- Language of mathematical morphology: set theory
- Sets $\equiv$ objects in an image
- Binary images: sets $\in Z^2$
- Gray-scale images: sets $\in Z^3$

9.1 Preliminaries

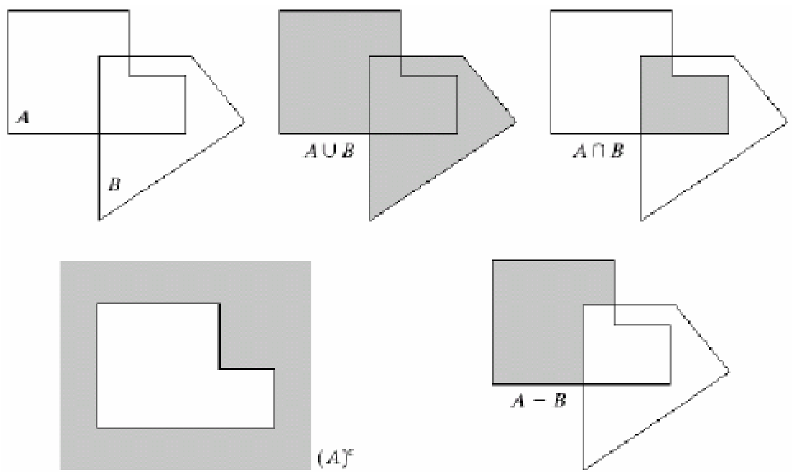
- Let A be a set in Z^2. If $a = (a_1, a_2)$ is an element of A, then we write $a \in A$
- Subset, union, intersection:

$$A \subseteq B,\ C = A \bigcup B,\ D = A \bigcap B$$

- Disjoint or mutually exclusive: $A \bigcap B = \emptyset$
- Complement: $A^c = \{w | w \notin A\}$
- Difference: $A - B = \{w | w \in A, w \notin B\} = A \bigcap B^c$

- **Reflection:** $\hat{B} = \{w | w = -b, \text{ for } b \in B\}$
- **Translation** of set A by point $z = (z_1, z_2)$: $(A)_z = \{c | c = a + z, \text{ for } a \in A\}$

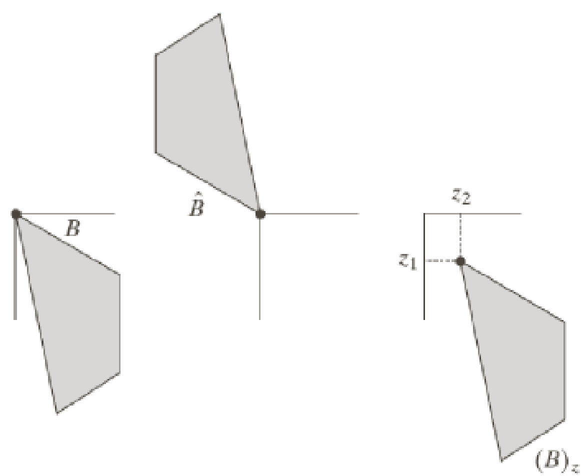
(Ed 2)



a b c
d e

FIGURE 9.1 (a) Two sets A and B. (b) The union of A and B. (c) The intersection of A and B. (d) The complement of A. (e) The difference between A and B.

(Ed 3)



a b c

FIGURE 9.1 (a) A set, (b) its reflection, and (c) its translation by z.

9.2 Erosion and dilation

These operations are fundamental to morphological processing

9.2.1 Erosion

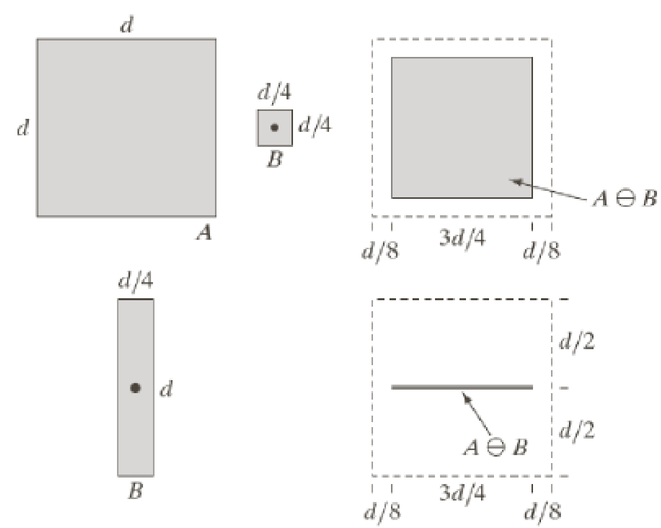
With A and B sets in Z^2, the erosion of A by B, is defined as

$$A \ominus B = \{z | (B)_z \subseteq A\}$$

or alternatively

$$A \ominus B = \left\{z | (B)_z \bigcap A^c = \emptyset\right\}$$

- **B is the SE**
- **Convolution process**



a b c
d e

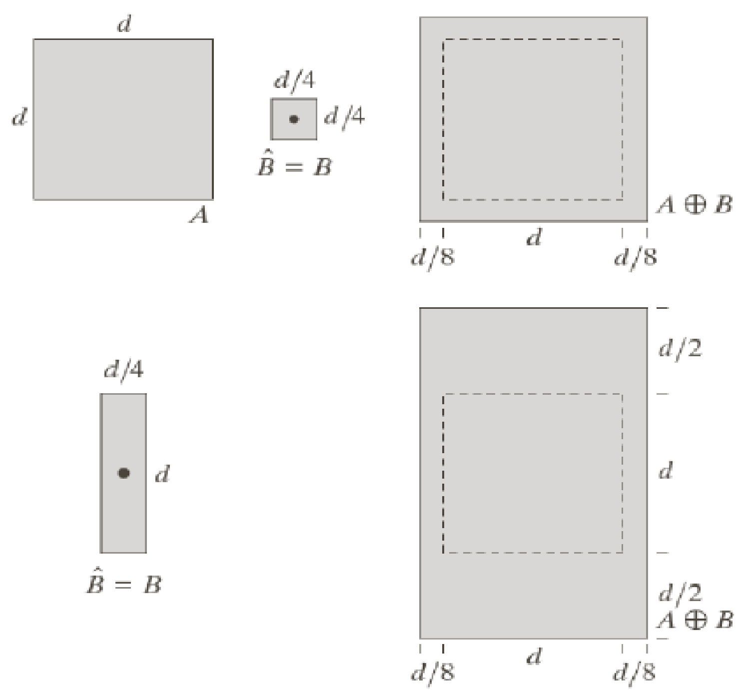
9.2.2 Dilation

With A and B sets in Z^2, the dilation of A by B, is defined as

$$A \oplus B = \left\{ z \middle| (\hat{B})_z \bigcap A \neq \emptyset \right\}$$

or alternatively

$$A \oplus B = \left\{ z \middle| \left[(\hat{B})_z \bigcap A \right] \subseteq A \right\}$$



a b c
d e

FIGURE 9.6 (a) Set A. (b) Square structuring element (the dot denotes the origin). (c) Dilation of A by B, shown shaded. (d) Elongated structuring element. (e) Dilation of A using this element. The dotted border in (c) and (e) is the boundary of set A, shown only for reference

9.2.3 Duality

Dilation and erosion are duals of each other with respect to set complementation and reflection, that is

$$\boxed{(A \ominus B)^c = A^c \oplus \hat{B}} \qquad (*)$$

and

$$\boxed{(A \oplus B)^c = A^c \ominus \hat{B}}$$

Proof of $(*)$:

$$\begin{aligned}
(A \ominus B)^c &= \{z | (B)_z \subseteq A\}^c \\
&= \{z | (B)_z \cap A^c = \emptyset\}^c \\
&= \{z | (B)_z \cap A^c \neq \emptyset\} \\
&= A^c \oplus \hat{B}
\end{aligned}$$

9.3 Opening and closing

USES

Opening: Smoothes the contour of an object
Breaks narrow isthmuses ("bridges")
Eliminates thin protrusions

Closing: Smoothes sections of contours
Fuses narrow breaks and long thin gulfs
Eliminates small holes in contours
Fills gaps in contours

Definitions

The opening of set A by structuring element B:

$$A \circ B = (A \ominus B) \oplus B$$

The closing of set A by structuring element B:

$$A \bullet B = (A \oplus B) \ominus B$$

Credits: Dr Hanno Coetzer, Department of Applied Mathematics, Stellenbosch University, S. Africa

Opening: $A \circ B = (A \ominus B) \oplus B$

Smoothes the contour of an object

Breaks narrow isthmuses ("bridges")

Eliminates thin protrusions

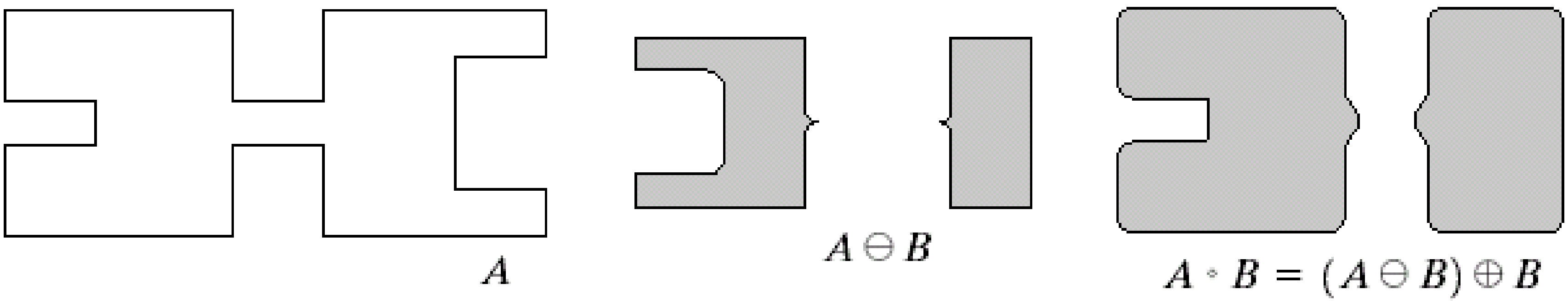
Closing: $A \bullet B = (A \oplus B) \ominus B$

Smoothes sections of contours

Fuses narrow breaks and long thin gulfs

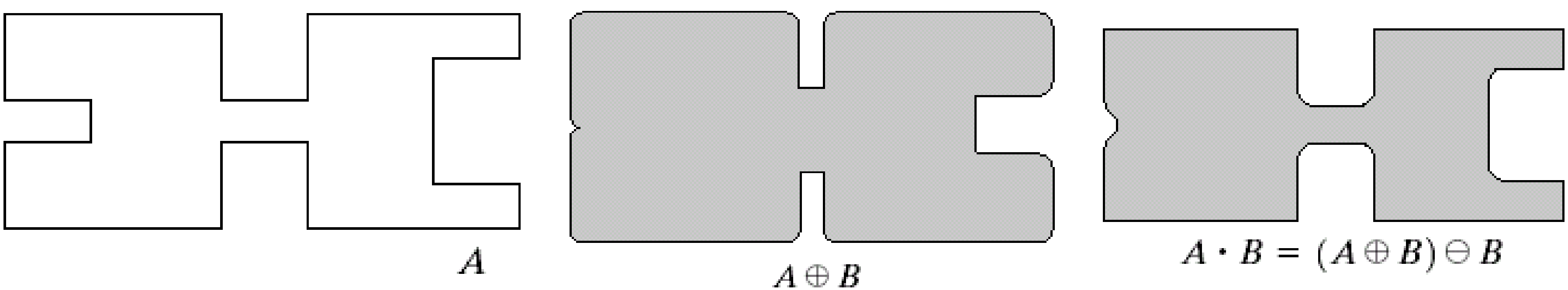
Eliminates small holes in contours

Fills gaps in contours



Original shape | After erosion | After dilation (**Opening**)

Outward pointing corners - rounded



Original shape | After dilation | After erosion (**Closing**)

Inward pointing corners - rounded

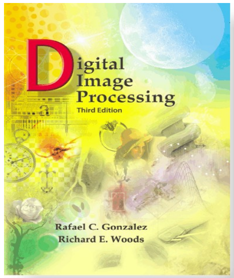


Illustration of opening…

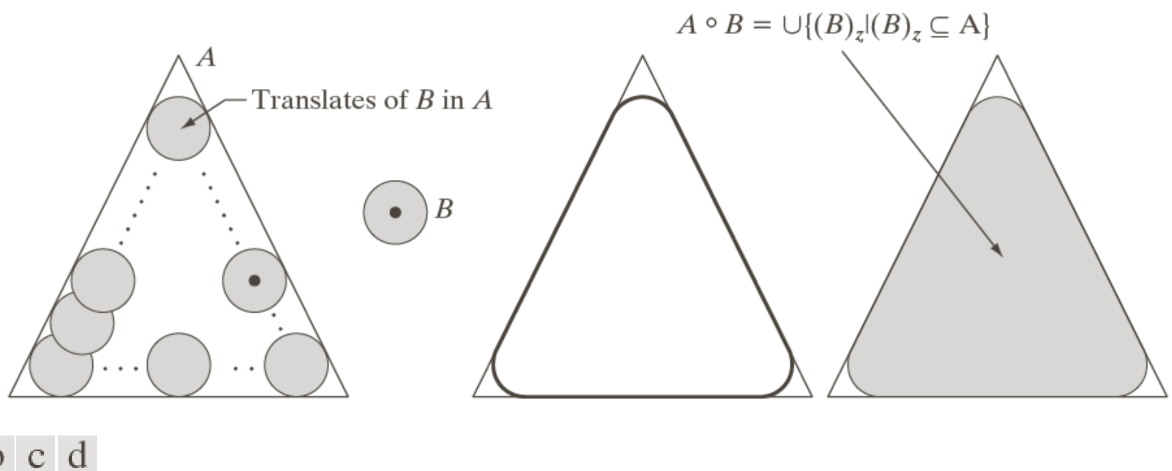


FIGURE 9.8 (a) Structuring element B "rolling" along the inner boundary of A (the dot indicates the origin of B). (b) Structuring element. (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded). We did not shade A in (a) for clarity.

Points in B that reach farthest into the boundary of A as B is rolled around inside of this boundary

Alternative definition for opening:

$$A \circ B = \bigcup \{(B)_z | (B)_z \subseteq A\}$$

Illustration of closing…

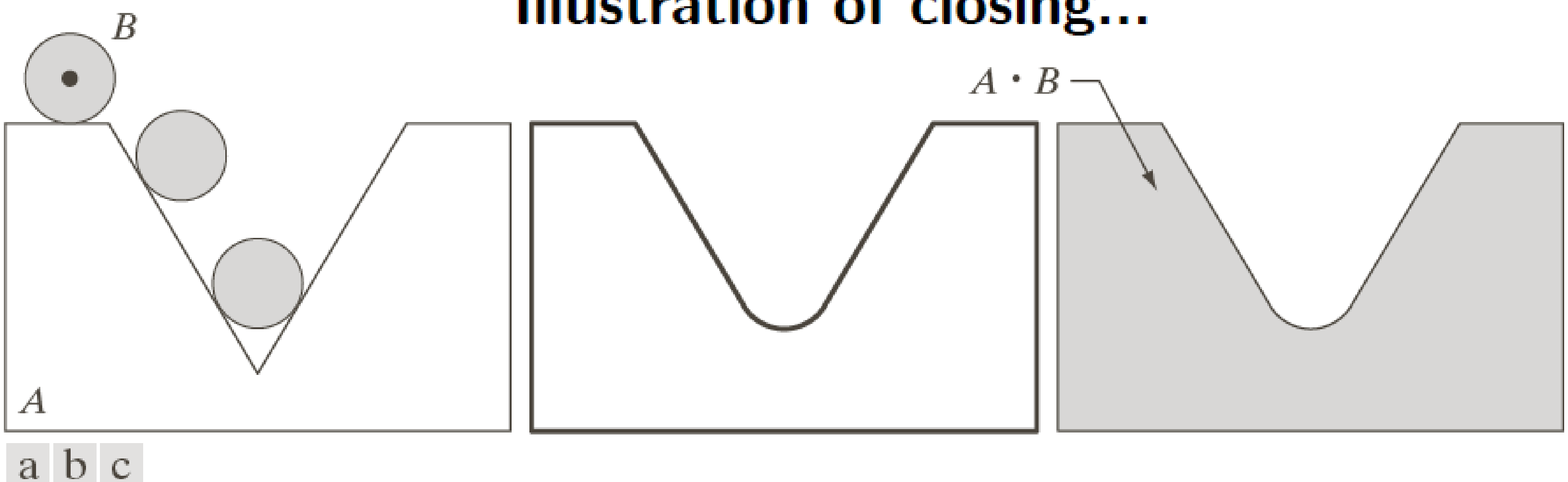
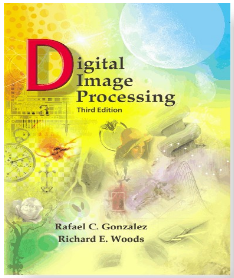


FIGURE 9.9 (a) Structuring element B "rolling" on the outer boundary of set A. (b) The heavy line is the outer boundary of the closing. (c) Complete closing (shaded). We did not shade A in (a) for clarity.

Alternative definition for closing:

A point w is an element of $A \bullet B$ if and only if $(B)_z \cap A \neq \emptyset$ for any translate of $(B)_z$ that contains w



Opening and closing are also duals of each other with respect to set complementation and reflection, that is

$$(A \bullet B)^c = A^c \circ \hat{B}$$

The opening operation satisfies the following properties:

(i) $A \circ B \subseteq A$ **(ii) If** $C \subseteq D$, **then** $C \circ B \subseteq D \circ B$ **(iii)** $(A \circ B) \circ B = A \circ B$

The closing operation satisfies the following properties:

(i) $A \subseteq A \bullet B$ **(ii) If** $C \subseteq D$, **then** $C \bullet B \subseteq D \bullet B$ **(iii)** $(A \bullet B) \bullet B = A \bullet B$

Hit or Miss Transformation

- **Objective is to find a disjoint region (set) in an image**
- **If B denotes the set composed of D and its background, the match/hit (or set of matches/hits) of B in A, is**

$$A \circledast B = (A \ominus D) \cap [A^c \ominus (W - D)]$$

This transform is considered as a basic tool for shape detection

- **Generalized notation:** $B = (B_1, B_2)$
  - B_1: **Set formed from elements of B associated with an object**
  - B_2: **Set formed from elements of B associated with the corresponding background**

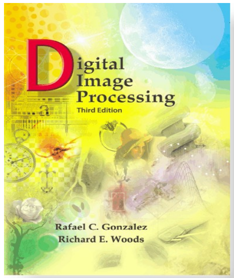
  [Preceding discussion: $B_1 = D$ **and** $B_2 = (W - D)$**]**

  - **More general definition:**

$$A \circledast B = (A \ominus B_1) - (A \oplus \hat{B}_2)$$

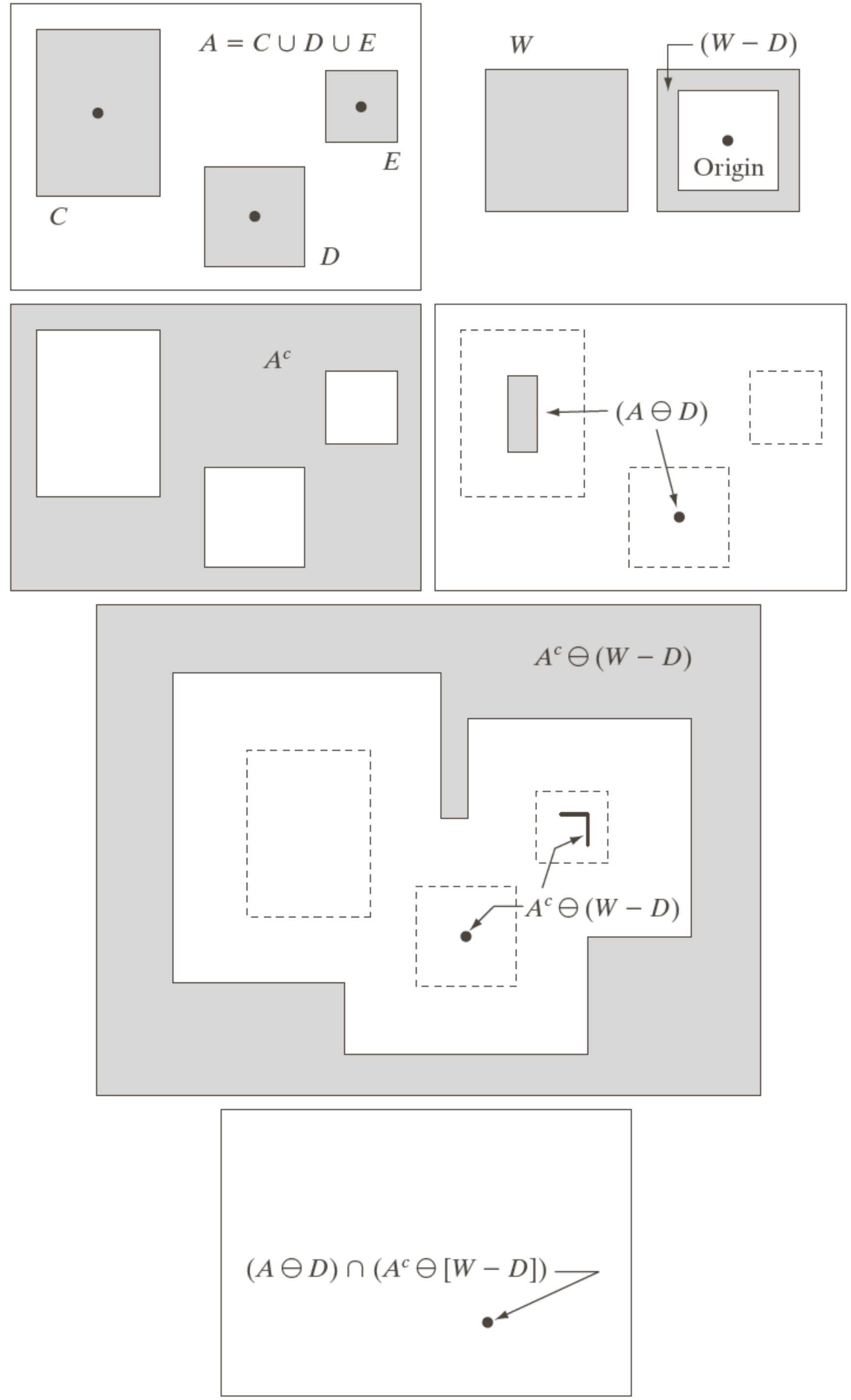
$$A \circledast B = (A \ominus B_1) \cap [A^c \ominus B_2]$$

  - $A \circledast B$ **contains all the origin points at which, simultaneously, B_1 found a hit in A and B_2 found a hit in A^c**



a b
c d
e
f

FIGURE 9.12
(a) Set A. (b) A window, W, and the local background of D with respect to W, $(W - D)$. (c) Complement of A. (d) Erosion of A by D. (e) Erosion of A^c by $(W - D)$. (f) Intersection of (d) and (e), showing the location of the origin of D, as desired. The dots indicate the origins of C, D, and E.



Morphological Algorithms

Principal applications of morphology is in extracting image components that are useful in the representation and description of shape.

We will consider morphological algorithms for extracting

– Boundaries, Connected Components, Convex hull , Skeleton of a region

We will also discuss some other methods like:

– Region filling, Thinning , Thickening, Pruning

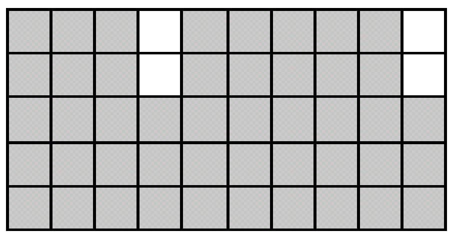
Credits: Dr. Brian Mac, Dublin Institute of Technology, Ireland

Boundary Extraction

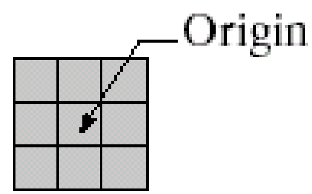
Extracting the boundary (or outline) of an object is often extremely useful

The boundary can be given simply as

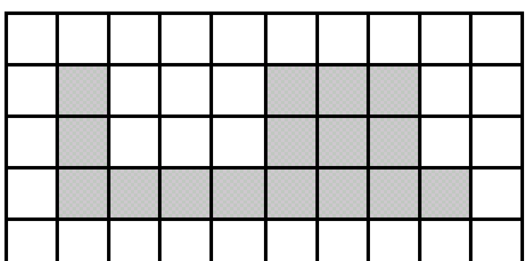
$$\beta(A) = A - (A \ominus B)$$



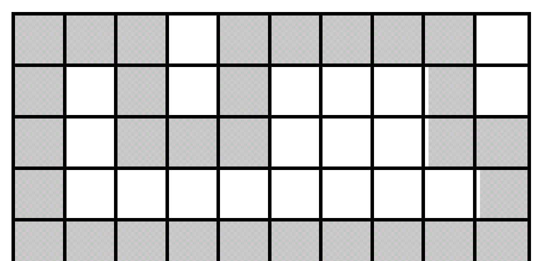
A



B



$A \ominus B$



$\beta(A)$

Region/Hole Filling

Region filling:

$$X_0 = P$$

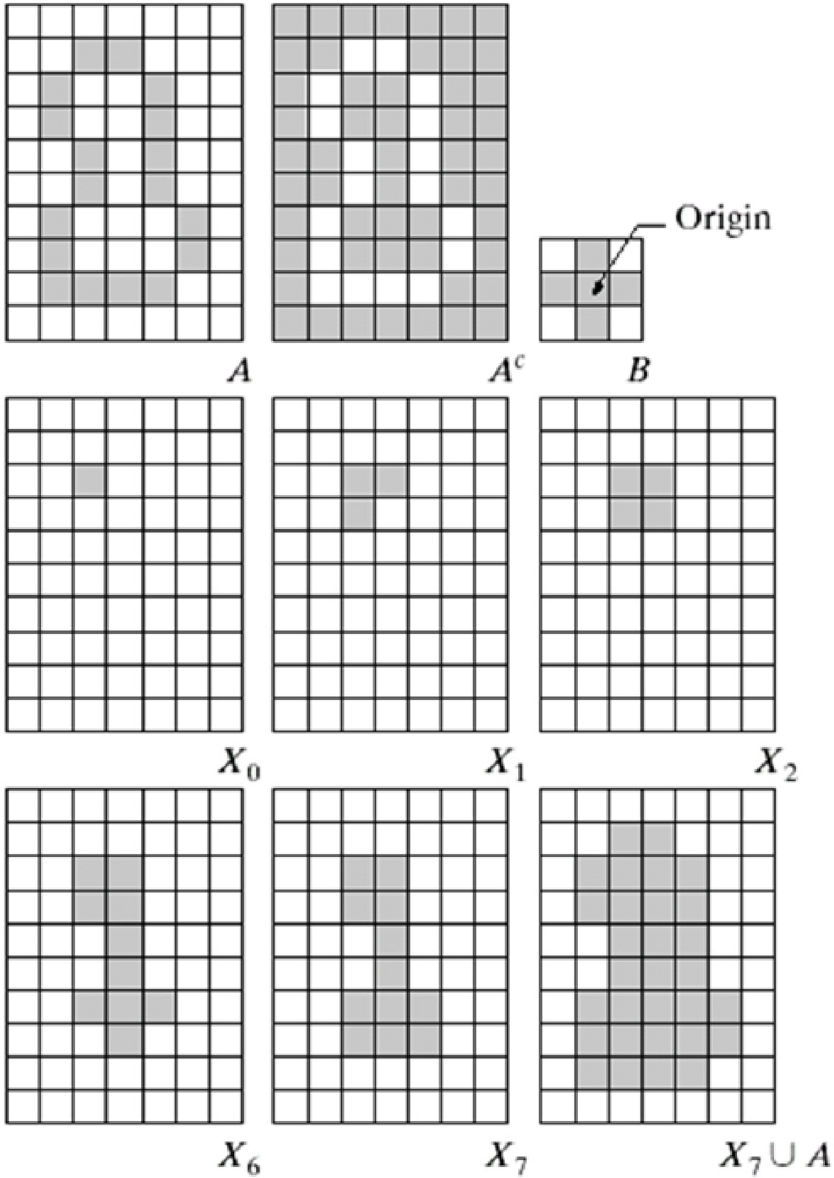
$$while\ X_k \neq X_{k-1}\ do$$

$$X_k = (X_{k-1} \oplus B) \cap A^C$$

$$X_F = X_k \cup A$$

The dilation would fill the whole area were it not for the intersection with Ac

→Conditional dilation



Extraction of Connected Components

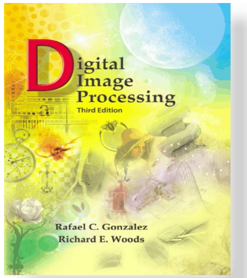
Let A be a set containing one or more connected components, and form an array X_0 (with the same size as A) whose elements are 0 (background), except at each location known to correspond to a point in each connected component in A, which is set to 1 (foreground)

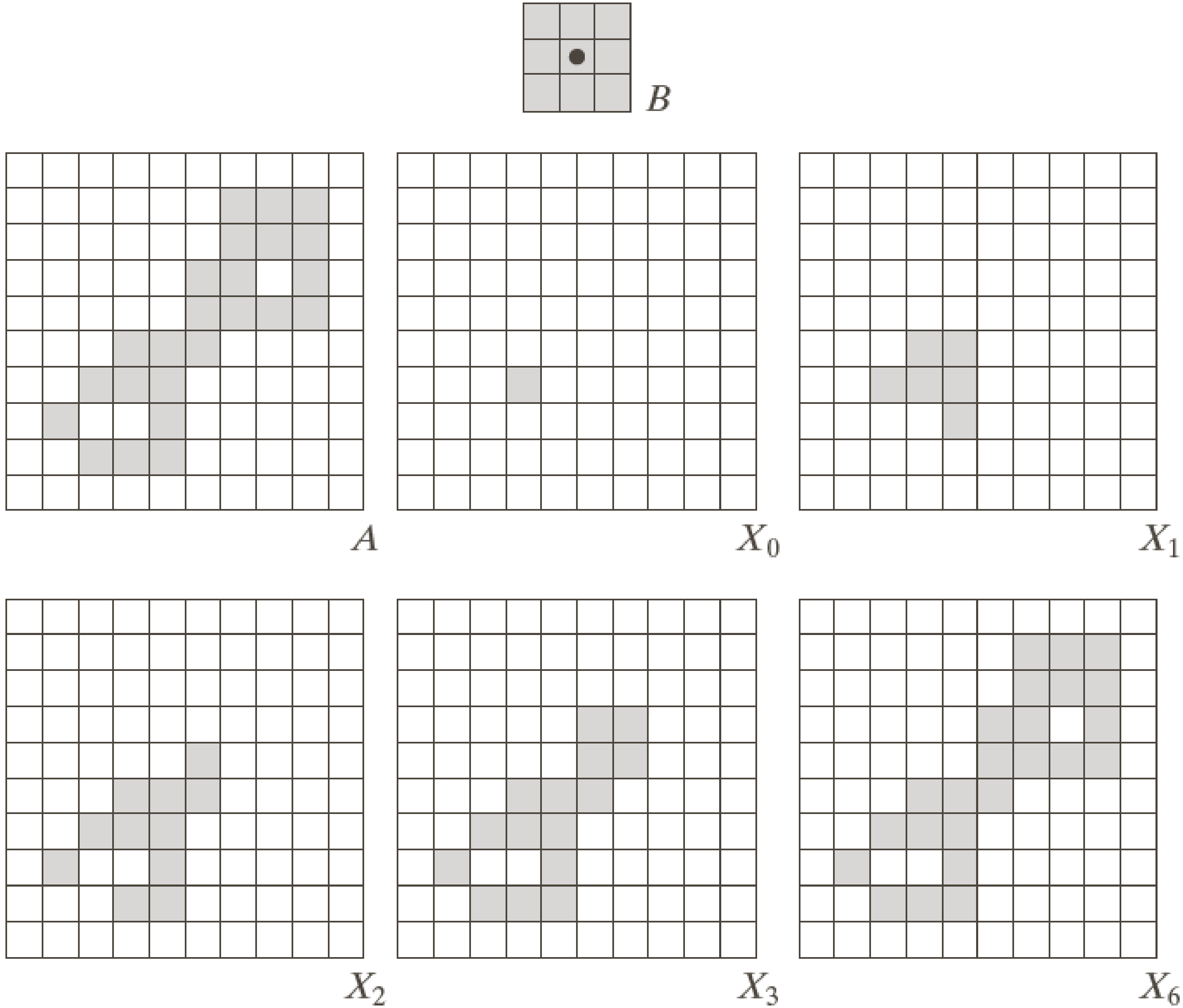
The following iterative procedure starts with X_0 and find all the connected components

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots,$$

where B is a suitable structuring element. When $X_k = X_{k-1}$, with X_k containing all the connected components, the procedure terminates

This algorithm is applicable to any finite number of sets of connected components contained in A, assuming that a point is known in each connected component





a
b c d
e f g

FIGURE 9.17 Extracting connected components. (a) Structuring element. (b) Array containing a set with one connected component. (c) Initial array containing a 1 in the region of the connected component. (d)–(g) Various steps in the iteration of Eq. (9.5-3).

Extraction of Connected Components

Hole filling

$$X_0 = P$$

$$while\ X_k \neq X_{k-1}\ do$$

$$X_k = (X_{k-1} \oplus B) \cap A^C$$

$$X_F = X_k \cup A$$

Extraction of connected components

$$X_0 = P$$

$$while\ X_k \neq X_{k-1}\ do$$

$$X_k = (X_{k-1} \oplus B) \cap A$$

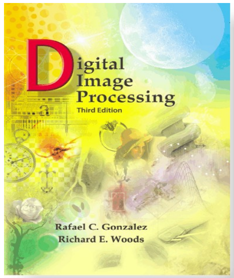
- **Intersection with A (not A^c) :** As we are looking for foreground points in CC-Extraction, but in Region-Filling, we were looking for background points

- Shape of structuring element used is based on 8-connectivity between pixels.

- This algorithm also assumes knowledge of the point within the connected component

Convex Hull

- A set A is said to be convex if the straight line segment joining any two points in A lies entirely within A
- The convex hull H of an arbitrary set S is the smallest convex set containing S
- H-S is called the convex deficiency of S
- The convex hull and convex deficiency are useful for object description, in some applications



Convex Hull

Morphological algorithm for obtaining the convex hull, $C(A)$, of a set A...

Let B^1, B^2, B^3 and B^4 represent the four structuring elements in Fig 9.19 (a), and then implement the equation ...

$$X_k^i = (X_{k-1} \circledast B^i) \cup A,\ \ i = 1,2,3,4,\ \ k = 1,2,\ldots,\ \ X_0^i = A$$

Now let $D^i = X_{\mathrm{conv}}^i$, where "conv" indicates convergence in the sense that $X_k^i = X_{k-1}^i$. Then the convex hull of A is

$$C(A) = \bigcup_{i=1}^{4} D^i$$

Here we are using simplified "Hit or miss" transform (no background match) i.e. erosion

Procedure illustrated in Fig 9.19: $\times$ entries indicate "don't care" conditions

Shortcoming of above algorithm: convex hull can grow beyond the minimum dimensions required to guarantee convexity

Possible solution: Limit growth so that it does not extend past the vertical and horizontal dimensions of the original set of points

Boundaries of greater complexity can be used to limit growth even further in images with more detail

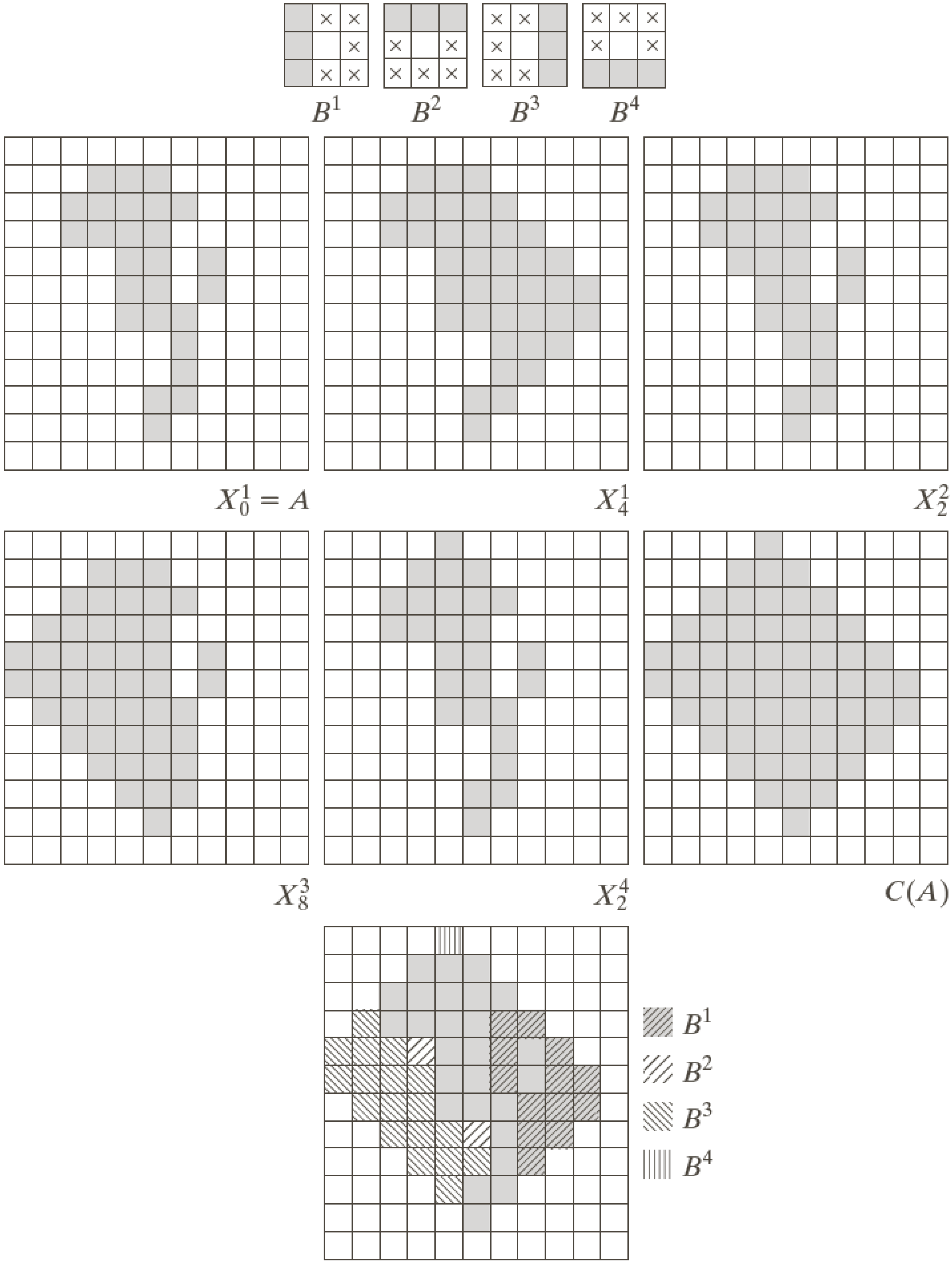


FIGURE 9.19 (a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

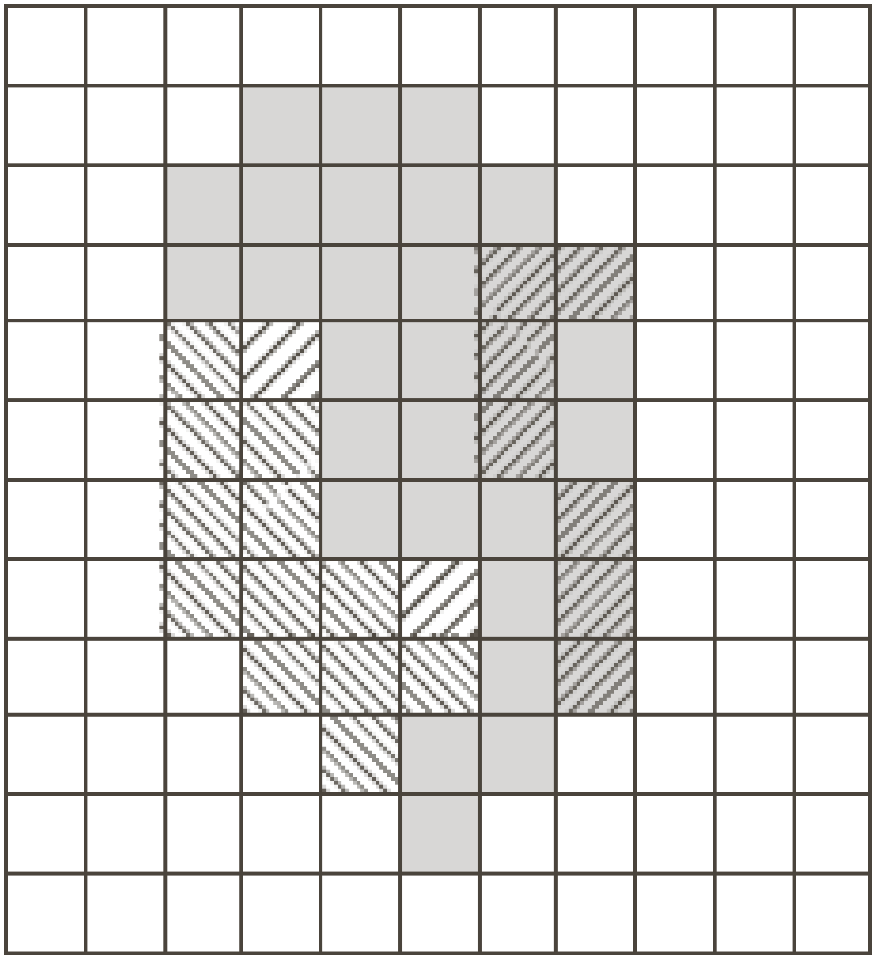
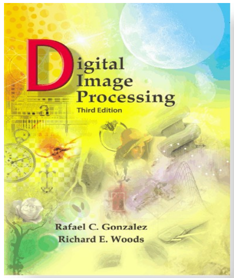
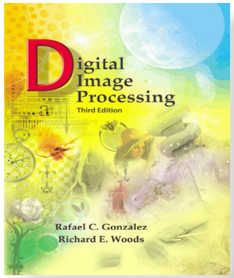


FIGURE 9.20 Result of limiting growth of the convex hull algorithm to the maximum dimensions of the original set of points along the vertical and horizontal directions.



Thinning

- Used to remove selected foreground pixels from binary images, somewhat like erosion or opening.
- Particularly useful for skeletonization. In this mode, it is commonly used to tidy up the output of edge detectors by reducing all lines to single pixel thickness.
- Most common use is to reduce the thresholded output of an edge detector such as the Sobel operator, to lines of a single pixel thickness, while preserving the full length of those lines (i.e. pixels at the extreme ends of lines should not be affected)
- Thinning is normally only applied to binary images, and produces another binary image as output.



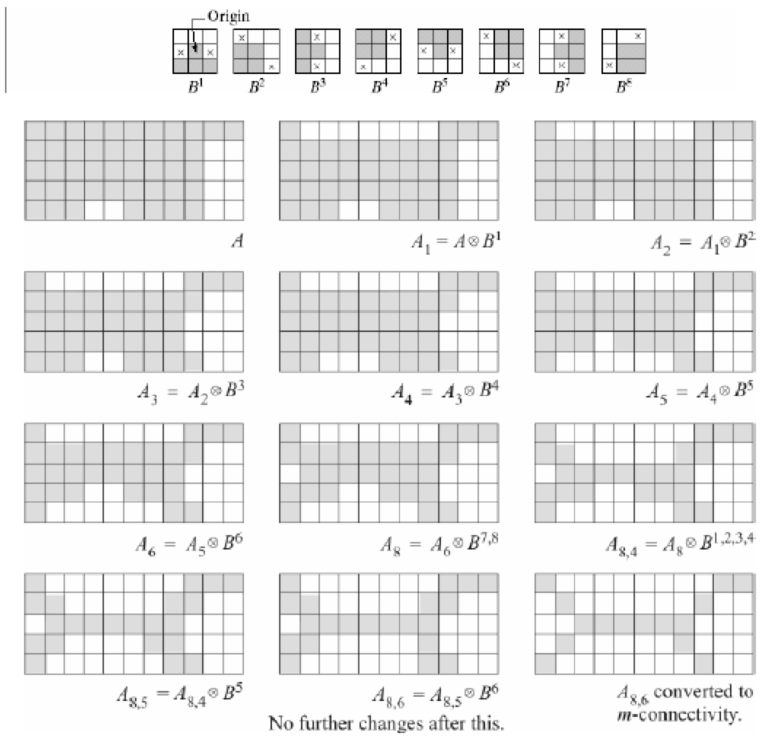
Thinning

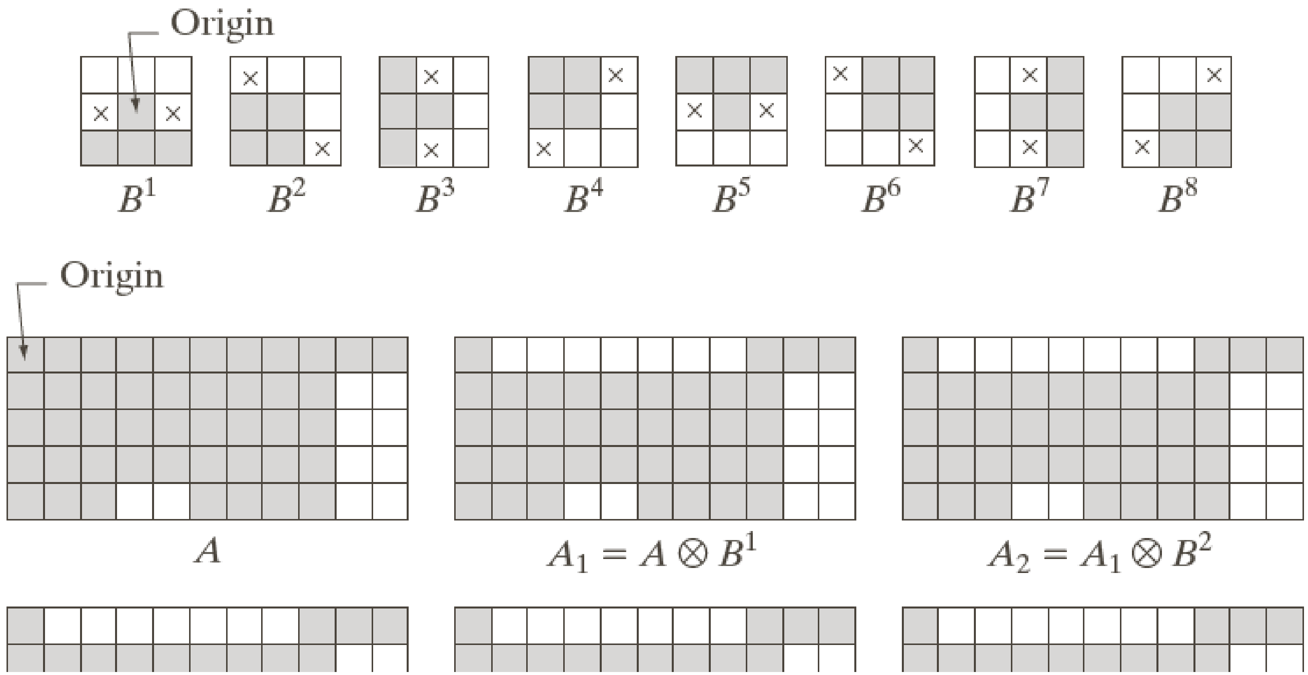
9.5.5 Thinning: The thinning of a set A by a structuring element B:

$$A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c$$

Symmetric thinning: Sequence of SEs, $\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$, where B^i is a rotated version of B^{i-1}

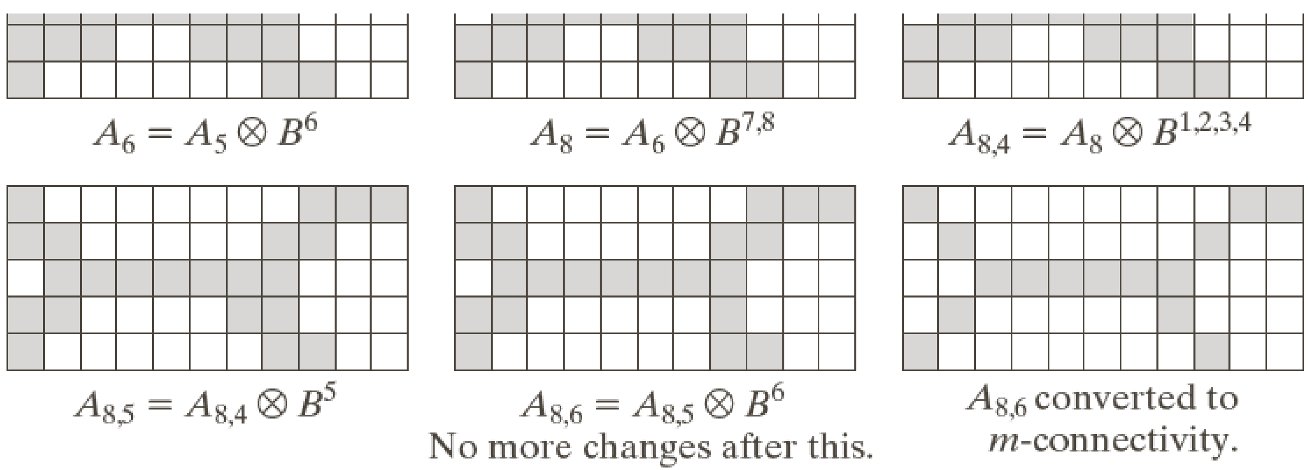
$$A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$$





	a	
b	c	d
e	f	g
h	i	j
k	l	m

FIGURE 9.21 (a) Sequence of rotated structuring elements used for thinning. (b) Set A. (c) Result of thinning with the first element. (d)–(i) Results of thinning with the next seven elements (there was no change between the seventh and eighth elements). (j) Result of using the first four elements again. (l) Result after convergence. (m) Conversion to m-connectivity.



Thinning

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0
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0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\rightarrow$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
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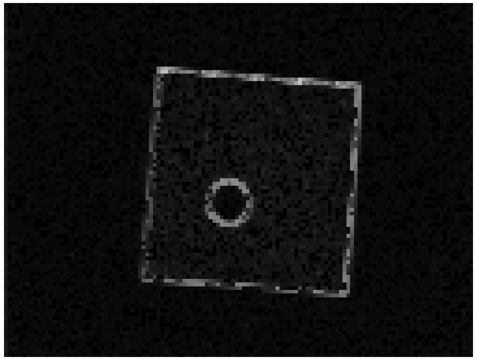
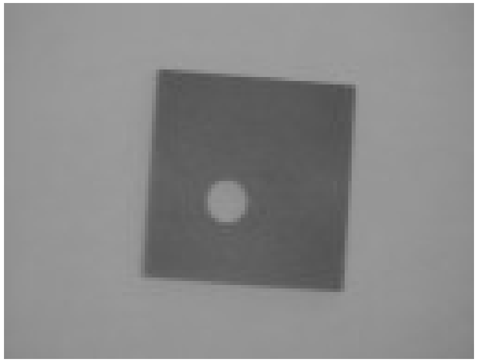
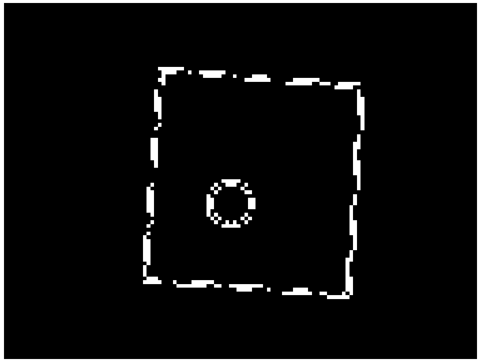
0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0



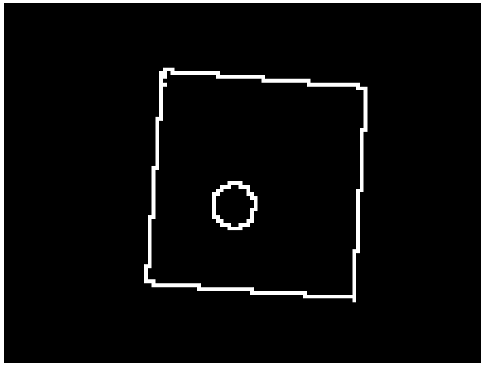
Given Image



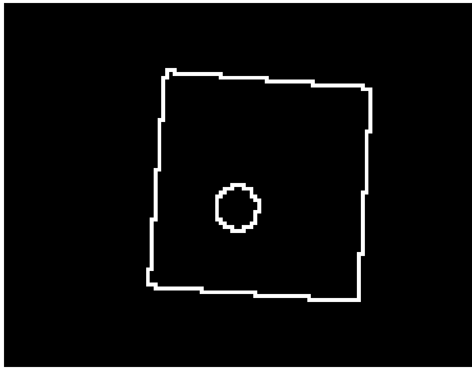
After applying
Sobel Operator



Thresholded



After Thining
(until convergence)



After Pruning

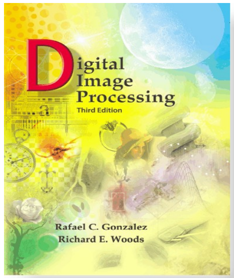
http://homepages.inf.ed.ac.uk/rbf/HIPR2/skeleton.htm

Thickening

- Used to grow selected regions of foreground pixels in binary images, somewhat like dilation or closing.

- It has several applications, including determining the approximate convex hull of a shape, and determining the skeleton by zone of influence.

- Thickening is normally only applied to binary images, and produces another binary image as output.



Thickening

9.5.6 Thickening: Thickening is the morphological dual of thinning and is defined by:

$$A \odot B = A \cup (A \circledast B),$$

where B is a structuring element

Similar to thinning: $A \odot \{B\} = ((\ldots((A \odot B^1) \odot B^2) \ldots) \odot B^n)$

Structuring elements for thickening are similar to those of Fig 9.21 (a), but with all 1's and 0's interchanged

A separate algorithm for thickening is seldom used in practice – we thin the background instead, and then complement the result

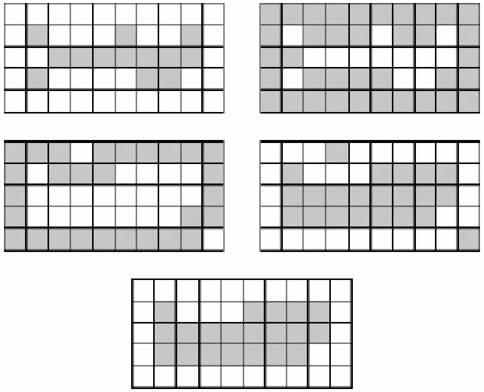
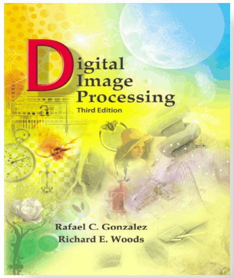


FIGURE 9.22 (a) Set A. (b) Complement of A. (c) Result of thinning the complement of A. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.



Thickening

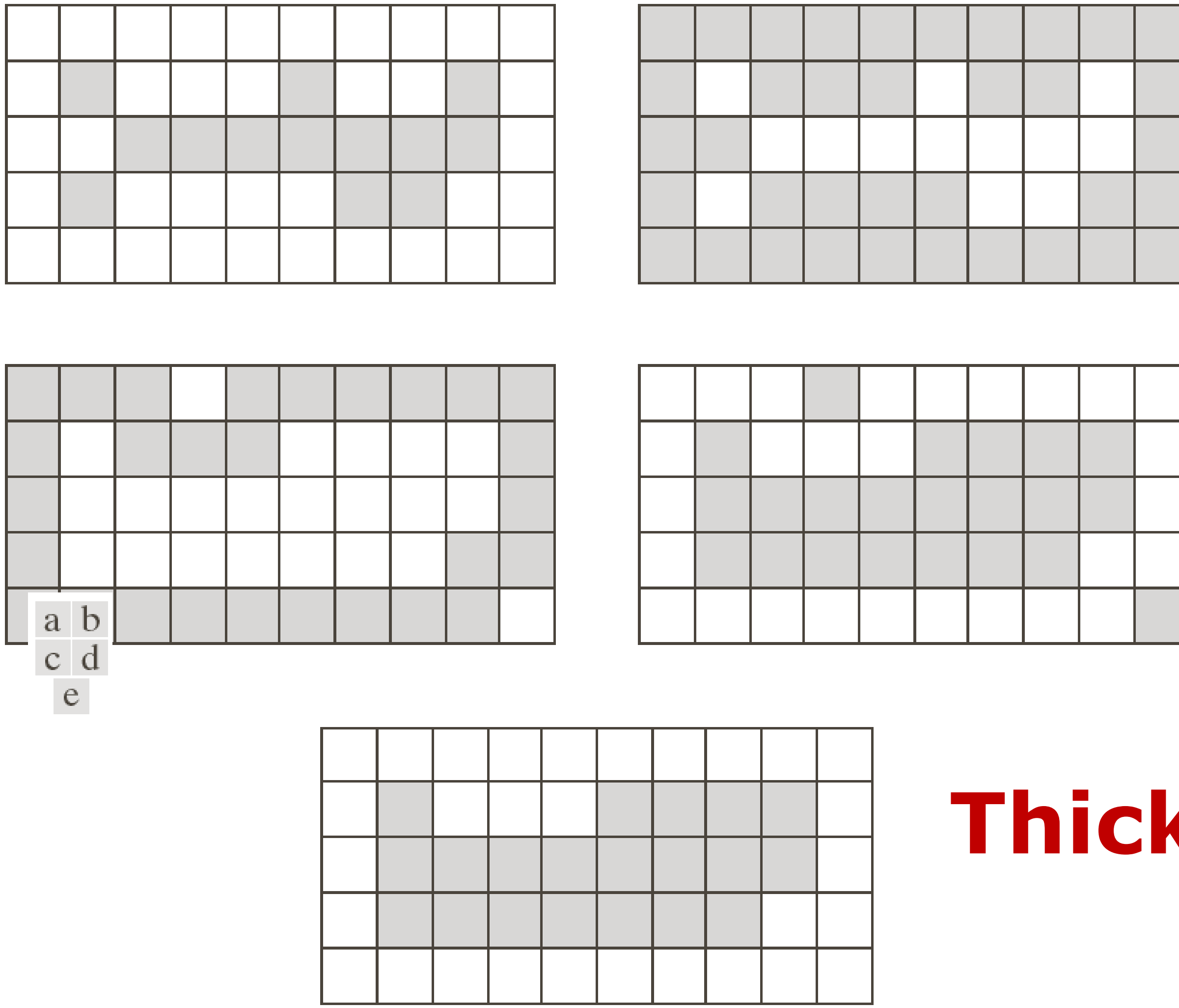
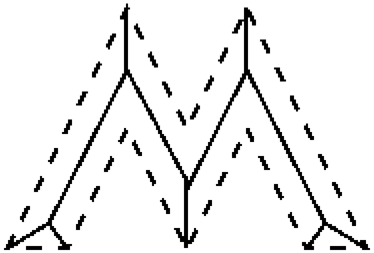


FIGURE 9.22 (a) Set A. (b) Complement of A. (c) Result of thinning the complement of A. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.

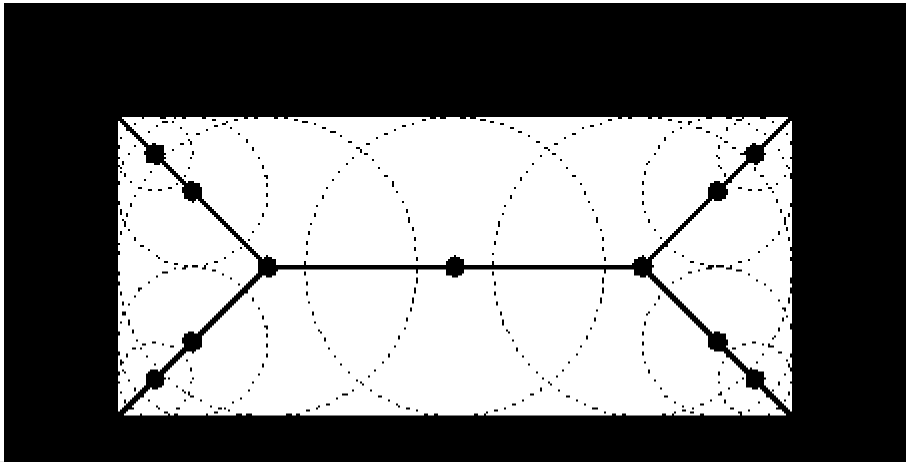
Skeletonization / Medial Axis Transform



• **Skeletonization** is a process for reducing foreground regions in a binary image to a **skeletal remnant that largely preserves the extent and connectivity of the original region** while throwing away most of the original foreground pixels.

•To see how this works, imagine that the foreground regions in the input binary image are made of some uniform slow-burning material. **Light fires simultaneously at all points along the boundary of this region** & watch the fire move into the interior. At points where the fire traveling from two different boundaries meets itself, the fire will extinguish itself and the points at which this happens form the so called `quench line'. This line is the skeleton. Under this definition it is clear that **thinning produces a sort of skeleton**.

•Another way to think about the skeleton is as **the loci of centers of bi-tangent circles that fit entirely within the foreground region being considered**.



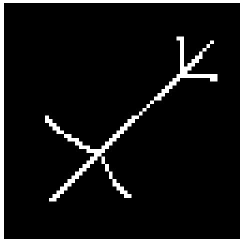
http://homepages.inf.ed.ac.uk/rbf/HIPR2/skeleton.htm

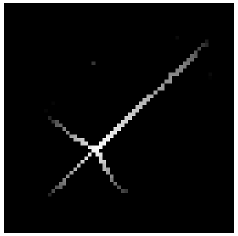
Skeletonization / Medial Axis Transform

• The terms medial axis transform (MAT) and skeletonization are often used interchangeably but we will distinguish between them slightly. The skeleton is simply a binary image showing the simple skeleton. The MAT on the other hand is a graylevel image where each point on the skeleton has an intensity which represents its distance to a boundary in the original object.

•The skeleton/MAT can be produced in two main ways.

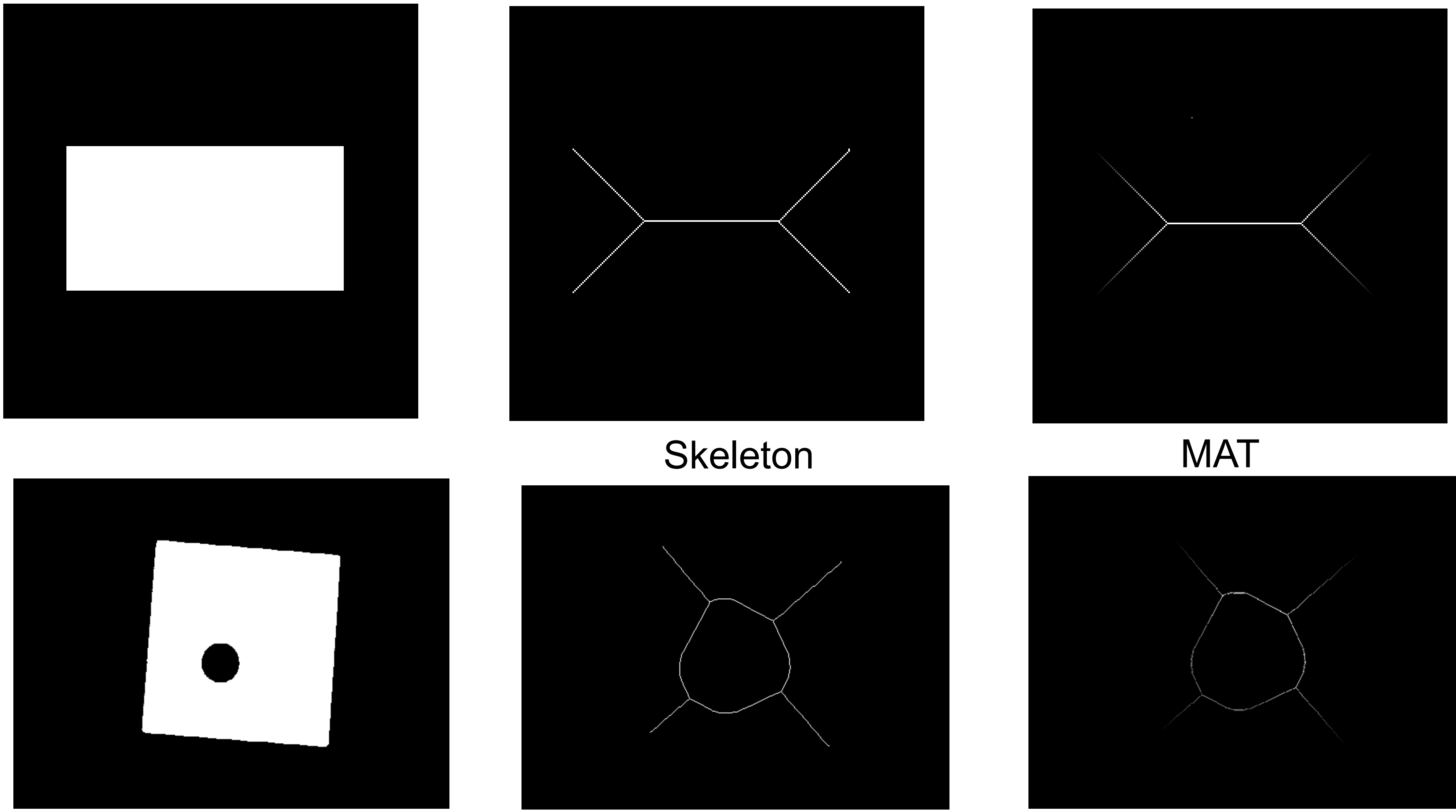
1. Use some kind of morphological thinning that successively erodes away pixels from the boundary (while preserving the end points of line segments) until no more thinning is possible, at which point what is left approximates the skeleton.
2. First calculate the distance transform of the image. The skeleton then lies along the singularities (i.e. creases or curvature discontinuities) in the distance transform. This latter approach is more suited to calculating the MAT since the MAT is the same as the distance transform but with all points off the skeleton suppressed to zero.





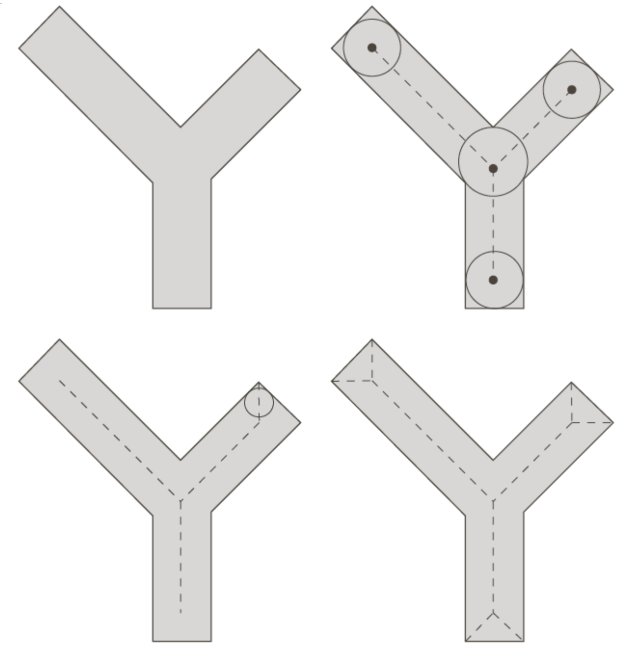
http://homepages.inf.ed.ac.uk/rbf/HIPR2/skeleton.htm

Skeletonization / Medial Axis Transform



http://homepages.inf.ed.ac.uk/rbf/HIPR2/skeleton.htm

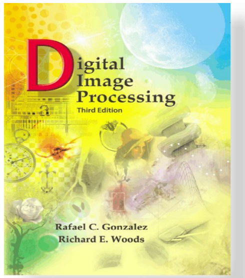
Skeletons

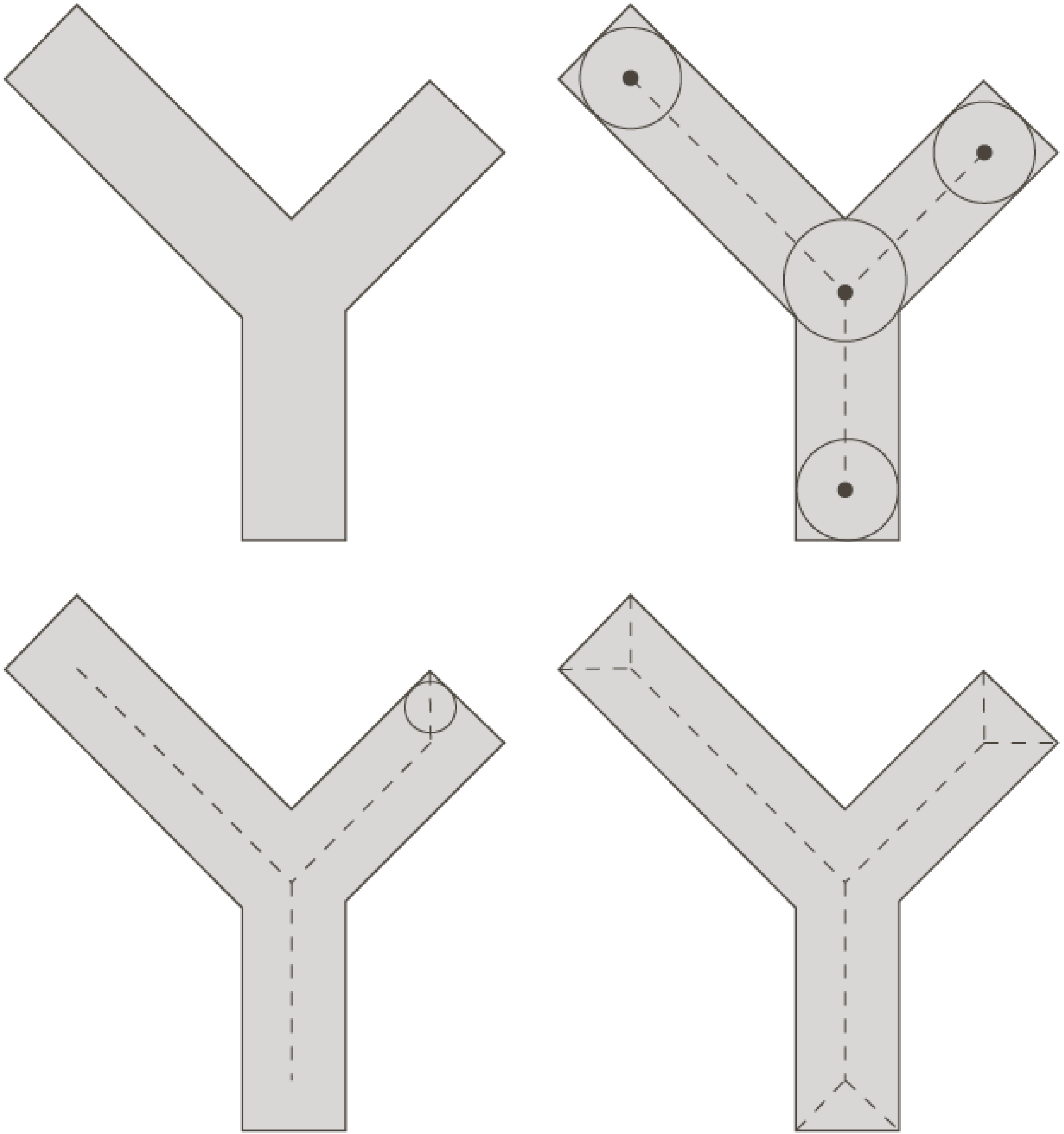


Given a set A, the following figure gives intuitively simple notion of a skeleton S(A) :

(a) If z is a point of $S(A)$ and $(D)_z$ is the largest disk centered at z and contained in A, one cannot find a larger disk (not necessarily centered at z) containing $(D)_z$ and included in A. The disk $(D)_z$ is called a *maximum disk*.

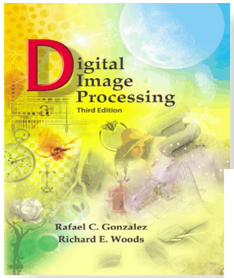
(b) The disk $(D)_z$ touches the boundary of A at two or more different places.





a	b
c	d

FIGURE 9.23 (a) Set A. (b) Various positions of maximum disks with centers on the skeleton of A. (c) Another maximum disk on a different segment of the skeleton of A. (d) Complete skeleton.



Skeletons

The skeleton of A can be expressed in terms of erosions and openings. That is, it can be shown (Serra [1982]) that

$$S(A) = \bigcup_{k=0}^{K} S_k(A) \qquad (9.5\text{-}11)$$

with

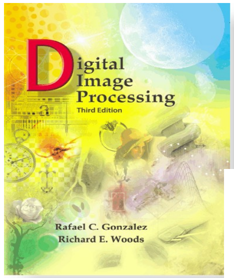
$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B \qquad (9.5\text{-}12)$$

where B is a structuring element, and $(A \ominus kB)$ indicates k successive erosions of A:

$$(A \ominus kB) = ((\ldots((A \ominus B) \ominus B) \ominus \ldots) \ominus B) \qquad (9.5\text{-}13)$$

k times, and K is the last iterative step before A erodes to an empty set. In other words,

$$K = \max\{k \mid (A \ominus kB) \neq \varnothing\} \qquad (9.5\text{-}14)$$



Skeletons

$$S(A) = \bigcup_{k=0}^{K} S_k(A) \tag{9.5-11}$$

with

$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B \tag{9.5-12}$$

The formulation given in Eqs. (9.5-11) and (9.5-12) states that $S(A)$ can be obtained as the union of the *skeleton subsets* $S_k(A)$. Also, it can be shown that A can be *reconstructed* from these subsets by using the equation

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB) \tag{9.5-15}$$

where $(S_k(A) \oplus kB)$ denotes k successive dilations of $S_k(A)$; that is,

$$(S_k(A) \oplus kB) = ((\ldots((S_k(A) \oplus B) \oplus B) \oplus \ldots) \oplus B) \tag{9.5-16}$$

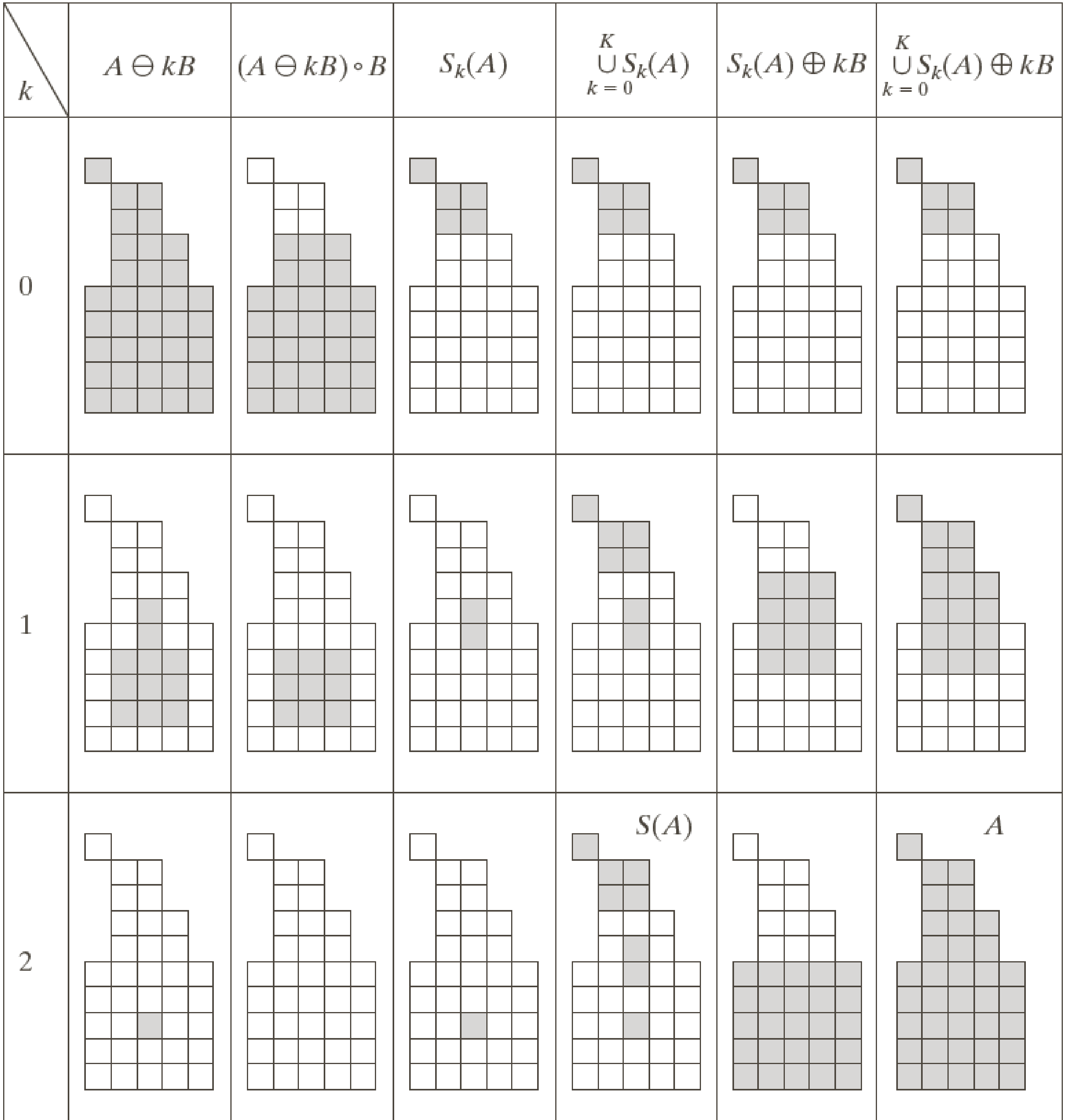
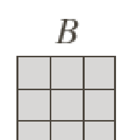


FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.



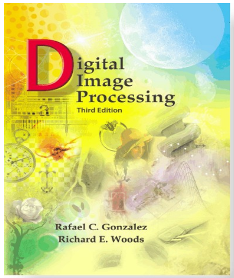
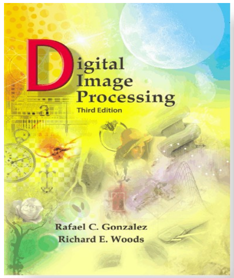


TABLE 9.1
Summary of morphological operations and their properties.

Operation	**Equation**	**Comments** (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Translation	$(B)_z = \{w \mid w = b + z, \text{ for } b \in B\}$	Translates the origin of B to point z.
Reflection	$\hat{B} = \{w \mid w = -b, \quad \text{for } b \in B\}$	Reflects all elements of B about the origin of this set.
Complement	$A^c = \{w \mid w \notin A\}$	Set of points not in A.
Difference	$A - B = \{w \mid w \in A, w \notin B\} = A \cap B^c$	Set of points that belong to A but not to B.
Dilation	$A \oplus B = \{z \mid (\hat{B}_z) \cap A \neq \varnothing\}$	"Expands" the boundary of A. (I)
Erosion	$A \ominus B = \{z \mid (B)_z \subseteq A\}$	"Contracts" the boundary of A. (I)
Opening	$A \circ B = (A \ominus B) \oplus B$	Smoothes contours, breaks narrow isthmuses, and eliminates small islands and sharp peaks. (I)

(Continued)

Operation	Equation	Comments (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Closing	$A \yen B = (A \oplus B) \ominus B$	Smoothes contours, fuses narrow breaks and long thin gulfs, and eliminates small holes. (I)
Hit-or-miss transform	$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$ $= (A \ominus B_1) - (A \oplus \hat{B}_2)$	The set of points (coordinates) at which, simultaneously, B_1 found a match ("hit") in A and B_2 found a match in A^c
Boundary extraction	$\beta(A) = A - (A \ominus B)$	Set of points on the boundary of set A. (I)
Hole filling	$X_k = (X_{k-1} \oplus B) \cap A^c$; $k = 1, 2, 3, \ldots$	Fills holes in A; $X_0 =$ array of 0s with a 1 in each hole. (II)
Connected components	$X_k = (X_{k-1} \oplus B) \cap A$; $k = 1, 2, 3, \ldots$	Finds connected components in A; $X_0 =$ array of 0s with a 1 in each connected component. (I)



Convex hull	$X_k^i = (X_{k-1}^i \circledast B^i) \cup A;$ $i = 1, 2, 3, 4;$ $k = 1, 2, 3, \ldots;$ $X_0^i = A;$ and $D^i = X_{\text{conv}}^i$	Finds the convex hull $C(A)$ of set A, where "conv" indicates convergence in the sense that $X_k^i = X_{k-1}^i$. (III)
Thinning	$A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c$ $A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$ $\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$	Thins set A. The first two equations give the basic definition of thinning. The last equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)
Thickening	$A \odot B = A \cup (A \circledast B)$ $A \odot \{B\} = ((\ldots(A \odot B^1) \odot B^2 \ldots) \odot B^n)$	Thickens set A. (See preceding comments on sequences of structuring elements.) Uses IV with 0s and 1s reversed.
Skeletons	$S(A) = \bigcup_{k=0}^{K} S_k(A)$ $S_k(A) = \bigcup_{k=0}^{K} \{(A \ominus kB) - [(A \ominus kB) \circ B]\}$ Reconstruction of A: $A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$	Finds the skeleton $S(A)$ of set A. The last equation indicates that A can be reconstructed from its skeleton subsets $S_k(A)$. In all three equations, K is the value of the iterative step after which the set A erodes to the empty set. The notation $(A \ominus kB)$ denotes the kth iteration of successive erosions of A by B. (I)