Formulation of Fibonacci rabbits with death

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Let I_n denote the number of immature rabbits at time n, M_n denote the number of mature rabbits at time n, and T_n denote the total number of rabbits at time n. Let us also assume that rabbits die after k months. For the standard Fibonacci sequence (i.e. $n < k$), we have

$$I_{n+1} = M_n, \tag{1}$$
$$M_{n+1} = M_n + I_n, \qquad (n < k), \tag{2}$$
$$T_n = M_n + I_n. \tag{3}$$

In plain language, equation (1) states that the number of immature rabbits at time $n+1$ is equal to the number of mature rabbits at time n, whilst equation (2) states that the number of mature rabbits at time $n+1$ is equal to the number of mature rabbits at time n plus the number of immature rabbits at time n. Equation (3) states that the total number of rabbits at time n is also equal to the number of mature rabbits at time n plus the number of immature rabbits at time n.

For $n \geq k$, equation (2) is redefined as

$$M_{n+1} = M_n + I_n - I_{n+1-k}, \qquad (n \geq k). \tag{4}$$

Substituting equations (1) and (4) into equation (3), we have

$$\begin{aligned} T_{n+1} &= M_{n+1} + I_{n+1}, \\ &= (M_n + I_n - I_{n+1-k}) + M_n, \\ &= (T_n - I_{n+1-k}) + M_n, \\ &= T_n - I_{n+1-k} + (M_{n-1} + I_{n-1} - I_{n-k}), \\ &= T_n - I_{n+1-k} + (T_{n-1} - I_{n-k}). \end{aligned} \tag{5}$$

Rearranging equation (5) and noticing from equation (1) that $I_{n+1-k} = M_{n-k}$, we have

$$T_{n+1} = T_n + T_{n-1} - (M_{n-k} + I_{n-k}),$$

which, upon substituting equation (3), gives our final recurrence relation

$$T_{n+1} = T_n + T_{n-1} - T_{n-k}. \tag{6}$$

The sequence is initialised as follows:

$$\begin{aligned} n < 0 : &\quad T_n = 0, \\ n = 0, 1, 2 : &\quad T_n = 1. \end{aligned}$$