Centrale MP 2021 Analyse.

531.

a) segments

b) i)

$$\forall p \in \mathbb{N}^*, \quad U_p = \frac{1}{p}\sum_{k=0}^{p-1} u^k .$$

Mq :

$A = \{x \in B \mid u(x) = x\}$.

$\subset$

Soit $x \in A$. On fixe $(x_p) \in B^{\mathbb{N}}$ tq $\forall p \in \mathbb{N}$, $U_p(x_p) = x$.
Soit $p \in \mathbb{N}$.

$$u(x) - x = u(U_p(x_p)) - U_p(x_p)$$

$$\text{''} \quad = u\left(\frac{1}{p}\sum_{k=0}^{p-1} u^k(x_p)\right) - U_p(x_p).$$

$$u(x) - x = \frac{1}{p}\sum_{k=0}^{p-1}\left[u^{k+1}(x_p) - U_p(x_p)\right].$$

$$\text{''} \quad = \frac{1}{p}\sum_{k=1}^{p}\left[u^k(x_p) - U_p(x_p)\right].$$

$$\text{''} \quad = \frac{1}{p}u^p(x_p) + \cancel{\frac{1}{p}\sum_{k=0}^{p-1} u^k(x_p)} - \frac{1}{p}x_p - \cancel{U_p(x_p)}$$

$$\text{''} \quad = \frac{1}{p}\left(u^p(x_p) - x_p\right) \xrightarrow[p \to +\infty]{} 0 \qquad (u^p(x_p) - x_p)_{p \in \mathbb{N}} \text{ est bornée.}$$

donc $u(x) - x = 0$. $u(x) = x$.

$\supset$ Réciproquement,

$$\forall x \in \{y \in B \mid u(y) = y\}, \quad x \in \bigcap_{p \in \mathbb{N}} U_p(B).$$

Donc $A = \{x \in B \mid x = u(x)\}$.

ii) Mq $A \neq \emptyset$. $A = \bigcap_{p \in \mathbb{N}} U_p(B)$.

Pour tout $p \in \mathbb{N}$, U_p est continue.

Donc

$\forall p \in \mathbb{N}$, $U_p(B)$ est compact (B compact).

Soit $m \in \mathbb{N}$, $k \in \mathbb{N}$. On pose $n = km$.
On mq $U_n(B) \subset U_m(B)$. Soit $y \in U_n(B)$.
On fixe $x \in B$ tq $U_n(x) = y$.

$$U_n(x) = \frac{1}{n}\sum_{\tilde{k}=0}^{n-1} u^{\tilde{k}}(x) = \frac{1}{km}\sum_{\tilde{k}=0}^{km-1} u^{\tilde{k}}(x)$$

$$u_n(x) = \frac{1}{km}\sum_{\tilde{k}=0}^{km-1} u^{\tilde{k}}(x) = \frac{1}{km}\sum_{j=0}^{k-1}\sum_{\ell=0}^{m-1} u^{mj+\ell}(x)$$

Puis $u_n(x) = \frac{1}{m}\sum_{\ell=0}^{m-1} u^{\ell}\left(\frac{1}{k}\sum_{j=0}^{k-1} u^{mj}(x)\right)$.

$\tilde{x} = \frac{1}{k}\sum_{j=0}^{k-1} u^{mj}(x)$. Or B est convexe et $\forall j \in [\![0, k-1]\!]$, $u^{mj}(x) \in B$.

Puis $\tilde{x} \in B$. donc $u_n(x) = \frac{1}{m}\sum_{\ell=0}^{m-1} u^{\ell}(\tilde{x}) = u_m(\tilde{x})$.

donc $U_n(B) \subset U_m(B)$. On considère $\tilde{A} = \bigcap_{n\in\mathbb{N}} U_{n!}(B)$.

$(U_{n!}(B))$ est une suite de compacts emboîtés non vides.
donc $\bigcap_{n\in\mathbb{N}} U_{n!}(B) \neq \emptyset$.

$A \subset \bigcap_{n\in\mathbb{N}} U_{n!}(B)$. Mq $\tilde{A} \subset A$.

$\forall n \in \mathbb{N}$, $U_{n!}(B) \subset U_n(B)$

donc

$$\bigcap_{n\in\mathbb{N}} U_{n!}(B) \subset \bigcap_{n\in\mathbb{N}} U_n(B).$$

Puis $A = \tilde{A}$. A est non vide.

Autre méthode :

Soit $y \in B$. $(u^p(y))_{p\in\mathbb{N}} \in B^{\mathbb{N}}$ $u^{\varphi(p)}(y) \xrightarrow[p\to+\infty]{} x$. $x \in B$.

$\forall p \in \mathbb{N}$, $(u - id_E)(u^{\varphi(p)}(y)) = \frac{1}{\varphi(p)}(u^{\varphi(p)}(y) - y)$.

Enfin $(u - id_E)(x) = 0$.