Sample Path Analysis of Flow Processes

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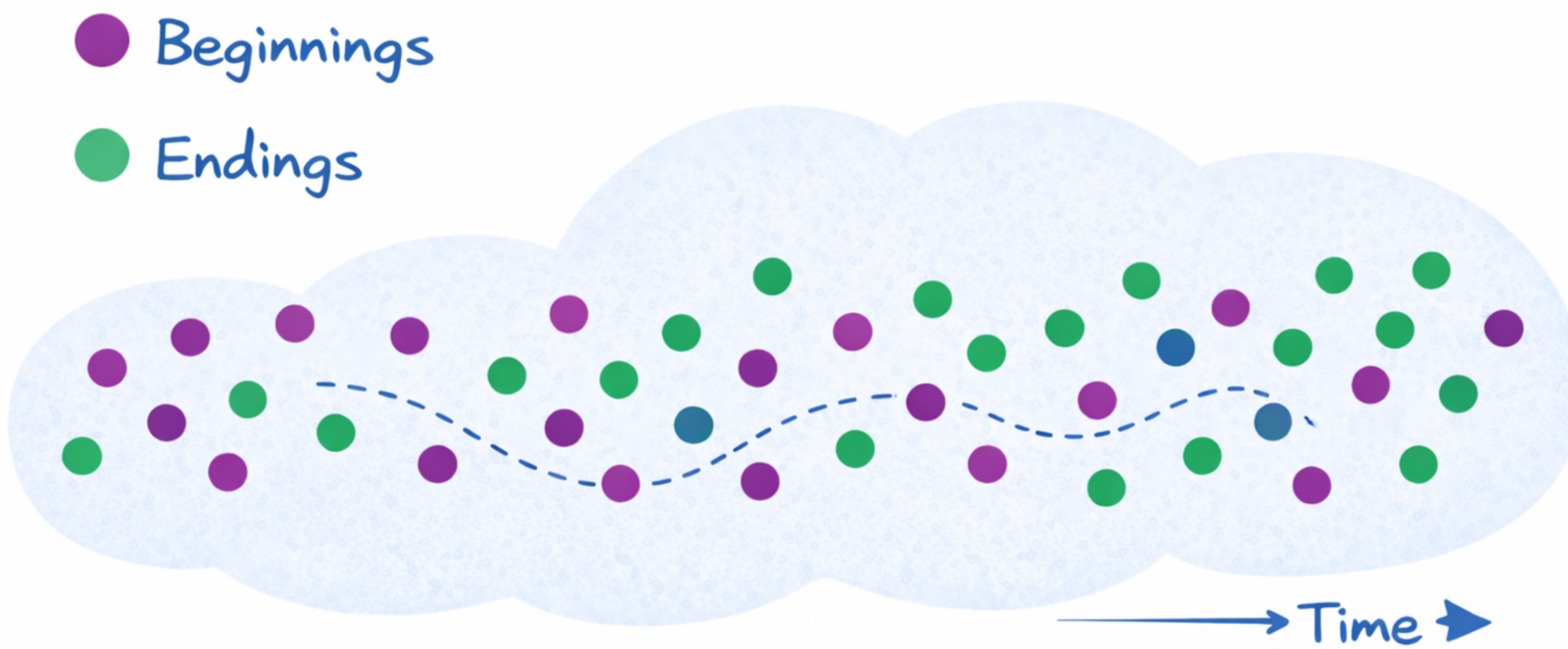
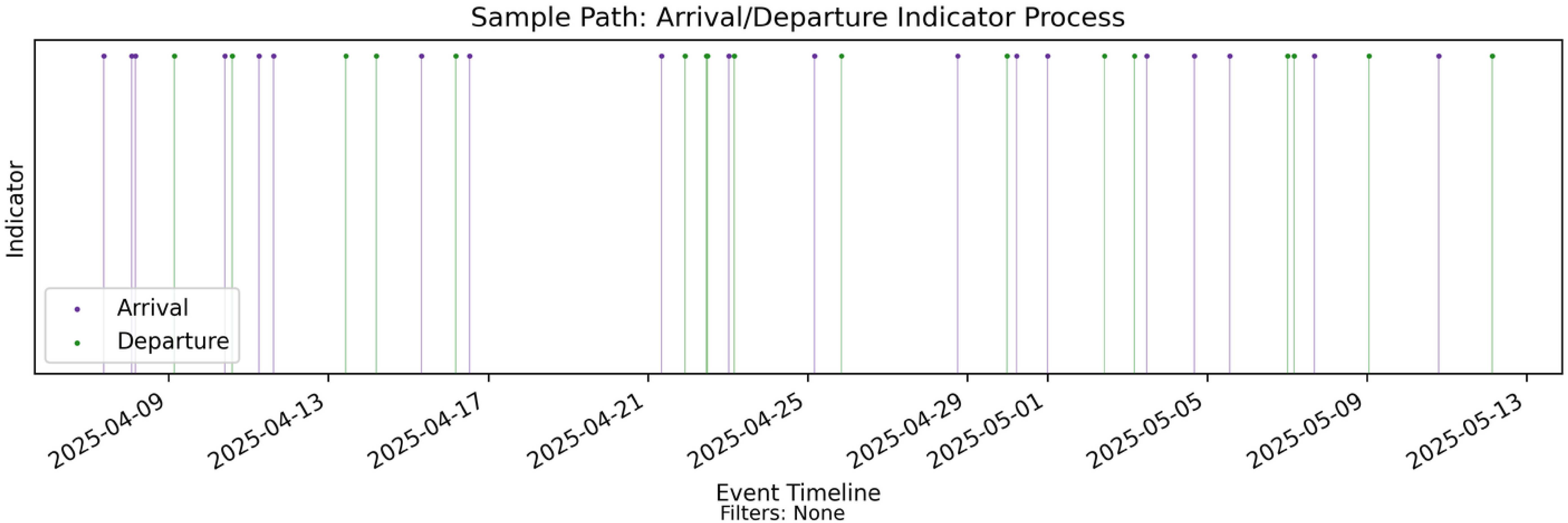
The Domain

Processes with well-defined beginnings and endings,
over a set of elements



Mathematically, these are marked point processes

(timestamps with marks attached)

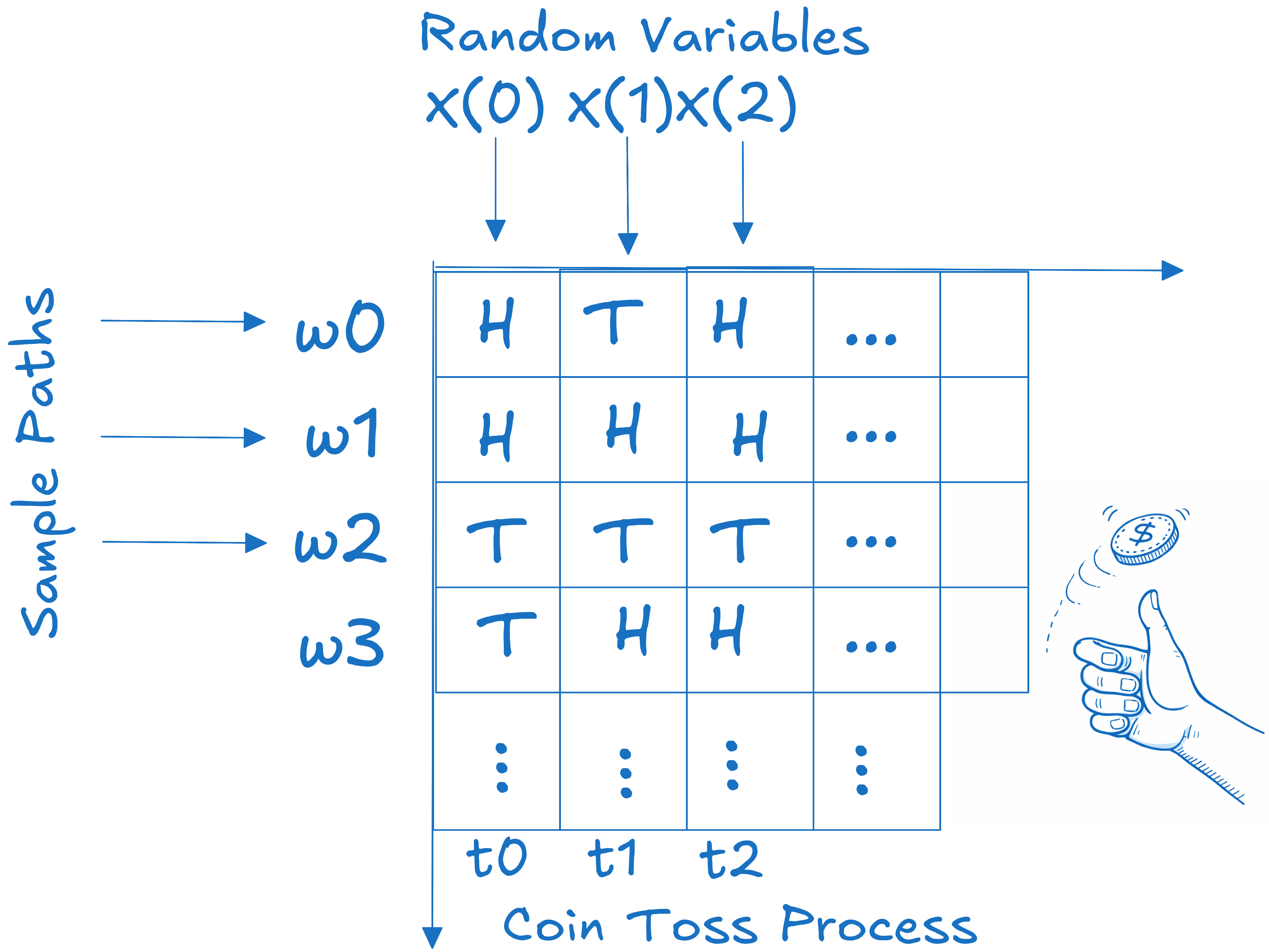


The diagram above is a sample path for a flow process

What is a sample path?

The term comes from stochastic process theory
(processes with randomness baked in)

Consider the simple coin toss



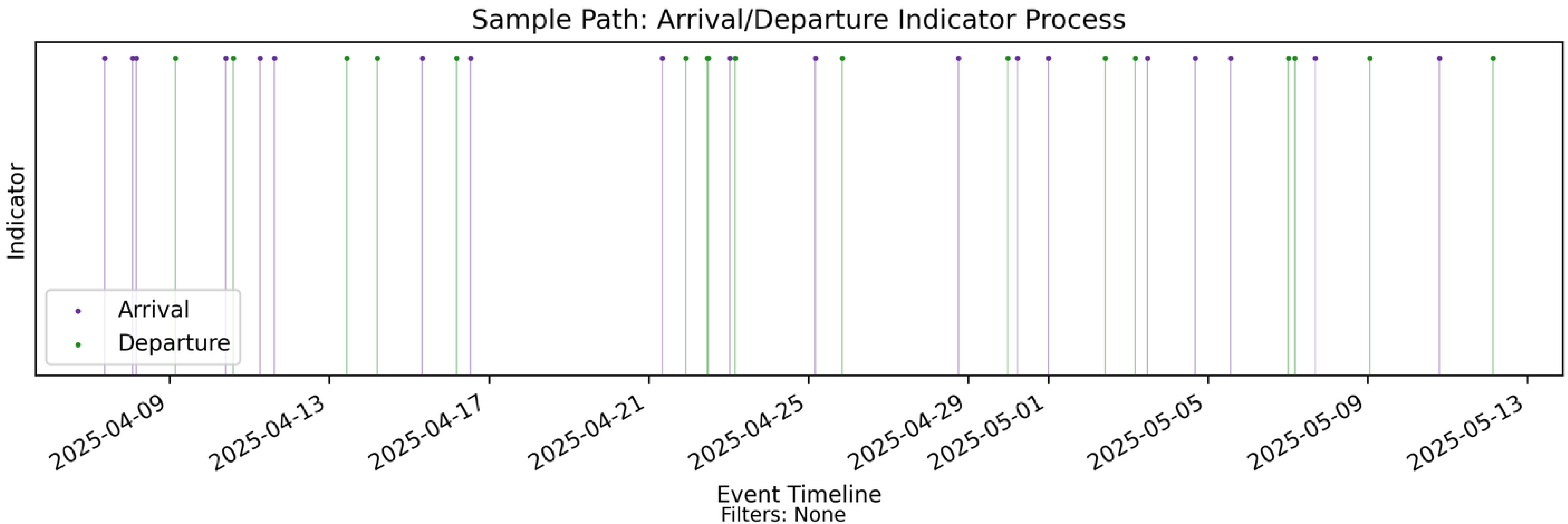
Each sequence of tosses is a sample path

Each toss is a random variable

A flow process has two dimensions of randomness

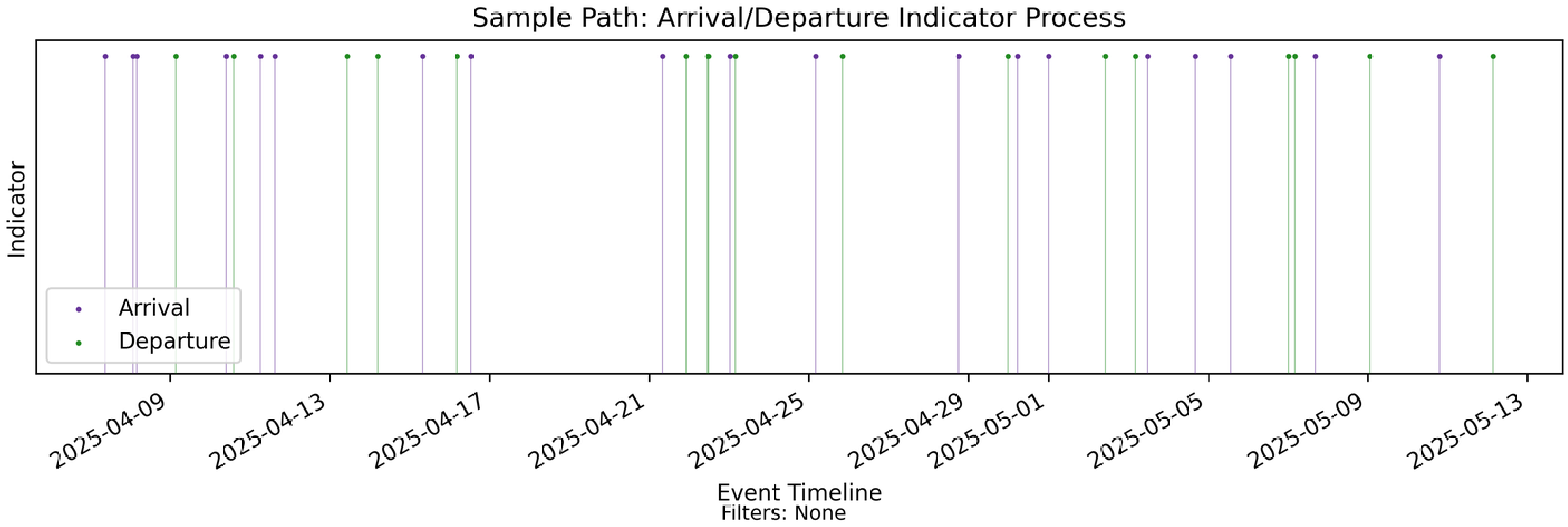
- The kind of event (the mark)
- The time between events

(One is a discrete outcome, other is time)

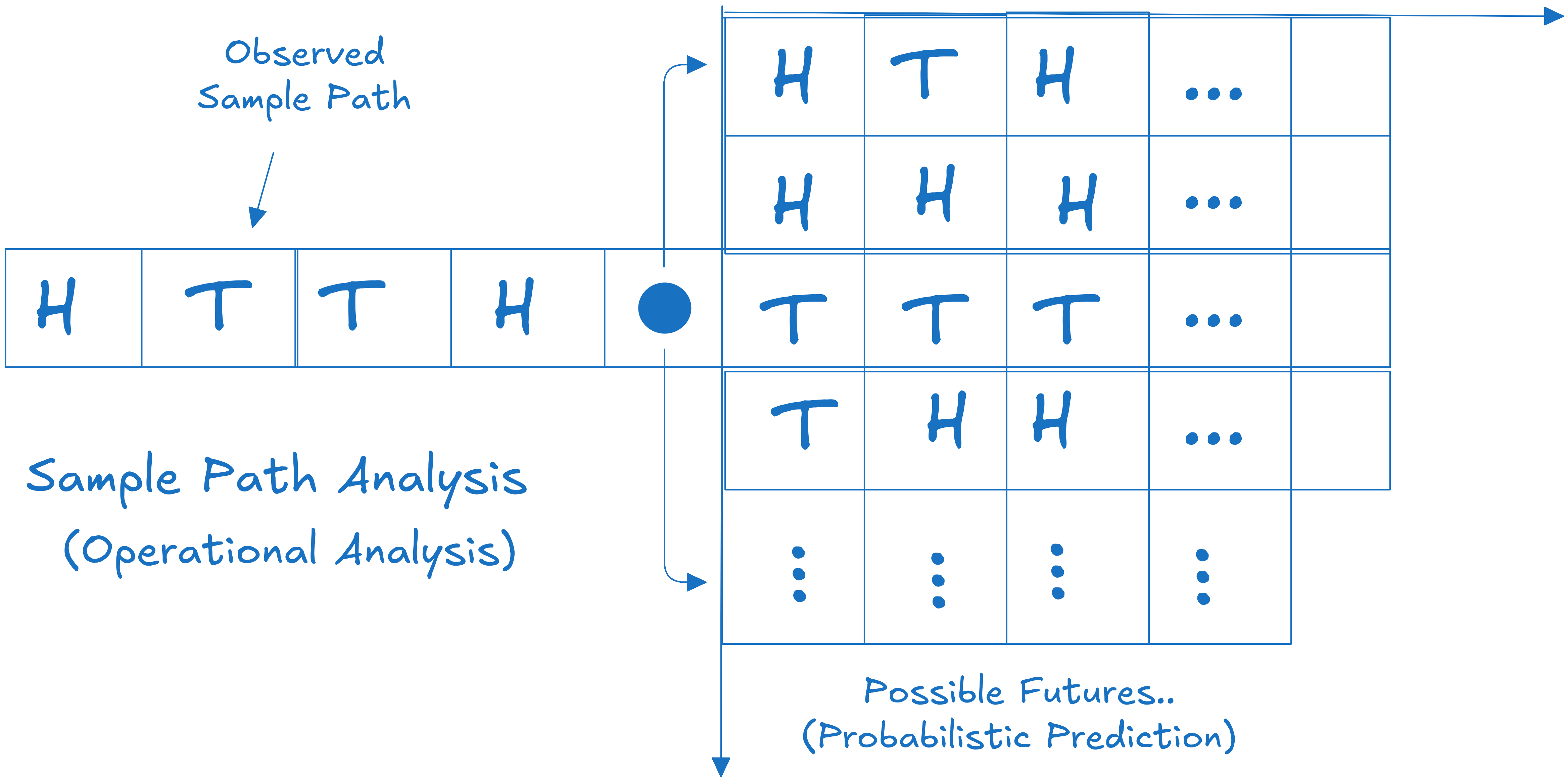


Key Idea

Once you have a sample path the only randomness is in the "next event"



Sample path analysis studies a single observed path



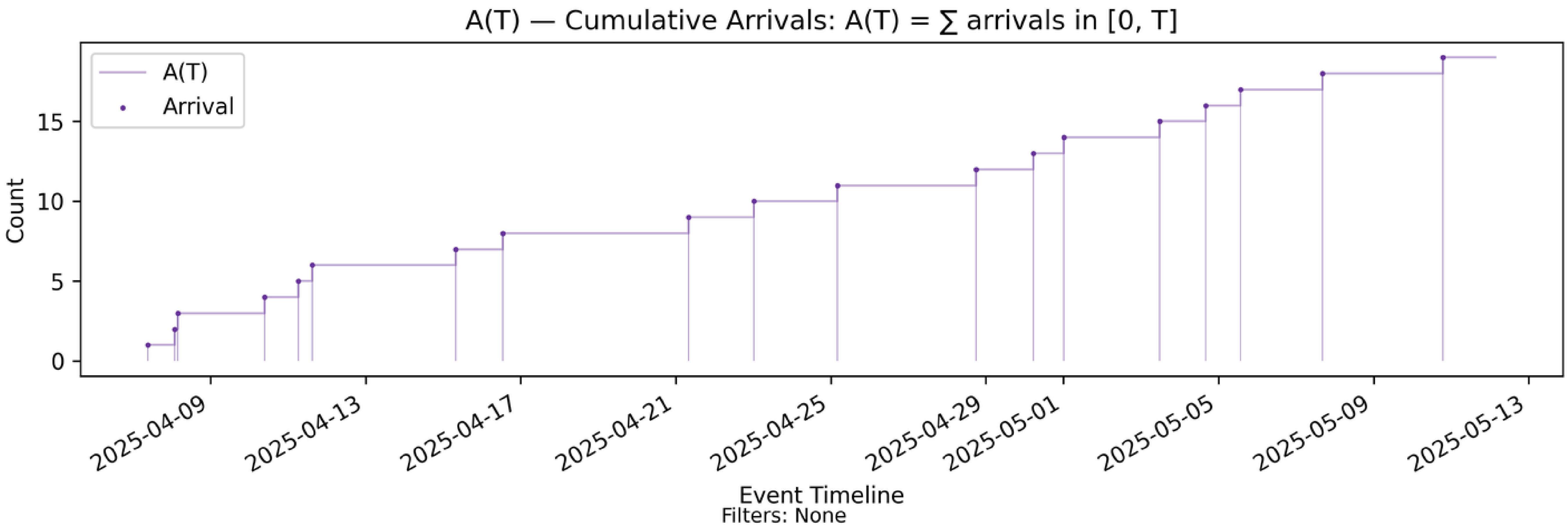
Probability theory studies properties of distributions over all possible paths

Key Idea in Sample Path Analysis

A measurement made
on an observed sample path
is completely deterministic provided
it does not introduce any additional randomness

Example: Number of arrivals seen up to time t.

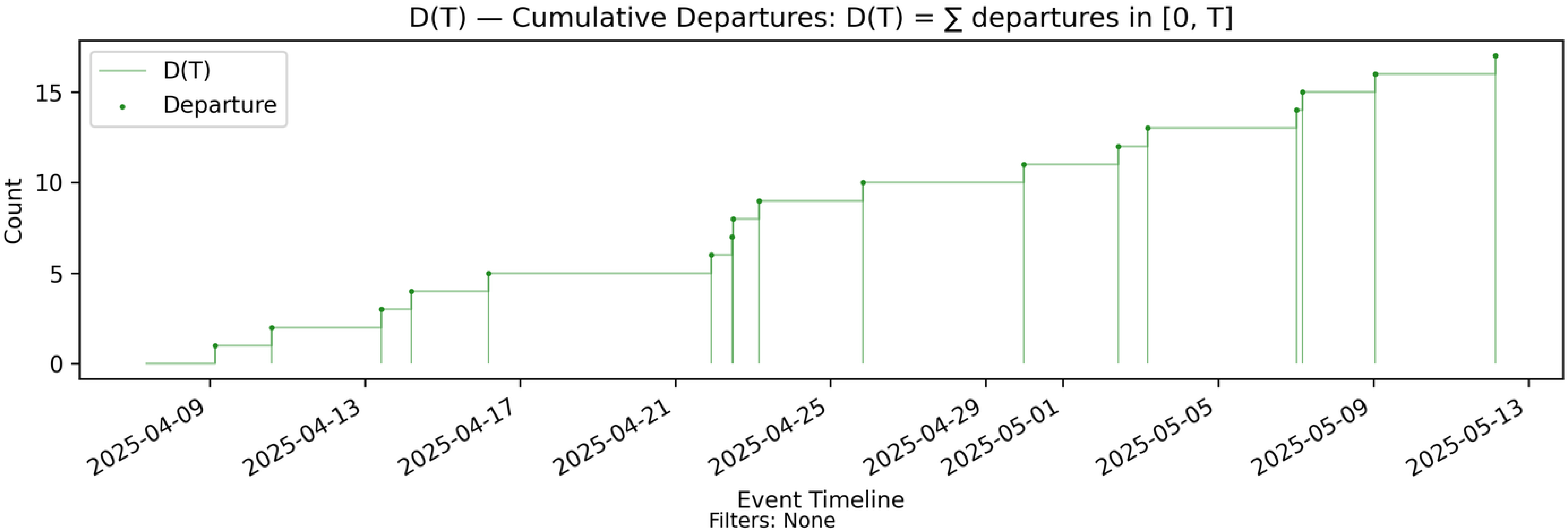
Step function that increments by 1 at each timestamp
iff the next event is an arrival



Given a prefix T of the sample path
this is a deterministic computation

Releated: Number of departures seen up to time t.

Step function that increments by 1 at each timestamp
iff the next event is a departure



Key Things

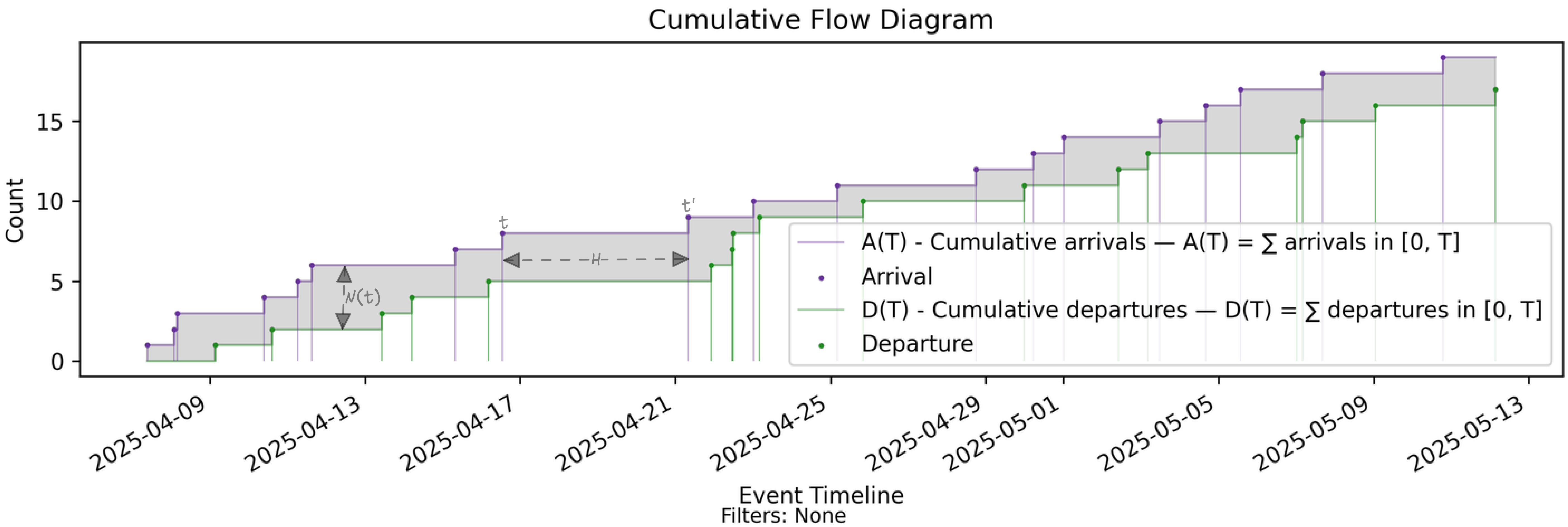
The metrics $A(T)$ and $D(T)$ are deterministic given a sample path

We also have deterministic
cause-effect relationships
between events and the metrics

Metrics can change only at event boundaries
Given the next event, the change in the metric
is fully determined.

Let's extend this to
all the key metrics in Little's Law....

Start with the CFD

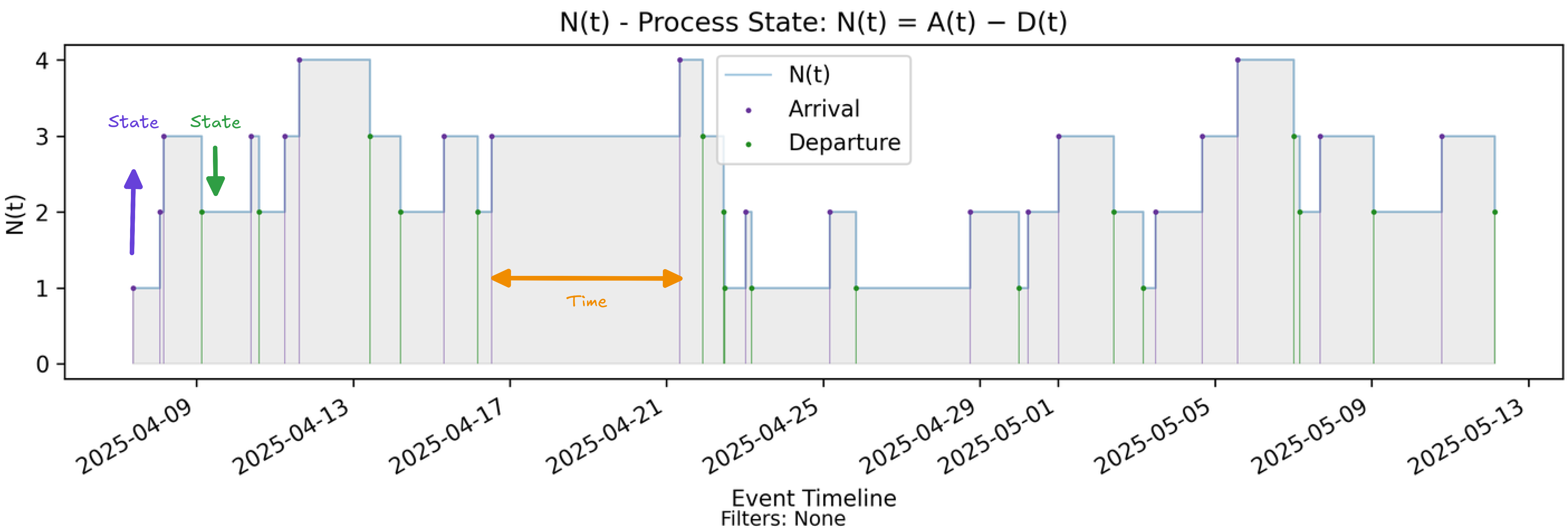


Let's quantify the grey area precisely.

The vertical distance between $A(T)$ and $D(T)$ is $N(t)$ - the instantaneous state of the process.

The horizontal area H swept between events is $N(t)$* time between events
Both are deterministic functions of the current sample path up to the next event

Lets chart N(t)

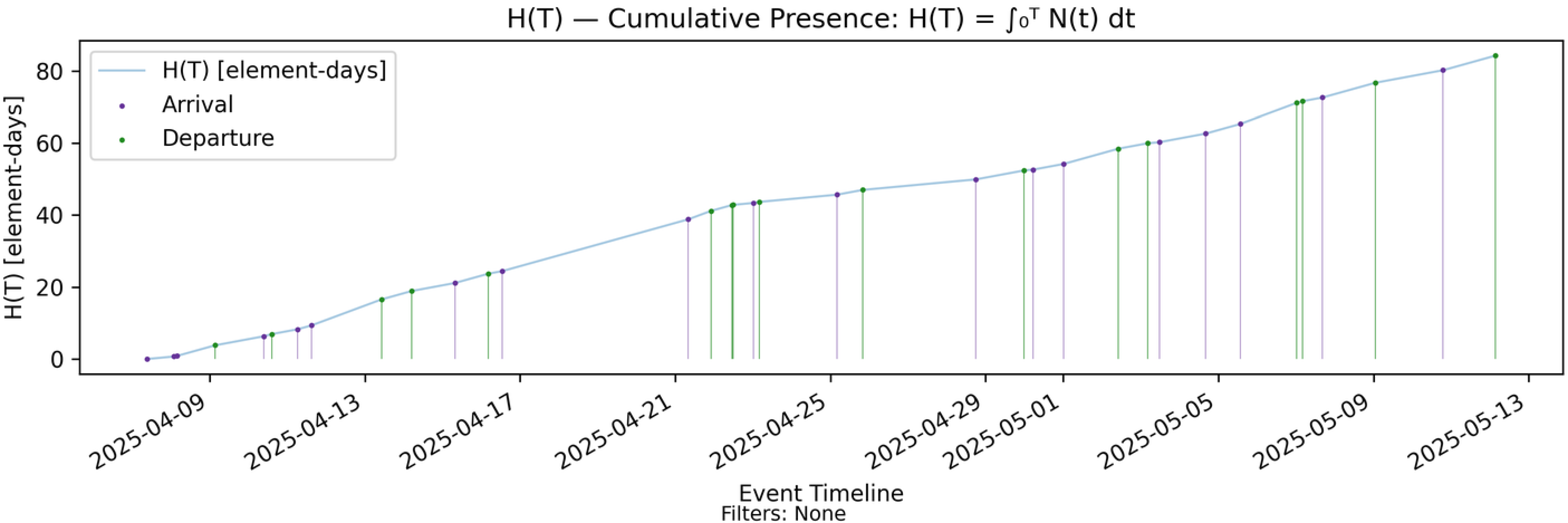


This is a step chart, where the steps
go up by 1 at each arrival and down by 1 at each departure
and stays constant in between event

Each horizontal segment represents the time the process remains in a given state between events.

The area of each rectangle represents the accumulated element-time spent in that state.

Let's measure the area (The Presence Mass)



$$H(T) = \int_0^T N(t)dt$$

The units of H(T) are in element-time, the product of state and time

The rate of growth of H(T) is proportional to N(t)
arrivals increase the rate, departures decrease the rate

$$\frac{dH(T)}{dt} = N(t)$$

Presence Mass H(T) is the fundamental flow metric

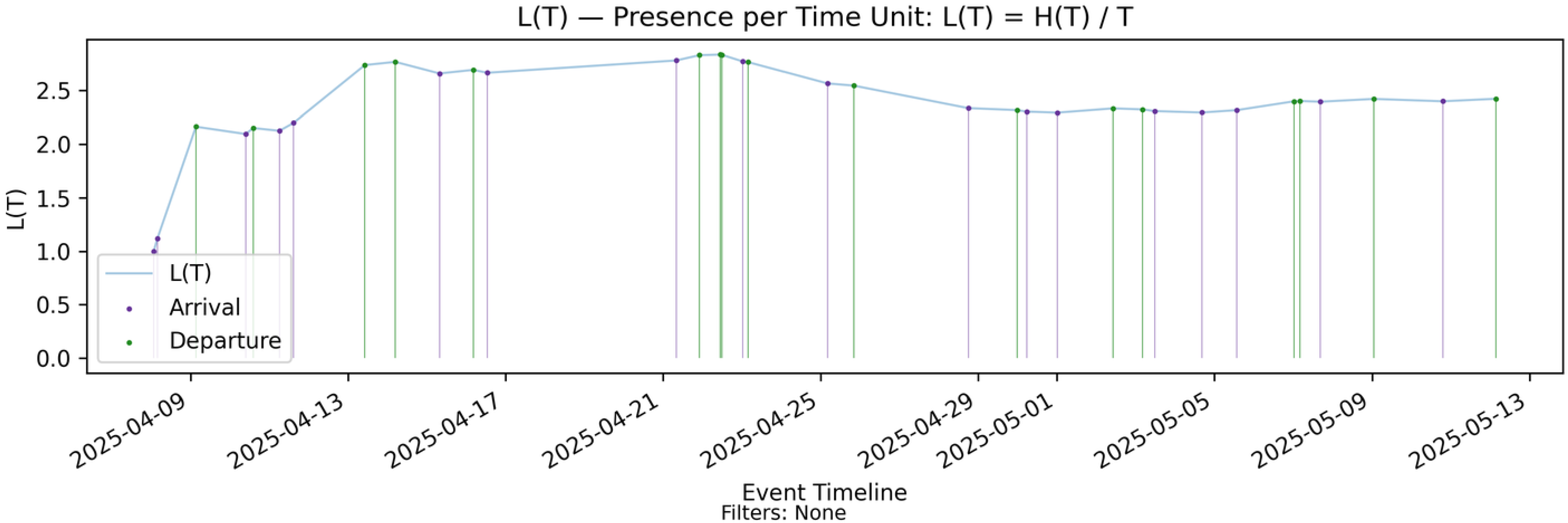
$$H(T) = \int_0^T N(t)dt$$

H(T) is the global state of the process, capturing its entire history up to time T

The dynamics of flow
(how H(T) changes over time)
are entirely captured by this equation

Little's Law is simply a consequence

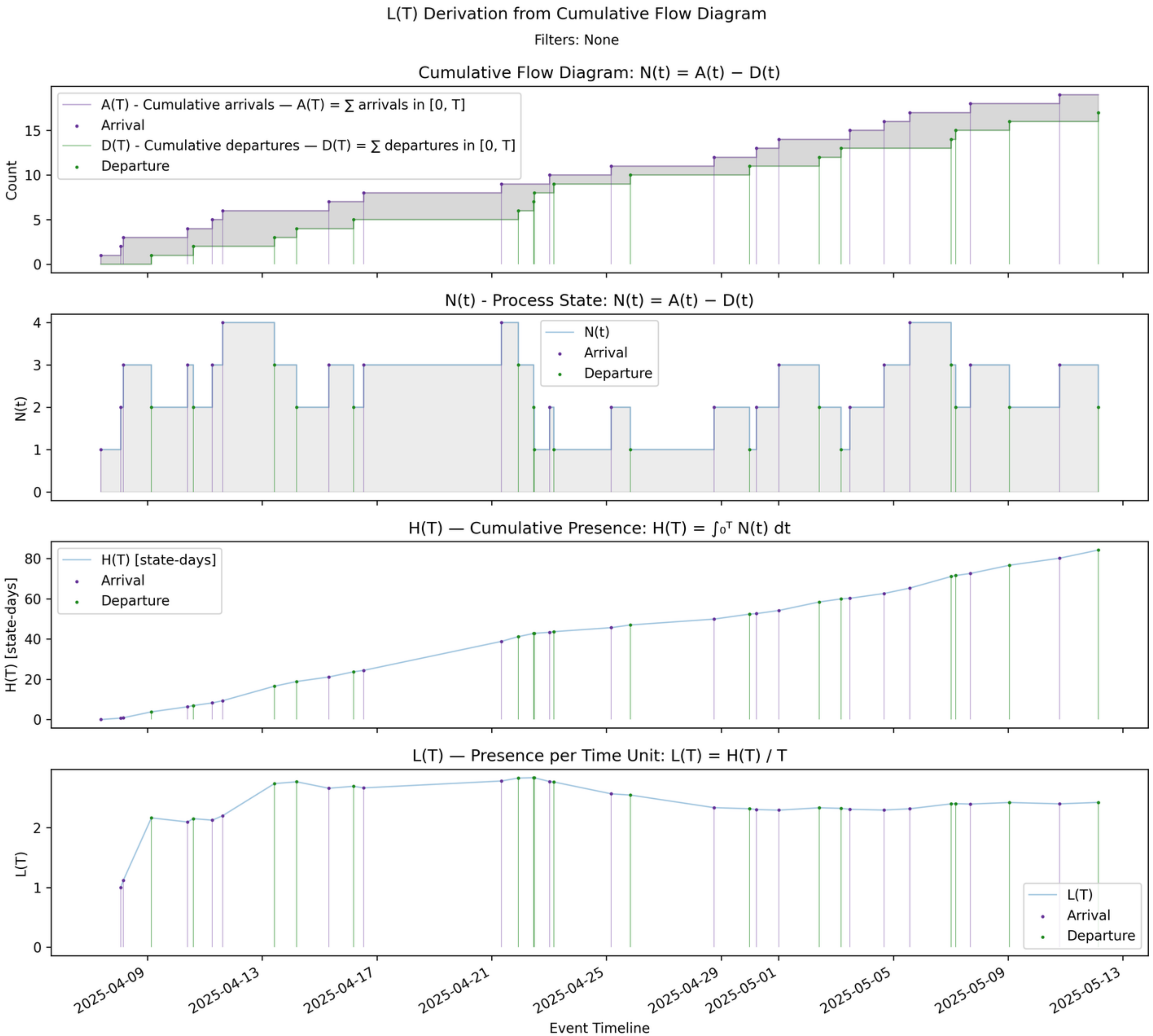
L(T) - The Left Hand Side of Little's Law: the time average of presence



This is accumulated presence normalized by time, expressed in units of elements

L(T) is the observable stability diagnostic for flow.

Putting it together



Instantaneous State

Global State

Time Normalized State

The Right Hand Side of Little's Law

$$L(T) = \frac{H(T)}{T} = \frac{A(T)}{T} \times \frac{H(T)}{A(T)} = \Lambda(T)\,.\,w(T)$$

Arrival Rate | Presence Mass Per Arrival | Arrival Rate | Residence Time per arrival

Note: The units of H(T)/A(T) are element-time/elements so it is a time quantity

$$L(T) = \Lambda(T).w(T)$$

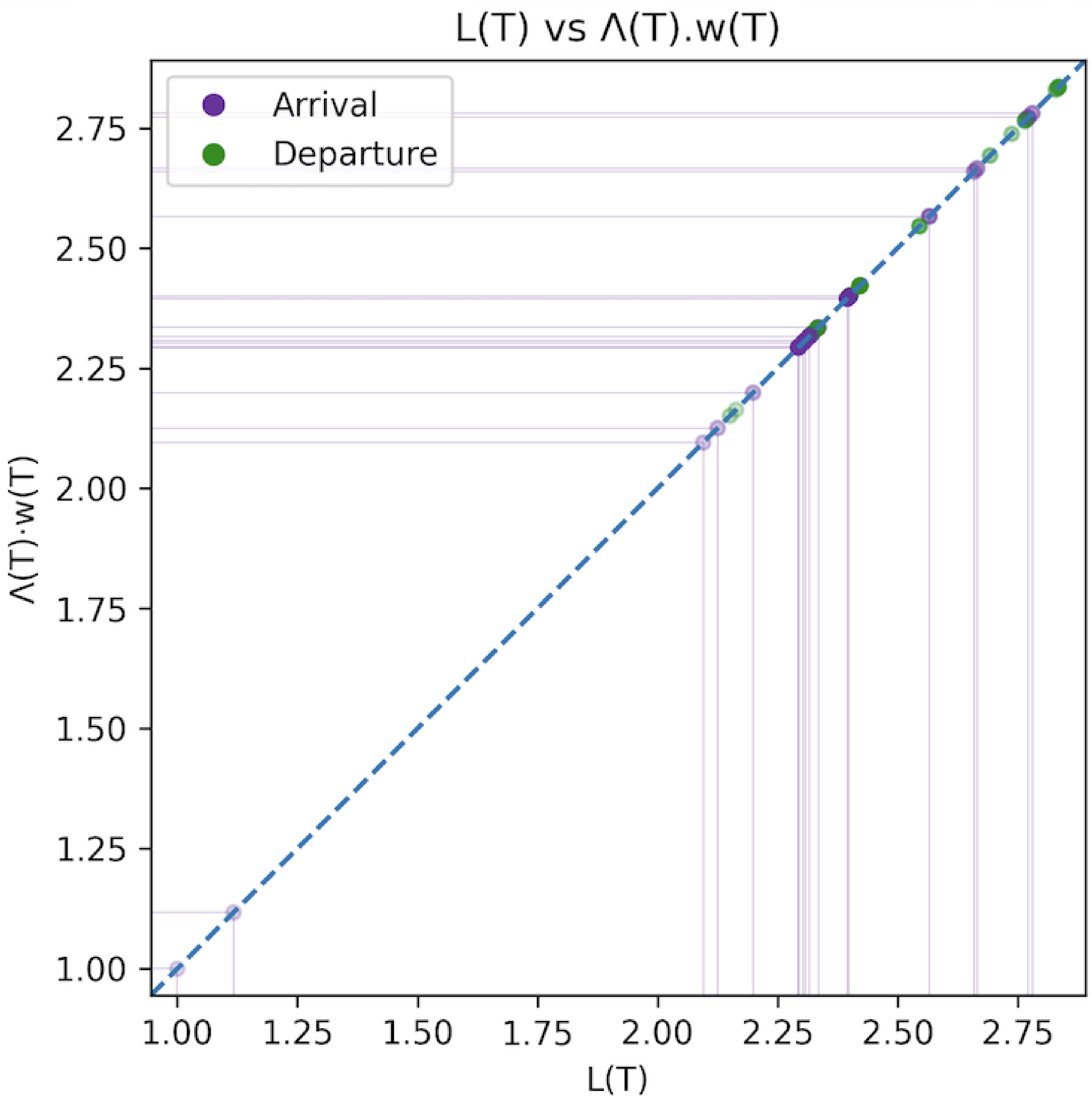
Looks like Little's Law but there are no conditions??

The key principle is Conservation of Presence Mass

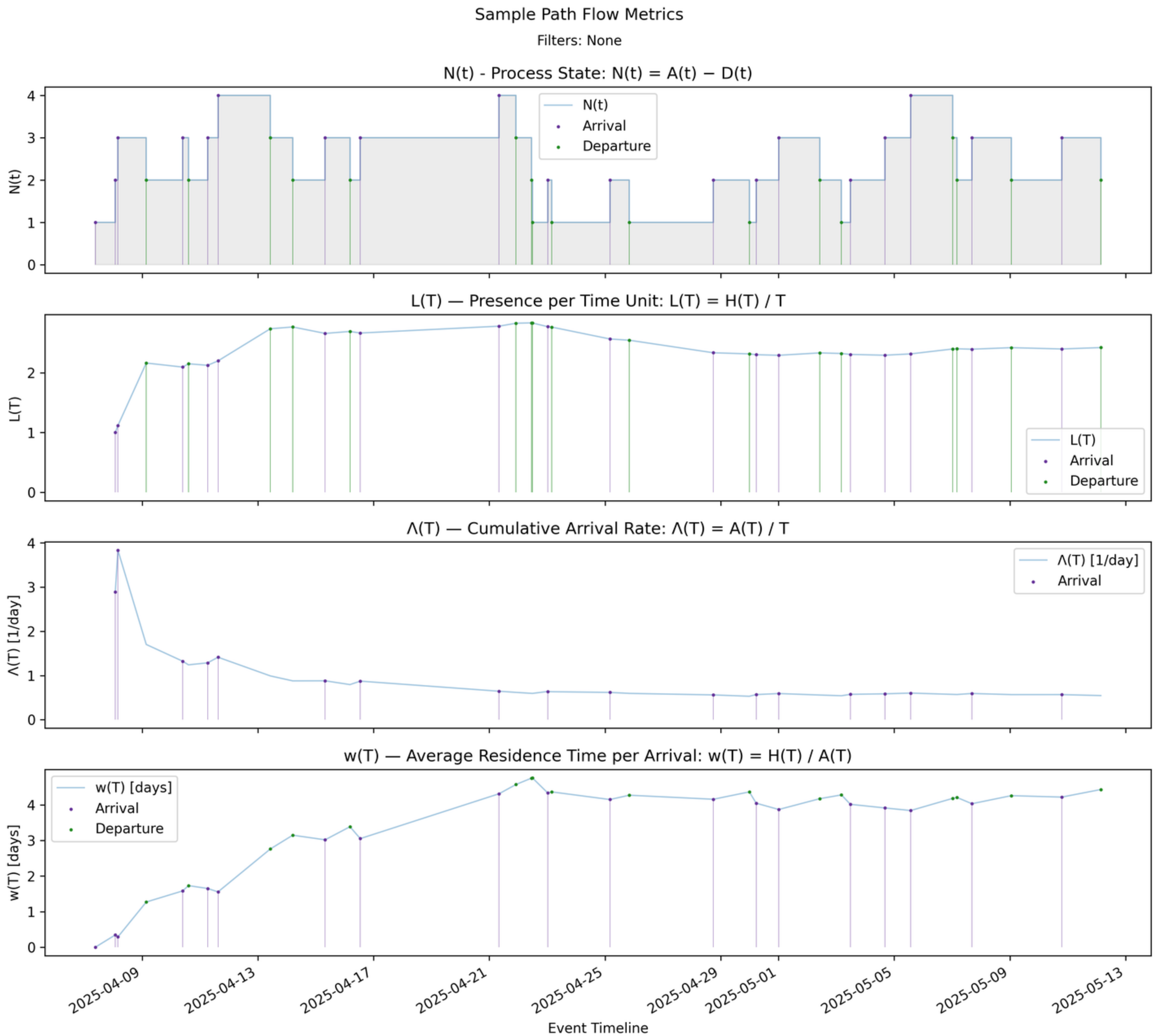
This is the finite version of Little's Law: aka The Presence Invariant.

Lets take it for a drive..

The Presence Invariant



The Sample Path Flow Metrics Dashboard



What about

Throughput, Cycle Time and...

$$L = \lambda W \;???$$

Throughput is Symmetrical to Arrival Rate

$$L(T) = \frac{H(T)}{T} = \underbrace{\frac{D(T)}{T}}_{\text{Departure Rate}} \times \underbrace{\frac{H(T)}{D(T)}}_{\text{Presence Mass Per Departure}} = \underbrace{\Theta(T)}_{\text{Throughput}} . \underbrace{w'(T)}_{\text{Residence Time per arrival}}$$

$$L(T) = \Theta(T) . w'(T)$$

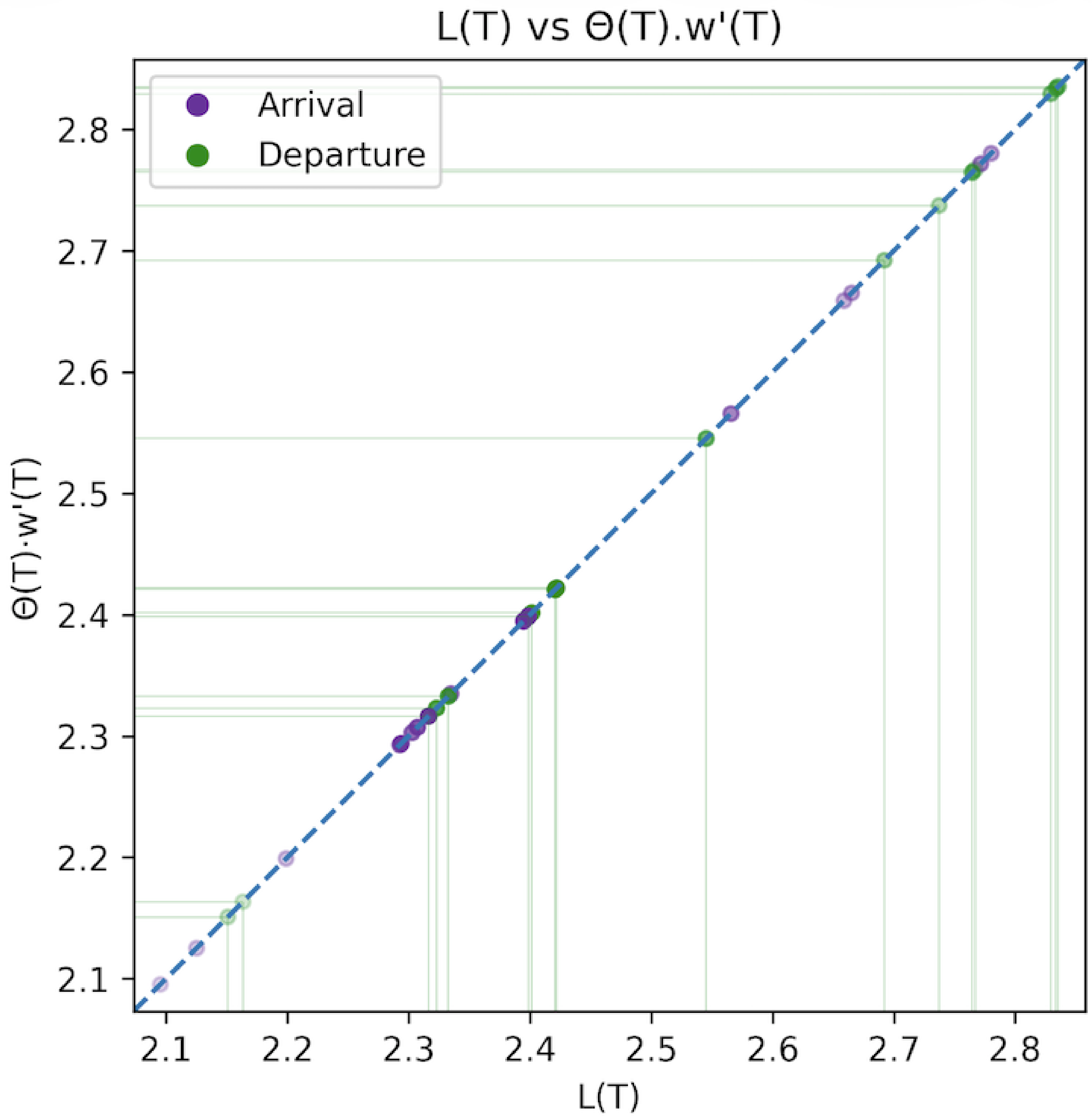
This is an alternative form of the Presence Invariant.

Since $L(T)$ is the same quantity on both equations, we have the stronger constraint

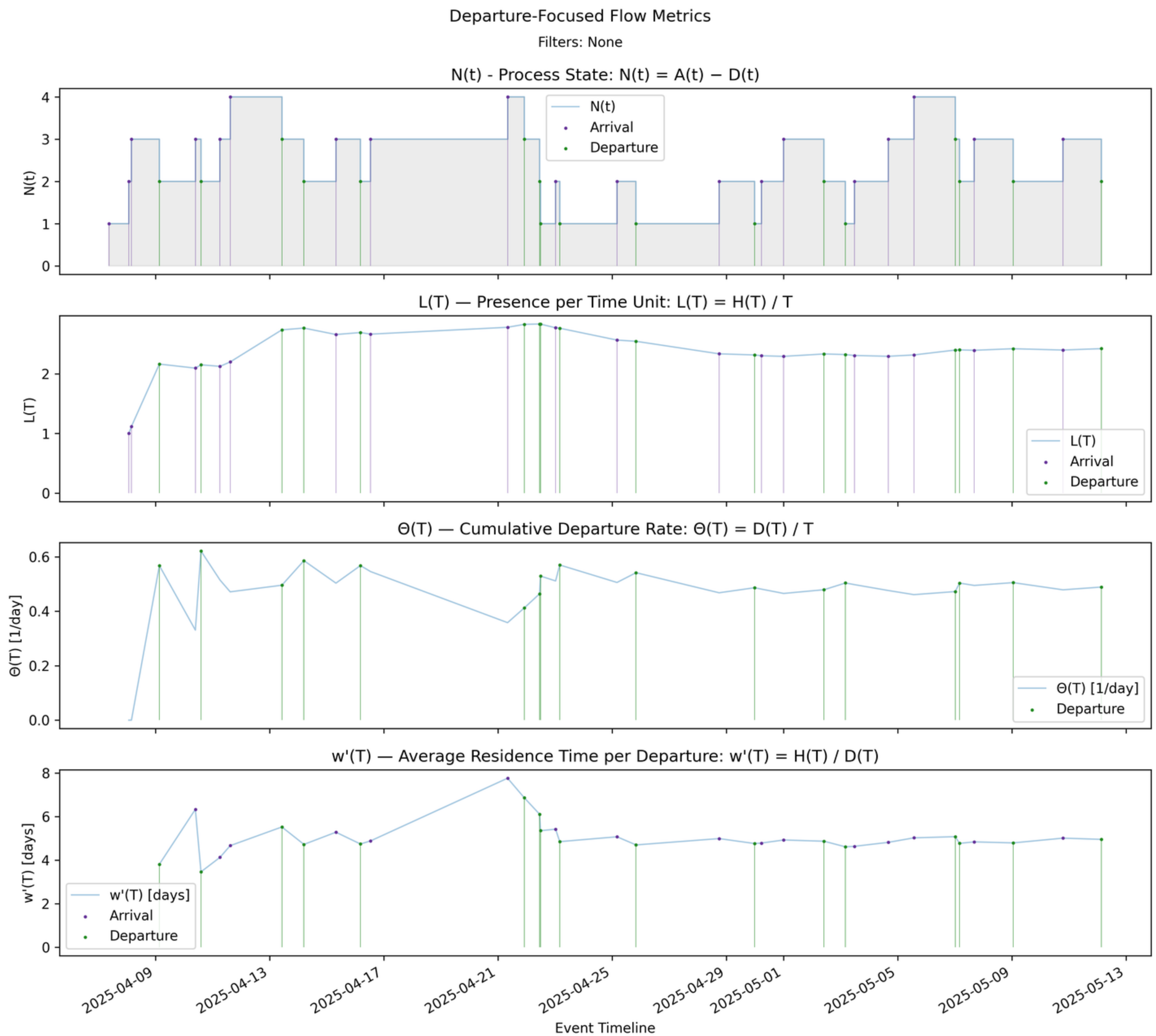
$$\Lambda(T) . w(T) = L(T) = \Theta(T) . w'(T)$$

At all points in time, unconditionally!

We can verify that too.

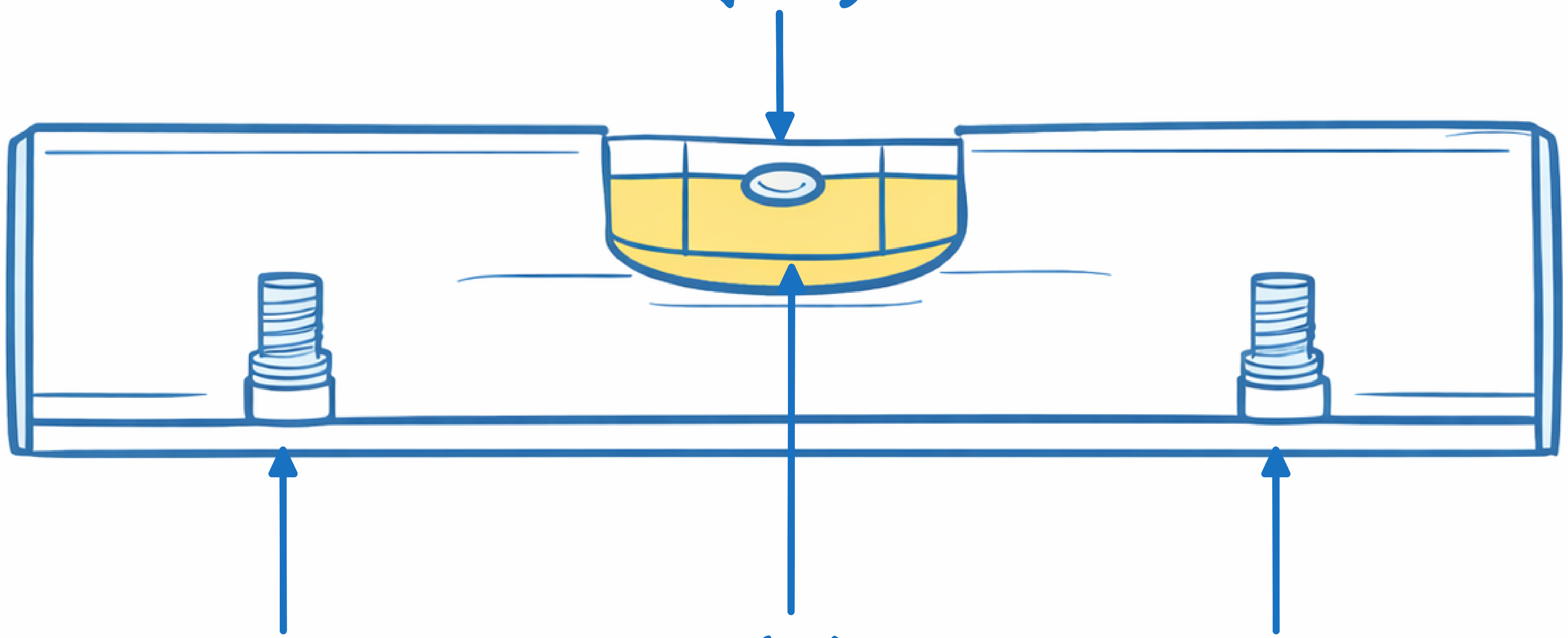


Departure Focused Sample Path Flow Metrics



Maintaining Balance

$L(T)$



State Control	$H(T)$ Presence Mass (State)	Time Control
How *much* presence accumulates		How *fast* presence accumulates
$w(T)$		$\lambda(T)$
Residence Time	Shouldnt it be flipped? NO!	Arrival Rate

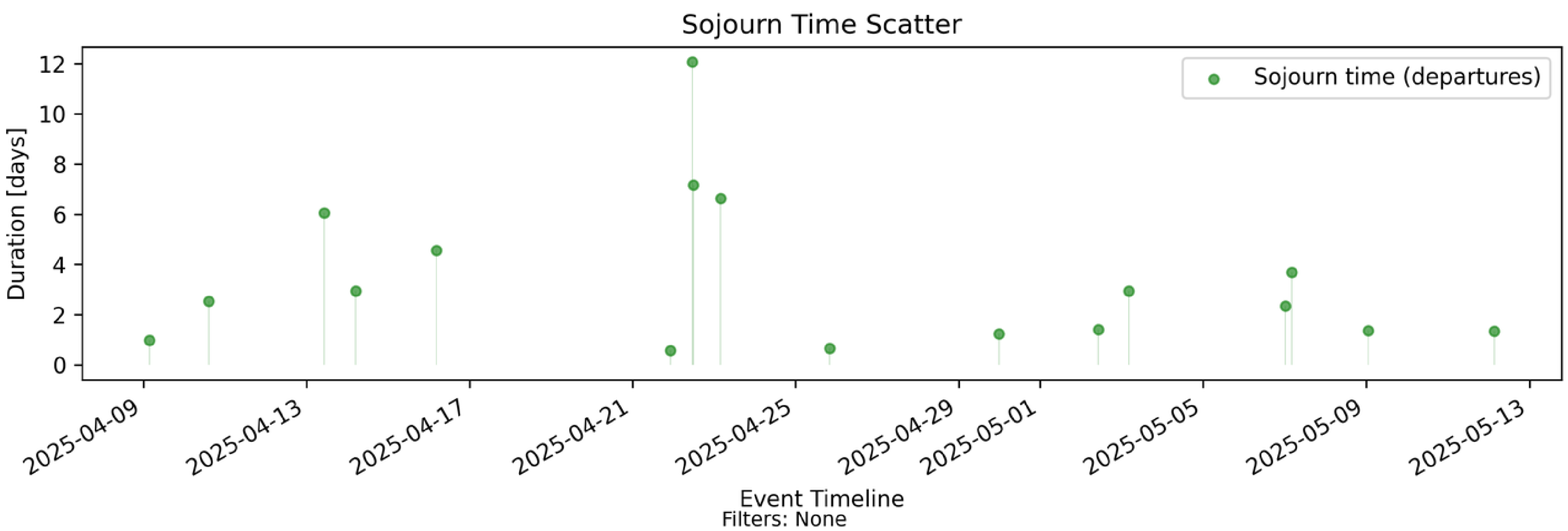
Stability

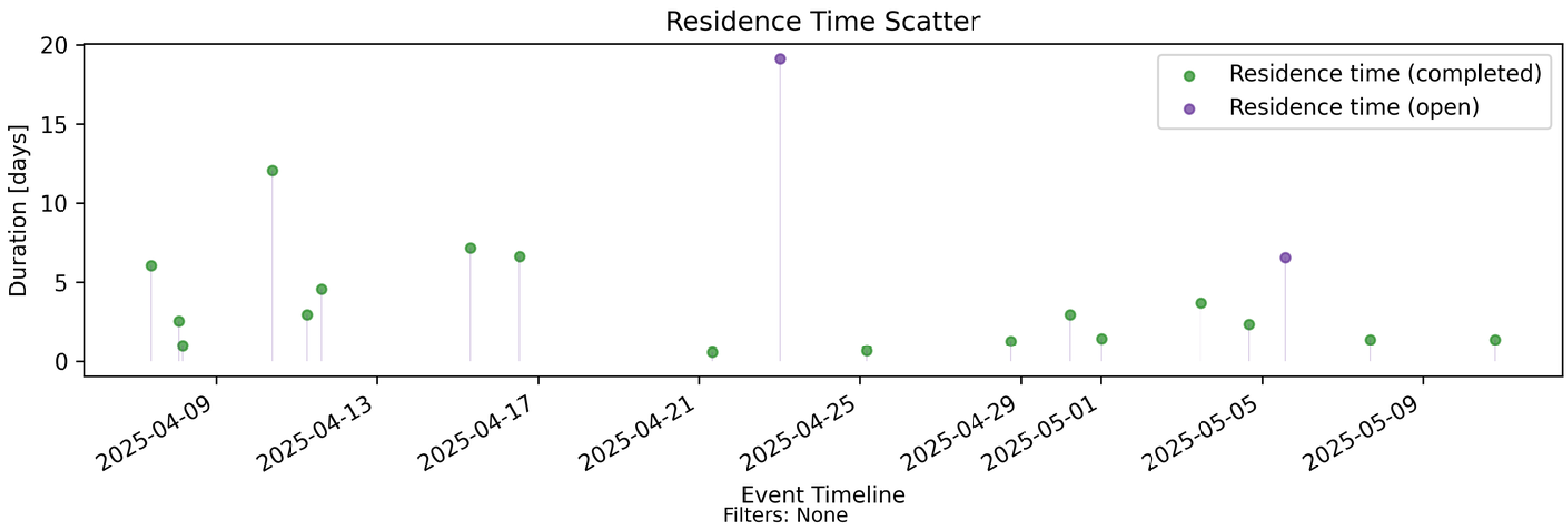
When $\Lambda(T)$ converges to a limit λ, and $w(T)$ converges to a limit W then $L(T)$ converges to a limit L

and

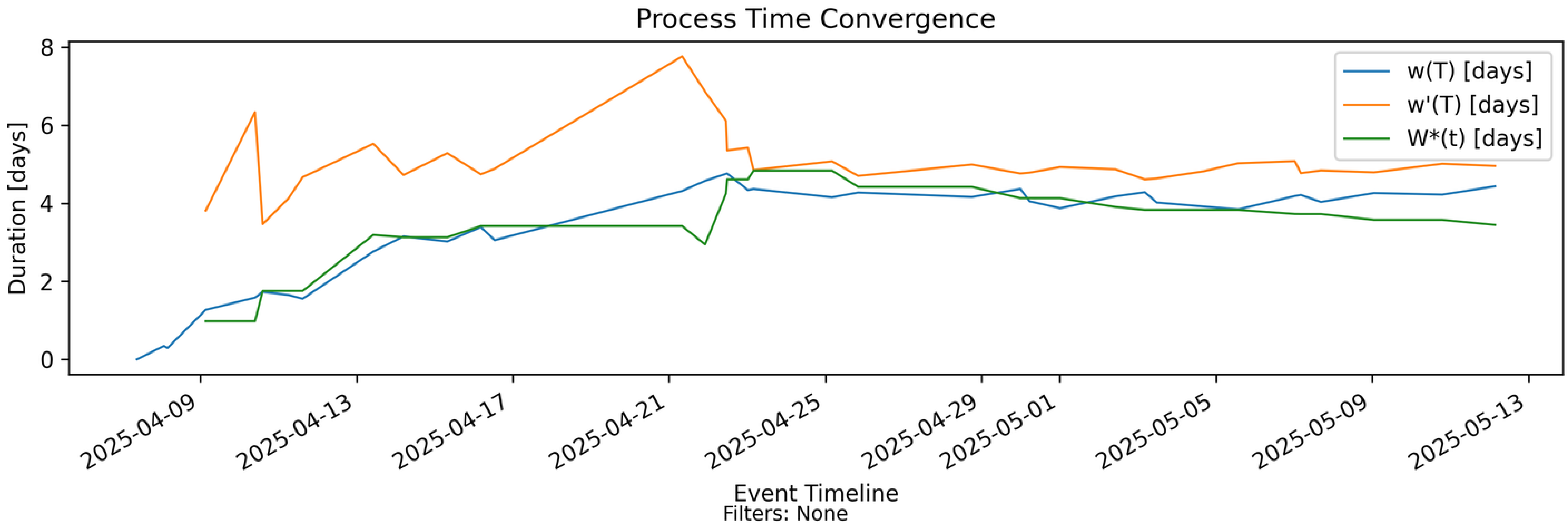
$$L = \lambda W$$

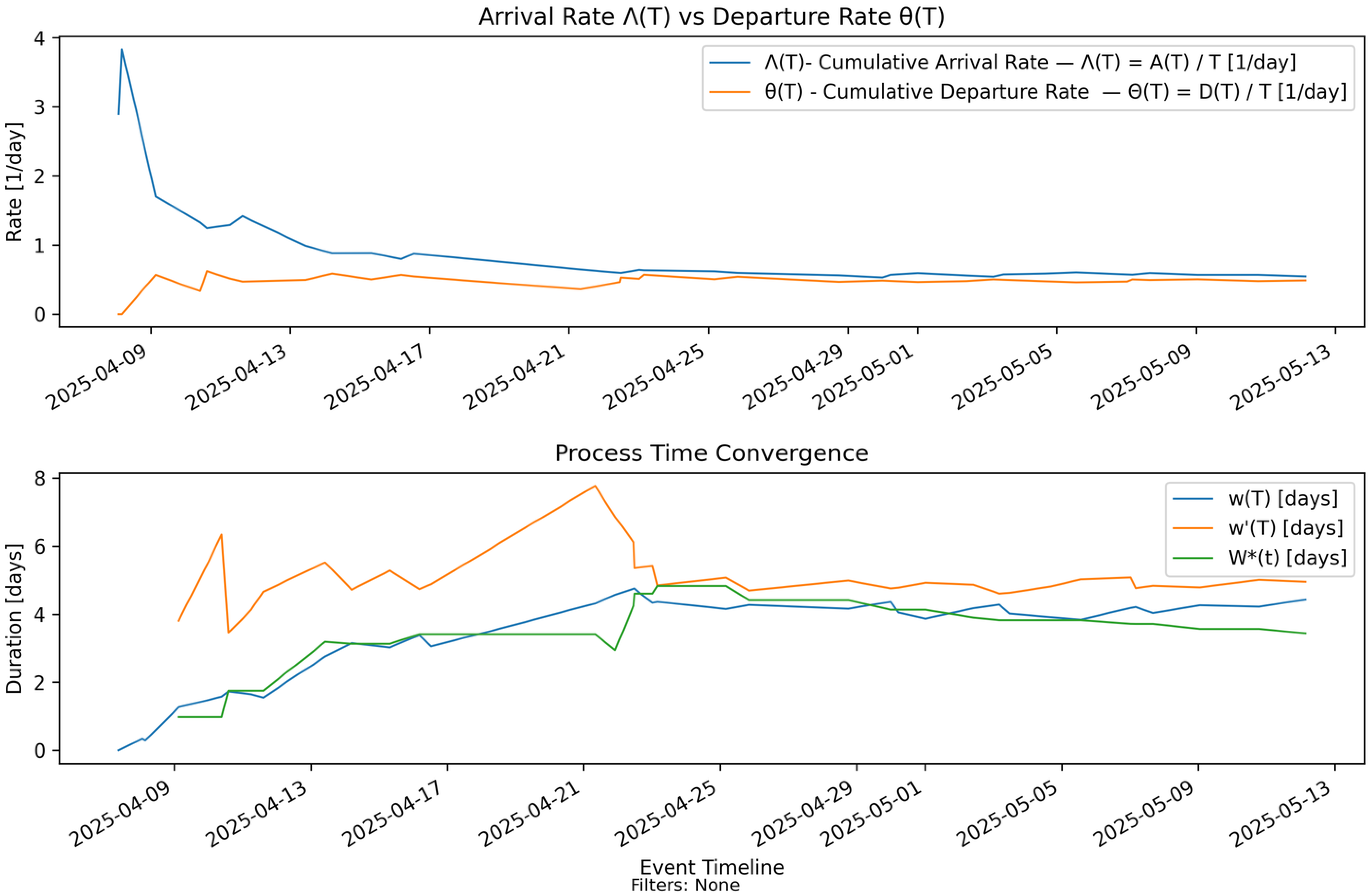
Residence Time vs Sojourn Time





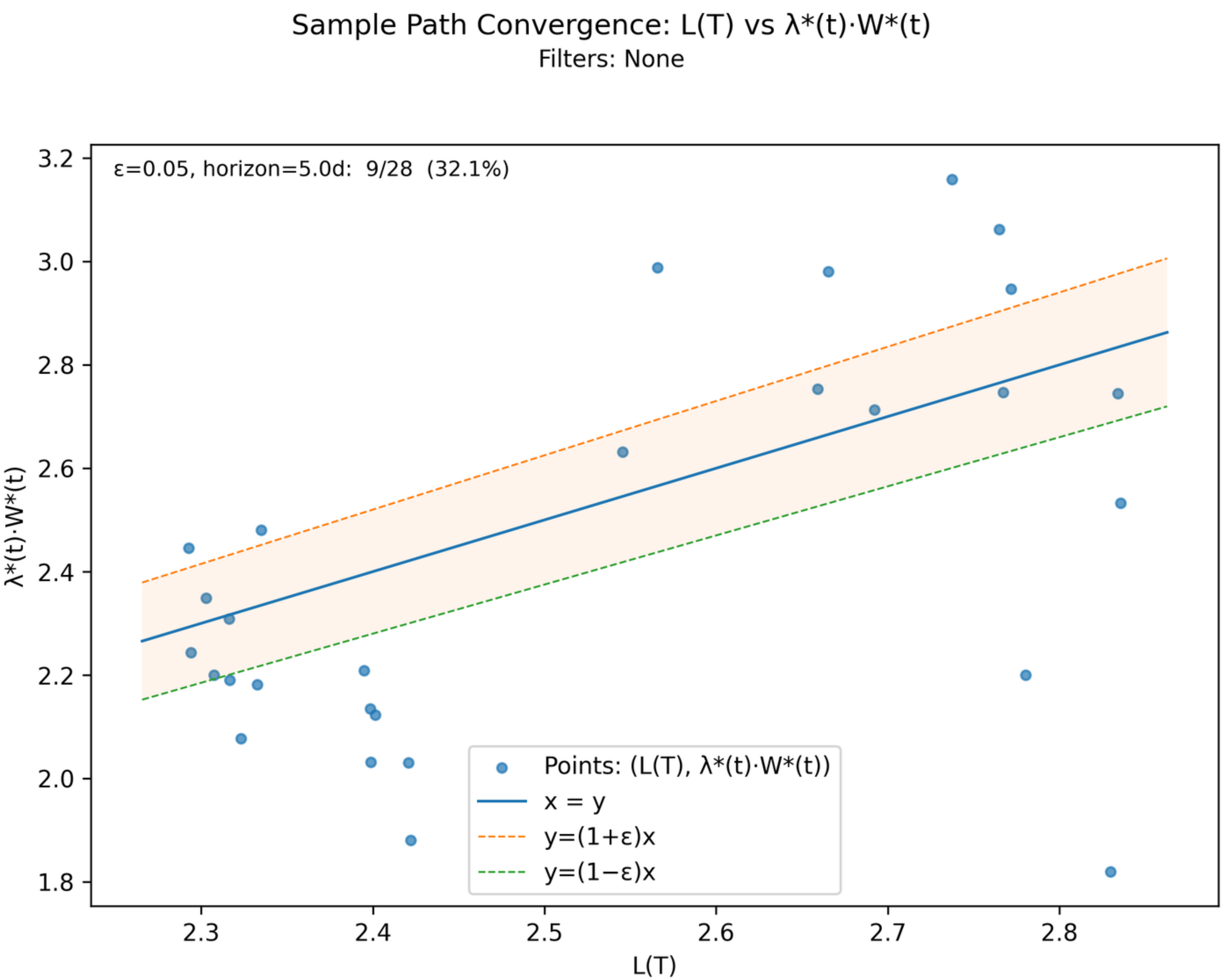
Process Time Convergence



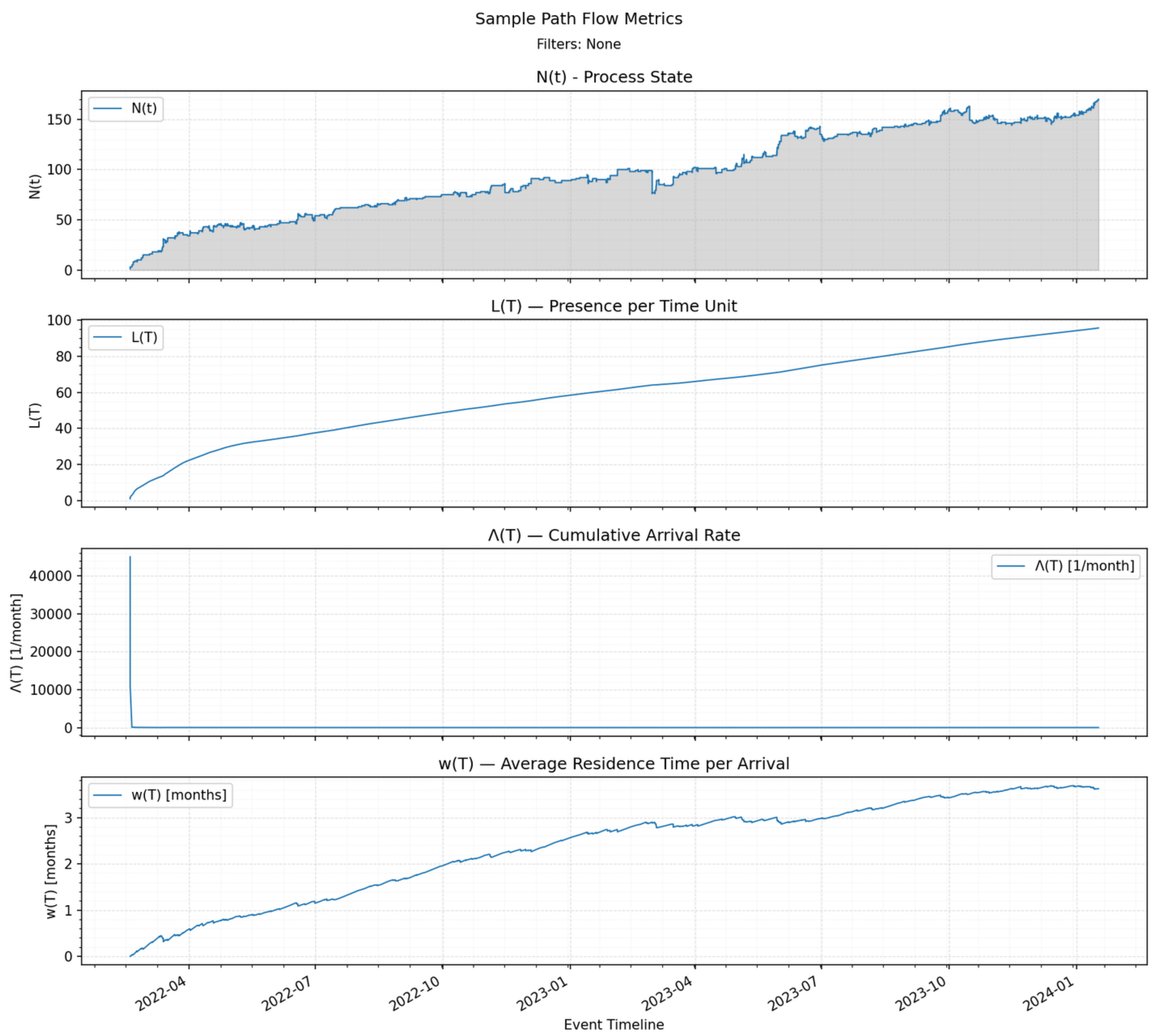


Arrival Rate Λ(T) vs Departure Rate θ(T)
Λ(T)- Cumulative Arrival Rate — Λ(T) = A(T) / T [1/day]
θ(T) - Cumulative Departure Rate — Θ(T) = D(T) / T [1/day]
Rate [1/day]
0
1
2
3
4
2025-04-09
2025-04-13
2025-04-17
2025-04-21
2025-04-25
2025-04-29
2025-05-01
2025-05-05
2025-05-09
2025-05-13
Process Time Convergence
w(T) [days]
w'(T) [days]
W*(t) [days]
Duration [days]
0
2
4
6
8
2025-04-09
2025-04-13
2025-04-17
2025-04-21
2025-04-25
2025-04-29
2025-05-01
2025-05-05
2025-05-09
2025-05-13
Event Timeline
Filters: None

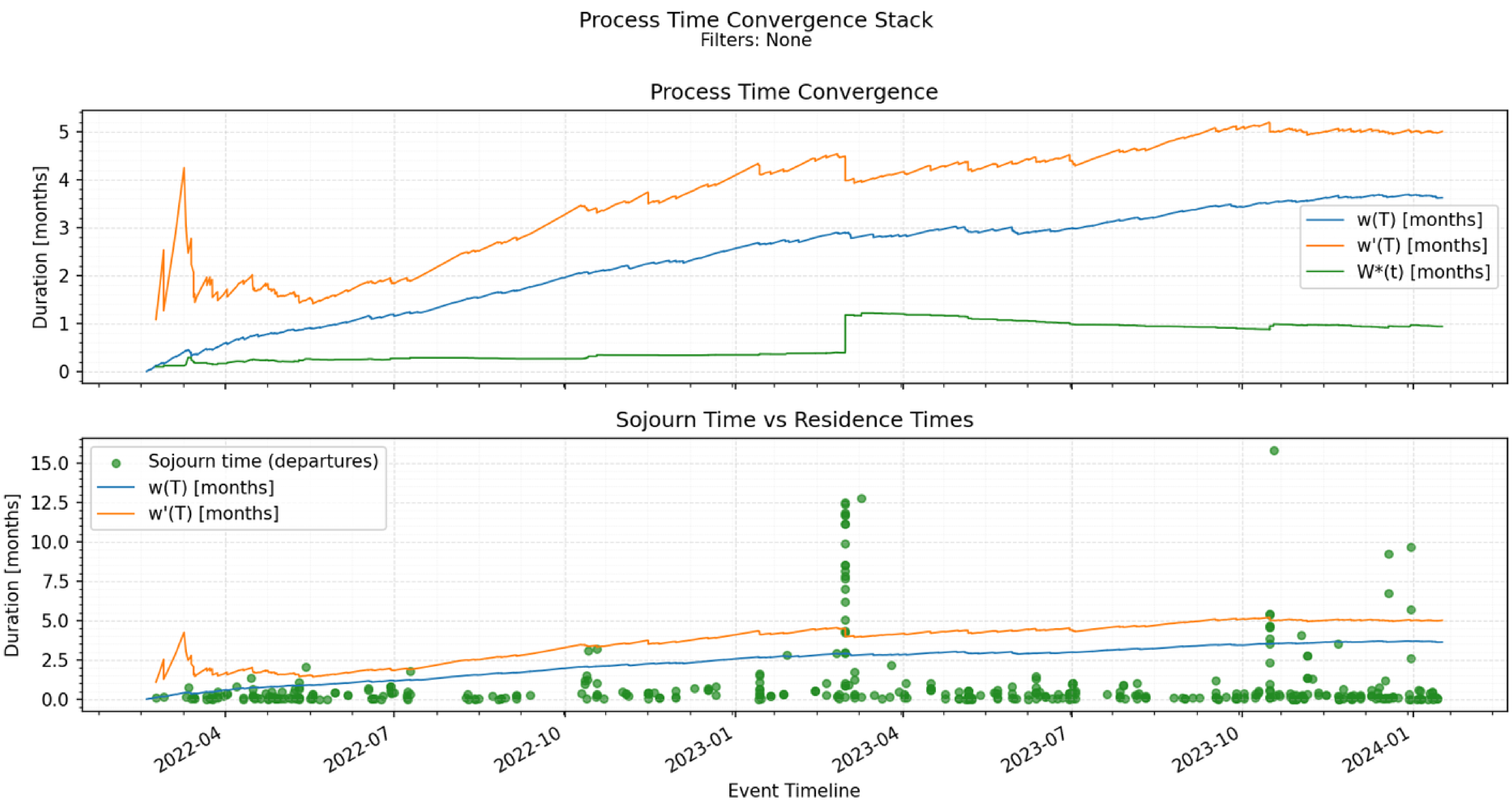
How far away are we from L = λW?



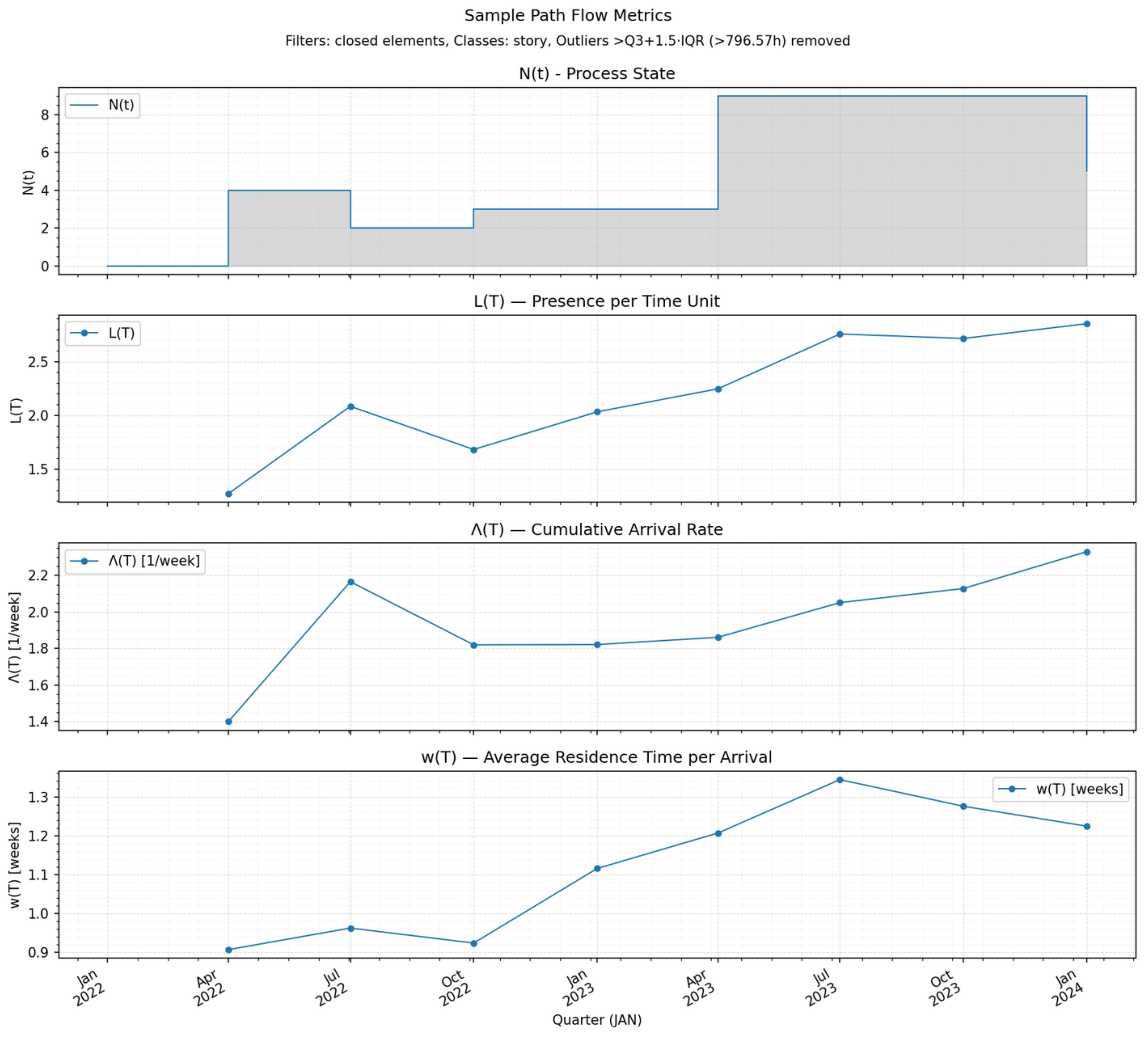
Divergence/Unstable



Process Time Divergence

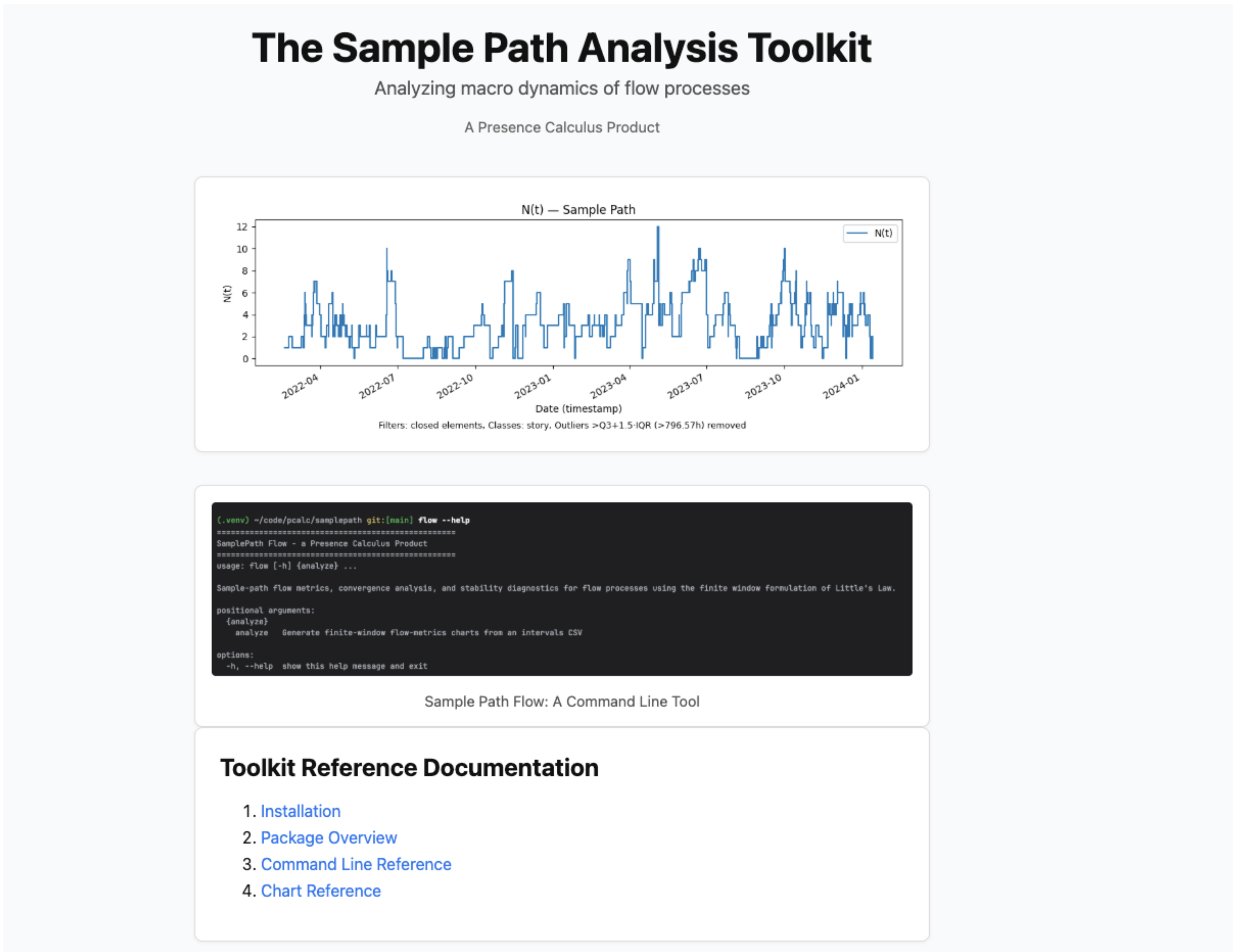


Subsampling



Try it out on your data

All it needs is a csv file with id, start_ts, and end_ts



https://samplepath.pcalc.org

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