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N3LO+PS matching with incoming hadrons

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Abstract: bla

Contents

1 Definitions

The second-order expansion of the CKKW-L weight is desired

$$w = \frac{x_0^+ f_{0+}\left(x_0^+, t_0\right)}{x_0^+ f_{0+}\left(x_0^+, t_1\right)} \frac{x_0^- f_{0-}\left(x_0^-, t_0\right)}{x_0^- f_{0-}\left(x_0^-, t_1\right)} \frac{x_1^+ f_{1+}\left(x_1^+, t_1\right)}{x_1^+ f_{1+}\left(x_1^+, \mu_f\right)} \frac{x_1^- f_{1-}\left(x_1^-, t_1\right)}{x_1^- f_{1-}\left(x_1^-, \mu_f\right)} \tag{1.1}$$

$$\frac{\alpha_s(t_1)}{\alpha_s(\mu_r)} \Pi_{0+}\left(x_0^+; t_0, t_1\right) \Pi_{0-}\left(x_0^-; t_0, t_1\right) \prod_{f \in FS} \Pi_f\left(x_0^+, x_0^-; t_0, t_1\right) \tag{1.2}$$

where

$$\Pi_{0\pm}\left(x; t_0, t_1\right) = \exp\Bigg(-\int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \sum_s \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t_{[0\pm,i\pm(s),r]}(\rho, z, m^2_{0\pm,r}))}{2\pi} \\ \frac{\frac{x}{z} f_{i\pm(s)}\left(\frac{x}{z}, \rho\right)}{x f_{0\pm}\left(x, \rho\right)} K_{0\pm \to i\pm(s)}\left(z, \rho, m^2_{i,r}\right) \Bigg) \tag{1.3}$$

where $\sum_r$ runs over all possible recoilers and $\sum_s$ runs over all possible (QCD, QED...) splittings. The phase-space boundaries $\Omega_{i,r,s}(x,\rho)$ may depend on the splitting, as can the argument of the strong coupling. The factors $\Pi_f\left(x_0^+, x_0^-; t_0, t_1\right)$ relate to final-state emissions, and can be written as

$$\Pi_f\left(x; t_0, t_1\right) = \exp\Bigg(-\int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_{r \notin \{+,-\}} \sum_s \int_{\Omega_{f,r,s}(\rho)} dz \frac{\alpha_s(t)}{2\pi} K_{f \to f'(s)}\left(z, \rho, m^2_{f,r}\right) \\ -\int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_{r \in \{+,-\}} \sum_s \int_{\Omega_{f,r,s}(\rho)} dz \frac{\alpha_s(t)}{2\pi} \frac{x'(z,\rho,m^2_{f,r}) f_r\left(x'(z,\rho,m^2_{f,r}), \rho\right)}{x f_r\left(x, \rho\right)} K_{f \to f'(s)}\left(z, \rho, m^2_{i,r}\right) \Bigg) \tag{1.4}$$

Now introduce the short-hands

$$\frac{x}{z} \widehat{f}_a\left(\frac{x}{z}, \mu\right) = \sum_{b=q,g} P_{a \leftarrow b}\left(z\right) \frac{x}{z} f_b\left(\frac{x}{z}, \mu\right) \tag{1.5}$$

$$\frac{x}{zz'} \widehat{\widehat{f}}_a\left(\frac{x}{zz'}, \mu\right) = \sum_{b=q,g} P_{a \leftarrow b}\left(z\right) \sum_{c=q,g} P_{b \leftarrow c}\left(z'\right) \frac{x}{zz'} f_c\left(\frac{x}{zz'}, \mu\right) \tag{1.6}$$

$$\frac{x}{z} \widetilde{f}_a\left(\frac{x}{z}, \mu\right) = \sum_s K_{a \to b(s)}\left(z\right) \frac{x}{z} f_{b(s)}\left(\frac{x}{z}, \mu\right) \approx \sum_b P_{b \leftarrow a}\left(z\right) \frac{x}{z} f_b\left(\frac{x}{z}, \mu\right) \tag{1.7}$$

with which we have

$$\Pi_{0\pm}\left(x;t_0,t_1\right) = \exp\left(-\int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t)}{2\pi} \frac{\frac{x}{z}\widetilde{f}_{0\pm}\left(\frac{x}{z},\rho\right)}{x f_{0\pm}\left(x,\rho\right)} \right) \tag{1.8}$$

2 Second-order expansion

For cases with incoming hadrons, the expansion in $\alpha_s(\mu_r)$ is understood as expansion at fixed μ_f. It makes sense to start with the second-order expansion of PDF ratios, since PDF ratios also appear in the no-emission probabilities. The strategy will be

- Extract $\mathcal{O}(\alpha_s^2(\mu_r))$ terms at fixed factorization scale μ_f
- For that, evolve both numerator and denominator to μ_f, and expand the fraction.

The $\alpha_s(\mu_r)$-expansion of a PDF, assuming the phase-space limits $C(x)$ (typically $C(x) = \{z > x \cap z < 1\}$) and using

$$\mathrm{sgn}\left(\mu-\mu_f\right) = \begin{cases} -1 & \text{if } \mu_f > \mu \\ 1 & \text{else} \end{cases}$$

is given by

$$\begin{aligned}
x f_a\left(x,\mu\right) &= x f_a\left(x,\mu_f\right) \\
&\quad + \mathrm{sgn}\left(\mu-\mu_f\right) \int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \frac{\alpha_s(\bar{\mu})}{2\pi} \int\limits_{C(x)} dz \frac{x}{z} \widehat{f}_a\left(\frac{x}{z},\bar{\mu}\right) \\
&= x f_a\left(x,\mu_f\right) \\
&\quad + \mathrm{sgn}\left(\mu-\mu_f\right) \int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \frac{\alpha_s(\bar{\mu})}{2\pi} \int\limits_{C(x)} dz \frac{x}{z} \widehat{f}_a\left(\frac{x}{z},\mu_f\right) \\
&\quad + \mathrm{sgn}\left(\mu-\mu_f\right) \int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \frac{\alpha_s(\bar{\mu})}{2\pi} \int\limits_{C(x)} dz\ \mathrm{sgn}\left(\bar{\mu}-\mu_f\right) \int\limits_{\min(\bar{\mu},\mu_f)}^{\max(\bar{\mu},\mu_f)} \frac{d\bar{\mu}'}{\bar{\mu}'} \frac{\alpha_s(\bar{\mu}')}{2\pi} \int\limits_{C(x/z)} dz' \frac{x}{zz'} \widehat{\widehat{f}}_a\left(\frac{x}{zz'},\bar{\mu}'\right) \\
&= x f_a\left(x,\mu_f\right) \\
&\quad + \mathrm{sgn}\left(\mu-\mu_f\right) \int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \frac{\alpha_s(\mu_r)}{2\pi} \int\limits_{C(x)} dz \frac{x}{z} \widehat{f}_{a,(0)}\left(\frac{x}{z},\mu_f\right) \\
&\quad + \mathrm{sgn}\left(\mu-\mu_f\right) \int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \frac{\beta_0}{2} \ln\left(\frac{\mu_r}{\bar{\mu}}\right) \int\limits_{C(x)} dz \frac{x}{z} \widehat{f}_{a,(0)}\left(\frac{x}{z},\mu_f\right)
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{sgn}\left(\mu-\mu_f\right)\int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)}\frac{d\bar{\mu}}{\bar{\mu}}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\int\limits_{C(x)}dz\frac{x}{z}\widehat{f}_{a,(1)}\left(\frac{x}{z},\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)}\frac{d\bar{\mu}}{\bar{\mu}}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\int\limits_{C(x)}dz\ \mathrm{sgn}\left(\bar{\mu}-\mu_f\right)\int\limits_{\min(\bar{\mu},\mu_f)}^{\max(\bar{\mu},\mu_f)}\frac{d\bar{\mu}'}{\bar{\mu}'}\int\limits_{C(x/z)}dz'\frac{x}{zz'}\widehat{\widehat{f}}_a\left(\frac{x}{zz'},\mu_f\right)\\
&+\mathcal{O}(\alpha_s^3(\mu_r))\\
=\;& xf_a\left(x,\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)}\frac{d\bar{\mu}}{\bar{\mu}}\frac{\alpha_s(\mu_r)}{2\pi}\int\limits_{C(x)}dz\sum_{b=q,g}P^{(0)}_{a\leftarrow b}\left(z\right)\frac{x}{z}f_b\left(\frac{x}{z},\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)}\frac{d\bar{\mu}}{\bar{\mu}}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\frac{\beta_0}{2}\ln\left(\frac{\mu_r}{\bar{\mu}}\right)\int\limits_{C(x)}dz\sum_{b=q,g}P^{(0)}_{a\leftarrow b}\left(z\right)\frac{x}{z}f_b\left(\frac{x}{z},\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)}\frac{d\bar{\mu}}{\bar{\mu}}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\sum_{b=q,g}P^{(1)}_{a\leftarrow b}\left(z\right)\frac{x}{z}f_b\left(\frac{x}{z},\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\int\limits_{\min(\mu,\mu_f)}^{\max(\mu,\mu_f)}\frac{d\bar{\mu}}{\bar{\mu}}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\ \mathrm{sgn}\left(\bar{\mu}-\mu_f\right)\int\limits_{\min(\bar{\mu},\mu_f)}^{\max(\bar{\mu},\mu_f)}\frac{d\bar{\mu}'}{\bar{\mu}'}\\
&\qquad\int\limits_{C(x)}dz\sum_{b=q,g}P^{(0)}_{a\leftarrow b}\left(z\right)\int\limits_{C(x/z)}dz'\sum_{c=q,g}P^{(0)}_{b\leftarrow c}\left(z'\right)\frac{x}{zz'}f_c\left(\frac{x}{zz'},\mu_f\right)\\
&+\mathcal{O}(\alpha_s^3(\mu_r))\\
=\;& xf_a\left(x,\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\ln\left[\frac{\max(\mu,\mu_f)}{\min(\mu,\mu_f)}\right]\frac{\alpha_s(\mu_r)}{2\pi}\int\limits_{C(x)}dz\sum_{b=q,g}P^{(0)}_{a\leftarrow b}\left(z\right)\frac{x}{z}f_b\left(\frac{x}{z},\mu_f\right)\\
&-\mathrm{sgn}\left(\mu-\mu_f\right)\frac{1}{2}\left[\ln^2\left[\frac{\max(\mu,\mu_f)}{\mu_r}\right]-\ln^2\left[\frac{\min(\mu,\mu_f)}{\mu_r}\right]\right]\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\\
&\qquad\frac{\beta_0}{2}\int\limits_{C(x)}dz\sum_{b=q,g}P^{(0)}_{a\leftarrow b}\left(z\right)\frac{x}{z}f_b\left(\frac{x}{z},\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\ln\left[\frac{\max(\mu,\mu_f)}{\min(\mu,\mu_f)}\right]\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\sum_{b=q,g}P^{(1)}_{a\leftarrow b}\left(z\right)\frac{x}{z}f_b\left(\frac{x}{z},\mu_f\right)\\
&+\mathrm{sgn}\left(\mu-\mu_f\right)\frac{1}{2}\left[\ln^2\left[\frac{\max(\mu,\mu_f)}{\mu_f}\right]-\ln^2\left[\frac{\min(\mu,\mu_f)}{\mu_f}\right]\right]\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\\
&\qquad\int\limits_{C(x)}dz\sum_{b=q,g}P^{(0)}_{a\leftarrow b}\left(z\right)\int\limits_{C(x/z)}dz'\sum_{c=q,g}P^{(0)}_{b\leftarrow c}\left(z'\right)\frac{x}{zz'}f_c\left(\frac{x}{zz'},\mu_f\right)
\end{aligned}
$$

$$+\mathcal{O}(\alpha_s^3(\mu_r))$$

The CKKW-L weight contains ratios of all-order factors. To expand a ratio of α_s-expansions, use

$$\frac{\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^n a_n}{\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^n b_n} = \frac{a_0}{b_0} + \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)\left(\frac{a_1}{b_0} - \frac{a_0}{b_0}\frac{b_1}{b_0}\right) + \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\left(\frac{a_2}{b_0} - \frac{a_0}{b_0}\frac{b_2}{b_0} + \frac{a_0}{b_0}\frac{b_1^2}{b_0^2} - \frac{a_1}{b_0}\frac{b_1}{b_0}\right) + \mathcal{O}(\alpha_s^3(\mu_r))$$

In the case of several numerator or denominator factors, cross-terms of their expansions will arise at $\mathcal{O}(\alpha_s^2)$. Thus, it is useful to apply this expression to the CKKW-L weight in question. In the expansion of the CKKW-L weight, it is safe to assume $a_0 = b_0 = x_0^+ f_{0+}\left(x_0^+, \mu_f\right) \cdot x_0^- f_{0-}\left(x_0^-, \mu_f\right) \cdot x_1^+ f_{1+}\left(x_1^+, \mu_f\right) \cdot x_1^- f_{1-}\left(x_1^-, \mu_f\right)$, since

$$\frac{\alpha_s(t_1)}{\alpha_s(\mu_r)} = 1 + \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)\frac{\beta_0}{2}\ln\left(\frac{\mu_r}{t_1}\right) + \left[\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)\frac{\beta_0}{2}\ln\left(\frac{\mu_r}{t_1}\right)\right]^2 + \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\frac{\beta_1}{4}\ln\left(\frac{\mu_r}{t_1}\right) + \mathcal{O}(\alpha_s^3(\mu_r)) \ ,$$

that each individual PDF ratios is of identical flavor/x-value, and that

$$\Pi_i\left(x; t_0, t_1\right) = 1 + \mathcal{O}(\alpha_s(\mu_r)) \ .$$

The coefficients a_1 and a_2 of the numerator expansion can be obtained by expanding all numerator factors, multiplying the expansions, and grouping the relevant terms. For the structure of the CKKW-L weight in question, and given that the third and fourth PDF ratios contain denominators already evaluated at μ_f, i.e. $b_{1\{3,4\}} = b_{2\{3,4\}} = 0$, this means that the expansion of the CKKW-L weight takes the form

$$\begin{aligned}
&\frac{(b_{01}+a_{11}x+a_{21}x^2)(b_{02}+a_{12}x+a_{22}x^2)(b_{03}+a_{13}x+a_{23}x^2)(b_{04}+a_{14}x+a_{24}x^2)\prod_{i=1}^{N}(1+w_{1i}x+w_{2i}x^2)}{(b_{01}+b_{11}x+b_{21}x^2)(b_{02}+b_{12}x+b_{22}x^2)b_{03}b_{04}}\\
&= 1 + x\left[\sum_{i=1}^{4}\frac{a_{1i}}{b_{0i}} + \sum_{i=1}^{N} w_{1i} - \sum_{i=1}^{2}\frac{b_{1i}}{b_{0i}}\right]\\
&+x^2\left[\left\{\sum_{i=1}^{4}\frac{a_{1i}}{b_{0i}}\left(\sum_{j=i+1}^{4}\frac{a_{1j}}{b_{0j}} + \sum_{j=1}^{N} w_{1j} - \sum_{j=1}^{2}\frac{b_{1j}}{b_{0j}}\right)\right\} + \left\{\sum_{i=1}^{N} w_{1i}\left(\sum_{j=i+1}^{N} w_{1j} - \sum_{j=1}^{2}\frac{b_{1j}}{b_{0j}}\right)\right\} + \left\{\sum_{i=1}^{2}\frac{b_{1j}}{b_{0j}}\left(\sum_{j=i+1}^{2}\frac{b_{1j}}{b_{0j}}\right)\right\}\right.\\
&\left.+\sum_{i=1}^{4}\frac{a_{2i}}{b_{0i}} - \sum_{j=1}^{2}\frac{b_{2j}}{b_{0j}} + \sum_{i=1}^{N} w_{2i} + \sum_{i=1}^{2}\frac{b_{1i}^2}{b_{0i}^2}\right]
\end{aligned}$$

To obtain a concrete expression, it is necessary to also expand the no-emission probabilities to second order:

$$\Pi_{0\pm}\left(x; t_0, t_1\right) = 1 - \int_{t_1}^{t_0}\frac{d\rho}{\rho}\sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t)}{2\pi}\frac{\frac{x}{z}\widetilde{f}_{0\pm}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}\left(x,\rho\right)} \tag{2.1}$$

$$+ \frac{1}{2}\left(\int_{t_1}^{t_0}\frac{d\rho}{\rho}\sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t)}{2\pi}\frac{\frac{x}{z}\widetilde{f}_{0\pm}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}\left(x,\rho\right)}\right)^2 + \text{“}\mathcal{O}(\alpha_s^3)\text{”} \tag{2.2}$$

$$= 1 - \int_{t_1}^{t_0}\frac{d\rho}{\rho}\sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t)}{2\pi}\frac{\frac{x}{z}\widetilde{f}_{0\pm,(0)}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}\left(x,\rho\right)} \tag{2.3}$$

$$- \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t)}{2\pi} \frac{\frac{x}{z}\widetilde{f}_{0\pm,(1)}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}(x,\rho)} \tag{2.4}$$

$$+ \frac{1}{2}\left(\int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(t)}{2\pi} \frac{\frac{x}{z}\widetilde{f}_{0\pm,(0)}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}(x,\rho)}\right)^2 + \text{“}\mathcal{O}(\alpha_s^3)\text{”} \tag{2.5}$$

Here, "$\mathcal{O}(\alpha_s^3)$" indicates that at present, this is not yet a consistent expansion in powers of $\alpha_s(\mu_r)$, since the remaining terms include integrals of running-coupling factors.

The PDF ratios in the no-emission rates need to be evolved to a common μ_f, using the PDF expansion formula in both numerator and denominator, and using the ratio expansion formula. Luckily, only the expansion up to $\mathcal{O}(\alpha_s^1(\mu_r))$ terms is required.

$$\begin{aligned}
\frac{\frac{x}{z}\widetilde{f}_{0\pm}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}(x,\rho)} &= = \frac{\sum_s K_{0\pm\to b(s)}(z)\,\frac{x}{z}f_{b(s)}\left(\frac{x}{z},\rho\right)}{xf_{0\pm}(x,\rho)} \\
&= \sum_s K_{0\pm\to b(s)}(z)\Bigg[\frac{\frac{x}{z}f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{xf_{0\pm}(x,\mu_f)} \\
&+ \operatorname{sgn}(\rho-\mu_f)\int_{\min(\rho,\mu_f)}^{\max(\rho,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}}\frac{\alpha_s(\mu_r)}{2\pi}\int_{C(x/z)} dz' \frac{\frac{x}{zz'}\widehat{f}_{b(s),(0)}\left(\frac{x}{zz'},\mu_f\right)}{xf_{0\pm}(x,\mu_f)} \\
&- \frac{\frac{x}{z}f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{xf_{0\pm}(x,\mu_f)}\operatorname{sgn}(\rho-\mu_f)\int_{\min(\rho,\mu_f)}^{\max(\rho,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}}\frac{\alpha_s(\mu_r)}{2\pi}\int_{C(x)} dz' \frac{\frac{x}{z'}\widehat{f}_{0\pm,(0)}\left(\frac{x}{z'},\mu_f\right)}{xf_{0\pm}(x,\mu_f)}\Bigg]
\end{aligned}$$

With this, the second-order expansion of a no-emission probability amounts to

$$\begin{aligned}
\Pi_{0\pm}(x;t_0,t_1) &= 1 \\
&- \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(\mu_r)}{2\pi} \frac{\frac{x}{z}\widetilde{f}_{0\pm,(0)}\left(\frac{x}{z},\mu_f\right)}{xf_{0\pm}(x,\mu_f)} \\
&- \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \frac{\beta_0}{2}\ln\left(\frac{\mu_r}{t}\right) \frac{\frac{x}{z}\widetilde{f}_{0\pm}\left(\frac{x}{z},\mu_f\right)}{xf_{0\pm}(x,\mu_f)} \\
&- \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \frac{\frac{x}{z}\widetilde{f}_{0\pm,(1)}\left(\frac{x}{z},\mu_f\right)}{xf_{0\pm}(x,\mu_f)} \\
&- \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \operatorname{sgn}(\rho-\mu_f)\Bigg\{ \\
&\int_{\min(\rho,\mu_f)}^{\max(\rho,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \int_{C(x/z)} dz' \frac{\sum_s K_{0\pm\to b(s)}(z)\,\frac{x}{zz'}\widehat{f}_{b(s),(0)}\left(\frac{x}{zz'},\mu_f\right)}{xf_{0\pm}(x,\mu_f)}
\end{aligned}$$

$$
- \frac{\sum_s K_{0\pm\to b(s)}(z)\, \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \int\limits_{\min(\rho,\mu_f)}^{\max(\rho,\mu_f)} \frac{d\bar{\mu}}{\bar{\mu}} \int\limits_{C(x)} dz' \frac{\frac{x}{z'} \widehat{f}_{0\pm,(0)}\left(\frac{x}{z'},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \Bigg\}
$$

$$
+ \frac{1}{2}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \left(\int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\frac{x}{z}\widetilde{f}_{0\pm,(0)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \right)^2
$$

$$
+ \mathcal{O}(\alpha_s^3(\mu_r))
$$

$$
= 1
$$

$$
- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(\mu_r)}{2\pi} \frac{\sum_s K^{(0)}_{0\pm\to b(s)}(z)\, \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)}
$$

$$
- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \frac{\beta_0}{2} \ln\left(\frac{\mu_r}{t}\right) \frac{\sum_s K^{(0)}_{0\pm\to b(s)}(z)\, \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)}
$$

$$
- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \frac{\sum_s K^{(1)}_{0\pm\to b(s)}(z)\, \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)}
$$

$$
- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \mathrm{sgn}\,(\rho-\mu_f) \ln\left[\frac{\max(\rho,\mu_f)}{\min(\rho,\mu_f}\right] \sum_s K^{(0)}_{0\pm\to b(s)}(z)
$$

$$
\left\{ \int\limits_{C(x/z)} dz' \frac{\sum_{c=q,g} P^{(0)}_{b(s)\leftarrow c}(z')\, \frac{x}{zz'} f_c\left(\frac{x}{zz'},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} - \frac{\frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \int\limits_{C(x)} dz' \frac{\sum_{c=q,g} P^{(0)}_{0\pm\leftarrow c}(z')\, \frac{x}{z'} f_c\left(\frac{x}{z'},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \right\}
$$

$$
+ \frac{1}{2}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \left(\int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\sum_s K^{(0)}_{0\pm\to b(s)}(z)\, \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \right)^2
$$

$$
+ \mathcal{O}(\alpha_s^3(\mu_r))
$$

$$
= 1
$$

$$
- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\alpha_s(\mu_r)}{2\pi} \sum_s K^{(0)}_{0\pm\to b(s)}(z) \frac{\frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)}
$$

$$
- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \sum_s K^{(0)}_{0\pm\to b(s)}(z) \frac{\frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)}
$$

$$
\left[\frac{\beta_0}{2} \ln\left(\frac{\mu_r}{t}\right) + \mathrm{sgn}\,(\rho-\mu_f) \ln\left[\frac{\max(\rho,\mu_f)}{\min(\rho,\mu_f}\right]\right.
$$

$$\cdot \left\{ \int\limits_{C(x/z)} dz' \frac{\sum\limits_{c=q,g} P^{(0)}_{b(s)\leftarrow c}(z') \frac{x}{zz'} f_c\left(\frac{x}{zz'},\mu_f\right)}{\frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)} - \int\limits_{C(x)} dz' \frac{\sum\limits_{c=q,g} P^{(0)}_{0\pm\leftarrow c}(z') \frac{x}{z'} f_c\left(\frac{x}{z'},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \right\} \Bigg]$$

$$- \int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \frac{\sum\limits_s K^{(1)}_{0\pm\to b(s)}(z) \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)}$$

$$+ \frac{1}{2}\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2 \left(\int\limits_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_r \int\limits_{\Omega_{0\pm,r,s}(x,\rho)} dz \frac{\sum\limits_s K^{(0)}_{0\pm\to b(s)}(z) \frac{x}{z} f_{b(s)}\left(\frac{x}{z},\mu_f\right)}{x f_{0\pm}(x,\mu_f)} \right)^2$$

$$+ \mathcal{O}(\alpha_s^3(\mu_r))$$

With this last piece, the second-order expansion of the CKKW-L weight amounts to

$$\frac{x_0^+ f_{0+}\left(x_0^+,t_0\right)}{x_0^+ f_{0+}\left(x_0^+,t_1\right)} \frac{x_0^- f_{0-}\left(x_0^-,t_0\right)}{x_0^- f_{0-}\left(x_0^-,t_1\right)} \frac{x_1^+ f_{1+}\left(x_1^+,t_1\right)}{x_1^+ f_{1+}\left(x_1^+,\mu_f\right)} \frac{x_1^- f_{1-}\left(x_1^-,t_1\right)}{x_1^- f_{1-}\left(x_1^-,\mu_f\right)}$$

$$\frac{\alpha_s(t_1)}{\alpha_s(\mu_r)} \Pi_{0+}\left(x_0^+;t_0,t_1\right) \Pi_{0-}\left(x_0^-;t_0,t_1\right) \prod_{f\in FS} \Pi_f\left(x_0^+,x_0^-;t_0,t_1\right)$$

$$\approx \frac{(b_{01}+a_{11}x+a_{21}x^2)(b_{02}+a_{12}x+a_{22}x^2)(b_{03}+a_{13}x+a_{23}x^2)(b_{04}+a_{14}x+a_{24}x^2)\prod_{i=1}^N(1+w_{1i}x+w_{2i}x^2)}{(b_{01}+b_{11}x+b_{21}x^2)(b_{02}+b_{12}x+b_{22}x^2)b_{03}b_{04}}$$

$$= 1 + x\left[\sum_{i=1}^4 \frac{a_{1i}}{b_{0i}} + \sum_{i=1}^N w_{1i} - \sum_{i=1}^2 \frac{b_{1i}}{b_{0i}}\right]$$

$$+x^2\left[\left\{\sum_{i=1}^4 \frac{a_{1i}}{b_{0i}}\left(\sum_{j=i+1}^4 \frac{a_{1j}}{b_{0j}} + \sum_{j=1}^N w_{1j} - \sum_{j=1}^2 \frac{b_{1j}}{b_{0j}}\right)\right\} + \left\{\sum_{i=1}^N w_{1i}\left(\sum_{j=i+1}^N w_{1j} - \sum_{j=1}^2 \frac{b_{1j}}{b_{0j}}\right)\right\} + \left\{\sum_{i=1}^2 \frac{b_{1j}}{b_{0j}}\left(\sum_{j=i+1}^2 \frac{b_{1j}}{b_{0j}}\right)\right\}\right.$$

$$\left. + \sum_{i=1}^4 \frac{a_{2i}}{b_{0i}} - \sum_{j=1}^2 \frac{b_{2j}}{b_{0j}} + \sum_{i=1}^N w_{2i} + \sum_{i=1}^2 \frac{b_{1i}^2}{b_{0i}^2}\right]$$

where

$$\frac{a_{11}}{b_{01}} - \frac{b_{11}}{b_{01}} + \frac{a_{12}}{b_{02}} - \frac{b_{12}}{b_{02}} + \frac{a_{13}}{b_{03}} + \frac{a_{14}}{b_{04}}$$

$$= \left(\mathrm{sgn}\left(t_0-\mu_f\right)\ln\left[\frac{\max(t_0,\mu_f)}{\min(t_0,\mu_f)}\right] - \mathrm{sgn}\left(t_1-\mu_f\right)\ln\left[\frac{\max(t_1,\mu_f)}{\min(t_1,\mu_f)}\right]\right) \int\limits_{C(x_0^+)} dz \frac{\frac{x_0^+}{z}\widehat{f}_{0+,(0)}\left(\frac{x_0^+}{z},\mu_f\right)}{x_0^+ f_{0+}\left(x_0^+,\mu_f\right)}$$

$$+\left(\mathrm{sgn}\left(t_0-\mu_f\right)\ln\left[\frac{\max(t_0,\mu_f)}{\min(t_0,\mu_f)}\right] - \mathrm{sgn}\left(t_1-\mu_f\right)\ln\left[\frac{\max(t_1,\mu_f)}{\min(t_1,\mu_f)}\right]\right) \int\limits_{C(x_0^-)} dz \frac{\frac{x_0^-}{z}\widehat{f}_{0-,(0)}\left(\frac{x_0^-}{z},\mu_f\right)}{x_0^- f_{0-}\left(x_0^-,\mu_f\right)}$$

$$+\mathrm{sgn}\left(t_1-\mu_f\right)\ln\left[\frac{\max(t_1,\mu_f)}{\min(t_1,\mu_f)}\right] \int\limits_{C(x_1^+)} dz \frac{\frac{x_1^+}{z}\widehat{f}_{1+,(0)}\left(\frac{x_1^+}{z},\mu_f\right)}{x_1^+ f_{1+}\left(x_1^+,\mu_f\right)}$$

$$+\mathrm{sgn}\left(t_1-\mu_f\right)\ln\left[\frac{\max(t_1,\mu_f)}{\min(t_1,\mu_f)}\right] \int\limits_{C(x_1^-)} dz \frac{\frac{x_1^-}{z}\widehat{f}_{1-,(0)}\left(\frac{x_1^-}{z},\mu_f\right)}{x_1^- f_{1-}\left(x_1^-,\mu_f\right)}$$

and where

$$
\begin{aligned}
\sum_{i=1}^{N} w_{1i} = & \sum_{i\in\{+,-\}} \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_{r} \int_{\Omega_{0i,r,s}(x,\rho)} dz \frac{\frac{x_0^i}{z} \widetilde{f}_{0i,(0)}\left(\frac{x_0^i}{z}, \mu_f\right)}{x_0^i f_{0i}\left(x_0^i, \mu_f\right)} \\
& - \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_{r\in\{+,-\}} \sum_{s} \int_{\Omega_{f,r,s}(\rho,x^{\pm})} dz \frac{x^{\pm'} f_{0r}\left(x^{\pm'}, \mu_f\right)}{x f_{0r}\left(x, \mu_f\right)} K_{f\to f'(s)}\left(z, \rho, m_{i,r}^2\right) \\
& - \int_{t_1}^{t_0} \frac{d\rho}{\rho} \sum_{r\notin\{+,-\}} \sum_{s} \int_{\Omega_{f,r,s}(\rho)} dz K_{f\to f'(s)}\left(z, \rho, m_{f,r}^2\right) \\
& + \frac{\beta_0}{2} \ln\left(\frac{\mu_r}{t_1}\right)
\end{aligned}
$$

Auxiliary formulae

$$\frac{a_0 + a_1\left(\frac{\alpha_s(\mu_r)}{2\pi}\right) + a_2\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2}{b_0 + b_1\left(\frac{\alpha_s(\mu_r)}{2\pi}\right) + b_2\left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2} = \frac{a_0}{b_0} + \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)\left(\frac{a_1}{b_0} - \frac{a_0}{b_0}\frac{b_1}{b_0}\right) + \left(\frac{\alpha_s(\mu_r)}{2\pi}\right)^2\left(\frac{a_2}{b_0} - \frac{a_0}{b_0}\frac{b_2}{b_0} + \frac{a_0}{b_0}\frac{b_1^2}{b_0^2} - \frac{a_1}{b_0}\frac{b_1}{b_0}\right)$$

$$\begin{aligned}
&\frac{a_0 + a_1 x + a_2 x^2}{(b_{01} + b_{11}x + b_{21}x^2)(b_{02} + b_{12}x + b_{22}x^2)(b_{03} + b_{13}x + b_{23}x^2)(b_{04} + b_{14}x + b_{24}x^2)} \\
&= \frac{a_0}{b_{01}b_{02}b_{03}b_{04}} \\
&+x\left(\frac{a_1}{(b_{01}b_{02}b_{03}b_{04})} - \frac{a_0}{(b_{01}b_{02}b_{03}b_{04})}\left(\frac{b_{11}}{b_{01}} + \frac{b_{12}}{b_{02}} + \frac{b_{13}}{b_{03}} + \frac{b_{14}}{b_{04}}\right)\right) \\
&+x^2\Big[\frac{a_0}{(b_{01}b_{02}b_{03}b_{04})}\left(\frac{b_{11}^2}{b_{01}^2} + \frac{b_{12}^2}{b_{02}^2} + \frac{b_{13}^2}{b_{03}^2} + \frac{b_{14}^2}{b_{04}^2} + \frac{b_{11}b_{12}}{(b_{01}b_{02})} + \frac{b_{11}b_{13}}{(b_{01}b_{03})} + \frac{b_{11}b_{14}}{(b_{01}b_{04})} + \frac{b_{12}b_{13}}{(b_{02}b_{03})} + \frac{b_{12}b_{14}}{(b_{02}b_{04})} + \frac{b_{13}b_{14}}{(b_{03}b_{04})}\right. \\
&\left.-\frac{b_{21}}{b_{01}} - \frac{b_{22}}{b_{02}} - \frac{b_{23}}{b_{03}} - \frac{b_{24}}{b_{04}}\right) - \frac{a_1}{(b_{01}b_{02}b_{03}b_{04})}\left(\frac{b_{11}}{b_{01}} + \frac{b_{12}}{b_{02}} + \frac{b_{13}}{b_{03}} + \frac{b_{14}}{b_{04}}\right) + \frac{a_2}{(b_{01}b_{02}b_{03}b_{04})}\Big]
\end{aligned}$$

relevant terms:

$$\begin{aligned}
&(b_{01} + a_{11}x + a_{21}x^2)(b_{02} + a_{12}x + a_{22}x^2)(b_{03} + a_{13}x + a_{23}x^2)(b_{04} + a_{14}x + a_{24}x^2)\prod_{i=1}^{N}(1 + w_{1i}x + w_{2i}x^2) \\
&= b_{01}b_{02}b_{03}b_{04} + x\left(\frac{a_{11}}{b_{01}} + \frac{a_{12}}{b_{02}} + \frac{a_{13}}{b_{03}} + \frac{a_{14}}{b_{04}} + \sum_{i=1}^{N} w_{1i}\right)b_{01}b_{02}b_{03}b_{04} \\
&+x^2\left(\frac{a_{11}}{b_{01}}\frac{a_{14}}{b_{04}} + \frac{a_{12}}{b_{02}}\frac{a_{14}}{b_{04}} + \frac{a_{13}}{b_{03}}\frac{a_{14}}{b_{04}} + \frac{a_{11}}{b_{01}}\frac{a_{13}}{b_{03}} + \frac{a_{12}}{b_{02}}\frac{a_{13}}{b_{03}} + \frac{a_{12}}{b_{02}}\frac{a_{11}}{b_{01}}\right. \\
&+\frac{a_{21}}{b_{01}} + \frac{a_{22}}{b_{02}} + \frac{a_{23}}{b_{03}} + \frac{a_{24}}{b_{04}} \\
&+\left(\sum_{i=1}^{N} w_{1i}\right)\left(\frac{a_{14}}{b_{04}} + \frac{a_{13}}{b_{03}} + \frac{a_{12}}{b_{02}} + \frac{a_{11}}{b_{01}}\right) \\
&\left.+\left(\sum_{i=1}^{N} w_{1i}\left(\sum_{j=i}^{N} w_{1j}\right)\right) + \sum_{i=1}^{N} w_{2i}\right)b_{01}b_{02}b_{03}b_{04}
\end{aligned}$$

$$\begin{aligned}
&\frac{(b_{01} + a_{11}x + a_{21}x^2)(b_{02} + a_{12}x + a_{22}x^2)(b_{03} + a_{13}x + a_{23}x^2)(b_{04} + a_{14}x + a_{24}x^2)\prod_{i=1}^{N}(1 + w_{1i}x + w_{2i}x^2)}{(b_{01} + b_{11}x + b_{21}x^2)(b_{02} + b_{12}x + b_{22}x^2)(b_{03} + b_{13}x + b_{23}x^2)(b_{04} + b_{14}x + b_{24}x^2)} \\
&= 1 + x\left(\frac{a_{11}}{b_{01}} + \frac{a_{12}}{b_{02}} + \frac{a_{13}}{b_{03}} + \frac{a_{14}}{b_{04}} + \sum_{i=1}^{N} w_{1i} - \frac{b_{11}}{b_{01}} - \frac{b_{12}}{b_{02}} - \frac{b_{13}}{b_{03}} - \frac{b_{14}}{b_{04}}\right) \\
&+x^2\left(\frac{a_{11}}{b_{01}}\frac{a_{14}}{b_{04}} + \frac{a_{12}}{b_{02}}\frac{a_{14}}{b_{04}} + \frac{a_{13}}{b_{03}}\frac{a_{14}}{b_{04}} + \frac{a_{11}}{b_{01}}\frac{a_{13}}{b_{03}} + \frac{a_{12}}{b_{02}}\frac{a_{13}}{b_{03}} + \frac{a_{12}}{b_{02}}\frac{a_{11}}{b_{01}}\right. \\
&+\frac{b_{11}^2}{b_{01}^2} + \frac{b_{12}^2}{b_{02}^2} + \frac{b_{13}^2}{b_{03}^2} + \frac{b_{14}^2}{b_{04}^2} + \frac{b_{11}b_{12}}{(b_{01}b_{02})} + \frac{b_{11}b_{13}}{(b_{01}b_{03})} + \frac{b_{11}b_{14}}{(b_{01}b_{04})} + \frac{b_{12}b_{13}}{(b_{02}b_{03})} + \frac{b_{12}b_{14}}{(b_{02}b_{04})} + \frac{b_{13}b_{14}}{(b_{03}b_{04})} \\
&+\frac{a_{21}}{b_{01}} + \frac{a_{22}}{b_{02}} + \frac{a_{23}}{b_{03}} + \frac{a_{24}}{b_{04}} - \frac{b_{21}}{b_{01}} - \frac{b_{22}}{b_{02}} - \frac{b_{23}}{b_{03}} - \frac{b_{24}}{b_{04}} \\
&+\left(\sum_{i=1}^{N} w_{1i}\right)\left(\frac{a_{14}}{b_{04}} + \frac{a_{13}}{b_{03}} + \frac{a_{12}}{b_{02}} + \frac{a_{11}}{b_{01}}\right) + \left(\sum_{i=1}^{N} w_{1i}\left(\sum_{j=i}^{N} w_{1j}\right)\right) + \sum_{i=1}^{N} w_{2i} \\
&\left.-\left(\frac{a_{11}}{b_{01}} + \frac{a_{12}}{b_{02}} + \frac{a_{13}}{b_{03}} + \frac{a_{14}}{b_{04}} + \sum_{i=1}^{N} w_{1i}\right)\left(\frac{b_{11}}{b_{01}} + \frac{b_{12}}{b_{02}} + \frac{b_{13}}{b_{03}} + \frac{b_{14}}{b_{04}}\right)\right)
\end{aligned}$$