the clock indicates 1 pm the next day?

Solution : It is given that 24 hours of actual time is equal to 24 hours 10 minutes on the clock. We shall convert this to minutes, as they are easier to work with. So, 1440 minutes of actual time = 1450 minutes of clock time.

When the clock shows 1 pm the next day, $24 + (13 - 8) = 29$ hours have passed on the clock. So, the clock time $= 29 \times 60 = 1740$ minutes.

Thus, Actual time $= \frac{1440}{1450} \times 1740 = 1728$ minutes.

This turns out to be 28 hours, 48 minutes after 8 am. Hence, the actual time is 12:48 pm the next day.

Chapter 2

Calendar

2.1 Zeller's Rule

Any problem regarding calendars can be solved using Zeller's Rule. The formula follows a particular order of consideration.

Leap years come once in 4. So that involves the last two digits of the year (4 divisibility). February is assumed to be the last month of the year to avoid confusion of 28 vs 29 days in the middle.

To calculate the day of any date, $dd/mm/YYyy$, take

(i) $K = dd$

(ii) $M = (mm - 2)$ if $mm > 2$, and $M = (mm + 10)$ if $mm \leq 2$

(iii) $D = yy$

(iv) $C = YY$

Then we compute F.

$$F = K + \left\lfloor \frac{13M - 1}{5} \right\rfloor + D + \left\lfloor \frac{D}{4} \right\rfloor + \left\lfloor \frac{C}{4} \right\rfloor - 2C$$

We take the F value and take the reminder when divided by 7.
From this reminder, we decide the day.
0 - Sunday
1 - Monday
2 - Tuesday
3 - Wednesday
4 - Thursday
5 - Friday
6 - Saturday

We will consider an example and find the day on which 10/06/1996 falls.

Here, $K = 10$, $M = 06 - 2 = 4$, $D = 96$, and $C = 19$.

So,

$$\begin{aligned} F &= 10 + \left\lfloor \frac{13 \times 4 - 1}{5} \right\rfloor + 96 + \left\lfloor \frac{96}{4} \right\rfloor + \left\lfloor \frac{19}{4} \right\rfloor - 2 \times 19 \\ &= 10 + \lfloor 10.2 \rfloor + 96 + \lfloor 24 \rfloor + \lfloor 4.75 \rfloor - 38 \\ &= 106 \end{aligned}$$

Now we calculate 106 (mod 7) = 1. And '1' corresponds to Monday. Hence, 10th June 1996 is on a Monday.

Note : When comparing two calendar years, it is preferable to use March 1 of those years as a reference to avoid confusion over leap years.

Chapter 3

Percentages and Interest

3.1 Percents

$$\frac{a}{b} = \left(\frac{a}{b} \times 100\right)\%$$

$$x\% = \frac{x}{100}$$

3.2 Relative Percentage

If A is $x\%$ more than B, then B is less than A by $\left[\frac{x}{(100+x)} \times 100\right]\%$.

If A is $x\%$ less than B, then B is more than A by $\left[\frac{x}{(100-x)} \times 100\right]\%$.

This is the case when the price of a commodity increases by $x\%$, and you need reduce the consumption so as to maintain the same expenditure. In this case, the consumption reduces by a factor of $\left[\frac{x}{(100+x)} \times 100\right]\%$.

3.3 Simple Interest

For a principal investment of P, the simple interest S for a rate of $R\%$ per annum, for n years, will be

$$S = \frac{P \times n \times R}{100}$$

The total amount A will be

$$A = P + S$$

3.4 Compound Interest

For a principal investment of P, rate $R\%$ per annum, and n years, total amount A in compound interest is given by

$$A = P \times \left(1 + \frac{R}{100}\right)^n$$

The compound interest C is given by

$$C = A - P$$

If the compound interest is computed half-yearly, then the amount A is given by

$$A = P \times \left(1 + \frac{R}{200}\right)^{2n}$$

If the compound interest is computed quarterly, then the amount A is given by

$$A = P \times \left(1 + \frac{R}{400}\right)^{4n}$$

The population growth, and product value depreciation, follows the rule of compound interest.

Chapter 4

Profit and Loss

4.1 Intro

The price at which an article is purchased is called its **Cost Price**, abbreviated as **C.P**.

The price at which an article is sold is called its **Selling Price**, abbreviated as **S.P**.

If S.P. is of an article is greater than its C.P., the seller gets a **Profit** or **Gain**.

If S.P. is less than the C.P., the seller attains a **Loss**.

4.2 Formulae

$$\text{Gain} = (\text{S.P.}) - (\text{C.P.})$$

$$\text{Loss} = (\text{C.P.}) - (\text{S.P.}) = -(\text{Gain})$$

$$\text{Gain\%} = \frac{\text{Gain} \times 100}{\text{C.P.}}$$

$$\text{Loss\%} = \frac{\text{Loss} \times 100}{\text{C.P.}}$$

$$\text{S.P.} = \frac{(100 + \text{Gain\%})}{100} \times \text{C.P.}$$

$$\text{C.P.} = \frac{100}{(100 + \text{Gain\%})} \times \text{S.P.}$$

4.3 Example Question

Question : An article is sold at a certain price. By selling it at 2/3rd of that price, one loses 10%. Find the gain percent at original price.

Solution :
Let the original S.P. be x. Then, the new S.P. is $\frac{2}{3}x$. Also, Loss = 10%.
The formula relating S.P., C.P., and Loss is this :

$$\text{C.P.} = \frac{100}{(100 + \text{Gain\%})} \times \text{S.P.}$$

Applying the data into the formula, we have

$$\text{C.P.} = \frac{100}{[100 + (-10)]} \times \frac{2}{3}x$$

$$= \frac{100}{90} \times \frac{2}{3}x$$

$$= \frac{20}{27}x$$

Since the original S.P. is x,

$$\text{Gain\%} = \left(\frac{\text{S.P.} - \text{C.P.}}{\text{C.P.}}\right) \times 100\%$$

$$= \left(\frac{x - \frac{20}{27}x}{\frac{20}{27}x}\right) \times 100\%$$

$$= \left(\frac{\frac{7}{27}x}{\frac{20}{27}x}\right) \times 100\%$$

$$= \frac{7}{20} \times 100\%$$

$$= 35\%$$

Chapter 5

Partnerships

When two or more persons run a business jointly, they are called **partners** and the deal is known as a **partnership**.

5.1 Division of Gains

Suppose A invests Rs. x in a business for p units of time, and B invests Rs. y in the same business for q units of time. Then

$$\frac{\text{A's share of profit}}{\text{B's share of profit}} = \frac{xp}{yq}$$

5.2 Example Problems

Question : Alfred started a business investing Rs. 45,000. After three months, Bruce joined him with a capital of Rs. 60,000. After another six months, Bruce's mother Martha added her share of Rs. 90,000 to the business. At the end of the year, they made a profit of Rs. 16,500. Find the share of each.

Solution :

Alfred invested in the business for 12 months, while Bruce and Martha invested in the business for 9 months and 3 months respectively.

$$\begin{aligned} \text{So, the ratio of their capitals} &= (45000 \times 12)\ :\ (60000 \times 9)\ :\ (90000 \times 3) \\ &= 540000 : 540000 : 270000 \\ &= 2 : 2 : 1 \end{aligned}$$

Alfred's share = Rs. $\left(16500 \times \frac{2}{5}\right)$ = Rs. 6600.

Bruce's share = Rs. $\left(16500 \times \frac{2}{5}\right)$ = Rs. 6600.

Martha's share = Rs. $\left(16500 \times \dfrac{1}{5}\right)$ = Rs. 3300.

Question : A, B and C enter into a partnership by investing in the ratio of 3 : 2 : 4. After one year, B invests another Rs. 2,70,000, and at the end of 2 years, C invests Rs. 2,70,000. At the end of three years, profits are shared in the ratio of 3 : 4 : 5. Find the initial investment of each.

Solution :

Let the initial investments of A, B and C be Rs. $3x$, Rs. $2x$ and Rs. $4x$ respectively. Then, the ratio of their profit shares is

$$(3x \times 36) : [(2x \times 12) + (2x + 270000) \times 24] : [(4x \times 24) + (4x + 270000) \times 12] = 3 : 4 : 5$$

$$\Rightarrow \qquad 108x : (72x + 6480000) : (144x + 3240000) = 3 : 4 : 5$$

Taking one part of it,

$$\frac{108x}{72x + 6480000} = \frac{3}{4}$$

$$\Rightarrow \qquad 432x = 216x + 19440000$$

$$\Rightarrow \qquad 216x = 19440000$$

$$\Rightarrow \qquad x = 90000$$

A's initial investment = $3x$ = Rs. 2,70,000.
B's initial investment = $2x$ = Rs. 1,80,000.
C's initial investment = $4x$ = Rs. 3,60,000.

Chapter 6

Progressions

6.1 Intro

Any sequence of numbers which follows a defined ordered rule constitutes a progression. The things you need to know are AP, GP, HP and some general series.

6.2 Arithmetic Progression

Any sequence in which the consecutive terms differ by a constant value, is called an Arithmetic Progression or AP. For example, $2, 5, 8, 11, 14, \ldots$ is an AP with common difference 3.

6.2.1 Representation of terms of an AP

If we assume that the first term of an AP is a, and the common difference is d, then any general n'th term of the AP can be written in terms of a and d as

$$a_n = a + (n-1)d$$

6.2.2 Sum of terms of an AP

The sum of the first n terms of an AP is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

6.2.3 Arithmetic Mean

The Arithmetic Mean of n numbers $a_1, a_2, \ldots, a_n$ is given by

$$\text{A.M} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

6.3 Geometric Progression

Any sequence in which the ratio of consecutive terms is a constant value, is called a Geometric Progression or GP. For example, $2, 4, 8, 16, 32, \ldots$ is a GP with common ratio 2.

6.3.1 Representation of terms of a GP

If we assume that the first term of a GP is a, and the common ratio is r, then any general n'th term of the GP can be written in terms of a and r as

$$a_n = a \times r^{n-1}$$

6.3.2 Sum of terms of a GP

The sum of the first n terms of a GP is given by

$$S_n = a \times \frac{r^n - 1}{r - 1}$$

6.3.3 Geometric Mean

The Geometric Mean of n numbers $a_1, a_2, \ldots, a_n$ is given by

$$\text{G.M} = (a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{\frac{1}{n}} = \sqrt[n]{a_1 \times a_2 \times a_3 \times \cdots \times a_n}$$

6.4 Harmonic Progression

Any sequence in which the reciprocal of the terms is in AP, is called a Harmonic Progression or HP. For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a HP because the reciprocal sequence, $1, 2, 3, 4, \ldots$ is an AP.

6.4.1 Harmonic Mean

The Harmonic Mean of n numbers $a_1, a_2, \ldots, a_n$ is given by

$$\text{H.M} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}}$$

6.5 AM-GM-HM Inequality

For any non-negative bunch of numbers $a_1, a_2, a_3, \ldots, a_n$,

$$\text{A.M} \geq \text{G.M} \geq \text{H.M}$$

The equality holds if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

6.6 Some General Series

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \ldots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$$

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

Chapter 7

Alligations and Mixtures

7.1 Intro

Alligation : It is the rule that enables us to find the ratio in which two ingredients of given price must be mixed to get a mixture of a desired price.

Mean Price : The cost price of a unit quantity of the mixture is called the mean price.

7.2 Rule of Alligation

If two ingredients are mixed, then

$$\left(\frac{\text{Quantity of cheaper}}{\text{Quantity of costlier}}\right) = \frac{\text{(C.P. of costlier)} - \text{(Mean Price)}}{\text{(Mean Price)} - \text{(C.P. of cheaper)}}$$

7.3 Replacing liquid with water

Suppose a container contains x units of pure liquid, from which y units are taken out and replaced with water. After n operations, the quantity of pure liquid left is

$$\text{Quantity of pure liquid} = \left[x\left(1 - \frac{y}{x}\right)^n\right] \text{ units}$$

7.4 Example Problems

Question : In what ratio must a grocer mix two varieties of pulses costing Rs. 15 and Rs. 20 per kg respectively, so as to get a mixture worth Rs. 16.50 per kg?

Solution :

We have that

- C.P. of cheaper = Rs. 15 per kg
- C.P. of costlier = Rs. 20 per kg
- Mean Price = Rs. 16.50 per kg

By the rule of alligation,

$$\left(\frac{\text{Quantity of cheaper}}{\text{Quantity of costlier}}\right) = \frac{20 - 16.50}{16.50 - 15}$$

$$= \frac{3.50}{1.50}$$

$$= \frac{7}{3}$$

Hence, the ratio of the cheaper to the costlier variety is 7 : 3.

Question : A container contains 40 litres of milk. From this container, 4 litres of milk was taken out and replaced by water. This was repeated further two times. How much milk is now contained by the container?

Solution :
We have that

- Amount of pure milk, $x = 40$ litres
- Amount replaced per try, $y = 4$ litres
- Number of replacements, $n = 3$

Hence, the amount of pure milk left is

$$\text{Quantity of pure liquid} = \left[x\left(1 - \frac{y}{x}\right)^n\right] \text{ litres}$$

$$= \left[40 \times \left(1 - \frac{4}{40}\right)^3\right] \text{ litres}$$

$$= \left[40 \times \left(\frac{9}{10}\right)^3\right] \text{ litres}$$

$$= 29.16 \text{ litres}$$

Chapter 8

Time and Work

8.1 Intro

If A can do a piece of work in n days, then A's work per day $= \dfrac{1}{n}$.

If A's one day's work $= \dfrac{1}{n}$, then A can complete the work in n days.

If A is k times better than B at completing a work, then :
Ratio of work done by A and B $= k : 1$.
Ratio of time taken by A and B to finish a work $= 1 : k$.

8.2 Pipes and Cisterns

If a pipe can fill a tank in x hours, then the part filled in 1 hour $= \dfrac{1}{x}$.

If a pipe can empty a tank in y hours, then part emptied per hour $= \dfrac{1}{y}$.

If a pipe can fill an empty tank in x hours, and another pipe can empty a full tank in y hours (where $y > x$), then on opening both pipes, the net part filled per hour $= \left(\dfrac{1}{x} - \dfrac{1}{y}\right)$.

In the above case, if $x > y$, then the net part emptied per hour $= \left(\dfrac{1}{y} - \dfrac{1}{x}\right)$.

8.3 Example Problems

Question : A and B undertake to do a piece of work for Rs. 600. A alone can do it in 6 days, while B alone can do it in 8 days. They recruit another person C for help, and together, they finish the work in 3 days. Find the share of each.

Solution :
A's work per day $= \frac{1}{6}$.

B's work per day $= \frac{1}{8}$.

Hence, A and B's work together per day $= \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$.

Total work done per day $= \frac{1}{3}$.

Hence, C's work per day $= \frac{1}{3} - \frac{7}{24} = \frac{1}{24}$.

Hence, work ratio of A, B and C $= \frac{1}{6} : \frac{1}{8} : \frac{1}{24} = 4 : 3 : 1$.

So,
A gets $\left(\frac{4}{8} \times 600\right) = 300$, B gets $\left(\frac{3}{8} \times 600\right) = 225$,
C gets $\left(\frac{1}{8} \times 600\right) = 75$

Question : Two pipes A and B together can fill a cistern in 4 hours. Had they been opened separately, then B would have taken 6 hours more than A to fill the cistern. How much time will be taken by A to fill the cistern separately?

Solution :
Let A take x hours to fill the cistern on its own.
Then, B will take $(x + 6)$ hours to fill the cistern independently.

We know that, together they can fill the cistern in 4 hours. Hence,

$$\frac{1}{x} + \frac{1}{x+6} = \frac{1}{4}$$

$$
\begin{aligned}
\Rightarrow \quad & 4[x+6+x] = x(x+6) \\
\Rightarrow \quad & x^2 + 6x = 8x + 24 \\
\Rightarrow \quad & x^2 - 2x - 24 = 0 \\
\Rightarrow \quad & (x-6)(x+4) = 0
\end{aligned}
$$

Hence, $x = 6$ or $x = -4$. But since x denotes hours, it cannot be negative. Hence, $x = 6$ is the solution.

This means that A can fill the cistern separately in 6 hours.

Question : Two taps A and B can fill a tank in 5 hours and 20 hours respectively. If the two taps are open, then due to a leakage, it took 30 minutes to fill the tank. Now, if the tank is full, how long will it take for the leakage alone to empty the tank?

Solution :
Amount filled per hour by A and B together $= \frac{1}{5} + \frac{1}{20} = \frac{4+1}{20} = \frac{1}{4}$.

Hence, A and B together takes 4 hours to fill the tank without leakage.

But with leakage, they take 30 minutes more. In other words, it takes 4.5 hours to fill the tank with leakage.

Suppose the leakage alone takes x hours to drain an empty tank. Then,

$$
\begin{aligned}
& \frac{1}{4} - \frac{1}{x} = \frac{1}{4.5} \\
\Rightarrow \quad & \frac{1}{x} = \frac{1}{4} - \frac{1}{4.5} \\
& = \frac{9-8}{36} \\
& = \frac{1}{36} \\
\Rightarrow \quad & x = 36
\end{aligned}
$$

Hence, the leakage alone takes 36 hours to empty a full tank.

Chapter 9

Relative Speeds

9.1 Trains

(i) a km/hr $= \left(a \times \dfrac{5}{18}\right)$ m/s (ii) b m/s $= \left(b \times \dfrac{18}{5}\right)$ km/hr

Time taken by a train of length l to pass a pole or a standing man is equal to the time taken by the train to cover the length l.

Time taken by a train of l metres to pass a stationary object of length b metres is the time taken by the train to cover $(l + b)$ metres.

Suppose two trains are moving in the same direction at u m/s and v m/s respectively. Then, their relative speed $= (u - v)$ m/s.

If the trains are moving in opposite direction, then their relative speed $= (u + v)$ m/s.

So, if two trains of length a and b are crossing each other, the total distance crossed by each train $= (a + b)$.

Suppose they move in the same direction at speeds u and v respectively $(u > v)$. Then, their relative speed $= (u - v)$.

Thus, the time taken to cross each other is

$$\text{Time to cross} = \frac{\text{Total distance}}{\text{Relative speed}} = \frac{(a + b)}{(u - v)}$$

9.2 Problems on Trains

Question : A train is moving at a speed of 132 km/hr. If the length of the train is 110 metres, how long will it take to cross a railway platform 165 metres long?

Solution :

$$\text{Speed of train} = 132 \text{ km/hr} = \left(132 \times \frac{5}{18}\right) \text{ m/s} = \frac{110}{3} \text{ m/s}.$$

Total distance to cross the platform $= (110 + 165)$ m $= 275$ m.

$$\text{Hence, time taken} = \frac{275 \text{ m}}{110/3 \text{ m/s}} = \frac{275 \times 3}{110} \text{ s} = 7.5 \text{ s}.$$

Question : Two trains are moving in the opposite directions at 60 km/hr and 90 km/hr. If the length of the trains are 1.1 km and 0.9 km respectively, then how long will the slower train take to cross the faster train?

Solution :
Time taken to cross is same for both slower and faster train.

$$\text{Speed of slower train} = 60 \text{ km/hr} = \left(60 \times \frac{5}{18}\right) \text{ m/s} = \frac{50}{3} \text{ m/s}.$$

$$\text{Speed of faster train} = 90 \text{ km/hr} = \left(90 \times \frac{5}{18}\right) \text{ m/s} = 25 \text{ m/s}.$$

$$\text{Relative speed} = \left(25 + \frac{50}{3}\right) \text{ m/s} = \left(\frac{75 + 50}{3}\right) \text{ m/s} = \frac{125}{3} \text{ m/s}.$$

Total length to cover $= (1.1 + 0.9)$ km $= 2$ km $= 2000$ m.

$$\text{Hence, time taken to cross} = \frac{\text{Total Length}}{\text{Relative Speed}} = \frac{2000 \text{ m}}{125/3 \text{ m/s}} = 48 \text{ s}.$$

9.3 Boats and Streams

In water, the direction along the stream is called **downstream**. The direction against the stream is called **upstream**.

Suppose the speed of the boat in still water be x km/hr, and the speed of the stream be y km/hr. Then,
Speed of the boat in downstream $= (x + y)$ km/hr.
Speed of the boat in upstream $= (x - y)$ km/hr.

If the speed of a boat downstream is u km/hr, and the speed of the boat upstream is v km/hr. Here, $u > v$. Then,

$$\text{Speed of the boat in still water} = \frac{(u + v)}{2} \text{ km/hr}.$$

$$\text{Speed of the stream} = \frac{(u - v)}{2} \text{ km/hr}.$$

9.4 Problems on Boats

Question : A man can row 18 kmph in still water. It takes thrice as long to row up as to row down the river. Find the rate of stream.

Solution :

Let the speed upstream be x km/hr. Then, the speed downstream is $3x$ km/hr.

Speed of the man in still water = 18 km/hr $= \dfrac{(3x + x)}{2} = \dfrac{4x}{2} = 2x$.

Hence, the speed upstream $= x = \dfrac{1}{2} \times 18$ km/hr = 9 km/hr.

The speed downstream $= 3x = 27$ km/hr.

Thus, the speed of stream $= \dfrac{(27 - 8)}{2}$ km/hr = 9 km/hr.

Question : In a stream running at 2 kmph, a motorboat goes 6 km upstream and comes back to the starting point in 33 minutes. Find the speed of the motorboat in still water.

Solution :

Let the speed of motorboat in still water be x km/hr.

Then, the speed upstream $= (x - 2)$ km/hr, and speed downstream $= (x + 2)$ km/hr.

Time taken to go upstream $= \dfrac{6}{x - 2}$ hr.

Similarly, time taken to go downstream $= \dfrac{6}{x + 2}$ hr.

Hence, total time taken $= \dfrac{6}{x - 2} + \dfrac{6}{x + 2} = \dfrac{33}{60}$ hr.

$$
\begin{aligned}
\Rightarrow \quad & 360 \cdot [x + 2 + x - 2] = 33 \cdot (x - 2) \cdot (x + 2) \\
\Rightarrow \quad & 720x = 33 \cdot (x^2 - 4) \\
\Rightarrow \quad & 11x^2 - 240x - 44 = 0 \\
\Rightarrow \quad & 11x^2 - 242x + 2x - 44 = 0 \\
\Rightarrow \quad & 11x \cdot (x - 22) + 2 \cdot (x - 22) = 0 \\
\Rightarrow \quad & (11x + 2) \cdot (x - 22) = 0
\end{aligned}
$$

Since x cannot be negative, $x = 22$ is the solution. Hence, the speed of the motorboat in still water = 22 km/hr.

Chapter 10

Races

10.1 Example Problems

Question : In a 1000 metre race, A beats B by 28 metres or 7 seconds. Find the time taken by A to run the race.

Solution :
From the data given, B can run 28 metres in 7 seconds.

Hence, speed of B $= \dfrac{28 \text{ m}}{7 \text{ s}} = 4$ m/s.

So, the time taken for B to complete the race $= \dfrac{1000 \text{ m}}{4 \text{ m/s}} = 250$ s.

Hence, A completes the race in $(250 - 7)$ seconds $= 243$ seconds.

Question : P, Q and R are three contestants in a 1 km race. P beats Q by 40 metres, and P beats R by 64 metres. Find the distance by which Q beats R.

Solution :
Suppose P covers 1000 metres in the race.
Then Q covers $(1000 - 40)$ m $= 960$ metres.
Also, R covers $(1000 - 64)$ m $= 936$ metres.

Thus, when Q covers 960 m, R covers 936 m. Hence, when Q covers 1000 m,

$$\text{Distance covered by R} = \left(\frac{936}{960}\right) \times 1000 \text{ m} = 975 \text{ m}$$

Hence, Q beats R by $(1000 - 975)$ metres $= 25$ metres.

Chapter 11

Set Theory

11.1 Intro

Set : A collection of items is called a set. For example, $\{1, 3, 4\}$ is a set. $\{\text{Monday}, \text{Wednesday}, \text{Thor}, \text{Rocket}\}$ is also a set.

In a set, the order of elements do not matter.

Element : Any item contained in a set is called an element of the set. For example, if $S = \{1, 6, 5\}$, then $6 \in S$, which is read as "6 belongs to S".

Equal Sets : Two sets A and B are equal if and only if both contain the same elements. In that case, we say that $A = B$.

Subset : Let A and B be two sets. We say that A is a subset of B, denoted by $A \subseteq B$, if and only if every element of A is also an element of B.

Proper Subset : A is a proper subset of B, only if A is a subset of B and $A \neq B$. The symbolic representation is $A \subset B$.

Union : Let A and B be two sets. Then, the union of A and B is a new set that contains elements from either set. Union of A and B is denoted by $A \cup B$.

Example : Let $A = \{1, 5, 4\}$, and $B = \{1, 7, 4, 9\}$. Then, $A \cup B = \{1, 5, 4, 7, 9\}$.

Intersection : Let A and B be two sets. Then, the intersection of A and B, denoted by $A \cap B$, is a new set that contains elements which are present in both sets.

Example : Consider the same example above, where $A = \{1, 5, 4\}$, and $B = \{1, 7, 4, 9\}$. Then, $A \cap B = \{1, 4\}$.

Difference : Let A and B be two sets. The set difference, denoted by $A - B$ or $A \setminus B$, is the set of all elements which are present in A, but not present in B.

Example : Consider the above case, where $A = \{1, 5, 4\}$, and $B = \{1, 7, 4, 9\}$. Then, $A \setminus B = \{5\}$. Also, $B \setminus A = \{7, 9\}$. So, $A \setminus B \neq B \setminus A$.

Null Set : A null set is a set with no elements. It is denoted by ϕ.

Universal Set : A universal set is a set that contains all elements. Every set we work with is a subset of the universal set. It is denoted by $\mathcal{U}$.

Power Set : Let A be a set. Then, the set of all subsets of A is called the power set of A, denoted by $\mathcal{P}(A)$. If a set has n elements, then its power set will have 2^n elements.

Example : Let $A = \{1, 3\}$. Then, $\mathcal{P}(A) = \{\phi, \{1\}, \{3\}, \{1, 3\}\}$.

11.2 Number of elements in Union

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) \\ &\qquad - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \end{aligned}$$

11.3 Example Problem

Question : In a class of 120 students numbered 1 to 120, all even numbered students opt for Physics, those whose numbers are divisible by 5 opt for Chemistry, and those whose numbers are divisible by 7 opt for Maths. How many opt for none of the three subjects?

Solution : Let the number of students who opt for none of the three subjects be x. Then,

$$\begin{aligned} x &= 120 - \left\lfloor \frac{120}{2} \right\rfloor - \left\lfloor \frac{120}{5} \right\rfloor - \left\lfloor \frac{120}{7} \right\rfloor + \left\lfloor \frac{120}{2 \times 5} \right\rfloor + \left\lfloor \frac{120}{5 \times 7} \right\rfloor + \left\lfloor \frac{120}{2 \times 7} \right\rfloor - \left\lfloor \frac{120}{2 \times 5 \times 7} \right\rfloor \\ &= 120 - 60 - 24 - 17 + 12 + 3 + 8 - 1 \\ &= 41 \end{aligned}$$

Chapter 12

Permutations and Combinations

12.1 Intro

Factorial : Factorial of n, denoted by $n!$, means

$$n! \;=\; n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

So, $3! = 6$, and $6! = 720$.

Permutations : The different arrangements of a given number of objects, by taking some or all at a time, are called permutations.

Number of Permutations : The number of permutations of n things taken r at a time, denoted by ${}^nP_r$, is given by

$${}^nP_r \;=\; n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \;=\; \frac{n!}{(n-r)!}$$

We can see from above, that ${}^nP_0 = 1$, and ${}^nP_n = n!$.

Combinations : Each of the different groups which can be formed by taking some or all objects at a time, are called combinations.

In permutation, the order of elements within a group matters, while in combination, the order of elements within a group does not matter.

Number of Combinations : The number of combinations of n things taken r at a time, denoted by ${}^nC_r$, is given by

$${}^nC_r \;=\; \frac{n!}{r! \cdot (n-r)!}$$

Hence, ${}^nC_0 = 1$, and ${}^nC_n = 1$.

12.2 Some Results on Combinatorics

$$^nP_r \;=\; r!\cdot {}^nC_r$$

$$^nC_r \;=\; {}^nC_{n-r}$$

$$^nC_r + {}^nC_{r+1} \;=\; {}^{n+1}C_{r+1}$$

$$(a+b)^n \;=\; \sum_{k=0}^{n} {}^nC_k \cdot a^k \cdot b^{n-k} \;=\; {}^nC_0 \cdot b^n + {}^nC_1 \cdot a \cdot b^{n-1} + \cdots + {}^nC_n \cdot a^n$$

$$\sum_{k=0}^{n} {}^nC_k \;=\; {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n \;=\; 2^n$$

Number of ways of distributing n identical objects among r different groups is ${}^{n+r-1}C_n$, which is the same as ${}^{n+r-1}C_{r-1}$. In this case, any of the r groups can have 0 objects.

12.3 Derangement

A derangement is a permutation of the elements of a set, such that no element appears in its original position. In other words, derangement is a permutation that has no fixed points.

The number of derangements, denoted by $!n$, is given by

$$!n \;=\; n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) \;=\; n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

12.4 Example Problem on Derangement

Question : In how many ways can you put 7 letters into their respective envelopes so that exactly 3 are addressed right?

Solution :

Number of ways in which 3 correct envelopes can be selected = ${}^7C_3 = 35$. The remaining 4 letters are deranged.

Thus, number of derangements = $!4 = 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$.

So, total number of ways = $35 \times 9 = 315$.

12.5 Circular Permutation

The order of arranging n objects in a circle is called a circular permutation.

The number of ways to arrange n objects in a circle is $(n-1)!$, if the clockwise and counter-clockwise orders **can** be distinguished.

The number of ways to arrange n objects in a circle is $\frac{(n-1)!}{2}$, if the clockwise and counter-clockwise orders **cannot** be distinguished.

12.6 Very Important Note

(i) Read the question properly. Understand the language, and what it means.

(ii) Have an idea of what you should compute and find.

(iii) Identify whether order is important or not (essentially P or C).

(iv) Identify whether repetitions are involved or not.

(v) Identify whether multiplications or additions are involved.

(vi) Then sequence the above data and solve.

Chapter 13

Probability

13.1 Intro

Experiment : An operation which can produce some well-defined outcomes is called an experiment.
Eg : Tossing a coin.

Random Experiment : An experiment in which all possible outcomes are known and the exact output cannot be predicted beforehand, is called a random experiment.
Eg : Rolling a dice, tossing a fair coin, etc.

Sample Space : When we perform an experiment, the set S of all possible outcomes is called the sample space.
Eg : In tossing a coin, $S = \{\text{H}, \text{T}\}$.

Event : Any subset E of a sample set is called an event.
Eg : In rolling a dice, $S = \{1, 2, 3, 4, 5, 6\}$. So, any $E \subseteq S$ can be an event. For example, $E = \{2, 3, 5\}$ is an event.

Probability of Occurrence of an Event : Let S be the sample space, and E be an event. So, $E \subseteq S$. Then, the probability of E occurring, denoted by $P(E)$, is

$$P(E) = \frac{n(E)}{n(S)}$$

13.2 Basic Probability Results

(i) $P(S) = 1$

(ii) $0 \leq P(E) \leq 1$

(iii) Let ϕ denote the null set. Then $P(\phi) = 0$

(iv) For any events A and B, we have :

$$P(A \cup B) \ + \ P(A \cap B) \ = \ P(A) \ + \ P(B)$$

(v) If $\overline{A}$ denotes Not-A, then $P(\overline{A}) = 1 - P(A)$

13.3 Example Problem

Question : Two cards are drawn at random from a pack of 52 cards. What is the probability that either both are black or both are queens?

Solution :
We have $n(S) = {}^{52}C_2 = \dfrac{(52 \times 51)}{(2 \times 1)} = 1326.$

Let A be the event of getting both black cards.
Let B be the event of getting both queens.
What we require is $A \cup B$, the event of getting either both black cards or both queens.
Then, $A \cap B$ is the event of getting queens of black cards.

There are ${}^{26}C_2$ ways of choosing two black cards, and ${}^{4}C_2$ ways of choosing two queens. But the number of ways of choosing two black queens is ${}^{2}C_2 = 1$.

Thus, $n(A) = {}^{26}C_2 = \dfrac{(26 \times 25)}{(2 \times 1)} = 325.$

$n(B) = {}^{4}C_2 = \dfrac{(4 \times 3)}{(2 \times 1)} = 6.$

$n(A \cap B) = {}^{2}C_2 = 1.$

W.k.t. $P(A) = \dfrac{n(A)}{n(S)}$, $P(B) = \dfrac{n(B)}{n(S)}$, and $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)}$.

We also know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence,

$$\begin{aligned} P(A \cup B) &= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \\ &= \frac{n(A) + n(B) - n(A \cap B)}{n(S)} \\ &= \frac{325 + 6 - 1}{1326} \ = \ \frac{330}{1326} \end{aligned}$$

13.4 Conditional Probability

Let A be any event with non-zero probability, and B be any event, both under the sample space S. Then, the conditional probability of B given A, denoted by $P(B \mid A)$, is the probability that B will occur when it is already known that A has occurred.

By definition,

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

We can also infer from the above that

$$P(A \cap B) \;=\; P(B \mid A) \cdot P(A) \;=\; P(A \mid B) \cdot P(B)$$

13.4.1 Independent Events

We can say that two events A and B are independent, if the occurrence of one event does not affect the probability of occurrence of the other event. Mathematically, this means that $P(B \mid A) = P(B)$, and $P(A \mid B) = P(A)$.

From the conditional probability formula, two non-zero events A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

13.4.2 Mutually Exclusive Events

Two events A and B are mutually exclusive if their intersection yields a null probability (i.e.) $P(A \cap B) = 0$.

13.5 Bayes Theorem

13.5.1 Partition of Sample Set

In the sample set S, let $E_1, E_2, E_3, \ldots, E_n$, be n pair-wise mutually exclusive events. That is, $P(E_i \cap E_j) = 0$ whenever $i \neq j$. Also, the events E_i are all non-zero probability events (i.e.) $P(E_i) > 0$.

We say that the set of E_i is **exhaustive**, if and only if

$$\bigcup_{i=1}^{n} E_i \;=\; E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n \;=\; S$$

13.5.2 Partition of Event

Let $E_1, E_2, \ldots, E_n$ be n non-zero mutually exclusive, and overall exhaustive events. Let A be an event in the sample space S. Then,

$$P(A) \;=\; \sum_{i=1}^{n} P(A \cap E_i) \;=\; P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_n)$$

But, we already know that $P(A \cap E_i) = P(A \mid E_i) \cdot P(E_i)$. Hence, we can re-write the above equation as

$$P(A) \;=\; \sum_{i=1}^{n} P(A \mid E_i) \cdot P(E_i)$$

13.5.3 Bayes Theorem

We have already seen before on conditional probability, and on how to partition an event. We can combine those two here.

Let $E_1, E_2, E_3, \ldots, E_n$ be non-zero, mutually exclusive, exhaustive events in the sample space. Let A be an event. We have seen from before that

$$P(E_k \mid A) \cdot P(A) \;=\; P(A \mid E_k) \cdot P(E_k)$$

Hence, for any event E_k,

$$P(E_k \mid A) \;=\; \frac{P(A \mid E_k) \cdot P(E_k)}{P(A)}$$

Replacing $P(A)$ with its partition formula, we get

$$P(E_k \mid A) \;=\; \frac{P(A \mid E_k) \cdot P(E_k)}{\sum\limits_{i=1}^{n} P(A \mid E_i) \cdot P(E_i)}$$

The above equation is the Bayes Theorem.

13.6 Question on Bayes Theorem

Question : A bag contains 7 red, 6 blue balls and another bag contains 6 red and 9 blue balls. A ball is drawn from the first bag and without knowing the colour is put in the second bag. A ball is drawn from the second bag. Find the probability that ball drawn is blue in colour.

Solution :
Let A be the event that the ball drawn from second bag is blue in colour.

Consider the following :
E_1 = Event that the ball drawn from first bag is red.
E_2 = Event that the ball drawn from first bag is blue.

Now, E_1 and E_2 are mutually exclusive. Also, they are exhaustive, because you cannot take out any other coloured ball from the first bag. And so, from the data,

$$P(E_1) = \frac{7}{13} \qquad\qquad P(E_2) = \frac{6}{13}$$

If event E_1 occurs, the second bag will contain 7 red and 9 blue balls (because we are adding a red ball). So, the probability of selecting a blue ball when E_1 is given, is

$$P(A \mid E_1) = \frac{9}{9+7} = \frac{9}{16}$$

Similarly, if E_2 occurs, the second bag will contain 6 red balls and 10 blue balls. In that case,

$$P(A \mid E_2) = \frac{10}{10+6} = \frac{10}{16}$$

Hence, by the Partition of Events formula, we get

$$\begin{aligned} P(A) &= P(A \mid E_1) \cdot P(E_1) \;+\; P(A \mid E_2) \cdot P(E_2) \\ &= \frac{9}{16} \times \frac{7}{13} \;+\; \frac{10}{16} \times \frac{6}{13} \\ &= \frac{63+60}{208} \;=\; \frac{123}{208} \end{aligned}$$

Chapter 14

Number Theory

14.1 Intro

As far as number theory is concerned, the preparation for aptitude is summed in just three words : divisibility, remainder, and cyclicity.

14.2 Divisibility

14.2.1 Euclid's Division Lemma

For any two integers a and b, $b > 0$, we can find unique integers q and r such that $a = b \cdot q + r$, where $0 \leq r < b$.

This can be used to make variable generalisation easier.

14.2.2 Fundamental Theorem of Arithmetic

Every integer greater than 1 is either a prime number itself, or can be represented as a product of primes. Moreover, this product of primes is unique, notwithstanding the order. For example, $60 = 2^2 \cdot 3 \cdot 5$.

More generally, $n = \prod_{i=1} p_i^{a_i}$, where n is the number factored, p_i is the i'th prime, and a_i its power.

14.2.3 Number of Factors

If $n = \prod_{i=1} p_i^{a_i}$, then its total number of factors, η, is $\eta = \prod_{i=1}(a_i + 1)$.

For example, since $60 = 2^2 \cdot 3 \cdot 5$, the number of factors is $\eta = (2 + 1) \cdot (1 + 1) \cdot (1 + 1) = 12$.

14.2.4 Sum of Factors

If $n = \prod_{i=1} p_i^{a_i}$, then the total sum of its factors, σ, is $\sigma = \prod_{i=1} \frac{p_i^{a_i+1} - 1}{p_i - 1}$.
For example, since $60 = 2^2 \cdot 3 \cdot 5$, the sum of factors is

$$\sigma = \frac{2^{2+1} - 1}{2 - 1} \cdot \frac{3^{1+1} - 1}{3 - 1} \cdot \frac{5^{1+1} - 1}{5 - 1} = \frac{7}{1} \cdot \frac{8}{2} \cdot \frac{24}{4} = 168.$$

14.2.5 Product of Factors

Product of factors of n is equal to $n^{\eta/2}$, where η denotes the number of factors of n.

14.3 Remainder

14.3.1 Modularity

Let a and b be two integers. Then, $a \mid b$ means that a divides b. Similarly, $a \nmid b$ means that a does not divide b.

Now, let n be a non-zero integer. Then, $a \equiv b \pmod{n}$ means that $(a-b)$ is divisible by n.

Some obvious properties include

(i) $a \equiv a \pmod{n}$.

(ii) If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.

(iii) If $a \equiv b \pmod{n}$, and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

14.3.2 Addition and Multiplication Modulo

The additive and multiplicative rule of modularity states that, for $n \neq 0$, suppose $a \equiv b \pmod{n}$, and $c \equiv d \pmod{n}$. Then

(i) $a + c \equiv b + d \pmod{n}$.

(ii) $a \cdot c \equiv b \cdot d \pmod{n}$.

(iii) $a^k \equiv b^k \pmod{n}$, for any positive integer k.

For example, $20 \equiv 8 \pmod{3}$, and $7 \equiv 10 \pmod{3}$.

We can see that $20 + 7 = 27 \equiv 8 + 10 = 18 \pmod{3}$, since $27 - 18 = 9$ is divisible by 3.

Similarly, $20 \cdot 7 = 140$, and $8 \cdot 10 = 80$. We can check that $140 \equiv 80 \pmod{3}$.

14.3.3 Euler's Totient Function

Let n be any positive integer. Then, the Euler's Totient Function, $\varphi(n)$, is the number of positive integers smaller than n that is co-prime to n. For example, let $n = 4$. We can see that 1 and 3 are the two numbers smaller than 4 that is co-prime to 4. Hence, $\varphi(4) = 2$.

Mathematically, suppose $n = \prod_{i=1} p_i^{a_i}$. Then $\varphi(n)$ is given by

$$\varphi(n) = n \cdot \prod_{i=1} \left(1 - \frac{1}{p_i}\right)$$

If two numbers m and n are co-prime (i.e.) $\gcd(m, n) = 1$, then

$$\varphi(mn) = \varphi(m) \cdot \varphi(n)$$

As an example, consider $n = 100 = 2^2 \cdot 5^2$. Now,

$$\varphi(100) = 100 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{5}\right) = 100 \cdot \frac{1}{2} \cdot \frac{4}{5} = 40$$

We can also verify this, by seeing that $\varphi(4) = 2$, and $\varphi(25) = 20$, and $\varphi(100) = \varphi(4) \cdot \varphi(25)$, since $\gcd(4, 25) = 1$.

14.3.4 Euler's Theorem

The main use of the Euler's Totient Function is in the Euler's Theorem. Now that you know basic modular arithmetic, expressing the theorem is really simple.

The theorem states that, for any pair of co-prime positive integers a and n (i.e.) $\gcd(a, n) = 1$,

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

Let's unpack the theorem with a couple of examples, shall we?

Okay, consider $a = 2$ and $n = 9$. Since $\gcd(2, 9) = 1$, we can use Euler's Theorem. From before, $\varphi(9) = 9 \cdot \left(1 - \frac{1}{3}\right) = 6$.

Next, we compute $a^{\varphi(n)} = 2^6 = 64$. Why is this number significant? Because $64 - 1 = 63$ is divisible by 9 (Since $63 = 9 \times 7$). Equivalently, we can say that $2^6 = 64 \equiv 1 \pmod{9}$.

Another example would be $a = 7$ and $n = 10$. We can see that $\varphi(10) = 4$, and so $7^4 = 2401 = 2400 + 1$. Hence, $7^4 \equiv 1 \pmod{10}$.

14.3.5 Applications of Euler's Theorem

Now, this might all seem superficial and useless. After all, what is an obscure mathematical formula of any use in aptitude? Well, as it turns out, a lot.

Let's rewind a bit :-

- We have seen that, if $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$, for any positive integer k.
- We also know that, for any positive pair of integers a and n with $\gcd(a, n) = 1$, $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Combining the above two, we can say that, if $\gcd(a, n) = 1$, then, for any positive k,

$$a^{k\varphi(n)} \equiv 1 \pmod{n}$$

This can be best illustrated with an example : Find the last two digits of 41^{81}. It looks very difficult on paper, but with the above theorem, it's a piece of cake.

First, we can see that finding the last two digits essentially means finding the remainder when the number is divided by 100. That means our $a = 41^{81}$ and $n = 100$.

We can compute $\varphi(n)$ using formula, and get $\varphi(100) = 40$. Also, $\gcd(41^{81}, 100) = 1$, so we're all clear to use the theorem.

Using the theorem, we can see that $41^{40} \equiv 1 \pmod{100}$. We can square both sides to get this : $(41^{40})^2 = 41^{80} \equiv (1)^2 = 1 \pmod{100}$. Now what?

It's almost done. Trivially, we can see that $41 \equiv 41 \pmod{100}$. We also have the previous result : $41^{80} \equiv 1 \pmod{100}$. Why not multiply the two?

When we do the multiplication, we get $41^{80} \cdot 41 = 41^{81} \equiv 41 \pmod{100}$. What this means is that $41^{81} - 41$ is divisible by 100. Or, in other words, 41 is the remainder when 41^{81} is divided by 100. Which is what we want, by the way. Which leads us to the next (and last) topic : Cyclicity.

14.4 Cyclicity

We can see from the previous example that numbers of the form $a^{k\varphi(n)}$ are cyclic : They all leave a remainder of 1 when divided by n. Indeed, we can say that $\varphi(n)$ is the period (corresponding to the cycle) of n.

So, for example, the period of 10 is $\varphi(10) = 4$. We can indeed verify this by checking on the powers of 2 :

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{2^n}$	2	4	8	16	32	64	128	256	512	1024	2048	4096

The last digit of each number reads, respectively : $2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6$. Each digit repeats after a period of 4, which confirms the periodicity of 10.

14.5 Finale

As a culmination of the concepts discussed here, let's do one last problem : Find the last two digits of 2^{155}.

In other words, find the remainder when $a = 2^{155}$ is divided by $n = 100$. We can see that $\varphi(100) = 40$, but 2 and 100 are not co-prime to each other. So what do we do now?

The key, here, is to decompose 100 into its prime factors. We know that $100 = 2^2 \cdot 5^2$. So, we can split 100 into 4 and 25, and find the remainder when each divides 2^{155}.

Straightaway we can see that 4 divides 2^{155}. Hence $2^{155} \equiv 0 \pmod 4$.

Next, we can apply the totient formula to get $\varphi(25) = 20$, which is the cyclicity of 25. And so, we can conclude that $(2^{20})^7 = 2^{140} \equiv 1 \pmod{25}$. So we only need to find the remainder when 2^{15} is divided by 25. How?

Well, here's where the modular arithmetic comes really handy. We know that $32 = 2^5 \equiv 7 \pmod{25}$. And so, we square both sides now, to get $2^{10} \equiv 49 \pmod{25}$. But, $49 = 50 - 1$, and 50 is divisible by 25. Hence we can rewrite the above as $2^{10} \equiv -1 \pmod{25}$.

Since $2^5 \equiv 7 \pmod{25}$, and $2^{10} \equiv -1 \pmod{25}$, we can multiply the two. Then, $2^{15} \equiv -7 \equiv -7 + 25 \equiv 18 \pmod{25}$. Hence $2^{155} \equiv 18 \pmod{25}$.

So, to recap the work done so far, $2^{155} \equiv 0 \pmod 4$, and $2^{155} \equiv 18 \pmod{25}$. Since we are finding the last two digits, the remainder should be less than 100.

The only numbers less than 100 which leaves a remainder of 18 when divided by 25, are $18, 43, 68, 93$. Out of these, only 68 is divisible by 4. So, only 68 satisfies the two conditions given above.

Hence, $2^{155} \equiv 68 \pmod{100}$, and so the last two digits of 2^{155} are 68.

Appendix A

Divisibility Tests

A.1 Divisibility by 2

A number is divisible by 2 if its unit's digit is divisible by 2 (i.e.) the last digit is 0, 2, 4, 6 or 8.

Example : 1234 is divisible by 2, while 123 is not.

A.2 Divisibility by 3

A number is divisible by 3 if the sum of its digits is divisible by 3.

Example : 192 is divisible by 3, because $1 + 9 + 2 = 12$, and 12 is divisible by 3 because $1 + 2 = 3$, which is divisible by 3.

A.3 Divisibility by 4

A number is divisible by 4 if the number formed by its last two digits is divisible by 4.

Example : Take 12396. The number formed by the last two digits is 96. Since 96 is divisible by 4, we can say that 12396 is also divisible by 4.

A.4 Divisibility by 5

A number is divisible by 5 if its last digit is divisible by 5 (i.e.) the unit's digit is either 0 or 5.

Example : 105 is divisible by 5, while 101 is not.

A.5 Divisibility by 6

A number is divisible by 6 if it is divisible by both 2 and 3. We can apply the aforementioned techniques to check this.

Example : 102 is divisible by 2, and it is also divisible by 3 (since $1+2=3$), hence it is divisible by 6.

A.6 Divisibility by 7

To find if a number is divisible by 7, separate the number into two parts, one with the units digit, and the other being the rest of the number. Subtract the double of this unit digit from the other part. If the resultant number is divisible by 7, then the original number is divisible by 7.

Example :
Consider the number 12348.
Separate it into two parts (i.e.) 1234 and 8.
Compute $(1234 - 2 \times 8)$, which equals to 1218.
Now again, separate the number into 121 and 8, and apply the rule. $121 - 2 \times 8 = 105$.
Again, $10 - 2 \times 5 = 0$.
Now, 0 is divisible by 7, and hence 12348 is also divisible by 7.

Another random example would be 75454.
We can separate them into 7545 and 4, and so $7545 - 2 \times 4 = 7537$.
Again, separate this new number into 753 and 7, and do $753-2\times 7 = 739$.
Separate them again into 73 and 9, and compute $73 - 2 \times 9 = 55$.
Now, separate them into 5 and 5, and hence $5 - 2 \times 5 = -5$.
Since -5 is not divisible by 7, 75454 is also not divisible by 7.

A.7 Divisibility by 8

A number is divisible by 8 if the number formed by its last three digits is divisible by 8.
Example : The number 123456 has the last three digits as 456. Since 456 is divisible by 8 (you have to do this by trial and error), we can safely say that the original number 123456 is also divisible by 8.

A.8 Divisibility by 9

A number is divisible by 9 if the sum of its digits is divisible by 9.
Example : The number 123 is not divisible by 9, because $1+2+3=6$ is not divisible by 9.
The number 9846 is divisible by 9, because $9+8+4+6=27$, and 27 is divisible by 9, because, again, $2+7=9$.

A.9 Divisibility by 10

A number is divisible by 10 if its last digit is 0.
Example : 12340 is divisible by 10, while 1234 is not.

A.10 Divisibility by 11

In a number, consider the sum of its digits at odd places, and the sum of its digits at even places. That is, sum the alternating digits. If the difference between these two sums is divisible by 11, then the number is divisible by 11.

Example : Consider 15631. The sum of the odd-placed digits is $1+6+1=8$, while the sum of even-placed digits is $5+3=8$. Thus, the difference between the sums $= 8-8=0$, and hence 15631 is divisible by 11.

Also, 1234 is not divisible by 11, because $1+3=4$, and $2+4=6$, and the difference between the sums $= 4-6=-2$, which is not divisible by 11.

A.11 Divisibility by 12

A number is divisible by 12 if it is divisible by both 3 and 4.

A.12 Divisibility by 14

A number is divisible by 14 if it is divisible by both 2 and 7.

A.13 Divisibility by 15

A number is divisible by 15 if it is divisible by both 3 and 5.

A.14 Divisibility by 16

A number is divisible by 16 if the number formed by its last 4 digits is divisible by 16.

Example : The number 123456 is divisible by 16, because the number formed by the last 4 digits (i.e.) 3456 is divisible by 16.

A.15 Divisibility by 18

A number is divisible by 18 if it is divisible by both 2 and 9.

A.16 Divisibility by 20

A number is divisible by 20 if it is divisible by both 4 and 5.

A.17 Divisibility by 25

A number is divisible by 25 if the number formed by its last 2 digits is divisible by 25 (i.e.) the number formed by its last two digits is any one of $\{00, 25, 50, 75\}$.

Example : 15675 is divisible by 25, because the last two digits form 75, which is divisible by 25.

A.18 Divisibility by 125

A number is divisible by 125, if the number formed by its last 3 digits is divisible by 125.

Example : 15625 is divisible by 125, because the last three digits form 625, which is divisible by 125.

A.19 Divisibility by any Composite Number

Suppose we are checking for divisibility by a composite number N. Let $N = L \times M$, where L and M are co-prime (i.e.) $\gcd(L, M) = 1$. Then, to check for divisibility by N, we need to only check for divisibility by both L and M.

If a number is divisible by both L and M, where $\gcd(L, M) = 1$, then the number is divisible by $L \times M = N$.

As we have seen before, if we need to check for divisibility by 12, we only need to check for divisibility by both 3 and 4. Also, to check for divisibility by 80, we should check for divisibility by both 5 and 16.