

# Graduate school project report

## Halo formation in the cosmic web

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### Abstract

abstract

## 1 Introduction

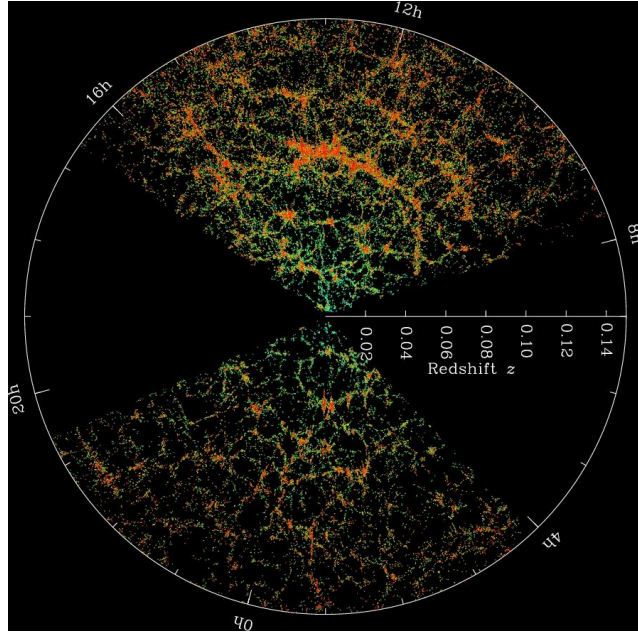
Cosmic Microwave Background (CMB) shows that the Universe was initially homogeneous with very small inhomogeneities. Thanks to the attractive gravitational force, those inhomogeneities led to the formation of galaxies. And the universe we see today has a lot of interesting structures well beyond the galactic scale. This foam-like large scale structure of the universe is called the cosmic web.

Understanding the statistical properties of the large scale structure and its evolution is crucial to test and constrain cosmological models. Computer simulations can be used to evolve initial inhomogeneities to the structure we see today and hence can be compared with sky survey observations.

## 2 Analytical tools

Though the evolution of large scale structure can be simulated, we need analytical tools to get a deeper understanding and also to make generic constraints that

can be tested by observations. Figure 1: *Large Scale Structure revealed by [SDSS]* On the other hand, simulations help in making and refining these analytical tools.



## 2.1 FLRW background

We assume the standard  $\Lambda$ CMD model without any curvature. The FLRW metric for the homogeneous background in comoving coordinates is

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \quad (1)$$

$$= a^2(\tau) (-d\tau^2 + d\vec{x}^2) \quad (2)$$

where  $\tau$  is defined as the conformal time.

Let  $\bar{\rho}_m$  and  $\bar{\rho}_\Lambda$  denote the mean matter density and mean dark energy density. Hubble parameter is defined as  $H \equiv \dot{a}/a$  where the dot denotes derivative with respect to time 't'. At the zeroth order, the Einstein equations reduces to the Friedmann equations,

$$\begin{aligned} H^2 &= \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \bar{\rho} \\ &= \frac{8\pi G}{3} (\bar{\rho}_m + \bar{\rho}_\Lambda) \end{aligned} \quad (3)$$

$$\begin{aligned} \dot{H} + H^2 &= \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) \\ &= -\frac{4\pi G}{3} [\bar{\rho}_m + \bar{\rho}_\Lambda + 3(0 - \bar{\rho}_\Lambda)] \\ &= -\frac{4\pi G}{3} [\bar{\rho}_m - 2\bar{\rho}_\Lambda] \end{aligned} \quad (4)$$

Assuming that the matter and dark energy are independently conserved,

$$\dot{\bar{\rho}}_m = -3H (\bar{\rho}_m + \bar{p}_m) = -3H \bar{\rho}_m \quad (5)$$

$$\dot{\bar{\rho}}_\Lambda = -3H (\bar{\rho}_\Lambda + \bar{p}_\Lambda) = 0 \quad (6)$$

Solving the above equations give  $\bar{\rho}_m \propto a^{-3}$  and  $\bar{\rho}_\Lambda$  is constant. Density parameters are defined as

$$\Omega_m \equiv \frac{8\pi G \bar{\rho}_m}{3H^2} \quad \Omega_\Lambda \equiv \frac{8\pi G \bar{\rho}_\Lambda}{3H^2} \quad (7)$$

so that the first Friedmann equation 3 reduces to  $\Omega_m + \Omega_\Lambda = 1$ . Let us now switch to conformal time ' $\tau$ ' and define conformal Hubble parameter  $\mathcal{H} \equiv \partial_\tau a/a = \dot{a}$ .

$$3H^2 \Omega_m = 8\pi G \bar{\rho}_m \quad (8)$$

$$3\mathcal{H}^2 \Omega_m = 8\pi G \bar{\rho}_m a^2 \quad (9)$$

The second Friedmann equation (4) becomes

$$-\frac{4\pi G}{3} [\bar{\rho}_m - 2\bar{\rho}_\Lambda] = \frac{\ddot{a}}{a} = \frac{\dot{\mathcal{H}}}{a} \quad (10)$$

$$-\frac{4\pi G}{3} [\bar{\rho}_m - 2\bar{\rho}_\Lambda] a^2 = \frac{d\mathcal{H}}{a^2 d\tau} a^2 \quad (11)$$

$$\mathcal{H}^2 \left[ -\frac{\Omega_m}{2} + \Omega_\Lambda \right] = \frac{d\mathcal{H}}{d\tau} \quad (12)$$

$$\Omega_\Lambda - \frac{\Omega_m}{2} = \frac{d\mathcal{H}}{d\tau} \mathcal{H}^{-2} = \frac{d\mathcal{H}}{\mathcal{H} d\tau} \mathcal{H}^{-1} \quad (13)$$

$$= \frac{d \ln \mathcal{H}}{d\tau} \left[ \frac{da}{a d\tau} \right]^{-1} = \frac{d \ln \mathcal{H}}{d \ln a} \quad (14)$$

It is useful to define another time evolution parameter  $y \equiv \ln a$ .

$$\Omega_\Lambda - \frac{\Omega_m}{2} = \frac{d \ln \mathcal{H}}{dy} \quad (15)$$

## 2.2 Growth of Structure

While the evolution of background cosmology can be studied fully analytically, the inhomogeneities responsible for structure formation can't be solved exactly without making any ansatz. We will look into different approaches but first let us setup the equations. Since a large fraction of the matter in the Universe is dark matter, gravity plays the major role in structure formation. While we can directly apply the Einstein equations of general relativity, we take a simpler approach by starting with the Newtonian fluid equations.

### 2.2.1 Newtonian equations in inhomogeneous $\Lambda$ CDM Universe

The matter density contrast can be quantified in terms of overdensity field  $\delta$ ,

$$\delta(\vec{x}, \tau) \equiv \frac{\rho_m(\vec{x}, \tau) - \bar{\rho}_m(\tau)}{\bar{\rho}_m(\tau)} = \frac{\rho_m(\vec{x}, \tau)}{\bar{\rho}_m(\tau)} - 1 \quad (16)$$

The velocity field is then defined as

$$\vec{v}(\vec{x}, \tau) \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(a\vec{x}) \quad (17)$$

$$= \frac{1}{a} \frac{d}{d\tau}(a\vec{x}) \quad (18)$$

$$= \frac{da/d\tau}{a} \vec{x} + \frac{d\vec{x}}{d\tau} \quad (19)$$

$$= \mathcal{H}(\tau) \vec{x} + \vec{u}(\vec{x}, \tau) \quad (20)$$

where  $\vec{u}(\vec{x}, \tau) \equiv d\vec{x}/d\tau$  is called the peculiar velocity, while the first term quantifies the Hubble flow. Due to the time dependance of  $\mathcal{H}$ , there is an associated acceleration purely due to hubble flow. That acceleration can be found by setting peculiar velocity to zero.

$$\frac{d}{dt}(\mathcal{H}(\tau) \vec{x}) = \frac{d}{dt} \frac{da}{dt} \vec{x} \quad (21)$$

$$= \ddot{a}(t) \vec{x} = \frac{1}{a} \ddot{a}(t) \vec{r} \quad (22)$$

This acceleration can be written in terms of a potential

$$\bar{\phi} \equiv -\frac{1}{2} a \ddot{a} |\vec{x}|^2 = -\frac{1}{2} \frac{\ddot{a}}{a} |\vec{r}|^2 \quad (23)$$

$$\implies \nabla_r \bar{\phi} = \frac{\ddot{a}}{a} \vec{r} \quad (24)$$

Let  $\phi$  denote the total gravitational potential in the presence of inhomogeneities. We can define the modified gravitaional potential as  $\Phi \equiv \phi - \bar{\phi}$ .

Evolution of the density inhomogeneity  $\delta$ , the peculiar velocity field  $\vec{u}$ , and the modified potential  $\Phi$  is described by the continuity equation, Euler equation and Poisson equation.

$$\partial_\tau \delta + \nabla \cdot [(1 + \delta) \vec{u}] = 0 \quad (25)$$

$$\partial_\tau \vec{u} + \mathcal{H} \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla \Phi \quad (26)$$

$$\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta \quad (27)$$

where  $\nabla$  is with respect to comoving coordinates.

### 2.2.2 Linear solutions to inhomogeneous CDM

If the inhomogeneities are small then we can consider them as perturbation to the homogeneous background. To the first order of perturbation, the equations are linear.

$$\partial_\tau \delta + \nabla \cdot \vec{u} = 0 \quad (28)$$

$$\partial_\tau \vec{u} + \mathcal{H} \vec{u} = -\nabla \Phi \quad (29)$$

$$\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta \quad (30)$$

Taking divergence of (29)

$$\partial_\tau \nabla \cdot \vec{u} + \mathcal{H} \nabla \cdot \vec{u} = -\nabla^2 \Phi \quad (31)$$

substituting for  $\nabla \cdot \vec{u}$  from (28) and for  $\nabla^2 \Phi$  from (30)

$$\partial_\tau (-\partial_\tau \delta) + \mathcal{H} (-\partial_\tau \delta) = -\frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta \quad (32)$$

$$\partial_\tau^2 \delta + \mathcal{H} \partial_\tau \delta = \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta \quad (33)$$

In the last equation, there is no spatial derivatives, so it is just a second order ordinary differential equation for  $\delta$ . In general, the solution is of the form,

$$\delta(\vec{x}, \tau) = A(\vec{x})D_1^{(+)}(\tau) + B(\vec{x})D_1^{(-)}(\tau) \quad (34)$$

where  $D_1^{(+)}(\tau)$  denotes growing mode and  $D_1^{(-)}(\tau)$  denotes decaying mode. The divergence of peculiar velocity is then

$$\theta(\vec{x}, \tau) \equiv \nabla \cdot \vec{u} = -\partial_\tau \delta \quad (35)$$

$$= -A(\vec{x})\frac{dD_1^{(+)}}{d\tau} - B(\vec{x})\frac{dD_1^{(-)}}{d\tau} \quad (36)$$

The growing mode and decaying mode can be obtained by directly solving the linear ODE (33). Before going into the full solution, let us look into the matter dominated era where  $\Omega_m(\tau) \approx 1$ . Inserting this in the background equation (9) we get,

$$\mathcal{H} \propto \sqrt{\bar{\rho}_m a^2} \propto a^{-1/2} \quad (37)$$

$$\implies \frac{da}{d\tau} \propto a^{1/2} \implies a^{-1/2} da \propto d\tau \quad (38)$$

We can solve for scale factor evolution by integrating the above equation.

$$a(\tau) \propto \tau^2 \implies \mathcal{H}(\tau) = \frac{2}{\tau} \quad (39)$$

$$H(\tau) = \frac{\dot{a}}{a} = \frac{\mathcal{H}}{a} \propto \tau^{-3} \quad (40)$$

Hence the linear perturbation equation (33) reduces to,

$$\partial_\tau^2 \delta + \mathcal{H} \partial_\tau \delta = \frac{3}{2} \mathcal{H}^2 \delta \quad (41)$$

$$\partial_\tau^2 \delta + \frac{2}{\tau} \partial_\tau \delta = \frac{3}{2} \left( \frac{2}{\tau} \right)^2 \delta \quad (42)$$

$$\partial_\tau^2 \delta + \frac{2}{\tau} \partial_\tau \delta - \frac{6}{\tau^2} \delta = 0 \quad (43)$$

By assuming power law solution, we get the growing mode  $D_1^{(+)}(\tau) \propto \tau^2 \propto a(\tau)$  and the decaying mode  $D_1^{(-)}(\tau) \propto \tau^{-3} \propto H(\tau)$ .

It turns out that even after dark energy starts to dominate, the decaying mode stays proportional to the Hubble parameter  $D_1^{(-)}(\tau) \propto H(\tau)$ . However, now the time dependence of Hubble parameter  $H(\tau)$  is different.

Similarly, the growing mode can be generalised to an integral involving  $H(\tau)$ ,

$$D_1^{(+)}(\tau) \propto H(\tau) \int^\tau \frac{1}{a(\tau') H^2(\tau')} d\tau' \quad (44)$$

$$\implies D_1^{(+)}(a) \propto H(a) \int^a \frac{1}{(a' H(a'))^3} da' \quad (45)$$

We can set the proportionality constant by setting  $D_1^{(+)}(a) = a$  during matter dominated phase.

$$D_1^{(+)}(a) = \frac{5}{2} \Omega_{m0} \frac{H(a)}{H_0} \int^a \frac{1}{(a' H(a')/H_0)^3} da' \quad (46)$$

### 2.2.3 Eulerian perturbation theory

We will switch to Fourier space with respect to spatial comoving coordinates. The density inhomogeneity parameter in the Fourier space is defined as

$$\delta_{\vec{k}}(\tau) \equiv \int d^3k e^{-i\vec{k} \cdot \vec{x}} \delta(\vec{x}, \tau), \quad \delta(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \delta_{\vec{k}}(\tau) \quad (47)$$

Similarly we can define Fourier space variables  $\theta_{\vec{k}}(\tau)$ ,  $\vec{u}_{\vec{k}}(\tau)$  &  $\Phi_{\vec{k}}(\tau)$  corresponding to  $\theta(\vec{x}, \tau)$ ,  $\vec{u}(\vec{x}, \tau)$  &  $\Phi(\vec{x}, \tau)$ .

The linear partial differential equations (28), (29) & (30) become linear ordinary differential equations in Fourier space and hence each wavemode evolves independently.

$$\partial_\tau \delta_{\vec{k}} + \theta_{\vec{k}} = 0 \quad (48)$$

$$\partial_\tau \theta_{\vec{k}} + \mathcal{H} \theta_{\vec{k}} - k^2 \Phi_{\vec{k}} = 0 \quad (49)$$

$$-k^2 \Phi_{\vec{k}} = \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta_{\vec{k}} \quad (50)$$

However when we include non-linear terms, Fourier wavemodes are coupled. Let us assume that there is no vorticity,  $\nabla \times \vec{u} = 0$  so that the divergence-less term is zero. Fourier transforming the continuity equation, we get

$$\partial_\tau \delta_{\vec{k}} + \theta_{\vec{k}} = - \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \frac{\vec{k}_1 \cdot (\vec{k}_1 + \vec{k}_2)}{k_1^2} \delta_{\vec{k}_2} \theta_{\vec{k}_1} \delta_D(\vec{k} - [\vec{k}_1 + \vec{k}_2]) \quad (51)$$

Fourier transforming the divergence of Euler equation, we get

$$\partial_\tau \theta_{\vec{k}} + \mathcal{H} \theta_{\vec{k}} - k^2 \Phi_{\vec{k}} = - \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \frac{(\vec{k}_1 \cdot \vec{k}_2) |\vec{k}_1 + \vec{k}_2|^2}{2k_1^2 k_2^2} \theta_{\vec{k}_2} \theta_{\vec{k}_1} \delta_D(\vec{k} - [\vec{k}_1 + \vec{k}_2]) \quad (52)$$

Since the poisson equation is linear in position space, there is no coupling terms in the Fourier space.

$$-k^2 \Phi_{\vec{k}} = \frac{3}{2} \mathcal{H}^2 \Omega_m(\tau) \delta_{\vec{k}} \quad (53)$$

### Perturbative expansion in the matter dominated era

We have seen that to the linear order,  $\delta$  grows as the scale factor 'a' in the matter dominated era. And then using continuity equation,  $\theta$  evolves as  $da/d\tau = \mathcal{H}a$ . The full solution can be expanded as a series.

$$\delta_{\vec{k}}(a) = \sum_{n=1}^{\infty} a^n \delta_{\vec{k}}^{(n)}, \quad \theta_{\vec{k}}(a) = \mathcal{H} \sum_{n=1}^{\infty} a^n \theta_{\vec{k}}^{(n)} \quad (54)$$

To the second order

$$\delta_{\vec{k}}(a) = a\delta_{\vec{k}}^{(1)} + a^2\delta_{\vec{k}}^{(2)}, \quad \theta_{\vec{k}}(a) = a\theta_{\vec{k}}^{(1)} + a^2\theta_{\vec{k}}^{(2)} \quad (55)$$

Inserting this into the equations (51) and (52), we get

$$\delta_{\vec{k}}^{(2)} = - \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} F^{(2)}(\vec{k}_1, \vec{k}_2) \delta_{\vec{k}_2} \delta_{\vec{k}_1} \delta_D \left( \vec{k} - [\vec{k}_1 + \vec{k}_2] \right) \quad (56)$$

$$\theta_{\vec{k}}^{(2)} = - \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} G^{(2)}(\vec{k}_1, \vec{k}_2) \delta_{\vec{k}_2} \delta_{\vec{k}_1} \delta_D \left( \vec{k} - [\vec{k}_1 + \vec{k}_2] \right) \quad (57)$$

where

$$F^{(2)}(\vec{k}_1, \vec{k}_2) = \frac{5}{7} + \frac{\vec{k}_1 \cdot \vec{k}_2}{2k_1 k_2} \left[ \frac{k_1}{k_2} + \frac{k_2}{k_1} \right] + \frac{2}{7} \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \quad (58)$$

$$G^{(2)}(\vec{k}_1, \vec{k}_2) = \frac{3}{7} + \frac{\vec{k}_1 \cdot \vec{k}_2}{2k_1 k_2} \left[ \frac{k_1}{k_2} + \frac{k_2}{k_1} \right] + \frac{4}{7} \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \quad (59)$$

Higher order corrections can be obtained similarly. This perturbative approach works as long the inhomogeneity is small ( $\delta \ll 1$ ).

#### 2.2.4 Lagrangian approach - Zel'dovich approximation

In Lagrangian approach, coordinates that move along the fluid is used instead of the usual position coordinates. We can define Fluid coordinates  $\vec{q} = (q_1, q_2, q_3)$  to be the background comoving coordinates at an initial time, that is  $\vec{q} = \vec{x}(\vec{q}, \tau_i)$ . At some later time, the displacement field is defined as displacement vector of fluid element.

$$\vec{\Psi}(\vec{q}, \tau) \equiv \vec{x}(\vec{q}, \tau) - \vec{x}(\vec{q}, \tau_i) \quad (60)$$

so that the mapping is,

$$\vec{x}(\vec{q}, \tau) = \vec{q} + \vec{\Psi}(\vec{q}, \tau) \quad (61)$$

$$x_i(\vec{q}, \tau) = q_i + \Psi_i(\vec{q}, \tau) \quad (62)$$

The Jacobian matrix of this transformation is

$$\mathbf{J}_{ij} \equiv \frac{\partial x_i}{\partial q_j} = \delta_{ij} + \Psi_{i,j} \quad (63)$$

The Jacobian is  $J = \det(\mathbf{J})$

### 2.2.5 Spherical collapse

## 2.3 Statistical properties

### 2.3.1 Correlation function power spectrum relation

$$P(\vec{k}) = \int \xi(\vec{r}) e^{i\vec{k}\vec{r}} d^3r \quad (64)$$

$$\xi(\vec{r}) = \frac{1}{(2\pi)^3} \int P(\vec{k}) e^{-i\vec{k}\vec{r}} d^3k \quad (65)$$

$$\xi(r) = \frac{4\pi}{(2\pi)^3} \int_0^\infty P(k) k^2 \frac{\sin(kr)}{kr} dk \quad (66)$$

$$\xi(r) = \frac{1}{2\pi^2} \int_{-\infty}^\infty P(k) k^3 \frac{\sin(kr)}{kr} d(\ln k) \quad (67)$$

$$\xi(r) = \int_{-\infty}^\infty \Delta^2(k) \frac{\sin(kr)}{kr} d(\ln k) \quad (68)$$

## 3 N-body simulations

## 4 Analysing a snapshot of a GADGET-2 simulation

[simulation]

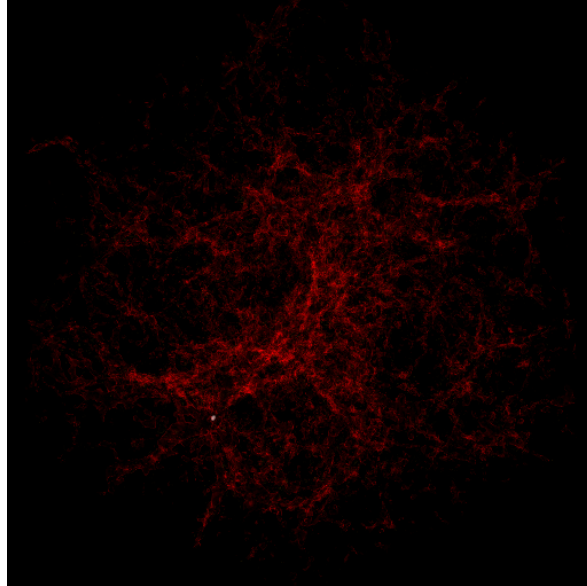


Figure 2: Density field from the snapshot      Volume rendered with yt-project



## 5 Conclusion and Future plan

### References

- [SDSS]        <https://www.sdss.org/science/>
- [simulation]    Aseem Paranjape, Shadab Alam, Voronoi volume function: a new probe of cosmology and galaxy evolution, Monthly Notices of the Royal Astronomical Society, Volume 495, Issue 3, July 2020, Pages 3233–3251, <https://doi.org/10.1093/mnras/staa1379>