Combinatorics

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PART

I

Graph Theory

SECTION 1

Isomorphism

Definition 1 Two graphs G and G' are isomorphic if there exists one to one correspondence from vertices in G to G' such that a pair of vertices are adjacent in G if and only if the corresponding vertices in G' are adjacent.

Some common ideas to check (or disprove) if two graphs are isomorphic:

- Number of vertices must be equal.
- Number of vertices for any given degree must be equal.
- Each subgraph in G must have an isomorphic subgraph in G'.
- Symmetries can be exploited to find matching vertices in two graphs.

Remark These set of conditions are not exhaustive to show if two graphs are isomorphic

SECTION 2

Edge Counting

Theorem 1 For any graph, sum of degrees of vertices is twice the number of edges in graph.

Some applications of above theorem:

- There must be even number of vertices with odd degree.
- If a graph has only two vertices with odd degree, there must be a path connecting those two vertices. Otherwise, they would be in different connected components, however such subgraphs cannot exist since they will contain only one (odd number) vertex of odd degree.

Problem 1 Two people start at locations A and Z which are at same elevation of a mountain range on opposite sides of a summit M. Prove that if there is no point lower than A or higher than M, then it is always possible for them to reach summit such that they are at the same level at every point of time.

Solution Construct a graph as follows:

- Vertices are tuples (P_L, P_R), with points P_L and P_R being on left and right side of summit M respectively and either one (or both) are local peaks or valleys.
- Edge exists between two vertices (P_L, P_R) and $(P'_L, P_{R'})$ if two people can walk from P_L to P'_L and P_R to $P_{R'}$ by walking in same direction (either up or down).

> The problem now reduces to showing that there exists a path from (A, Z) to (M, M).
> It can be shown that the vertices (A, Z) and (M, M) have degree one (one can only go in one of the two directions from these points and stop when one reaches a local peak or valley) and all other vertices have degrees 2 or 4.
> Then there must exist a path between the two vertices since they are the only two vertices of odd degree.

SECTION 3

Bipartite Graphs

Definition 2 A graph is bipartite if we can partition it into two sets such that no two vertices in same set are adjacent.

Theorem 2 A graph is bipartite if and only if each cycle has even length (length here means number of edges in graph).

SECTION 4

Planar Graphs

Definition 3 A graph is planar if it can be drawn on a plane with no two edge intersecting.

SUBSECTION 4.1

Circle-cord method

A method to check if a graph is planar. Note that this method will not always disprove if a graph is planar.

First, find a cycle in graph which covers all the vertices, and represent it as a circle.

Now pick an edge and draw it as in-circle or out-circle chord (doesn't matter which). This will force some edges to be out-circle or in-circle chord. Draw these edges. Keep doing this. If we find an edge which cannot be either in-circle or out-circle, the graph is not planar. If the process terminate, we have found a representation of the graph as a planar graph.

Definition 4 The graph K_5 is the complete graph with five vertices, that is every vertex is connected to every other vertex.

The graph $K_{3,3}$ is the complete bipartite graph with 3 vertices in each partition, that is, every vertex in one partition is adjacent to every vertex in other partition.

A graph G' is a configuration of another graph G if it is obtained by adding vertices in between edges, that is an edge ab is converted to two edges ac and cb by introducing a new vertex c.

Theorem 3 **Kuratowski, 1930**
A graph is planar if and only if it does not contain a subgraph that is a configuration of K_5 or $K_{3,3}$

Definition 5 Region in a planar representation of graph is an area bounded by edges. This definition also includes areas which extend to infinity. For example, in the graph C_3 (cycle of length 3), there are two regions, one inside the cycle and one outside.

Theorem 4 **Euler's Formula (1752)**
For a connected planar graph G with r regions, v vertices and e edges, the following holds:

$$r = e - v + 2 \tag{4.1}$$

Proof We can construct an inductive proof. The theorem is true for a connected graph with two vertices and one edge. Consider it is true for all graphs having $n - 1$ edges.
Now consider adding another edge. If it connects two vertices x, y (they must be in same region) which were in the graph, it adds new region and new edge, and hence formula is still satisfied. If a new vertex is added, the number of regions remain the same but an edge and a vertex increases, and hence formula remains valid. We cannot add two new vertices and connect them using new edge as otherwise the graph will not be connected. □

Theorem 5 For a general connected planar graph with more than one edge, the following inequality must hold:

$$e \leq 3v - 6 \tag{4.2}$$

And for bipartite connected planar graph, we get a tighter constraint:

$$e \leq 2v - 4 \tag{4.3}$$

Proof Consider degree of region as number of edges incident to region. An edge may be counted twice in degree if the region is on both sides of the edge.
Each region must have degree at least 3 (since region of degree two is possible only in graph with one edge). Also, sum of degrees of regions is equal to $2e$ (each edge is counted twice, one in each of the two boundaries on the two sides of the edge.) Hence, (d represented degree), $d = 2e \geq 3r \implies r \leq 2e/3$. Using Euler's formula, $2e/3 \geq r = e - v + 2 \implies e \leq 3v - 6$.
A similar argument for bipartite graph where each cycle has even length and hence every region must have degree at least 4 gives the second inequality. □

Section 5

Euler Cycles

Definition 6 Under the modified definition of a cycle which allows us to visit a vertex more than once, an Euler cycle is a cycle that contains all the edges in the graph and visits each vertex at least once. This definition is also valid for multigraphs which allows multiple edges to exist between two vertices.

Theorem 6 An undirected multigraph has an Euler cycle if and only if it is connected and has all the vertices of even degree.

Note: If we remove the constraint that all the vertices have to be visited once at least, we may have graphs with isolated vertices which still have an Euler cycle. Hence the constraint that the graph needs to be connected may be violated if considering the alternate definition of Euler cycle.

Proof Graph having Euler cycle will be connected and will have all the vertices of even degree.
For the reverse direction of proof, consider a graph which satisfies the given conditions. Pick a vertex a, and go along a trail (path with repeated vertices allowed). Since every vertex has even degree, you can exit any vertex you entered and hence the trail must end at a. Let the cycle formed be C and let the remaining graph by removing the edges in trail be G'. Let b be the common vertex of C and G' (it must exist since the graph is connected). Construct a trail from b in G' which terminates at b and call it C'. Joining C and C' gives us a bigger cycle and a smaller subgraph G''. Repeating this procedure gives us the Euler cycle. □

Section 6

Hamiltonian Cycles

Definition 7 A Hamiltonian cycle is a cycle which includes all the vertices in the graph.

A few rules to use when building/determining if a graph has a Hamiltonian cycle:

- If a vertex x has degree 2, both of the edges incident to x must be in Hamiltonian cycle
- No proper subcycle can be formed when building a Hamiltonian cycle.
- Once the Hamiltonian cycle is required to use 2 edges at a vertex, all other edges at the vertex must be removed from consideration.

Theorem 7 **Dirac, 1952**
A graph with $n > 2$ vertices has a Hamiltonian cycle if the degree of each vertex is at least $n/2$

Theorem 8 **Chvatal, 1972**
Let G be a connected graph with, and let the vertices $x_1, x_2, \ldots, x_n$ be sorted according to their degrees. If for each $k \leq n/2$, either $\deg(x_k) > k$ or $\deg(x_{n-k}) \geq n - k$, G has a Hamiltonian cycle.

Theorem 9 **Grinberg, 1968**
For a planar graph G having Hamiltonian cycle H drawn with a planar depiction, let r_i denote the number of regions inside the Hamiltonian cycle bounded by i edges in the depiction. Also let r'_i be number of regions outside the cycle bounded by i edges. Then the following equation holds:

$$\sum_i (i-2)(r_i - r'_i) = 0 \tag{6.1}$$

Definition 8 A **tournament** is a directed graph where there is either an edge uv or vu (not both as we are considering simple graphs) for any pair of vertices u, v

Theorem 10 Any tournament has a path of length of $n-1$, where n is the number of vertices, that is a path including all the vertices.

PROOF Proof by induction:
The statement is true for $n = 1$ and $n = 2$. Now assume the statement is true for all graphs with $n - 1$ vertices or less. Consider a tournament with n nodes. Pick a vertex x and remove it from the graph. The smaller graph will then have a path $a \to b$ including all the vertices. If an edge oriented xa or bx were to exist, we could include the vertex x in the path ($x \to a \to b$ or $a \to b \to x$).
For the case when the edges were oriented ax and xb, there must exist vertices a', b' adjacent in the path with a' coming before b' such that the edges with x are oriented $a'x$ and xb' (the direction of edges must flip somewhere along the path as they oppositely oriented at the endpoints). Then we can add the vertex in the path as follows: $a \to a' \to x \to b' \to b$, and hence a path including all the vertices exist in the graph. Hence by induction, the theorem holds. □

When using induction on structure of graph, it may be incorrect to construct a bigger graph from a smaller graph which satisfies the hypothesis. The correct way is to reduce a bigger graph, and show the hypothesis remains true by increasing the size from the smaller graph.