Joseph Fourier

Joseph's father was a tailor in Auxerre
Joseph was the ninth of twelve children
His mother died when he was nine and his father died the following year

Fourier demonstrated talent on math at the age of 14.
In 1787 Fourier decided to train for the priesthood - a religious life or a mathematical life?
In 1793, Fourier joined the local Revolutionary Committee

Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France

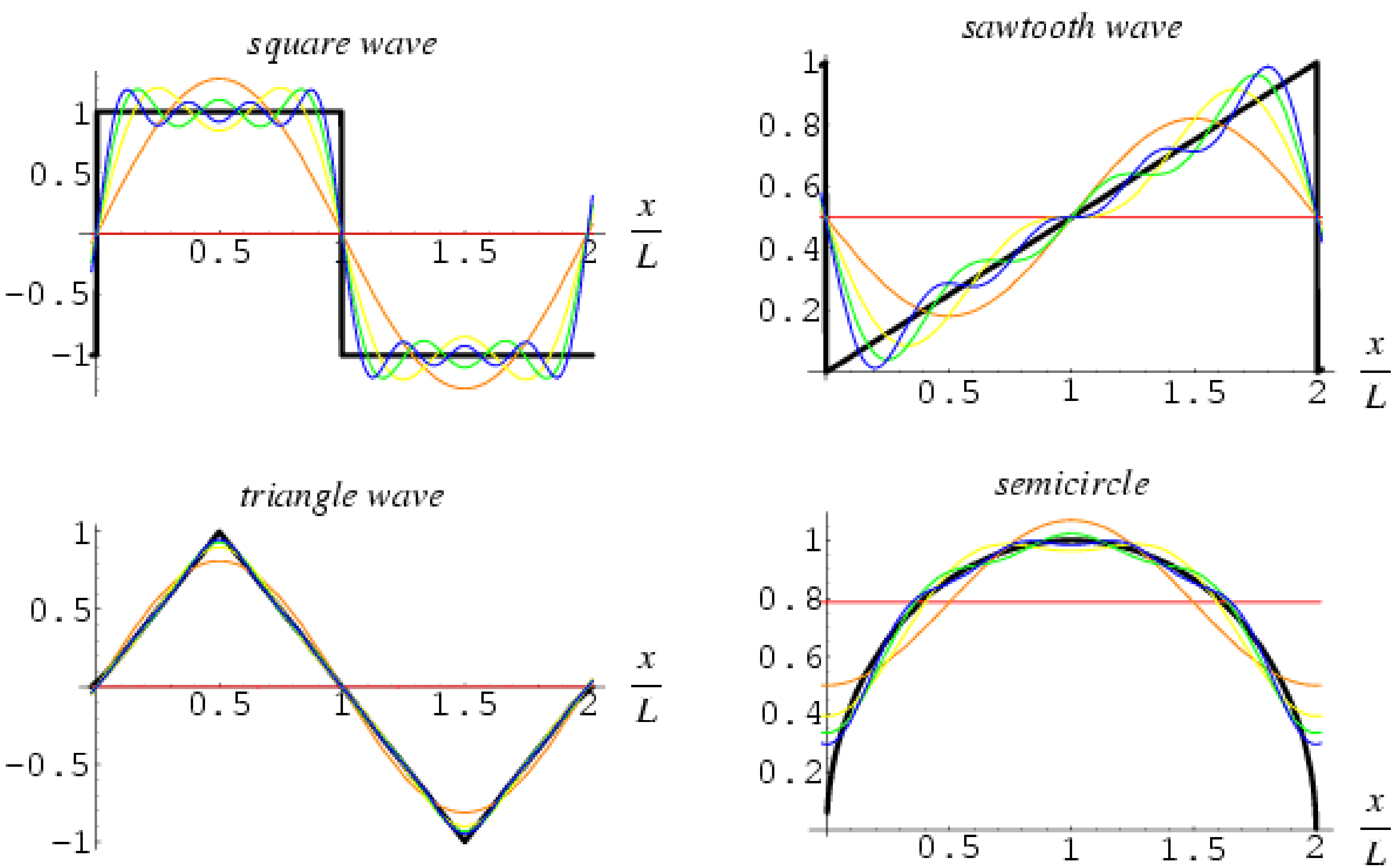
Fourier's "Controversy" Work

Fourier did his important mathematical work on the theory of heat (highly regarded memoir *On the Propagation of Heat in Solid Bodies*) from 1804 to 1807

This memoir received objection from Fourier's mentors (Laplace and Lagrange) and not able to be published until 1815

Napoleon awarded him a pension of 6000 francs, payable from 1 July, 1815. However Napoleon was defeated on 1 July and Fourier did not receive any money

Examples

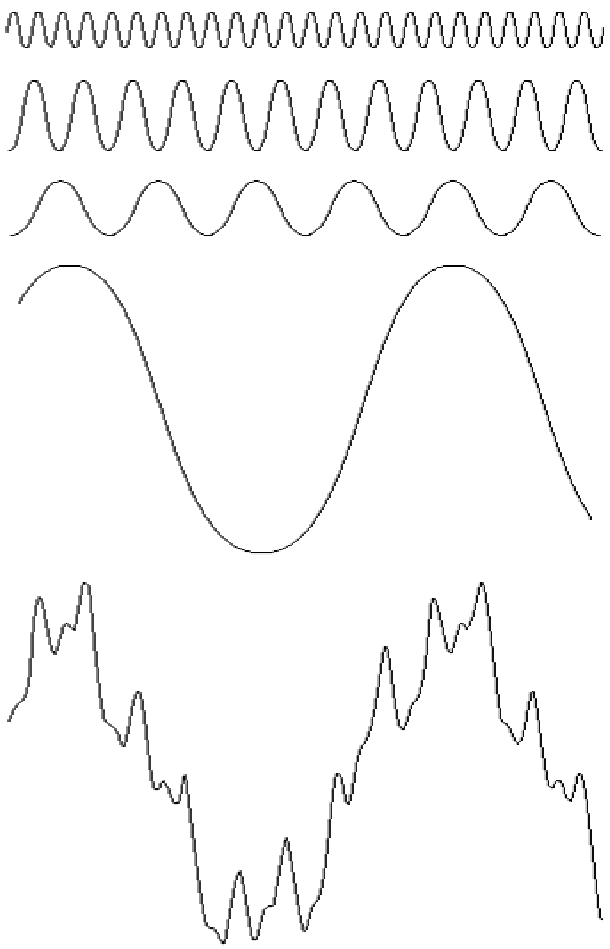


Credits: Xin Li Professor, Lane Dept of CSEE, West Virginia University

Fourier Transform: a review

Basic ideas:

- A periodic function can be represented by the sum of sines/cosines functions of different frequencies, multiplied by a different coefficient.
- Non-periodic functions can also be represented as the integral of sines/cosines multiplied by weighing function.

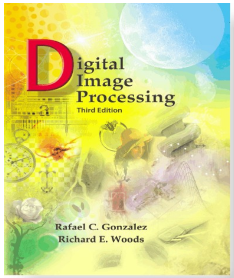


Filtering in the Frequency Domain

Filter: A device or material for suppressing or minimizing waves or oscillations of certain frequencies.

Frequency: The number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable.

Webster’s New Collegiate Dictionary



4.1 Background

Fourier Series and Transform

Jean-Baptiste Fourier (1768)

• Any periodic function can be expressed as a weighted sum of sines and/or cosines (*Fourier series*)

• Other functions can be expressed as an integral of sines and/or cosines multiplied by a weighing function (*Fourier Transform*)

• Functions can be recovered by the inverse operation with *no loss of information*

• Applications to Image Enhancement

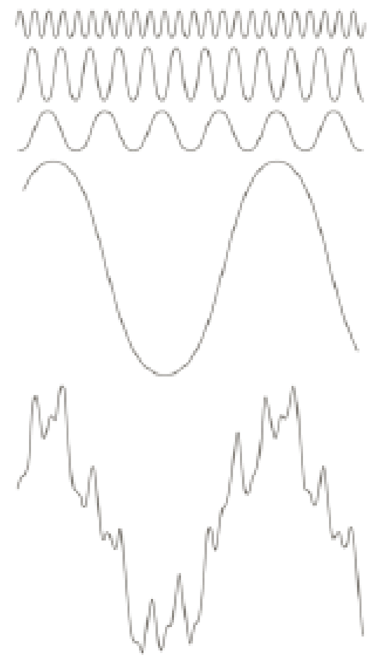


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

4.2 Preliminary Concepts

4.2.1 Complex Numbers

4.2.2 Fourier Series

If $f(t)$ is a periodic function of a continuous variable t, with period T

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}}$$

Where:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt$$

4.2.3 Impulses and their Shifting Property

Unit impulse of a continuous variable *t* located at *t=0* :

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Sifting property :

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \qquad \int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

(if *f(t)* continuous at *t=0*)

However, Convolution of f(t) and δ(t) = f(t)

Unit discrete impulse located at $x=0$:

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

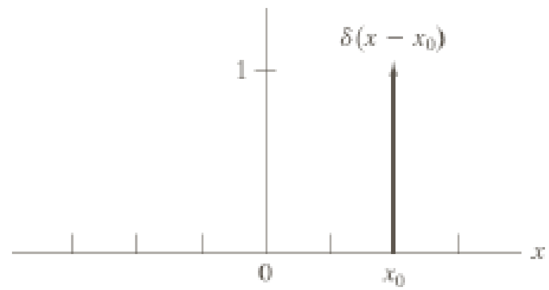


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Sifting property :

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0) \qquad \sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Impulse train : sum of infinitely many periodic impulses ΔT units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

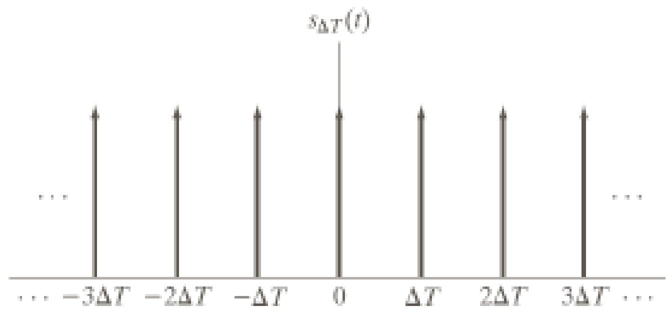


FIGURE 4.3 An impulse train.

4.2.4 The Fourier Transform of Functions of One Continuous Variable

Fourier Transform of a continuous function *f(t)* of a continuous variable *t* :

$$FT\,[f(t)] = \quad F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Inverse Fourier Transform:

$$FT^{-1}\,[F(\mu)] = \quad f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

4.2.4 The Fourier Transform of Functions of One Continuous Variable

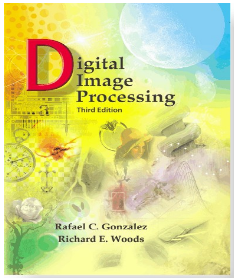
Fourier Transform of a continuous function $f(t)$ of a continuous variable t :

$$FT\,[f(t)] = \quad F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Fourier transform of $f(t) + g(t) \;=\; F(\mu) + G(\mu)$

Fourier transform of $a * f(t) \;=\; a * F(\mu)$

Fourier transform of $f(t-t_0) \;=\; F(\mu)\exp(-j2\pi\mu t_0)$



Chapter 4

Filtering in the Frequency Domain

Example : $$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt = \int_{-W/2}^{W/2} Ae^{-j2\pi\mu t}dt$$

$$= AW\frac{\sin(\pi\mu W)}{(\pi\mu W)} \qquad = AW\,sinc(\pi\mu W)$$

It is customary for display purposes to work with the magnitude of the transform, called **Fourier spectrum** or the **frequency spectrum**

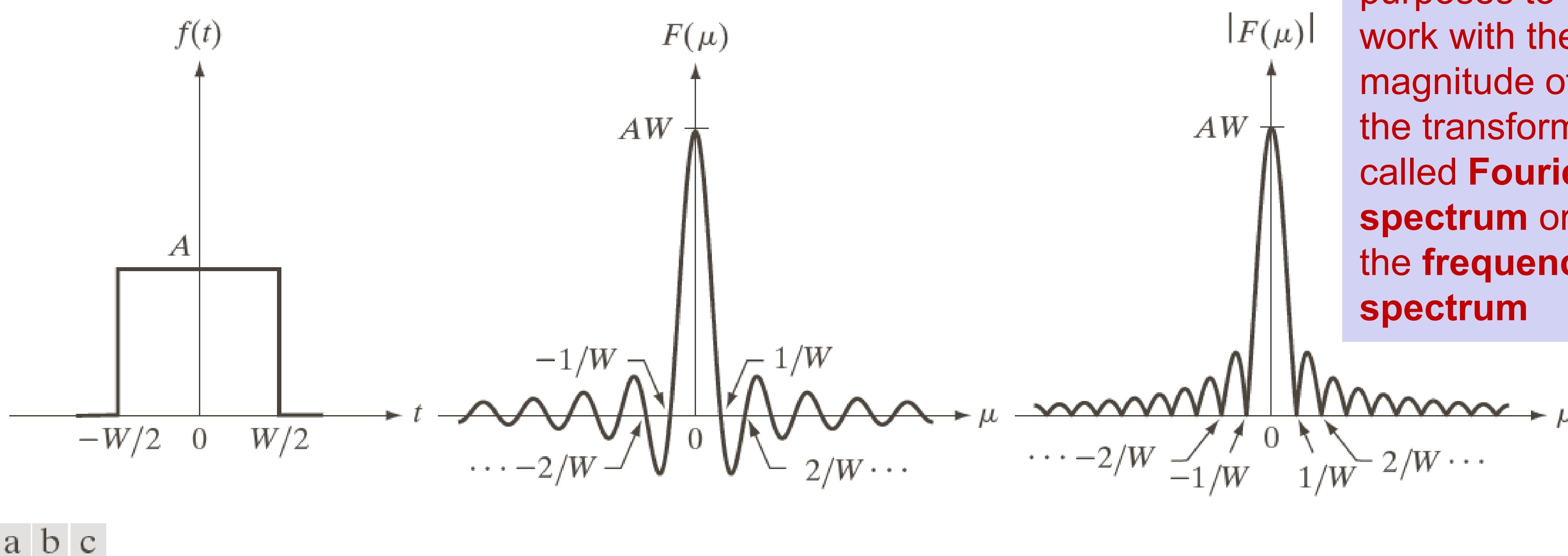
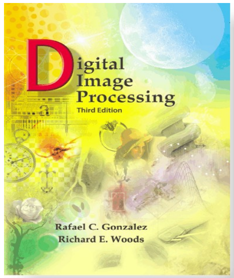


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.



Example 2 : Fourier Transform of a unit impulse located at the origin:

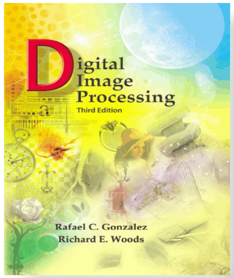
$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt \quad = e^{-j2\pi\mu 0} = e^{0} = 1$$

Fourier Transform of a unit impulse located at $t = t_0$:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0} \quad = \cos(2\pi\mu t_0) - j sin(2\pi\mu t_0)$$

if a function

$f(t)$ has the Fourier transform $F(\mu)$, then the latter function evaluated at t, that is, $F(t)$, must have the transform $f(-\mu)$. Using this *symmetry* property and given, as we showed above, that the Fourier transform of an impulse $\delta(t - t_0)$ is $e^{-j2\pi\mu t_0}$, it follows that the function $e^{-j2\pi t_0 t}$ has the transform $\delta(-\mu - t_0)$. By letting $-t_0 = a$, it follows that the transform of $e^{j2\pi at}$ is $\delta(-\mu + a) = \delta(\mu - a)$, where the last step is true because δ is not zero only when $\mu = a$, which is the same result for either $\delta(-\mu + a)$ or $\delta(\mu - a)$, so the two forms are equivalent.



Example 2 : Fourier Transform $S(\mu)$ of an impulse train with period ΔT :

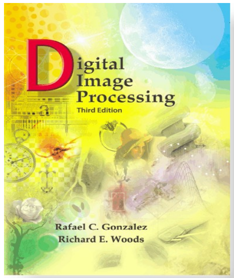
Periodic function with period ΔT =>

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}}$$

Where : $$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}$$

=> $$S(\mu) = FT\left[s_{\Delta t}(t)\right] = \frac{1}{\Delta T} FT\left[\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{T}}\right] = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

Also an *impulse train*, with period $1/\Delta T$



4.2.5 Convolution

Convolution of functions $f(t)$ and $h(t)$, of one continuous variable t :

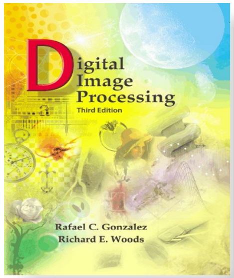
$$f(t)\bigstar h(t)=\int_{-\infty}^{\infty}f(\tau)h(t-\tau)d\tau$$

$$FT\left\{f(t)\bigstar h(t)\right\}=H(\mu)F(\mu)$$

Fourier Transform pairs:

$$f(t)\bigstar h(t)\Longleftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t)\Longleftrightarrow H(\mu)\bigstar F(\mu)$$



$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j2\pi\mu t}\,dt\right]d\tau$$

$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)\,e^{-j2\pi\mu\tau}\right]d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)\,e^{-j2\pi\mu\tau}\,d\tau$$

$$= H(\mu)\,F(\mu)$$

4.3 Sampling and the Fourier Transform of Sampled Functions

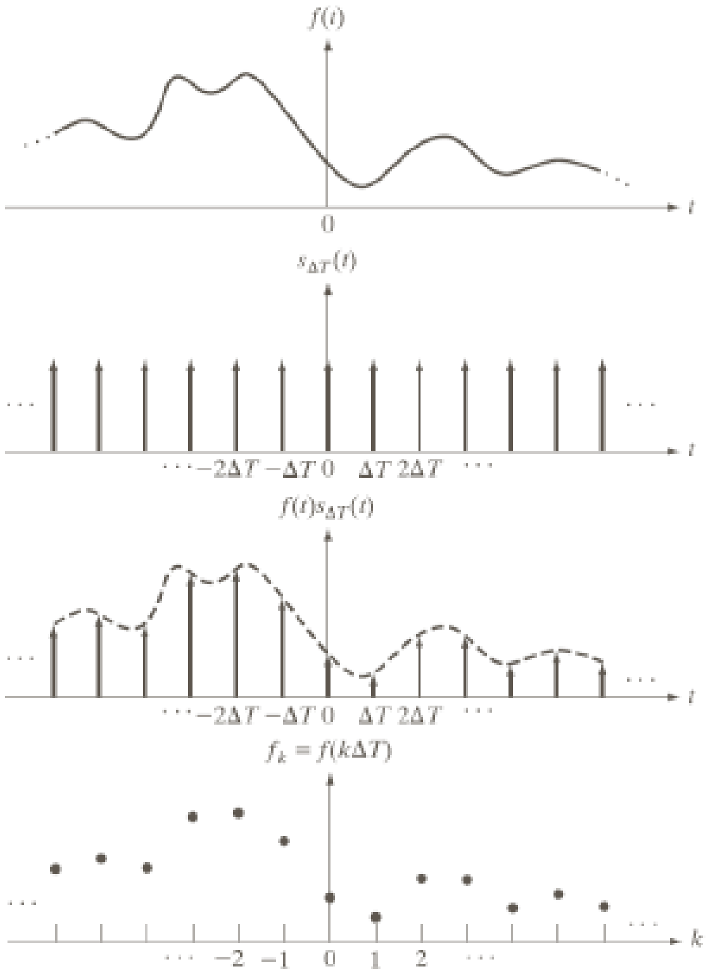
4.3.1 Sampling

Model of the sampled function: multiplication of *f(t)* by a sampling function equal to a train of impulses *ΔT* units apart

$$\tilde{f}(t) = f(t)s_{\Delta T} = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt$$

$$= f(k\Delta T)$$



a
b
c
d

FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

Value of each sample – given by weighted impulse **(weight f_k)**

4.3.2 The Fourier Transform of Sampled Functions

$$\tilde{F}(\mu) = FT\left[\tilde{f}(t)\right] = F(\mu) \bigstar S(\mu) \quad = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

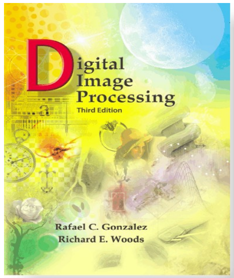
Infinite, periodic sequence of *copies* of *F(μ), continuous*

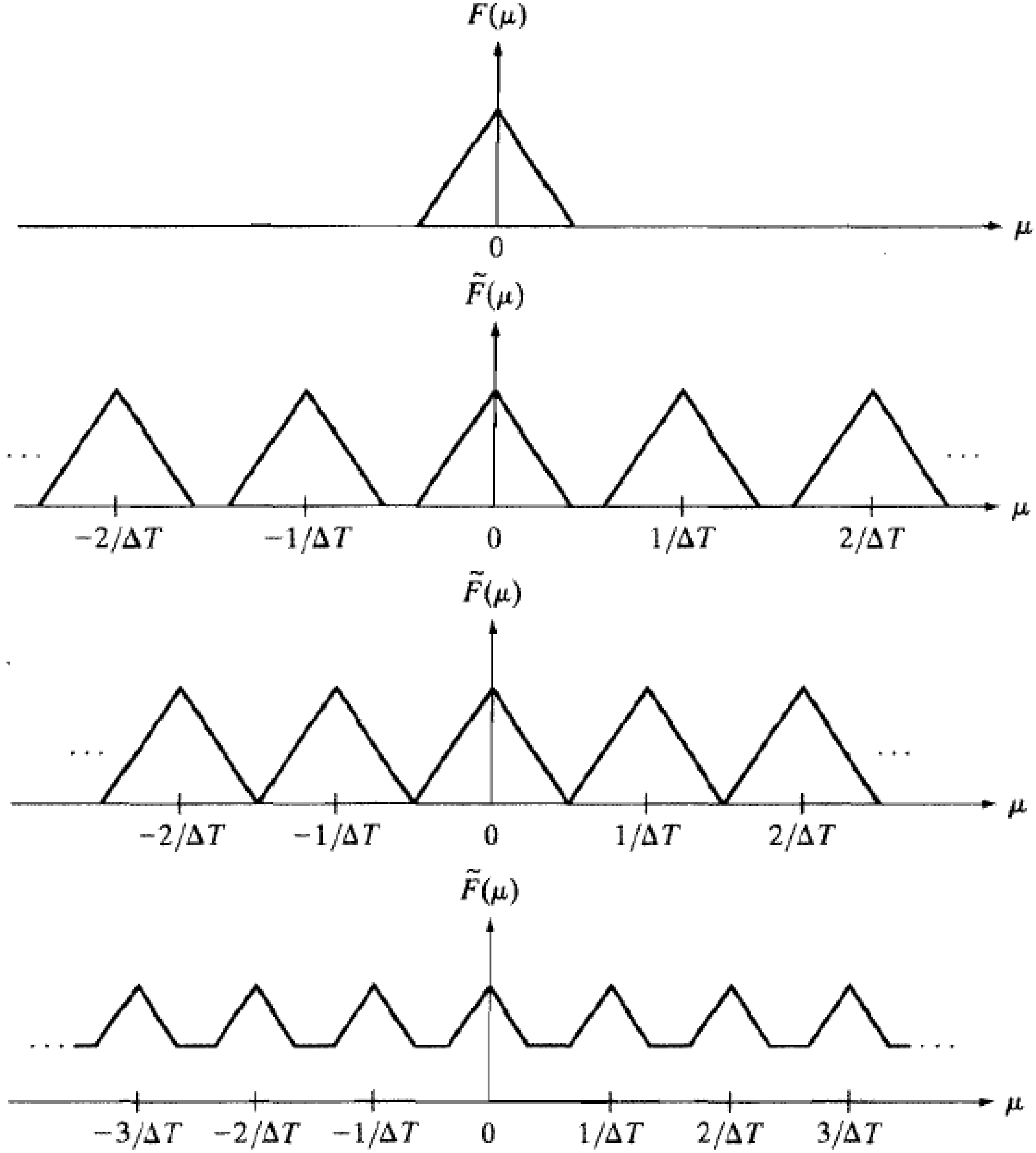
Separation between copies determined by *1/ΔT*

$$\tilde{F}(\mu) = F(\mu) \bigstar S(\mu) \quad = \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau)\, d\tau$$

$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \quad = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\, \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$



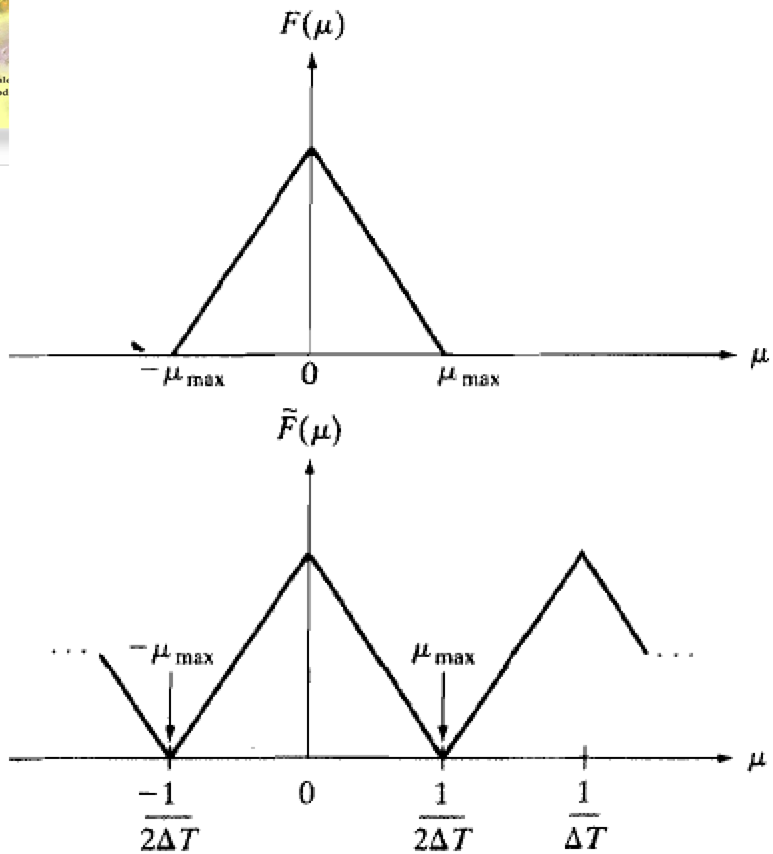


a
b
c
d

FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

A function $f(t)$ whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{\max}, \mu_{\max}]$ about the origin is called a *band-limited* function. Figure 4.7(a), which is a magnified section of Fig. 4.6(a), is such a function. Similarly, Fig. 4.7(b) is a more detailed view of the transform of a critically-sampled function shown in Fig. 4.6(c). A lower value of $1/\Delta T$ would cause the periods in $\tilde{F}(\mu)$ to merge; a higher value would provide a clean separation between the periods.

We can recover $f(t)$ from its sampled version if we can isolate a copy of $F(\mu)$ from the periodic sequence of copies of this function contained in $\tilde{F}(\mu)$, the transform of the sampled function $\tilde{f}(t)$. Recall from the discussion in the previous section that $\tilde{F}(\mu)$ is a *continuous*, *periodic* function with period $1/\Delta T$. Therefore, all we need is one complete period to characterize the entire transform. This implies that we can recover $f(t)$ from that single period by using the inverse Fourier transform.

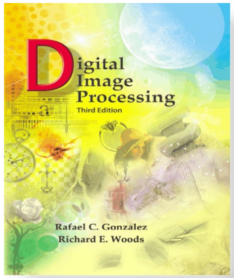


a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

Extracting from $\tilde{F}(\mu)$ a single period that is equal to $F(\mu)$ is possible if the separation between copies is sufficient (see Fig. 4.6). In terms of Fig. 4.7(b), sufficient separation is guaranteed if $1/2\Delta T > \mu_{max}$ or

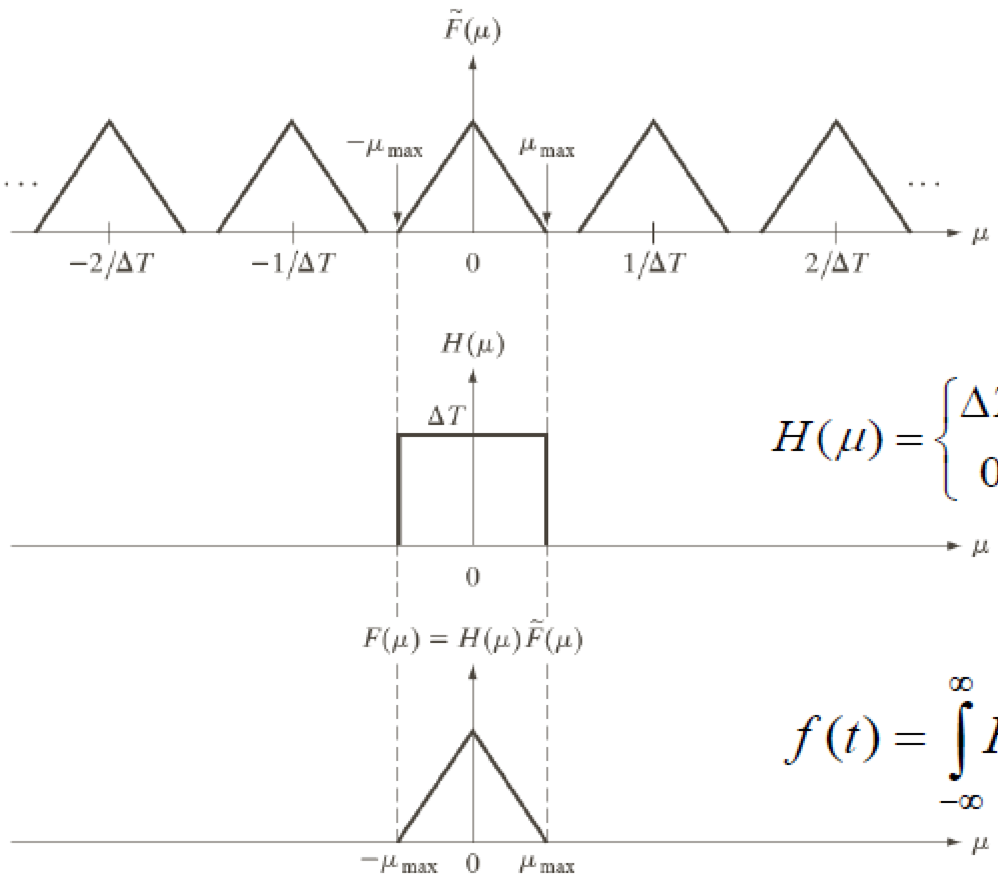
$$\frac{1}{\Delta T} > 2\mu_{max} \tag{4.3-6}$$



$$\frac{1}{\Delta T} > 2\mu_{\max} \qquad (4.3\text{-}6)$$

This equation indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function. This result is known as the *sampling theorem*.† We can say based on this result that no information is lost if a continuous, band-limited function is represented by samples acquired at a rate greater than twice the highest frequency content of the function. Conversely, we can say that the *maximum* frequency that can be "captured" by sampling a signal at a rate $1/\Delta T$ is $\mu_{\max} = 1/2\Delta T$. Sampling at the Nyquist rate sometimes is sufficient for perfect function recovery, but there are cases in which this leads to difficulties, as we illustrate later in Example 4.3. Thus, the sampling theorem specifies that sampling must exceed the Nyquist rate.

Sampling Theorem



$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & otherwise \end{cases}$$

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{i2\pi\mu t} d\mu$$

Note: If f(t) is a **band-limited function,** this implies f(t) must extend from –inf, to +inf.
H(μ) is called a low-pass filter

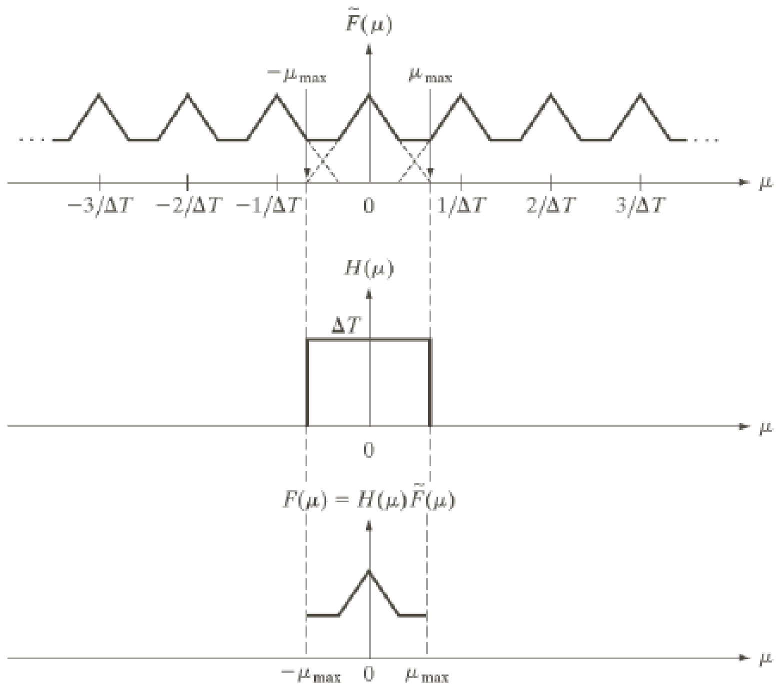
4.3.4 Aliasing

Effect of under-sampling a function

Transform corrupted by frequencies from adjacent periods

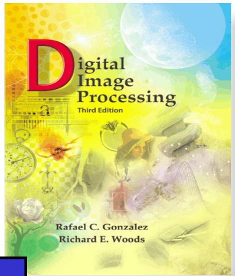
NB: the effect of aliasing can be reduced by smoothing the input function to attenuate its higher frequencies: *anti-aliasing*.

Has to be done *before* the sampling



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.



Unfortunately, except for some special cases mentioned below, aliasing is always present in sampled signals because, even if the original sampled function is band-limited, infinite frequency components are introduced the moment we limit the duration of the function, which we always have to do in practice. For example, suppose that we want to limit the duration of a band-limited function $f(t)$ to an interval, say $[0, T]$. We can do this by multiplying $f(t)$ by the function

$$h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{4.3-10}$$

This function has the same basic shape as Fig. 4.4(a) whose transform, $H(\mu)$, has frequency components extending to infinity, as Fig. 4.4(b) shows. From the convolution theorem we know that the transform of the product of $h(t)f(t)$ is the convolution of the transforms of the functions. Even if the transform of $f(t)$ is band-limited, convolving it with $H(\mu)$, which involves sliding one function across the other, will yield a result with frequency components extending to infinity. Therefore, no function of finite duration can be band-limited. Conversely, a function that is band-limited must extend from $-\infty$ to ∞.[†]

The period, P, of $\sin(\pi t)$ is 2 s, and its frequency is $1/P$, or $1/2$ cycles/s. According to the sampling theorem, we can recover this signal from a set of its samples if the sampling rate, $1/\Delta T$, exceeds twice the highest frequency of the signal. This means that a sampling rate greater than 1 sample/s $[2 \times (1/2) = 1]$, or $\Delta T < 1$ s, is required to recover the signal. Observe that sampling this signal at *exactly* twice the frequency (1 sample/s), with samples taken at $t = \ldots -1, 0, 1, 2, 3 \ldots$, results in $\ldots \sin(-\pi), \sin(0), \sin(\pi), \sin(2\pi), \ldots$, which are all 0. This illustrates the reason why the sampling theorem requires a sampling rate that exceeds twice the highest frequency, as mentioned earlier.

4.3.4 Aliasing

Illustration: sampling a sine wave of period 2s

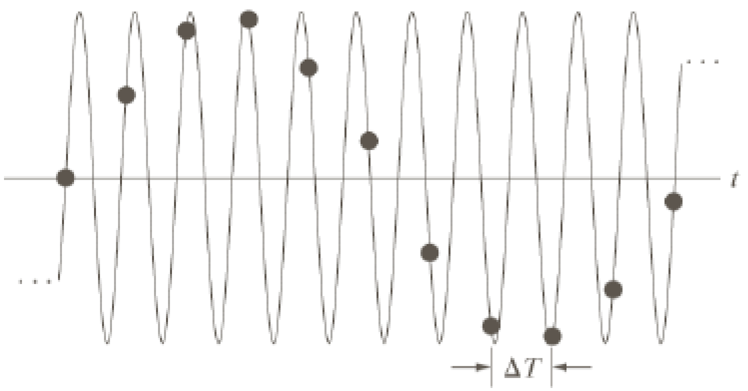


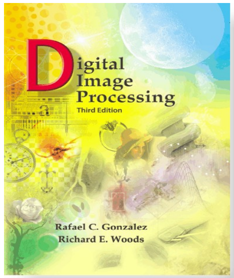
FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

4.3.5 Function Reconstruction (Recovery) from Sampled Data

Recovery expressed in the spatial domain:

$$F(\mu) = H(\mu)\tilde{F}(\mu) \Rightarrow \quad f(t) = FT^{-1}\left[F(\mu)\right] \; = h(t) \star \tilde{f}(t)$$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T) sinc\left[(t - n\Delta T)/ \; \Delta T\right]$$



4.5 Extension to Function of Two Variables

4.5.1 The 2-D impulse and its Sifting Property

Impulse of two continuous variables t and z :

$$\delta(t,z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

Impulse of two continuous variables t and z :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

4.5 Extension to Function of Two Variables

4.5.1 The 2-D impulse and its Sifting Property

Sifting property: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0,z-z_0)\,dt\,dz = f(t_0,z_0)$$

4.5 Extension to Function of Two Variables

4.5.1 The 2-D impulse and its Sifting Property

2-D discrete impulse: $$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Sifting property: $$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0) = f(x_0,y_0)$$

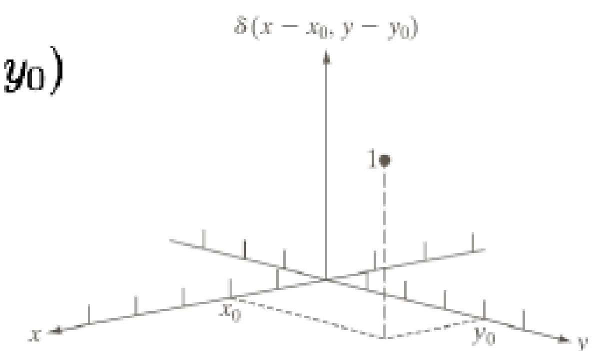
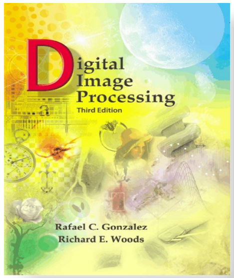


FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).



4.5 Extension to Function of Two Variables

4.5.2 The 2-D Continuous Fourier Transform Pair

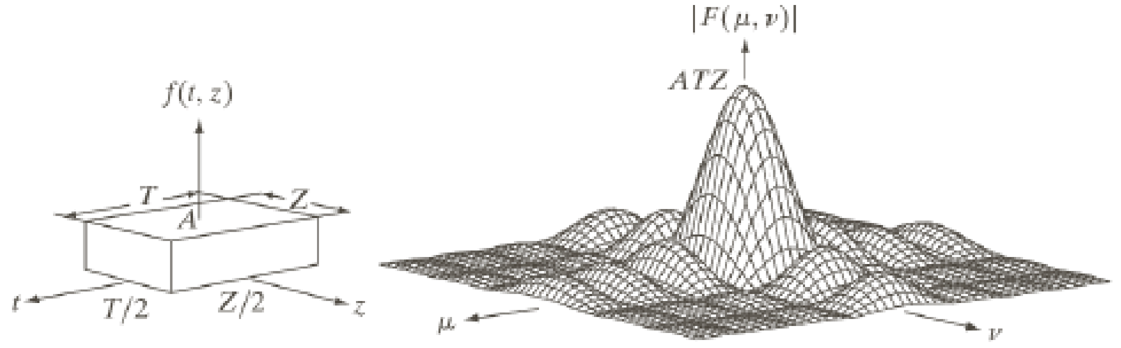
f(t,z) continuous function of two continuous variables *t* and *z*, the 2-D continuous Fourier transform pair is given by:

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)}\, dt\, dz$$

and

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)}\, d\mu\, d\nu$$

Example: Obtaining the 2-D Fourier transform of a simple function

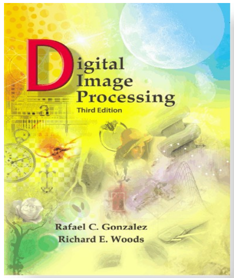


a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$



4.5.3 Two-Dimensional Sampling and the 2-D Sampling Theorem

Sampling in 2-D can be modeled using the sampling function (2-D impulse train):

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

Multiplying *f(t,z)* by $s_{\Delta T \Delta Z}(t,z)$ yields the sampled function

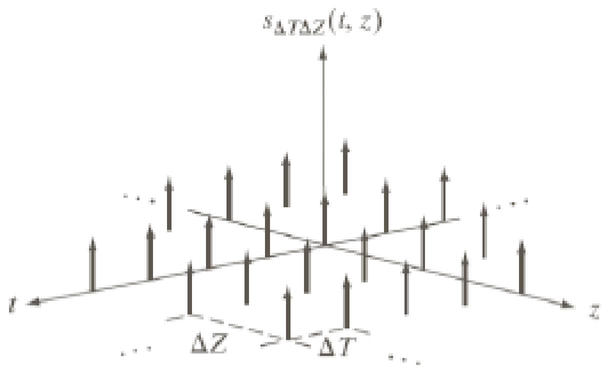


FIGURE 4.14 Two-dimensional impulse train.

f(t,z) is said to be *band-limited* if its Fourier transform is 0 outside of a rectangle $[-\mu_{max}, \mu_{max}]$ and $[-\nu_{max}, \nu_{max}]$

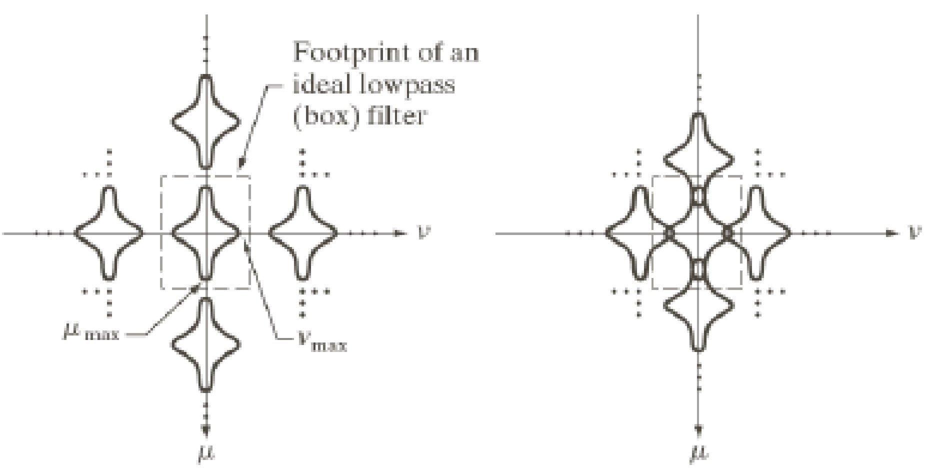
2-D sampling theorem:

A continuous, band-limited function *f(t,z)* can be recovered with no error from a set of its sample if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{max}}$$

and

$$\Delta Z < \frac{1}{2\nu_{max}}$$



a b

FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

4.4 The Discrete Fourier Transform (DFT) of One Variable

4.4.1 Obtaining the DFT from the Continuous Transform of a Sampled Function

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

f_n : discrete function