Shear Relaxation Equation

$$\tau_\pi\left(\Delta_{\mu\nu\alpha\beta}D\pi^{\alpha\beta}+C\pi_{\mu\nu}\Theta\right)+\pi_{\mu\nu}=2\eta\sigma_{\mu\nu}$$

- $\Delta_{\mu\nu\alpha\beta}=\frac{1}{2}\left(\Delta_{\mu\alpha}\Delta_{\nu\beta}+\Delta_{\mu\beta}\Delta_{\nu\alpha}-\frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right)$, $\Delta_{\mu\nu}=g_{\mu\nu}-u_\mu u_\nu$

- $\sigma_{\mu\nu}=\Delta_{\mu\nu\alpha\beta}\nabla^\alpha u^\beta$

$$=\left(\Delta_{\mu\alpha}\Delta_{\nu\beta}+\Delta_{\mu\beta}\Delta_{\nu\alpha}-\frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right)\nabla^\alpha u^\beta$$

$$=\Delta_{\mu\alpha}\Delta_{\nu\beta}\left(\nabla^\alpha u^\beta+\nabla^\beta u^\alpha\right)-\frac{2}{3}\Delta_{\mu\nu}\left(\Delta_{\alpha\beta}\nabla^\alpha u^\beta\right)$$

$$=\left(g_{\mu\alpha}-u_\mu u_\alpha\right)\left(g_{\nu\beta}-u_\nu u_\beta\right)\left(\nabla^\alpha u^\beta+\nabla^\beta u^\alpha\right)-\frac{2}{3}\Delta_{\mu\nu}\left(\Delta_{\alpha\beta}\nabla^\alpha u^\beta\right)$$

(*) $\longrightarrow u_\beta\nabla_\alpha u^\beta=u_\beta\left(\underbrace{\partial_\alpha u^\beta}_{0}+\Gamma^\beta_{\alpha\lambda}u^\lambda\right)=u_\beta u^\lambda\Gamma^\beta_{\alpha\lambda}$

$$=\frac{1}{2}u_\beta u^\lambda g^{\beta\sigma}\left(\partial_\alpha g_{\sigma\lambda}+\partial_\lambda g_{\sigma\alpha}-\partial_\sigma g_{\alpha\lambda}\right)=0$$

$$\Rightarrow 2\eta\sigma^{\mu\nu}=\eta\Big[\left(\nabla_\mu u_\nu+\nabla_\nu u_\mu\right)-g_{\mu\alpha}u_\nu u_\beta\big(\underbrace{\nabla^\alpha u^\beta}_{0}+\nabla^\beta u^\alpha\big)-g_{\nu\beta}u_\mu u_\alpha\big(\nabla^\alpha u^\beta+\underbrace{\nabla^\beta u^\alpha}_{0}\big)$$

$$+u_\mu u_\alpha u_\nu u_\beta\big(\underbrace{\nabla^\alpha u^\beta}_{0}+\underbrace{\nabla^\beta u^\alpha}_{0}\big)\Big]-\frac{2\eta}{3}\Delta_{\mu\nu}\left(\nabla_\alpha u^\alpha\right)$$

$$=\eta\left[\nabla_\mu u_\nu+\nabla_\nu u_\mu-u_\nu Du_\mu-u_\mu Du_\nu\right]-\frac{2\eta}{3}\Delta_{\mu\nu}\left(\nabla_\alpha u^\alpha\right)$$

$$=\eta\left(\partial_\mu u_\nu+\partial_\nu u_\mu-2u_\lambda\Gamma^\lambda_{\mu\nu}\right)-\eta\left(\gamma u_\nu\frac{d}{d\tau}u_\mu+\gamma u_\mu\frac{d}{d\tau}u_\nu\right)$$

$$+\eta\left(-\frac{2}{3}\Delta_{\mu\nu}\Theta+u_\nu u^\beta\Gamma^\lambda_{\beta\mu}u_\lambda+u_\mu u^\beta\Gamma^\lambda_{\beta\nu}u_\lambda\right)$$

(*) Three terms:

$$u^\sigma u^\lambda\partial_\alpha g_{\sigma\lambda}=u^\sigma\partial_\alpha\left(u^\lambda g_{\sigma\lambda}\right)-u^\sigma g_{\sigma\lambda}\partial_\alpha u^\lambda$$

$$=u^\sigma\partial_\alpha u_\sigma-u_\lambda\partial_\alpha u^\lambda=0$$

$$u^\sigma u^\lambda\partial_\lambda g_{\sigma\alpha}-u^\sigma u^\lambda\partial_\sigma g_{\alpha\lambda}$$

$$=u^\sigma u^\lambda\left(\partial_\lambda g_{\sigma\alpha}-\partial_\lambda g_{\alpha\sigma}\right)=0$$

$$\Theta\equiv\nabla_\alpha u^\alpha$$

N.B.: $\underline{\nabla_\alpha g_{\mu\nu}=0}$

- $\Gamma^\lambda_{\mu\nu}=\Gamma^\lambda_{\nu\mu}$

- $u_\nu u_\beta\nabla^\beta u_\mu=u_\nu u^\beta\nabla_\beta u_\mu$

$$=u_\nu u^\beta\left(\partial_\beta u_\mu-\Gamma^\lambda_{\beta\mu}u_\lambda\right)$$

$$2\eta\sigma^{\mu\nu} = \eta\left(\partial_\mu u_\nu + \partial_\nu u_\mu - 2u_\lambda \Gamma^\lambda_{\mu\nu}\right) - \eta\left(\gamma u_\nu \frac{d}{d\tau}u_\mu + \gamma u_\mu \frac{d}{d\tau} u_\nu\right)$$

$$+ \left(-\frac{2}{3}\Delta_{\mu\nu}\Theta + u_\nu u^\beta \Gamma^\lambda_{\beta\mu} u_\lambda + u_\mu u^\beta \Gamma^\lambda_{\beta\nu} u_\lambda\right)\eta$$

$$= \eta\left[\partial_\mu u_\nu + \partial_\nu u_\beta - \gamma\left(u_\nu \frac{du_\mu}{d\tau} + u_\mu \frac{du_\nu}{d\tau}\right)\right] - \frac{2}{3}\Delta^{\mu\nu}\Theta$$

$$+ u^\alpha u_\lambda\left(\underbrace{u_\nu \Gamma^\lambda_{\alpha\mu}} + u_\mu \Gamma^\lambda_{\alpha\nu}\right)\eta$$

$$\rightarrow u_\nu \Gamma^\lambda_{\alpha\mu} = u_\nu\left(\tau\delta^\lambda_3\delta^3_\alpha\delta^0_\mu + \tau\delta^\lambda_3\delta^0_\alpha\delta^3_\mu + \frac{1}{\tau}\delta^\lambda_0\delta^3_\alpha\delta^3_\mu\right)$$

$$\Rightarrow u^\alpha u_\lambda\left(u_\mu \Gamma^\lambda_{\alpha\nu} + u_\nu \Gamma^\lambda_{\alpha\mu}\right)$$

$$= u^\alpha u_\lambda\Big[u_\mu\left(\tau\delta^\lambda_3\delta^3_\alpha\delta^0_\nu + \tau\delta^\lambda_3\delta^0_\alpha\delta^3_\nu + \frac{1}{\tau}\delta^\lambda_0\delta^3_\alpha\delta^3_\nu\right)$$

$$+ u_\nu\left(\tau\delta^\lambda_3\delta^3_\alpha\delta^0_\mu + \tau\delta^\lambda_3\delta^0_\alpha\delta^3_\mu + \frac{1}{\tau}\delta^\lambda_0\delta^3_\alpha\delta^3_\mu\right)\Big]$$

$$= \tau u^3 u_3 u_\mu \delta^0_\nu + \tau u^0 u_3 u_\mu \delta^3_\nu + \frac{1}{\tau} u_0 u^3 u_\mu \delta^3_\nu$$

$$+ u_\nu\left(\tau u^3 u_3 \delta^0_\mu + \tau u^0 u_3 \delta^3_\mu + \frac{1}{2} u_0 u^3 \delta^3_\mu\right) = 0$$

for boost-invariant system ($u_3 = 0$)

Thus $2\eta\sigma^{\mu\nu} = \eta\left(\partial_\mu u_\nu + \partial_\nu u_\mu - 2\gamma\tau\delta^3_\mu\delta^3_\nu\right) - \eta\left(\gamma u_\nu \frac{d}{d\tau}u_\mu + \gamma u_\mu \frac{d}{d\tau}u_\nu\right) - \frac{2\eta}{3}\Delta_{\mu\nu}\Theta$

$$u_\lambda \Gamma^\lambda_{\mu\nu} = u_0 \Gamma^0_{\mu\nu} + u_3 \Gamma^3_{\mu\nu}$$

$$= \gamma\tau\delta^3_\mu\delta^3_\nu + \cancelto{0}{u_3 \Gamma^3_{\mu\nu}}$$

for boost-invariant system ($u_3 = 0$)

$$\tau_\pi \Delta_{\mu\nu\alpha\beta} D\pi^{\alpha\beta} = \tau_\pi D\underbrace{(\Delta_{\mu\nu\alpha\beta}\pi^{\alpha\beta})}_{\equiv \pi_{\mu\nu}} - \tau_\pi \pi^{\alpha\beta} D(\Delta_{\mu\nu\alpha\beta})$$

- $$\pi^{\alpha\beta} D(\Delta_{\mu\nu\alpha\beta}) = \frac{1}{2}\pi^{\alpha\beta} D\left(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\mu\beta}\Delta_{\nu\alpha} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right)$$

$$= \pi^{\alpha\beta} D\left(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right)$$

$$= \pi^{\alpha\beta} D(\Delta_{\mu\alpha}\Delta_{\nu\beta}) - \frac{1}{3}\pi^{\alpha\beta}\left(\Delta_{\alpha\beta} D\Delta_{\mu\nu} + \Delta_{\mu\nu} D\Delta_{\alpha\beta}\right)$$

$$\rightarrow \Delta_{\alpha\beta}\pi^{\alpha\beta} = (\cancelto{0}{g_{\alpha\beta}} - \cancelto{0}{u_\alpha u_\beta})\pi^{\alpha\beta} = 0$$

$$\rightarrow \pi^{\alpha\beta} D\Delta_{\alpha\beta} = \pi^{\alpha\beta} u^\lambda \nabla_\lambda (\cancelto{0}{g_{\alpha\beta}} - u_\alpha u_\beta)$$

$$= -u^\lambda \pi^{\alpha\beta}(\cancelto{0}{u_\beta \nabla_\lambda u_\alpha} + \cancelto{0}{u_\alpha \nabla_\lambda u_\beta}) = 0$$

$$\Rightarrow \pi^{\alpha\beta} D(\Delta_{\mu\nu\alpha\beta}) = \pi^{\alpha\beta} D(\Delta_{\mu\alpha}\Delta_{\nu\beta})$$

$$= \pi^{\alpha\beta}(\Delta_{\mu\alpha} D\Delta_{\nu\beta} + \Delta_{\nu\beta} D\Delta_{\mu\alpha})$$

$$= -\left(\pi_\mu{}^\beta D(u_\nu u_\beta) + \pi^\alpha{}_\nu D(u_\mu u_\alpha)\right)$$

$$= -\left(u_\nu \pi^\alpha_\mu + u_\mu \pi^\alpha_\nu\right) D u_\alpha$$

$$= -\gamma\left(u_\nu \pi^\alpha_\mu + u_\mu \pi^\alpha_\nu\right)\frac{du_\alpha}{d\tau}$$

- $$D g_{\mu\nu} = u^\lambda \nabla_\lambda g_{\mu\nu} = 0$$

- $$D u_\alpha = u^\lambda \nabla_\lambda u_\alpha = u^\lambda(\partial_\lambda u_\alpha - \Gamma^\sigma_{\lambda\alpha} u_\sigma)$$

$$= \gamma \frac{du_\alpha}{d\tau} - u^\lambda u_\sigma \Gamma^\sigma_{\lambda\alpha}$$

$$= \gamma \frac{du_\alpha}{d\tau} - u^\lambda u_\sigma\left(\tau \delta^\sigma_3(\delta^0_\lambda \delta^3_\alpha + \delta^3_\lambda \delta^0_\alpha) + \frac{1}{\tau}\delta^0_\sigma \delta^3_\lambda \delta^3_\alpha\right)$$

$$= \gamma \frac{du_\alpha}{d\tau} - \left(\tau u_3(u^0 \delta^3_\alpha + u^3 \delta^0_\alpha) + \frac{1}{\tau} u^3 u_0 \delta^3_\alpha\right)$$

$$= \gamma \frac{du_\alpha}{d\tau}$$ ~~for~~ boost-invariant system

$$\Rightarrow \pi^{\alpha\beta} D(\Delta_{\mu\nu\alpha\beta}) = -\gamma\left(u_\mu \pi^{\alpha}_{\ \nu} + u_\nu \pi^{\alpha}_{\ \mu}\right)\frac{du_\alpha}{d\tau}$$

$$= -\gamma\left(u_\mu \pi^{0}_{\ \nu} + u_\nu \pi^{0}_{\ \mu}\right)\frac{d\gamma}{d\tau} - \gamma\left(u_\mu \pi^{i}_{\ \nu} + u_\nu \pi^{i}_{\ \mu}\right)\frac{du_i}{d\tau}$$

$$= \gamma\left(u_\mu \pi^{0}_{\ \nu} + u_\nu \pi^{0}_{\ \mu}\right) v^k \frac{du_k}{d\tau} - \gamma\left(u_\mu \pi^{i}_{\ \nu} + u_\nu \pi^{i}_{\ \mu}\right)\frac{du_i}{d\tau}$$

$$= \gamma\left[\left(u_\mu \pi^{0}_{\ \nu} + u_\nu \pi^{0}_{\ \mu}\right) v^k \frac{du_k}{d\tau} - \left(u_\mu \pi^{i}_{\ \nu} + u_\nu \pi^{i}_{\ \mu}\right)\frac{du_i}{d\tau}\right]$$

$$D\pi_{\mu\nu} = u^\alpha \nabla_\alpha \pi_{\mu\nu} = \gamma \frac{d\pi_{\mu\nu}}{d\tau} - u^\alpha\left(\Gamma^\lambda_{\alpha\mu}\pi_{\lambda\nu} + \Gamma^\lambda_{\alpha\nu}\pi_{\lambda\mu}\right)$$

$$= \gamma \frac{d\pi_{\mu\nu}}{d\tau} - u^0\left(\Gamma^\lambda_{0\mu}\pi_{\lambda\nu} + \Gamma^\lambda_{0\nu}\pi_{\lambda\mu}\right)$$

$$= \gamma \frac{d\pi_{\mu\nu}}{d\tau} - \frac{\gamma}{\tau}\left(\delta^3_\mu \pi_{3\nu} + \delta^3_\nu \pi_{3\mu}\right)$$

Thus: $\Delta_{\mu\nu\alpha\beta} D\pi^{\alpha\beta} = D\pi_{\mu\nu} - \pi^{\alpha\beta} D\Delta_{\mu\nu\alpha\beta}$

$$= \gamma \frac{d\pi_{\mu\nu}}{d\tau} - \frac{\gamma}{\tau}\left(\delta^3_\mu \pi_{3\nu} + \delta^3_\nu \pi_{3\mu}\right) - \gamma\left[\left(u_\mu \pi^{0}_{\ \nu} + u_\nu \pi^{0}_{\ \mu}\right) v^k \frac{du_k}{d\tau} - \left(u_\mu \pi^{i}_{\ \nu} + u_\nu \pi^{i}_{\ \mu}\right)\frac{du_i}{d\tau}\right]$$

$$= \gamma\left[\frac{d\pi_{\mu\nu}}{d\tau} + \left(u_\mu \pi^{i}_{\ \nu} + u_\nu \pi^{i}_{\ \mu}\right)\frac{du_i}{d\tau} - \left(u_\mu \pi^{0}_{\ \nu} + u_\nu \pi^{0}_{\ \mu}\right) v^k \frac{du_k}{d\tau} - \frac{1}{\tau}\left(\delta^3_\mu \pi_{3\nu} + \delta^3_\nu \pi_{3\mu}\right)\right]$$

Finally, the shear relaxation equation becomes:

$$\tau_\pi\left(\Delta_{\mu\nu\alpha\beta} D\pi^{\alpha\beta} + C\pi_{\mu\nu}\Theta\right) + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu}$$

$$\Rightarrow \tau_\pi\gamma\left[\frac{d\pi_{\mu\nu}}{d\tau} + \left(u_\mu\pi^i{}_\nu + u_\nu\pi^i{}_\mu\right)\frac{du_i}{d\tau} - \left(u_\mu\pi^0{}_\nu + u_\nu\pi^0{}_\mu\right)v^k\frac{du_k}{d\tau} - \frac{1}{\tau}\left(\delta^3_\mu\pi_{3\nu} + \delta^3_\nu\pi_{3\mu}\right)\right]$$

$$+\pi_{\mu\nu}\left(1 + C\tau_\pi\left(\partial\cdot u + \frac{\gamma}{\tau}\right)\right)$$

$$= \eta\left(\partial_\mu u_\nu + \partial_\nu u_\mu - 2\gamma\tau\delta^3_\mu\delta^3_\nu\right) - \eta\left(\gamma u_\nu\frac{d}{d\tau}u_\mu + \gamma u_\mu\frac{d}{d\tau}u_\nu\right) - \frac{2\eta}{3}\left(g_{\mu\nu} - u_\mu u_\nu\right)\left(\partial\cdot u + \frac{\gamma}{\tau}\right)$$

Specializing to the case with $\mu=i, \nu=j$:

$$\tau_\pi\gamma\frac{d\pi_{ij}}{d\tau} = \eta\left[\partial_i u_j + \partial_j u_i - \gamma\left(u_i\frac{du_j}{d\tau} + u_j\frac{du_i}{d\tau}\right) + \frac{2}{3}\left(\delta_{ij} + u_i u_j\right)\Theta\right] - \pi_{ij}\left(1 + C\tau_\pi\Theta\right)$$

$$-\tau_\pi\gamma\left[\left(u_i\pi^k{}_j + u_j\pi^k{}_i\right)\frac{du_k}{d\tau} - \left(u_i\pi^0{}_j + u_j\pi^0{}_i\right)v^k\frac{du_k}{d\tau}\right]$$

$$\Rightarrow \frac{d\pi_{ij}}{d\tau} = \frac{\eta}{\tau_\pi\gamma}\left(\partial_i u_j + \partial_j u_i - \gamma\left(u_i\frac{du_j}{d\tau} + u_j\frac{du_i}{d\tau}\right) + \frac{2}{3}\left(\delta_{ij} + u_i u_j\right)\Theta\right) - \frac{\pi_{ij}}{\tau_\pi\gamma}\left(1 + C\tau_\pi\Theta\right)$$

$$-\left(u_i\pi^k{}_j + u_j\pi^k{}_i\right)\frac{du_k}{d\tau} + \left(u_i\pi^0{}_j + u_j\pi^0{}_i\right)v^k\frac{du_k}{d\tau}$$

Although this result is sufficiently developed to be included in the code, to match it to other sets of notes requires expanding Θ as follows:

$$\Theta = \nabla_\alpha u^\alpha = \partial_\alpha u^\alpha + \frac{\gamma}{\tau} = \frac{d\gamma}{d\tau} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{\gamma}{\tau}$$

$$= \frac{\gamma}{\tau} - v^k \frac{du_k}{d\tau} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} = \underline{\gamma\left(\frac{1}{\tau} - \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) - v^k \frac{du_k}{d\tau}}$$

This gives:

$$\frac{d\pi_{ij}}{d\tau} = \frac{\eta}{\tau_\pi \gamma}\left(\partial_i u_j + \partial_j u_i - \gamma\left(u_i \frac{du_j}{d\tau} + u_j \frac{du_i}{d\tau}\right) + \frac{2}{3}(\delta_{ij} + u_i u_j)\Theta\right) - \frac{\pi_{ij}}{\tau_\pi \gamma}(1 + C\tau_\pi \Theta)$$

$$- (u_i \pi^k{}_j + u_j \pi^k{}_i)\frac{du_k}{d\tau} + (u_i \pi^0_j + u_j \pi^0_i) v^k \frac{du_k}{d\tau}$$

$$= \frac{\eta}{\tau_\pi \gamma}\left(\partial_i u_j + \partial_j u_i - \gamma\left(u_i \frac{du_j}{d\tau} + u_j \frac{du_i}{d\tau}\right)\right) - \frac{\pi_{ij}}{\tau_\pi \gamma} - (u_i \pi^k{}_j + u_j \pi^k{}_i)\frac{du_k}{d\tau} + (u_i \pi^0_j + u_j \pi^0_i) v^k \frac{du_k}{d\tau}$$

$$+ \left[\frac{2\eta}{3\tau_\pi \gamma}(\delta_{ij} + u_i u_j) - \frac{C\pi_{ij}}{\gamma}\right]\left[\gamma\left(\frac{1}{\tau} - \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) - v^k \frac{du_k}{d\tau}\right]$$

Raising all indices except for those on the gradients $(\partial_i u^j, \partial_j u^i)$ gives:

$$\frac{d\pi^{ij}}{d\tau} = -\frac{1}{\tau_\pi \gamma}\left[\pi^{ij} + \eta\left(\partial_i u^j + \partial_j u^i + \gamma\left(u^i \frac{du^j}{d\tau} + u^j \frac{du^i}{d\tau}\right)\right)\right] + (u^i \pi^{kj} + u^j \pi^{ki})\frac{du^k}{d\tau}$$

$$- \left[u^i \pi^{0j} + u^j \pi^{0i} - \frac{2\eta}{3\tau_\pi \gamma}(\delta^{ij} + u^i u^j) + \frac{C\pi^{ij}}{\gamma}\right] v^k \frac{du^k}{d\tau}$$

$$+ \left(C\pi^{ij} - \frac{2\eta}{3\tau_\pi}(\delta^{ij} + u^i u^j)\right)\left(\frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau}\right)$$

All of this agrees with the code if we make the following identifications (several of which we already knew were correct):

$$\text{Sigl} \equiv \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau}$$

$$\text{Ipi} \equiv C\pi^{ij} - \frac{2\eta}{3\tau_\pi}\left(\delta^{ij} + u^i u^j\right)$$

$$\text{ubpi} \equiv u^i \pi^{0j}$$

$$\text{vduk} \equiv v^k \frac{du^k}{d\tau}$$

$$\text{pimin} \equiv \pi^{ij}$$

$$\text{setas} \equiv \eta$$

$$\text{partU} \equiv \partial_i u^j + \partial_j u^i$$

$$\text{gamt} = \frac{1}{\gamma \tau_\pi}$$

$$\text{eta\_o\_tau} \equiv \frac{\eta}{\tau_\pi}$$

$$\text{ududt} \equiv u^i \frac{du^j}{d\tau}$$

$$\text{dpidtsub} \equiv \left(u^i \pi^{kj} + u^j \pi^{ki}\right)\frac{du^k}{d\tau}$$

This makes it clear where several of the remaining discrepancies are.