Graduate school project report
Halo formation in the cosmic web

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Abstract

abstract

1 Introduction

Cosmic Microwave Background (CMB) shows that the Universe was initially homogenous with very small inhomogeneities. Thanks to the attractive gravitational force, those inhomogeneities led to the formation of galaxies. And the universe we see today has a lot of interesting structures well beyond the galactic scale. This foam-like 1 large scale structure of the universe is called the cosmic web.

Understanding the statistical properties of the large scale structure and its evolution is crucial to test and constrain cosmological models. Computer simulations can be used to evolve initial inhomogeneities to the structure we see today and hence can be compared with sky survey observations.

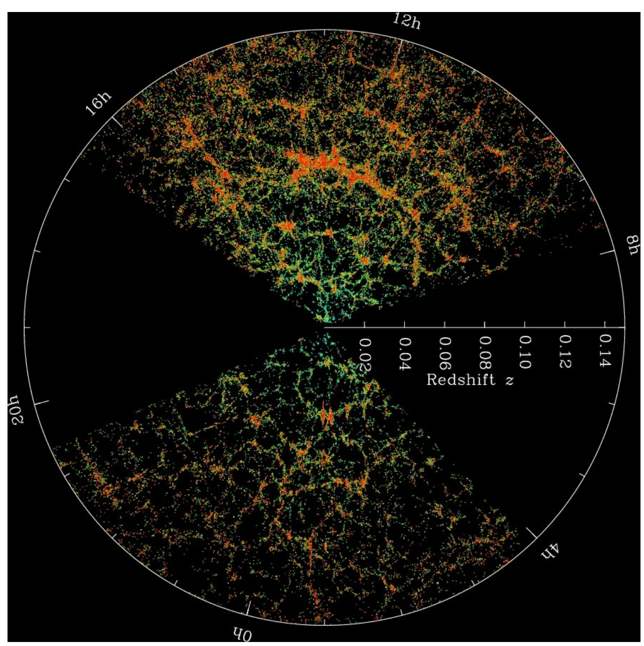

Figure 1: Large Scale Structure (LSS) revealed by [SDSS]

2 Analytical tools

Though the evolution of large scale structure can be simulated, we need analytical tools to get a deeper understanding and also to make generic constraints that can be tested by observations.

On the other hand, simulations help in making and refining these analytical tools. A large fraction of the matter in the Universe is dark matter and it interacts only by gravity. Let us consider the standard ΛCMD model without any curvature.

2.1 FLRW background

The background FLRW metric in comoving coordinates is

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)d\vec{x}^2 && (1)\\ &= a^2(\tau)\left(-d\tau^2 + d\vec{x}^2\right) && (2) \end{aligned}$$

where τ is defined as the conformal time.
Let $\bar{\rho}_m$ and $\bar{\rho}_\Lambda$ denote the mean matter density and mean dark energy density. Hubble parameter is defined as $H \equiv \dot{a}/a$ where the dot denotes derivative with respect to time 't'. At the zeroth order, the Einstein equations reduces to the Friedmann equations,

$$\begin{aligned} H^2 = \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\bar{\rho} \\ &= \frac{8\pi G}{3}\left(\bar{\rho}_m + \bar{\rho}_\Lambda\right) && (3)\\ \dot{H} + H^2 = \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\bar{\rho} + 3\bar{p}\right) \\ &= -\frac{4\pi G}{3}\left[\bar{\rho}_m + \bar{\rho}_\Lambda + 3(0 - \bar{\rho}_\Lambda)\right] \\ &= -\frac{4\pi G}{3}\left[\bar{\rho}_m - 2\bar{\rho}_\Lambda\right] && (4) \end{aligned}$$

Assuming that the matter and dark energy are independently conserved,

$$\begin{aligned} \dot{\bar{\rho}}_m &= -3H\left(\bar{\rho}_m + \bar{p}_m\right) = -3H\bar{\rho}_m && (5)\\ \dot{\bar{\rho}}_\Lambda &= -3H\left(\bar{\rho}_\Lambda + \bar{p}_\Lambda\right) = 0 && (6) \end{aligned}$$

Solving the above equations give $\bar{\rho}_m \propto a^{-3}$ and $\bar{\rho}_\Lambda$ is constant.
Density parameters are defined as

$$\Omega_m \equiv \frac{8\pi G\bar{\rho}_m}{3H^2} \quad \Omega_\Lambda \equiv \frac{8\pi G\bar{\rho}_\Lambda}{3H^2} \qquad (7)$$

so that the first Friedmann equation 3 reduces to $\Omega_m + \Omega_\Lambda = 1$. Let us now switch to conformal time 'τ' and define conformal Hubble parameter $\mathcal{H} \equiv \partial_\tau a/a = \dot{a}$.

$$\begin{aligned} 3H^2\Omega_m &= 8\pi G\bar{\rho}_m && (8)\\ 3\mathcal{H}^2\Omega_m &= 8\pi G\bar{\rho}_m a^2 && (9) \end{aligned}$$

The second Friedmann equation (4) becomes

$$-\frac{4\pi G}{3}\left[\bar{\rho}_m - 2\bar{\rho}_\Lambda\right] = \frac{\ddot{a}}{a} = \frac{\dot{\mathcal{H}}}{a} \tag{10}$$

$$-\frac{4\pi G}{3}\left[\bar{\rho}_m - 2\bar{\rho}_\Lambda\right] a^2 = \frac{d\mathcal{H}}{a^2 d\tau} a^2 \tag{11}$$

$$\mathcal{H}^2\left[-\frac{\Omega_m}{2} + \Omega_\Lambda\right] = \frac{d\mathcal{H}}{d\tau} \tag{12}$$

$$\Omega_\Lambda - \frac{\Omega_m}{2} = \frac{d\mathcal{H}}{d\tau}\mathcal{H}^{-2} = \frac{d\mathcal{H}}{\mathcal{H} d\tau}\mathcal{H}^{-1} \tag{13}$$

$$= \frac{d\ln\mathcal{H}}{d\tau}\left[\frac{da}{a d\tau}\right]^{-1} = \frac{d\ln\mathcal{H}}{d\ln a} \tag{14}$$

It is useful to define another time evolution parameter $y \equiv \ln a$.

$$\Omega_\Lambda - \frac{\Omega_m}{2} = \frac{d\ln\mathcal{H}}{dy} \tag{15}$$

2.2 Growth of Structure

While the evolution of background cosmology can be studied fully analytically, the inhomogenities responsible for structure formation can't be solved exactly without making any ansatz. We will look into different approaches but first let us setup the equations.

2.2.1 Newtonian equations for inhomogeneities in ΛCDM Universe

The matter density contrast can be quantified in terms of overdensity parameter δ,

$$\delta(\vec{x},\tau) \equiv \frac{\rho_m(\vec{x},\tau) - \bar{\rho}_m(\tau)}{\bar{\rho}_m(\tau)} \quad = \frac{\rho_m(\vec{x},\tau)}{\bar{\rho}_m(\tau)} - 1 \tag{16}$$

The velocity field is then defined as

$$\vec{v}(\vec{x},\tau) \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(a\vec{x}) \tag{17}$$

$$= \frac{1}{a}\frac{d}{d\tau}(a\vec{x}) \tag{18}$$

$$= \frac{da/d\tau}{a}\vec{x} + \frac{d\vec{x}}{d\tau} \tag{19}$$

$$= \mathcal{H}(\tau)\ \vec{x} + \vec{u}(\vec{x},\tau) \tag{20}$$

where $\vec{u}(\vec{x},\tau) \equiv d\vec{x}/d\tau$ is called the peculiar velocity, while the first term quantifies the Hubble flow. Due to the time dependance of $\mathcal{H}$, there is an associated acceleration purely due to hubble flow. That acceleration can be found by setting peculiar velocity to zero.

$$\frac{d}{dt}\left(\mathcal{H}(\tau)\ \vec{x}\right) = \frac{d}{dt}\frac{da}{dt}\ \vec{x} \tag{21}$$
$$= \ddot{a}(t)\ \vec{x} = \frac{1}{a}\ddot{a}(t)\ \vec{r} \tag{22}$$

This acceleration can be written in terms of a potential

$$\bar{\phi} \equiv -\frac{1}{2}a\ddot{a}\ \ |\vec{x}|^2 = -\frac{1}{2}\frac{\ddot{a}}{a}\ \ |\vec{r}|^2 \tag{23}$$
$$\implies \nabla_r\bar{\phi} = \frac{\ddot{a}}{a}\ \vec{r} \tag{24}$$

Let ϕ denote the total gravitational potential in the presence of inhomeneities. We can define the modified gravitaional potential as $\Phi \equiv \phi - \bar{\phi}$.

Evolution of the density inhomogeneity δ, the peculiar velocity field $\vec{u}$, and the modified potential Φ is described by the continuity equation, Euler equation and Poisson equation.

$$\partial_\tau\delta + \nabla\cdot[(1+\delta)\vec{u}] = 0 \tag{25}$$
$$\partial_\tau\vec{u} + \mathcal{H}\vec{u} + (\vec{u}\cdot\nabla)\,\vec{u} = -\nabla\Phi \tag{26}$$
$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega_m(\tau)\delta \tag{27}$$

where ∇ is with respect to comoving coordinates.

2.2.2 Linear solutions to inhomogeneous CDM

If the inhomogeneities are small then we can consider them as perturbation to the homogenous background. To the first order of perturbation, we get

$$\partial_\tau\delta + \nabla\cdot\vec{u} = 0 \tag{28}$$
$$\partial_\tau\vec{u} + \mathcal{H}\vec{u} = -\nabla\Phi \tag{29}$$
$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\Omega_m(\tau)\delta \tag{30}$$

Taking divergence of (29)

$$\partial_\tau\nabla\cdot\vec{u} + \mathcal{H}\nabla\cdot\vec{u} = -\nabla^2\Phi \tag{31}$$

substituting for $\nabla\cdot\vec{u}$ from (28) and for $\nabla^2\Phi$ from (30)

$$\partial_\tau(-\partial_\tau\delta) + \mathcal{H}(-\partial_\tau\delta) = -\frac{3}{2}\mathcal{H}^2\Omega_m(\tau)\delta \tag{32}$$
$$\partial_\tau^2\delta + \mathcal{H}\partial_\tau\delta = \frac{3}{2}\mathcal{H}^2\Omega_m(\tau)\delta \tag{33}$$

In the last equation, there is no spatial derivatives, so it is just a second order ordinary differential equation for δ. In general, the solution is of the form,

$$\delta(\vec{x},\tau) = A(\vec{x})D_1^{(+)}(\tau) + B(\vec{x})D_1^{(-)}(\tau) \tag{34}$$

where $D_1^{(+)}(\tau)$ denotes growing mode and $D_1^{(-)}(\tau)$ denotes decaying mode. The divergence of peculiar velocity is then

$$\theta(\vec{x},\tau) \equiv \nabla\cdot\vec{u} = -\partial_\tau\delta \tag{35}$$

$$= -A(\vec{x})\frac{dD_1^{(+)}}{d\tau} - B(\vec{x})\frac{dD_1^{(-)}}{d\tau} \tag{36}$$

The growing mode and decaying mode can be obtained by directly solving the linear ODE (33). Before going into the full solution, let us look into the matter dominated era where $\Omega_m(\tau) \approx 1$. Inserting this in the background equation (9) we get,

$$\mathcal{H} \propto \sqrt{\bar{\rho}_m a^2} \propto a^{-1/2} \tag{37}$$

$$\implies \frac{da}{d\tau} \propto a^{1/2} \implies a^{-1/2}da \propto d\tau \tag{38}$$

We can solve for scale factor evolution by integrating the above equation.

$$a(\tau) \propto \tau^2 \quad \implies \mathcal{H}(\tau) = \frac{2}{\tau} \tag{39}$$

$$H(\tau) = \frac{\dot{a}}{a} = \frac{\mathcal{H}}{a} \propto \tau^{-3} \tag{40}$$

Hence the linear perturbation equation (33) reduces to,

$$\partial_\tau^2\delta + \mathcal{H}\partial_\tau\delta = \frac{3}{2}\mathcal{H}^2\delta \tag{41}$$

$$\partial_\tau^2\delta + \frac{2}{\tau}\partial_\tau\delta = \frac{3}{2}\left(\frac{2}{\tau}\right)^2\delta \tag{42}$$

$$\partial_\tau^2\delta + \frac{2}{\tau}\partial_\tau\delta - \frac{6}{\tau^2}\delta = 0 \tag{43}$$

By assuming power law solution, we get the growing mode $D_1^{(+)}(\tau) \propto \tau^2 \propto a(\tau)$ and the decaying mode $D_1^{(-)}(\tau) \propto \tau^{-3} \propto H(\tau)$.
It turns out that even after dark energy starts to dominate, the decaying mode stays proportional to the Hubble parameter $D_1^{(-)}(\tau) \propto H(\tau)$. However, now the time dependence of Hubble parameter $H(\tau)$ is different.
Similarly, the growing mode can be generalised to an integral involving $H(\tau)$,

$$D_1^{(+)}(\tau) \propto H(\tau)\int^\tau \frac{1}{a(\tau')H^2(\tau')}d\tau' \tag{44}$$

$$\implies D_1^{(+)}(a) \propto H(a)\int^a \frac{1}{(a'H(a'))^3}da' \tag{45}$$

We can set the proportionality constant by setting $D_1^{(+)}(a) = a$ during matter dominated phase.

$$D_1^{(+)}(a) = \frac{5}{2}\Omega_{m0}\frac{H(a)}{H_0}\int^a \frac{1}{(a'H(a')/H_0)^3}da' \tag{46}$$

2.2.3 Eulerian perturbation theory

We will switch to Fourier space with respect to spatial comoving coordinates. The density inhomogeneity parameter in the Fourier space is defined as

$$\delta_{\vec{k}}(\tau) \equiv \int d^3k\ e^{-i\vec{k}\cdot\vec{x}}\ \delta(\vec{x},\tau), \qquad \delta(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3}\ e^{i\vec{k}\cdot\vec{x}}\ \delta_{\vec{k}}(\tau) \tag{47}$$

Similarly we can define Fourier space variables $\theta_{\vec{k}}(\tau)$, $\vec{u}_{\vec{k}}(\tau)$ & $\Phi_{\vec{k}}(\tau)$ corresponding to $\theta(\vec{x},\tau), \vec{u}(\vec{x},\tau)$ & $\Phi(\vec{x},\tau)$.

The linear partial differential equations (28), (29) & (30) become linear ordinary differential equations in Fourier space and hence each wavemode evolves independently.

$$\partial_\tau \delta_{\vec{k}} + \theta_{\vec{k}} = 0 \tag{48}$$

$$\partial_\tau \theta_{\vec{k}} + \mathcal{H}\theta_{\vec{k}} - k^2\ \Phi_{\vec{k}} = 0 \tag{49}$$

$$-k^2\ \Phi_{\vec{k}} = \frac{3}{2}\mathcal{H}^2\Omega_m(\tau)\ \delta_{\vec{k}} \tag{50}$$

However when we include non-linear terms, Fourier wavemodes are coupled. Let us assume that there is no vorticity, $\nabla \times \vec{u} = 0$ so that the divergence-less term is zero. Fourier transforming the continuity equation, we get

$$\partial_\tau \delta_{\vec{k}} + \theta_{\vec{k}} = -\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\ \frac{\vec{k}_1\cdot\left(\vec{k}_1+\vec{k}_2\right)}{k_1^2}\ \delta_{\vec{k}_2}\ \theta_{\vec{k}_1}\ \delta_D\left(\theta_{\vec{k}} - \left[\vec{k}_1+\vec{k}_2\right]\right) \tag{51}$$

Fourier transforming the divergence of Euler equation, we get

$$\partial_\tau \theta_{\vec{k}} + \mathcal{H}\theta_{\vec{k}} - k^2\ \Phi_{\vec{k}} = -\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\ \frac{\left(\vec{k}_1\cdot\vec{k}_2\right)\left|\vec{k}_1+\vec{k}_2\right|^2}{2k_1^2k_2^2}\ \theta_{\vec{k}_2}\ \theta_{\vec{k}_1}\ \delta_D\left(\theta_{\vec{k}} - \left[\vec{k}_1+\vec{k}_2\right]\right) \tag{52}$$

Since the poisson equation is linear in position space, there is no coupling terms in the Fourier space.

$$-k^2\ \Phi_{\vec{k}} = \frac{3}{2}\mathcal{H}^2\Omega_m(\tau)\ \delta_{\vec{k}} \tag{53}$$

Perturbative expansion in the matter dominated era

2nd order solution

2.2.4 Lagrangian approach - Zel'dovich approximations

2.2.5 Spherical collapse

2.3 Statistical propetries

2.3.1 Correlation function power spectrum relation

$$P(\vec{k}) = \int \xi(\vec{r})\ e^{i\vec{k}\vec{r}}\ d^3r \tag{54}$$

$$\xi(\vec{r}) = \frac{1}{(2\pi)^3} \int P(\vec{k})\ e^{-i\vec{k}\vec{r}}\ d^3k \tag{55}$$

$$\xi(r) = \frac{4\pi}{(2\pi)^3} \int_0^\infty P(k)\ k^2\ \frac{\sin(kr)}{kr} dk \tag{56}$$

$$\xi(r) = \frac{1}{2\pi^2} \int_{-\infty}^\infty P(k)\ k^3\ \frac{\sin(kr)}{kr} d(\ln k) \tag{57}$$

$$\xi(r) = \int_{-\infty}^\infty \Delta^2(k)\ \frac{\sin(kr)}{kr} d(\ln k) \tag{58}$$

3 N-body simulations

4 Analysing a snapshot of a GADGET-2 simulation

[simulation]

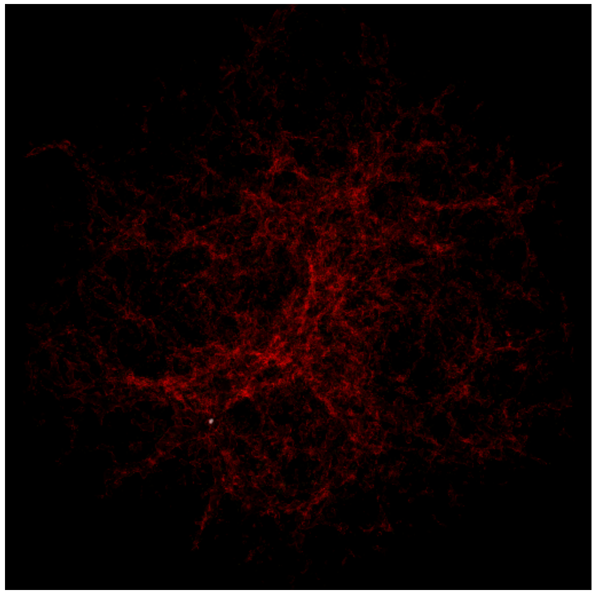

Figure 2: Density field from the snapshot   Volume rendered with yt-project

5 Conclusion and Future plan

References

[SDSS] `https://www.sdss.org/science/`

[simulation] Aseem Paranjape, Shadab Alam, Voronoi volume function: a new probe of cosmology and galaxy evolution, Monthly Notices of the Royal Astronomical Society, Volume 495, Issue 3, July 2020, Pages 3233–3251, `https://doi.org/10.1093/mnras/staa1379`