Assumptions

Our Bitcoin pricing model is derived from three major assumptions.

Assumption 1: Value (purchasing power) cannot be created or destroyed and is finite in substance. Hence the entire financial eco-system is predicated on **ZERO**-sum game dynamics. For example, for one asset to increase in market capitalisation there needs to be another asset (or multiple assets) that equivalently decrease in market capitalisation.

Assumption 2: In short time horizons, the aggregate change of value for all assets within the financial ecosystem must equate to **ZERO** in order to maintain finite substance condition in assumption 1. Changes in the value of Bitcoin, can only be occurring via some form of substitution phenomenon. Substitution itself must be bi-directional representing the rate of value absorption/dispersion from alternative asset within its own class as well as assets outside of its class.

Assumption 3 : Substitution is driven by Markowitz portfolio optimisations from which multiple rational economic agents are competing to maximise individual investor utility whilst minimising their individual portfolio risk as close to **ZERO** as physically possible. Assumption 3 suggests economic agents tend to embrace Bitcoin as a superior store of value (especially considering longer time horizons) which can combat inflation and provide a unique hedge against systemic risk in the traditional financial system and/or traditional currencies

Representation of a New Asset Class via Substitution

First, consider the case of 2 assets. Then, the Wronskian of the functions M_1 and M_2 will be

$$W(M_1, M_2) = \begin{vmatrix} M_1 & M_2 \\ M_1' & M_2' \end{vmatrix} = M_1 \cdot M_2' - M_1' \cdot M_2.$$

On the other hand, we know that

$$\frac{\partial (M_1 + M_2)}{\partial t} = M_1' + M_2' = 0, \tag{1}$$

leading to

$$M_1' = -M_2'.$$

Substituting this into the expression for $W(M_1, M_2)$, we obtain that

$$W(M_1, M_2) = M_1 \cdot M_2' - (-M_2') \cdot M_2 = M_1 \cdot M_2' + M_2' \cdot M_2 = M_2' \cdot (M_1 + M_2).$$

Apparently, $W(M_1, M_2) \neq 0$ at least for some t, providing that M_1 and M_2 are linearly independent functions of time. However, it is important to note that in view of the equality (1), M_1' and M_2' are not linearly independent.

The linear independence of M_1 and M_2 means that any new asset M_3 in the vector space with basis functions M_1 and M_2 can be represented as

$$M_3 = \omega_1 \cdot M_1 + \omega_2 \cdot M_2,$$

where ω_1 and ω_2 are not simultaneously 0.

Evidently, for any $\omega_3 \neq -1$, we can write

$$\omega_1 \cdot M_1 + \omega_2 \cdot M_2 + \omega_3 M_3 \neq 0$$

for the above ω_1 and ω_2. Therefore, M_1, M_2 and M_3 are linearly independent functions forming a basis for a new, higher-dimensional vector space, in which, any new asset M_4 can be represented as

$$M_4 = \omega_1 \cdot M_1 + \omega_2 \cdot M_2 + \omega_3 M_3.$$

In a similar fashion, one can show that in the n-dimensional space with basis functions M_1, M_2, …, M_n, the new asset M_{BTC} can be represented as

$$M_{\text{BTC}} = \sum_{k=1}^{n} \omega_k \cdot M_k \tag{2}$$

with $\omega_k \neq -1$ which is satisfied in our case since all $\omega_k > 0$.

Hereinafter, M_k is regarded as the total capitalization of Market k, and ω_k– as the substitution of Market k.

The Governing Equation

We start with the equation of exchange below:

$$P_{\mathrm{BTC}} \cdot Q_{\mathrm{BTC}} = M_{\mathrm{BTC}} \cdot V_{\mathrm{BTC}},$$

where P_{BTC} is the price of asset, Q_{BTC} is the output power, M_{BTC} is the market capitalization, and V_{BTC} is the velocity of the asset class.

We consider the general case when Q_{BTC}, M_{BTC} and V_{BTC} are functions of time variable t, i.e., in what follows, $Q_{\mathrm{BTC}} = Q_{\mathrm{BTC}}(t)$, $M_{\mathrm{BTC}} = M_{\mathrm{BTC}}(t)$ and $V_{\mathrm{BTC}} = V_{\mathrm{BTC}}(t)$.

Evidently, taking natural logarithm in both sides of the previous equality, we derive

$$\ln P_{\mathrm{BTC}} + \ln Q_{\mathrm{BTC}} = \ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}}.$$

Differentiating both sides of this expression with respect to t, we obtain

$$\frac{\partial}{\partial t} \left[\ln P_{\mathrm{BTC}} + \ln Q_{\mathrm{BTC}} - (\ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}})\right] = 0.$$

or

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial t} = \frac{\partial}{\partial t} \left[\ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]. \tag{3}$$

For the sake of simplicity, we denote

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial t} = \Pi_{\mathrm{BTC}}.$$

Therefore:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t} \left[\ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].$$

A Particular Model for Capitalization

We now assume that

$$M_k = P_k \cdot U_k, \tag{4}$$

where P_k is the price and U_k is the unit of the kth asset.

Then,

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t} \left[\ln \left(\sum_{k=1}^{n} \omega_k \cdot P_k \cdot U_k\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]. \tag{5}$$

A Particular Model for Substitution

In this section, we consider a specific model for the absorption rate ω_k as follows

$$\omega_k = \frac{U_{s_k}}{U_k} \tag{6}$$

for $k = 1, 2, \dots, n$ and where U_{sk} is the substitution unit of the kth asset. Substituting it into (5), we derive

$$\begin{aligned}\Pi_{\mathrm{BTC}} &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_k \cdot \frac{U_{s_k}}{U_k}\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].\end{aligned}$$

A Particular Model for Velocity

Consider the following model for V_{BTC}:

$$V_{\mathrm{BTC}} = \frac{1}{m}\sum_{j=1}^{m} T'_j, \tag{7}$$

where T_j represents the transactions. Substituting it into the final expression of Π_{BTC} leads us to

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln Q_{\mathrm{BTC}}\right].$$

A Particular Model for Output

Now, we assume that

$$Q_{\mathrm{BTC}} = \frac{b \cdot h}{d}, \tag{8}$$

where b, h and d are time-dependent production parameters. Therefore, for Π_{BTC}, we will have

$$\begin{aligned}\Pi_{\mathrm{BTC}} &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln\left(\frac{b \cdot h}{d}\right)\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - (\ln b + \ln h - \ln d)\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right].\end{aligned}$$

The Absorption Consideration

Assume that

$$U_{s_k} = \alpha_k \cdot R_k, \tag{9}$$

where α_k is the absorption rate of the market k, R_k, $k = 1, 2, \cdot n$, are time-dependent.

Then, Π_{BTC} will obtain the following form:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot \alpha_k \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right].$$

Hence the Zero Theorem governing equation is:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} \alpha_k \cdot P_k \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right].$$

Investigating Alpha

In this section, we are going to investigate the time-dependent behavior of the absorption rate (α) in two principal cases: when the absorption rate is assumed to be the same for all markets and when it is unique for each market.

The Case of Market Specific Alpha (α_k)

In this case, we derive a single equation for all α_k:

$$\sum_{k=1}^{n} P_k \cdot \alpha_k \cdot R_k = \frac{P_{\mathrm{BTC}} \cdot b \cdot h}{d \cdot \sum_{j=1}^{m} T'_j}. \tag{10}$$

Note that (10) is a single functional equation with respect to n unknowns α_1, α_2, $\ldots$, α_n. Apparently, unlike the previous case, it can not be solved exactly for continuous t. Therefore, we need to consider (10) at discrete values of t where the data measurements are made.

In order to describe the algorithm of determination of α_1, α_2, $\ldots$, α_n, let us denote

$$F(\alpha_1, \alpha_2, \ldots, \alpha_n; t) = \sum_{k=1}^{n} P_k \cdot \alpha_k \cdot R_k$$

and

$$S(t) = \frac{P_{\mathrm{BTC}} \cdot b \cdot h}{d \cdot \sum_{j=1}^{m} T'_j}.$$

Then, (10) can be written as

$$F(\alpha_1, \alpha_2, \ldots, \alpha_n; t) = S(t). \tag{11}$$

It is important to recognize that in (11), α_1, α_2, $\ldots$, α_n, P_{BTC}, b, h, d and T'_j are all functions of time.

Now assume that the raw data measurements have been made at given instances t_i, $i = 1, 2, \ldots, N$. For example, when data are measured on daily basis, then t_1, t_2, $\ldots$, t_N represent days. Evaluating both sides of (11) at instances t_i, $i = 1, 2, \dots, N$, we will have

$$\begin{aligned} F\left(\alpha_1\left(t_1\right), \alpha_2\left(t_1\right), \dots, \alpha_n\left(t_1\right); t_1\right) &= S\left(t_1\right), \\ F\left(\alpha_1\left(t_2\right), \alpha_2\left(t_2\right), \dots, \alpha_n\left(t_2\right); t_2\right) &= S\left(t_2\right), \\ &\vdots \\ F\left(\alpha_1\left(t_N\right), \alpha_2\left(t_N\right), \dots, \alpha_n\left(t_N\right); t_N\right) &= S\left(t_N\right). \end{aligned} \tag{12}$$

Note that (12) is *linear* with respect to each $\alpha_k\left(t_i\right)$, $k = 1, 2, \dots, n$, $i = 1, 2, \dots, N$. Nonetheless, (12) contains N equations with respect to $n \cdot N$ unknowns. At this, $N \gg n$ and the system is under-determined. Therefore, direct methods can not be applied for solving (12). On the other hand, in order to determined unknowns $\alpha_k\left(t_i\right)$, $k = 1, 2, \dots, n$, $i = 1, 2, \dots, N$, we can apply either efficient numerical methods of linear programming, choosing, e.g., P_{BTC} as a cost function, or we can solve the equivalent problem of numerical minimization

$$\left|\left|F\left(\alpha_1\left(t_i\right), \alpha_2\left(t_i\right), \dots, \alpha_n\left(t_i\right); t_i\right) - S\left(t_i\right)\right|\right| \to \min_{0 \le \alpha_k\left(t_i\right) \le 1}, \qquad k = 1, 2, \dots, n \quad i = 1, 2, \dots, N.$$

Here, $||\cdot||$ denotes an appropriate norm. In this case, it can be the l^2-norm. Then, the problem formulation will be: determine the solution to the following numerical minimization problem:

$$\left|F\left(\alpha_1\left(t_i\right), \alpha_2\left(t_i\right), \dots, \alpha_n\left(t_i\right); t_i\right) - S\left(t_i\right)\right|^2 \to \min_{0 \le \alpha_k\left(t_i\right) \le 1}, \qquad k = 1, 2, \dots, n \quad i = 1, 2, \dots, N.$$

The Case of Single Alpha (α)

In the case of single absorption, consideration of particular models leads to the following expression for Π_{BTC}:

$$\begin{aligned} \Pi_{\mathrm{BTC}} &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} \alpha \cdot P_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T_j'\right) - \ln b - \ln h + \ln d\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\alpha \cdot \sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T_j'\right) - \ln b - \ln h + \ln d\right] = \\ &= \frac{\partial}{\partial t}\left[\ln \alpha + \ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T_j'\right) - \ln b - \ln h + \ln d\right]. \end{aligned}$$

Similarly, in this case, we have

$$\ln P_{\mathrm{BTC}} = \ln \alpha + \ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d.$$

Numerical Analysis

In this section, we are going to consider the determination of single and individual absorption rates. To this end, we are going to consider a specific database of 11 markets, i.e., $n = 11$. Based on the database structure, the time variable has the following form:

$$t_i = 7 + \frac{i}{365},$$

with $N = 1592$ data points. In this particular case, $m = 1$, so that there is only T_1'.

In this case, we have a system of 1592 equations with respect to $11 \cdot 1592 = 17512$ unknowns, which are computed by minimizing the l^1-norm of the residue:

$$\mathrm{Err}\left(t_i\right) = \left|F\left(\alpha_1\left(t_i\right), \alpha_2\left(t_i\right), \ldots, \alpha_n\left(t_i\right); t_i\right) - S\left(t_i\right)\right| \to \min_{0 \le \alpha_k\left(t_i\right) \le 1},$$

with $k = 1, 2, \ldots, 11$, and $i = 1, 2, \ldots, 1592$.

The numerical values of $\alpha_k\left(t_i\right)$ are plotted in Figures 1 and 2 below. The corresponding error function Err is plotted in Figure 11, as well.

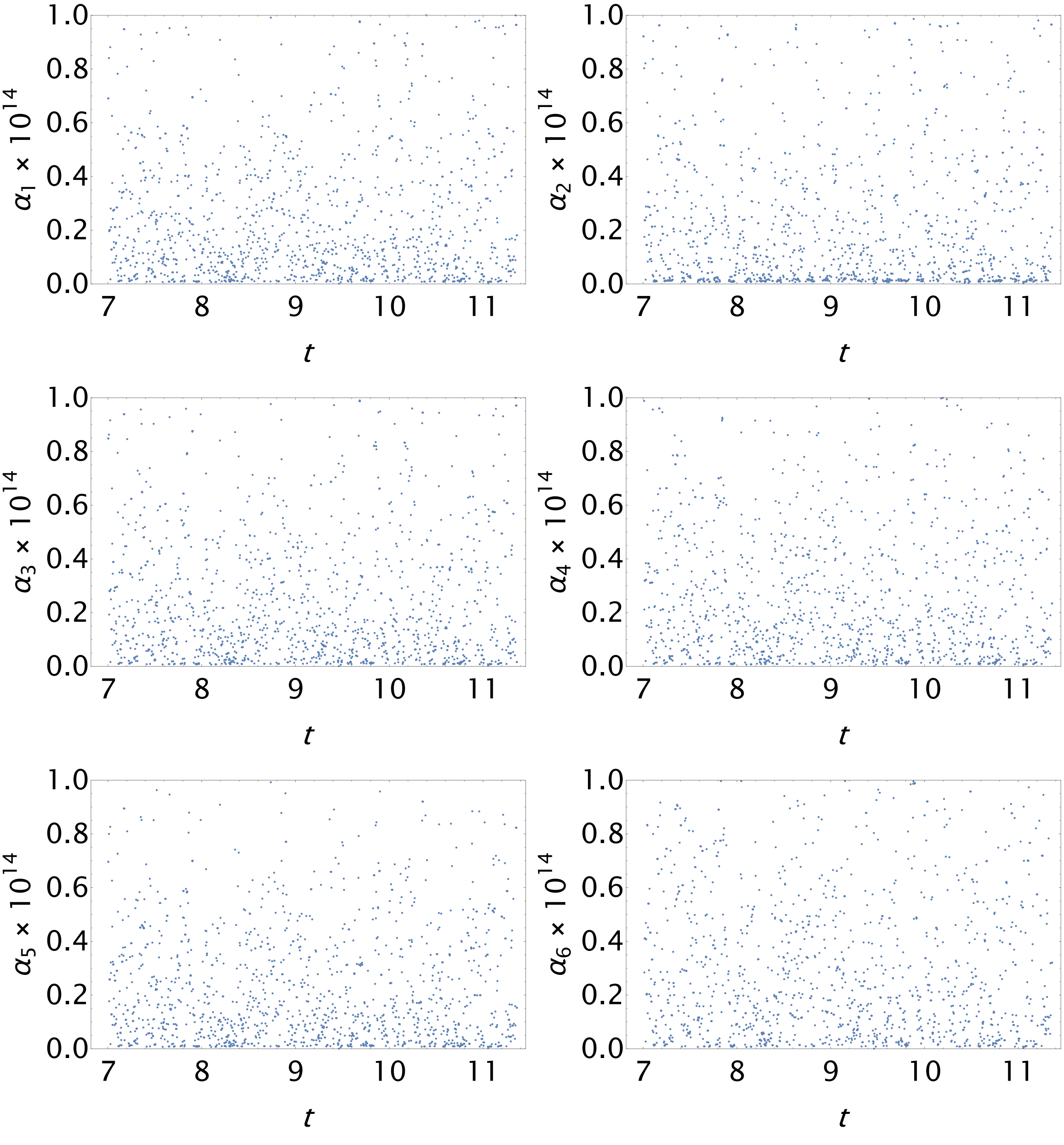

Figure 1: Dependence of α_1, ..., α_6 on t according to (12)

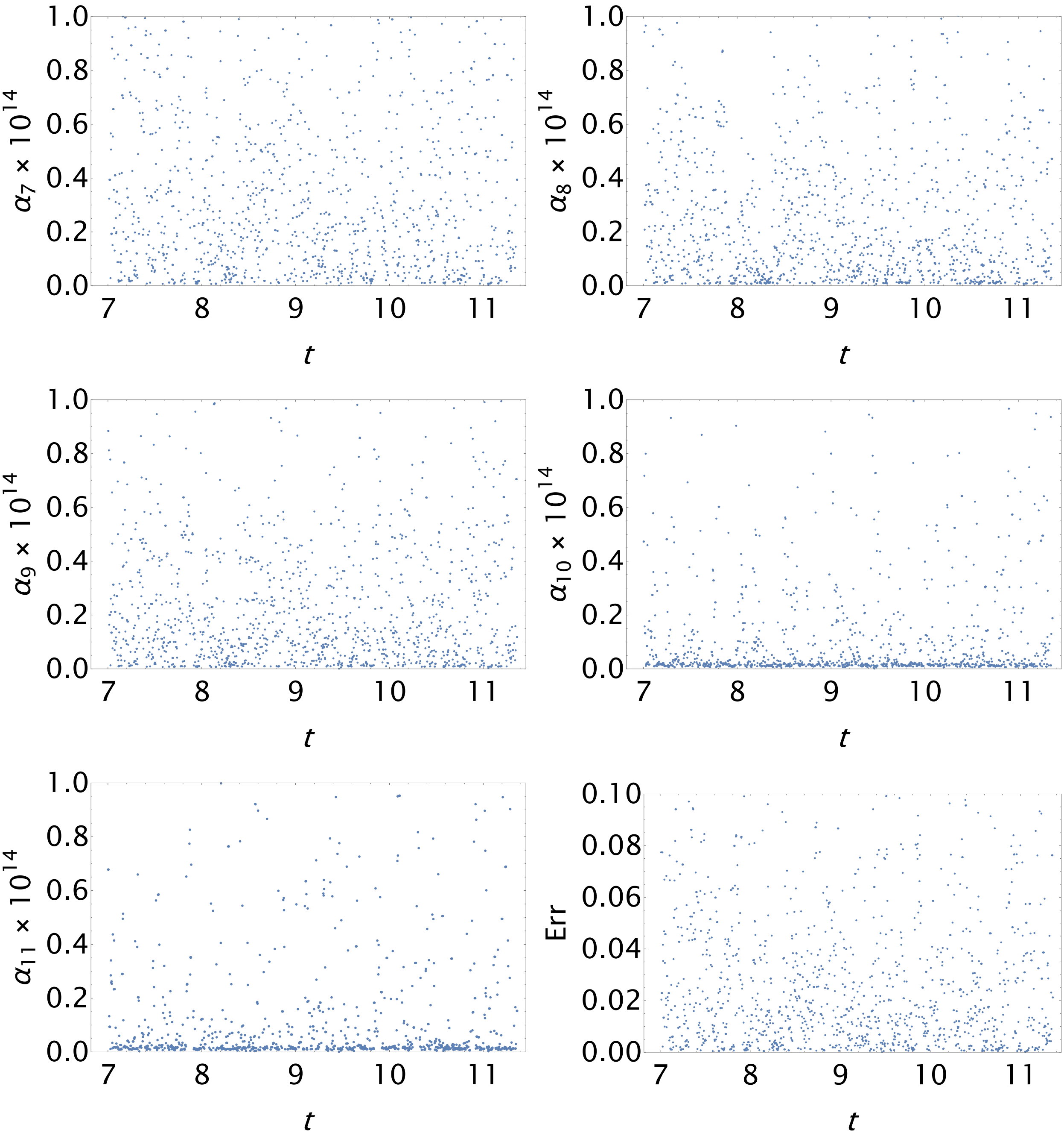

Figure 2: Dependence of α_7, ..., α_{11} and Err on t according to (12)

Thus, the numerical analysis in the above particular case shows similar behavior and range of values for α_1, α_2, ..., α_{11} making the dependence of the absorption rate on a specific market very weak. This apparently suggests that, in general, for n markets, we can make an assumption that

$$\alpha_1 \approx \alpha_2 \approx \ldots \approx \alpha_n := \alpha,$$

which will simplify the rigorous analysis significantly.

Our aim now is to show that the case of individual absorption rates can be, to some extent, approximated by a single absorption rate for all markets. In other words, we are going to assume that the absorption rate introduced in (9) does not depend on the subscript k. Then,

$$\sum_{k=1}^{n} \alpha_k \cdot P_k \cdot R_k = \alpha \cdot \sum_{k=1}^{n} P_k \cdot R_k,$$

where P_k and R_k are given, α_k are determined in the previous section and α is unknown determined as follows:

$$\alpha = \frac{\sum_{k=1}^{n} \alpha_k \cdot P_k \cdot R_k}{\sum_{k=1}^{n} P_k \cdot R_k}. \tag{13}$$

Figure 3 below shows the dependence of α and $\ln \alpha$ on t.

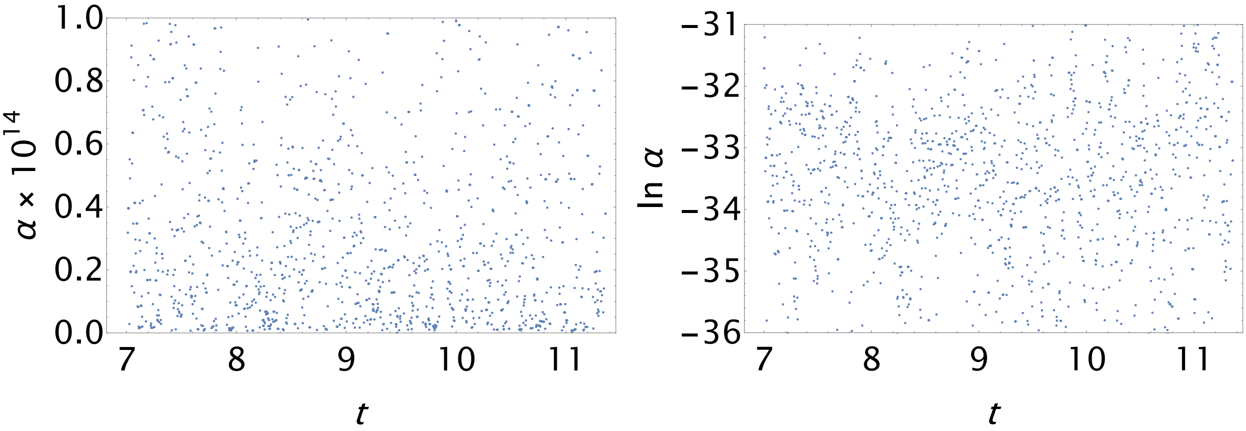

Figure 3: Dependence of α and $\ln \alpha$ on t according to (13)

In order to estimate the error of approximation by single absorption rate, we measure the error

$$\mathrm{MSE} = \frac{1}{2}\left(\alpha \sum_{k=1}^{n} P_k \cdot R_k - S\right)^2. \tag{14}$$

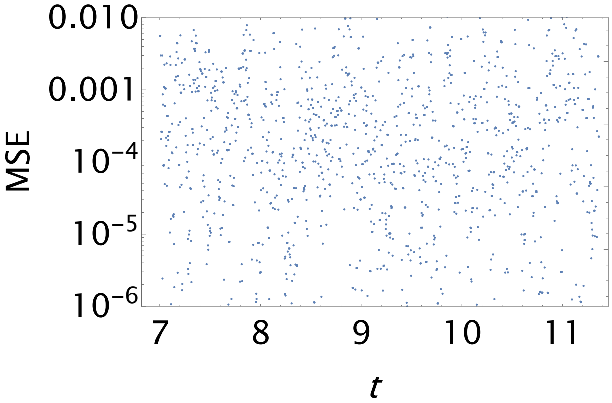

Figure 4: Dependence of MSE (14) with respect to t

Sensitivity Analysis - General Case

In this section, we carry out sensitivity analysis for Π_{BTC} with respect to all parameters included in it. We start with the general case and the incorporate some specific models and assumptions.

The Case of the Generalised Model

First, consider the general case when

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].$$

Now, before proceeding to the sensitivity analysis, we assume that there is no correlation between f' and f for none of the functions entering the expression of Π_{BTC}.

Detailed calculation of derivatives used for the sensitivity analysis carried out in this section are presented at the end of this Appendix.

Sensitivity with Respect to Velocity

In this section, we study the sensitivity of Π_{BTC} with respect to V_{BTC} and V'_{BTC}. First note that since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V_{\mathrm{BTC}}} = -\frac{V'_{\mathrm{BTC}}}{V^2_{\mathrm{BTC}}}.$$

Taking into account that the denominator is always positive, we come to the evident conclusion that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V_{\mathrm{BTC}}} > 0 \quad \text{when} \quad V'_{\mathrm{BTC}} < 0$$

and

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V_{\mathrm{BTC}}} < 0 \quad \text{when} \quad V'_{\mathrm{BTC}} > 0.$$

In other words, when V_{BTC} is a decreasing (increasing) function of t, then Π_{BTC} is an increasing (decreasing) function of V_{BTC}.

On the other hand, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V'_{\mathrm{BTC}}} = \frac{1}{V_{\mathrm{BTC}}}$$

which is apparently always positive, we conclude that Π_{BTC} is an increasing (linear) function of V'_{BTC} for all values of t.

Sensitivity with Respect to Output

According to the derivative expression derived at the end of this section,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q_{\mathrm{BTC}}} = \frac{Q'_{\mathrm{BTC}}}{Q^2_{\mathrm{BTC}}}.$$

Taking into account that Q_{BTC} is an increasing function of t, implying that $Q'_{\mathrm{BTC}} > 0$, we conclude that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q_{\mathrm{BTC}}} > 0$$

for all t. In other words, Π_{BTC} is always an increasing function of Q_{BTC}.

Similarly, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q'_{\mathrm{BTC}}} = -\frac{1}{Q_{\mathrm{BTC}}},$$

we see that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q'_{\mathrm{BTC}}} < 0$$

for all t. Therefore, Π_{BTC} is a decreasing function of Q'_{BTC}.

Sensitivity with Respect to Asset Prices

Taking into account that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P'_k} = \frac{U_{s_k}}{S},$$

in which

$$S = \sum_{k=1}^{n} P_k \cdot U_{s_k},$$

it becomes evident that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P'_k} > 0$$

for all t. Thence, Π_{BTC} is an increasing function of P'_k.

On the other hand, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} = \frac{U_{s_k}}{S} \cdot \frac{\partial}{\partial t}\left[\ln\left(\frac{U_{s_k}}{S}\right)\right],$$

we observe that when $\frac{U_{s_k}}{S}$ is an increasing function of t, then

$$\frac{\partial}{\partial t}\left[\ln\left(\frac{U_{s_k}}{S}\right)\right] > 0$$

implying that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k} > 0.$$

Similarly, when $\frac{U_{s_k}}{S}$ is a decreasing function of t, then

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k} < 0.$$

Thence, when $\frac{U_{s_k}}{S}$ is an increasing (decreasing) function of t, Π_{BTC} is an increasing (decreasing) function of P_k.

Sensitivity with Respect to Absorption (U_{s_k})

Since

$$\frac{\partial \Pi_{\text{BTC}}}{\partial U'_{s_k}} = \frac{P_k}{S} > 0$$

for all t. Therefore, Π_{BTC} is an increasing function of U'_{s_k}.

On the other hand,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial U_{s_k}} = \frac{P_k}{S} \cdot \frac{\partial}{\partial t}\left[\ln\left(\frac{P_k}{S}\right)\right]$$

providing that when $\frac{P_k}{S}$ is an increasing (decreasing) function of t, Π_{BTC} is an increasing (decreasing) function of U_{s_k}.

Sensitivity Analysis - Market Specific Alpha Case

In this section, we are going to study the sensitivity of Π_{BTC} with respect to parameters introduced when considering the particular models above. To this end, we are going to make use of the following expression for Π_{BTC}:

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} \alpha_k \cdot P_k \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right].$$

Sensitivity Analysis with Respect to Volume

Since

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{W_k}{R} \cdot \frac{\partial}{\partial t}\left[\ln \frac{W_k}{R}\right],$$

where

$$W_k = P_k \cdot \alpha_k, \quad R = \sum_{k=1}^{n} P_k \cdot R_k,$$

then, we straightforwardly conclude that when $\frac{W_k}{R}$ is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of R_k.

On the other hand,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R'_k} = \frac{W_k}{R},$$

leading to the conclusion that Π_{BTC} is an increasing function of R'_k.

Sensitivity with Respect to Velocity

Taking into account that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} = -\frac{1}{\left(T'_j\right)^2} \cdot \sum_{l=1}^{m} T''_l,$$

and taking into account that the denominator is positive for all t, then it is obvious that

$$\sum_{m=1}^{n} T''_m < 0$$

implies that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial T_j'} > 0,$$

i.e., Π_{BTC} is an increasing function of T_j'. On the other hand,

$$\sum_{m=1}^{n} T_m'' > 0$$

implies that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial T_j'} < 0,$$

i.e., Π_{BTC} is a decreasing function of T_j'.

Sensitivity with Respect to Output Parameters

Taking into account that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial b} = \frac{b'}{b^2},$$

the denominator of which is positive for all t, we conclude that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial b} > 0$$

when $b' > 0$. Similarly,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial b} < 0$$

when $b' < 0$.

Hence, when b is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of b.

Using the expression

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial h} = \frac{h'}{h^2},$$

we conclude that, when h is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of h.

However, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial d} = -\frac{d'}{d^2},$$

we come to the conclusion that when d is an increasing (decreasing) function of t, then Π_{BTC} is a decreasing (increasing) function of d.

Sensitivity Analysis - Single Alpha Case

In this section, we consider the sensitivity analysis in the single α , i.e., we assume that the absorption rate is the same for all markets. Here we will use actual raw market data to determine relational effects.

In the case of single absorption, consideration of particular models leads to the following expression for Π_{BTC}:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\alpha + \ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln b - \ln h + \ln d\right].$$

Therefore, we have

$$\ln P_{\mathrm{BTC}} = \ln\alpha + \ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d.$$

Sensitivity of $\ln P_{\mathrm{BTC}}$

Using the simple formulas obtained in Appendix 1, we come to the following conclusions.

1. $\ln P_{\mathrm{BTC}}$ is an increasing function of α for all values of t. See Figure 5 (a).

2. $\ln P_{\mathrm{BTC}}$ is an increasing function of P_k and R_k for all values of k and t.

3. $\ln P_{\mathrm{BTC}}$ is always an increasing function of P. See Figure 5 (b).

4. $\ln P_{\mathrm{BTC}}$ is an increasing function of T'_j for all values of j and t. See Figure 5 (c).

5. $\ln P_{\mathrm{BTC}}$ is a decreasing function of b and h, and it is an increasing function of d for all values of t. See Figure 5 (d-f).

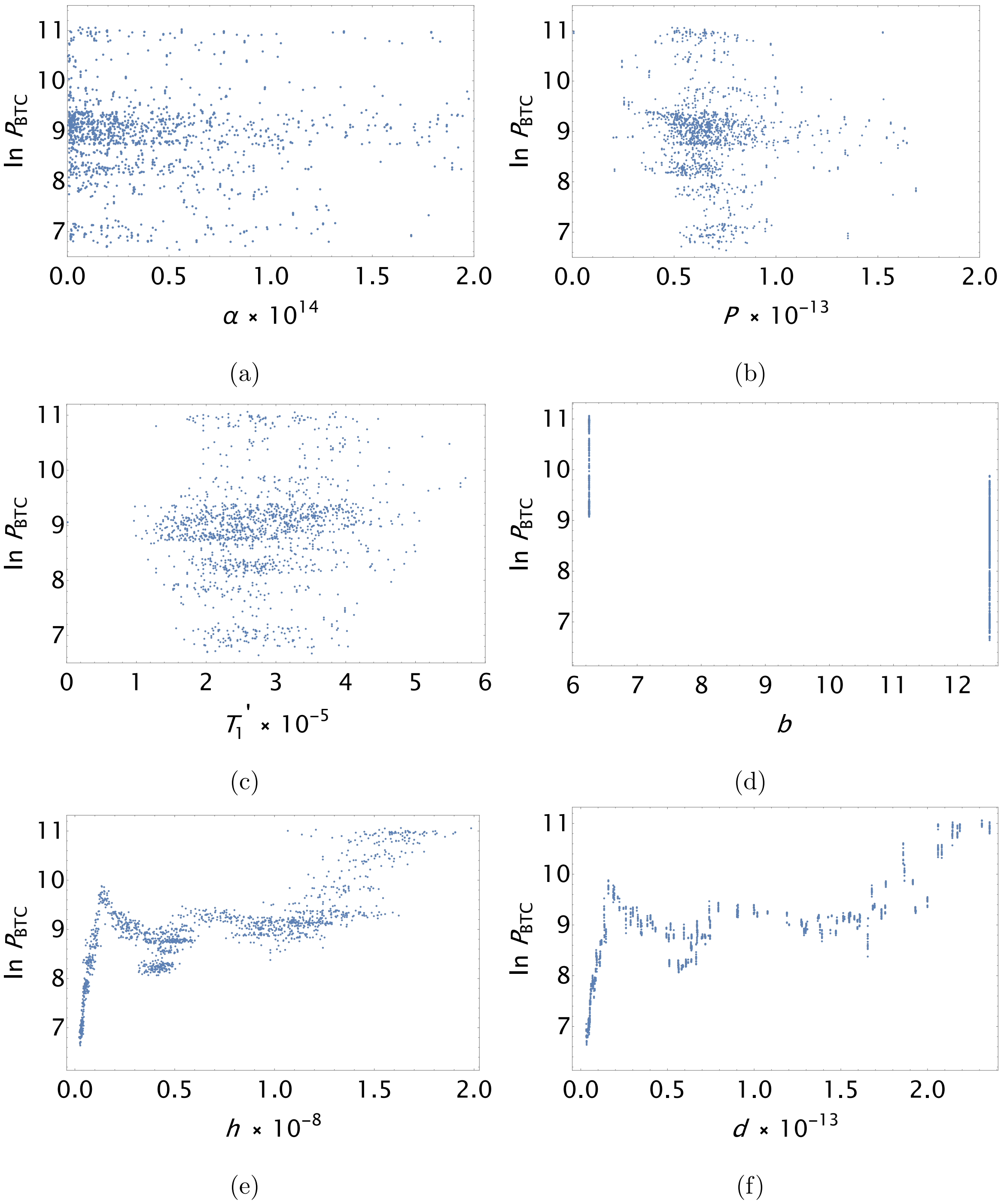

Figure 5: Sensitivity plots of $\ln P_{\mathrm{BTC}}$ with respect to system parameters

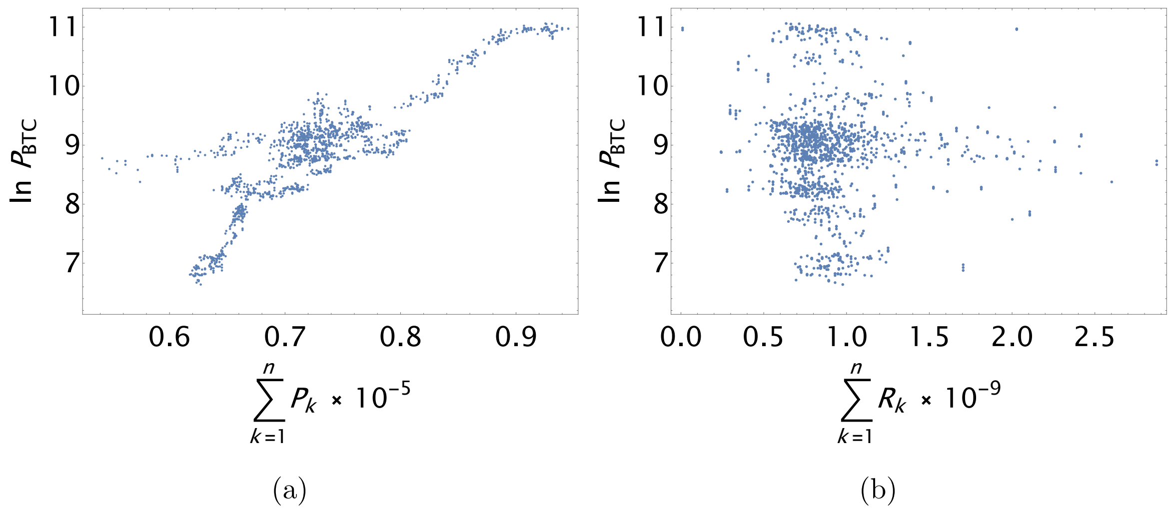

Figure 6: Sensitivity plots of $\ln P_{\mathrm{BTC}}$ with respect to sum of P_k (a) and R_k (b)

Sensitivity of Π_{BTC}

Sensitivity with Respect to Absorption Rate

In this case,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial \alpha} = -\frac{\alpha'}{\alpha^2},$$

leading us to the conclusion that when α is an increasing function of t, i.e., when $\alpha' > 0$, Π_{BTC} is a decreasing function of α. And, when α is a decreasing function of t, i.e., when $\alpha' < 0$, Π_{BTC} is an increasing function of α.

On the other hand,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial \alpha'} = \frac{1}{\alpha}.$$

Since $\alpha > 0$ for all t, we conclude that Π_{BTC} is an increasing function of α' for all values of t.

Sensitivity with Respect to Price and Asset Volume

Due to symmetry of Π_{BTC} with respect to P_k and R_k, we derive similar formulas for

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} = \frac{R_k}{P}\left[1 - P'\right]$$

and

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_k} = \frac{P_k}{P}\,[1 - P'],$$

where

$$P = \sum_{m=1}^{n} P_m \cdot R_m.$$

Apparently, the sign of both $\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k}$ and $\frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_k}$ depends on the sign of the expression $1 - P'$. More specifically, when $1 - P' > 0$, we get

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} > 0 \quad \text{and} \quad \frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_k} > 0,$$

meaning that Π_{BTC} is an increasing function of P_k and R_k simultaneously. On the other hand, $1 - P' < 0$ leads to

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} < 0 \quad \text{and} \quad \frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_k} < 0,$$

meaning that Π_{BTC} is a decreasing function of P_k and R_k simultaneously.

Note that Π_{BTC} is symmetric also with respect to P'_k and R'_k with

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P'_k} = \frac{R_k}{P}$$

and

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial R'_k} = \frac{P_k}{P}.$$

Apparently, both expressions are positive for all values of t leading to the conclusion that Π_{BTC} is an increasing function of both P'_k and R'_k.

Sensitivity with Respect to Sum

On the basis of the expression

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P} = -\frac{P'}{P^2},$$

we conclude that when P is an increasing (decreasing) function of time, i.e., $P' > 0$ ($P' < 0$, respectively), then Π_{BTC} is a decreasing (increasing) function of P.

Sensitivity with Respect to Transactions

The sensitivity of Π_{BTC} with respect to transactions is more complicated to explore, since the corresponding derivative is given by

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} = -\frac{1}{\left(T'_j\right)^2} \cdot \sum_{l=1}^{m} T''_l.$$

Therefore, taking into account that $\left(T'_j\right)^2$ is positive for all values of t, we come to the conclusion that when

$$\sum_{l=1}^{m} T''_l < 0,$$

then Π_{BTC} is an increasing function of T'_j, and when

$$\sum_{l=1}^{m} T''_l > 0,$$

evidently, Π_{BTC} is a decreasing function of T'_j.

Sensitivity with Respect to Output Parameters

In Appendix 1 we derive that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b} = \frac{b'}{b^2}, \quad \frac{\partial \Pi_{\text{BTC}}}{\partial h} = \frac{h'}{h^2}$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial d} = -\frac{d'}{d^2}.$$

And for the derivatives,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b'} = -\frac{1}{b}, \quad \frac{\partial \Pi_{\text{BTC}}}{\partial h'} = -\frac{1}{h}$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial d'} = \frac{1}{d}.$$

Note that $b > 0$, $h > 0$ and $d > 0$ for all values of t.

Therefore, we straightforwardly conclude:

1. When $b' > 0$ ($b' < 0$), i.e., b is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of b.

2. When $h' > 0$ ($h' < 0$), i.e., h is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of h.

3. When $d' > 0$ ($d' < 0$), i.e., d is a decreasing (increasing) function of t, then Π_{BTC} is a decreasing (increasing) function of d.

4. Π_{BTC} is a decreasing function of b' and h' and it is an increasing function of d' for all values of t.

Numerical Analysis for Particular Data

In this section, we are going to verify the conclusions obtained in the previous section on a particular database. To this aim, in Figure 7, we plot the dependence of corresponding derivative of Π_{BTC} with respect to sensitivity parameter.

Specifically, Figure 7 (a) shows that when α increases, α' becomes negative. Therefore, since

$$\frac{\partial \Pi_{\text{BTC}}}{\partial \alpha} = -\frac{\alpha'}{\alpha^2},$$

we conclude that for large α, Π_{BTC} is an increasing function of α.

An important conclusion is based on Figure 7 (b). Evidently, $1 - P'$ becomes positive when P increases. Therefore, taking into account that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k} = \frac{R_k}{P} \cdot (1 - P'), \quad \frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{P_k}{P} \cdot (1 - P'),$$

we obtain that Π_{BTC} increases with respect to P_k and R_k for all k when P increases.

Similar behavior is observed for P as shown on Figure 7 (c). Since P' becomes negative for large values of P, and taking into account that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P} = -\frac{P'}{P^2},$$

we conclude that for large values of P, Π_{BTC} is an increasing function of P.

Figure 7 (c) shows similar behavior for T_1' leading to the conclusion that for large values of T_1', Π_{BTC} is an increasing function of T_1'.

On the other hand, since

$$\frac{\partial \Pi_{\text{BTC}}}{\partial d} = -\frac{d'}{d^2},$$

and as Figure 7 (e) shows, d' is positive for large values of d, we come to the conclusion that for large values of d, Π_{BTC} is an increasing function of d.

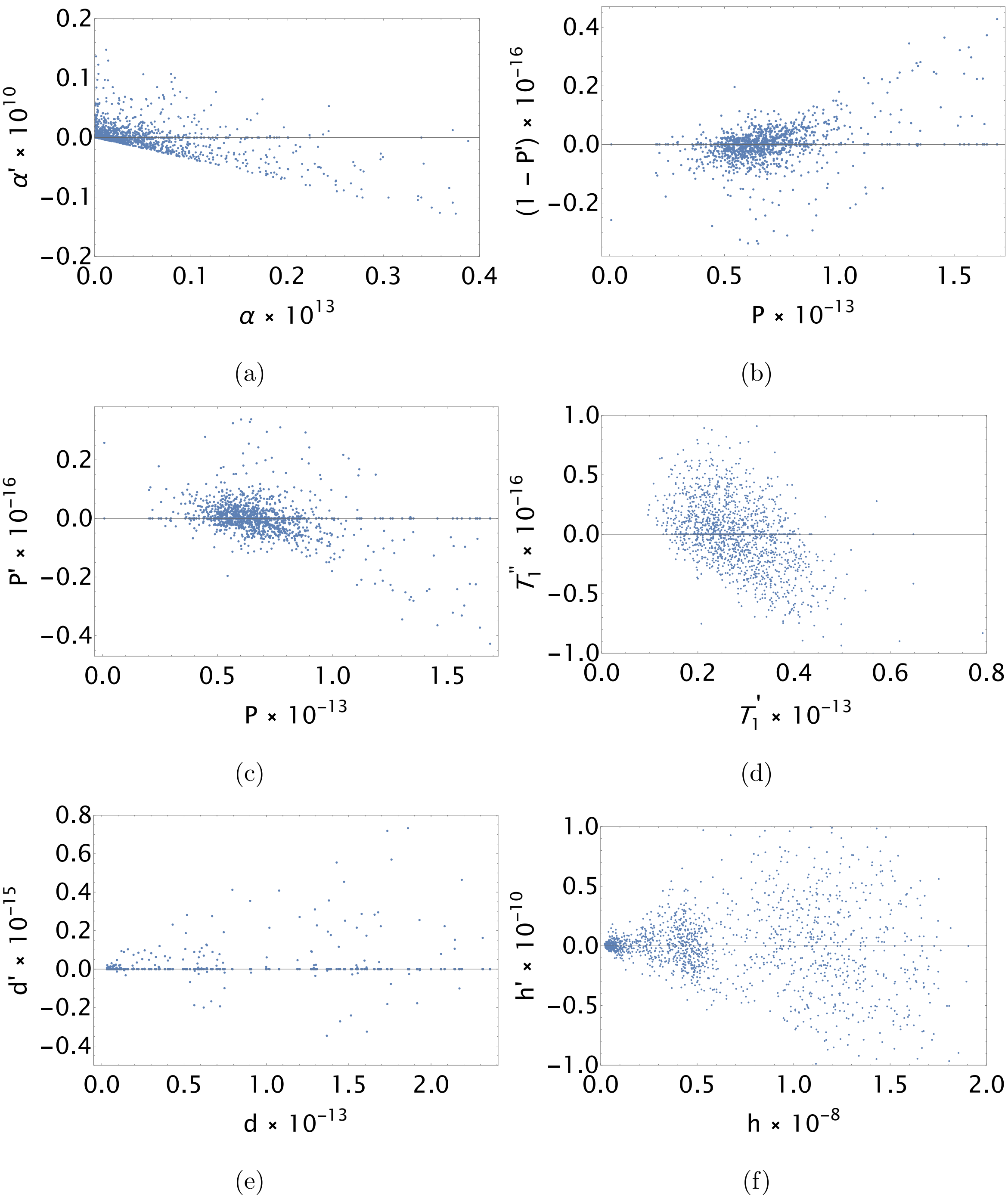

Figure 7: Sensitivity plots of α' (a), $1 - P'$ (b), P' (c), T_1' (d), d' (e) and h' (f)

Derivations

We collect the derivative calculations that are used in sensitivity analysis with respect to system parameters.

Derivative of $\ln P_{\text{BTC}}$

We now proceed with evaluation of derivatives of the expression

$$\ln P_{\text{BTC}} = \ln \alpha + \ln \left(\sum_{k=1}^{n} P_k \cdot R_k \right) + \ln \left(\frac{1}{m} \sum_{j=1}^{m} T'_j \right) - \ln b - \ln h + \ln d.$$

Apparently, $\ln P_{\text{BTC}}$ does not depend on parameter derivatives explicitly.

With Respect to α

We easily compute

$$\frac{\partial \ln P_{\text{BTC}}}{\partial \alpha} = \frac{\partial \ln \alpha}{\partial \alpha} = \frac{1}{\alpha}.$$

With Respect to Price and Volume

It is easy to see that $\ln P_{\text{BTC}}$ is symmetric with respect to P_k and R_k, leading to

$$\frac{\partial \ln P_{\text{BTC}}}{\partial P_k} = \frac{1}{P} \frac{\partial}{\partial P_k} \left(P_k \cdot R_k \right) = \frac{R_k}{P}$$

and

$$\frac{\partial \ln P_{\text{BTC}}}{\partial R_k} = \frac{1}{P} \frac{\partial}{\partial R_k} \left(P_k \cdot R_k \right) = \frac{P_k}{P}.$$

With Respect to Sum

In this case,

$$\frac{\partial \ln P_{\text{BTC}}}{\partial P} = 1.$$

With Respect to Transactions

Due to linear dependence of $\ln P_{\text{BTC}}$ on T'_j, we obtain

$$\frac{\partial \ln P_{\text{BTC}}}{\partial T'_j} = \frac{1}{m} \frac{\partial}{\partial T'_j} \left(\sum_{j=1}^{m} T'_j \right) = \frac{1}{m}.$$

With Respect to Output Parameters

In the case of output parameters, we easily derive

$$\begin{aligned}
\frac{\partial \ln P_{\mathrm{BTC}}}{\partial b} &= -\frac{\partial \ln b}{\partial b} = -\frac{1}{b}, \\
\frac{\partial \ln P_{\mathrm{BTC}}}{\partial h} &= -\frac{\partial \ln h}{\partial h} = -\frac{1}{h}, \\
\frac{\partial \ln P_{\mathrm{BTC}}}{\partial d} &= \frac{\partial \ln d}{\partial d} = \frac{1}{d}.
\end{aligned}$$

Derivative of Π_{BTC}

We consider the expression

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln \alpha + \ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d\right].$$

With Respect to α

Apparently, only the first term in the expression for Π_{BTC} depends on α explicitly. Therefore, we get

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\frac{\alpha'}{\alpha}\right] = -\frac{\alpha'}{\alpha^2}.$$

On the other hand,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial \alpha'} = \frac{\partial}{\partial \alpha'}\left[\frac{\alpha'}{\alpha}\right] = \frac{1}{\alpha}.$$

With Respect to Price

In this case,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} = \frac{\partial}{\partial P_k}\left[\frac{\sum_{k=1}^{n}(P_k \cdot R_k)'}{\sum_{k=1}^{n} P_k R_k}\right] = \frac{\partial}{\partial P_k}\left[\frac{\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k')}{\sum_{k=1}^{n} P_k R_k}\right] =$$

$$= -\frac{1}{P^2}\cdot\frac{\partial P}{\partial P_k}\cdot\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k') + \frac{1}{P}\frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k')\right] =$$

$$= -\frac{1}{P^2}\cdot R_k \cdot \sum_{m=1}^{n}(P_m' \cdot R_m + P_m \cdot R_m') + \frac{1}{P}\cdot R_k =$$

$$= \frac{R_k}{P}\left[-\sum_{m=1}^{n}(P_m' \cdot R_m + P_m \cdot R_m') + 1\right] = \frac{R_k}{P}\left[1 - P'\right].$$

Here,

$$P = \sum_{k=1}^{n} P_k \cdot R_k.$$

On the other hand, Π_{BTC} is a linear function with respect to P_k' with the linear coefficient

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k'} = \frac{\partial}{\partial P_k'}\left[\frac{1}{P}\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k')\right] = \frac{R_k}{P}.$$

With Respect to Volume

And in a similar way we compute

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_k} = \frac{\partial}{\partial R_k}\left[\frac{\sum_{k=1}^{n}(P_k \cdot R_k)'}{\sum_{k=1}^{n} P_k R_k}\right] = \frac{\partial}{\partial R_k}\left[\frac{\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k')}{\sum_{k=1}^{n} P_k R_k}\right] =$$

$$= -\frac{1}{P^2}\cdot\frac{\partial P}{\partial R_k}\cdot\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k') + \frac{1}{P}\frac{\partial}{\partial R_k}\left[\sum_{k=1}^{n}(P_k' \cdot R_k + P_k \cdot R_k')\right] =$$

$$= -\frac{1}{P^2}\cdot P_k \cdot \sum_{m=1}^{n}(P_m' \cdot R_m + P_m \cdot R_m') + \frac{1}{P}\cdot P_k =$$

$$= \frac{P_k}{P}\left[-\sum_{m=1}^{n}(P_m' \cdot R_m + P_m \cdot R_m') + 1\right] = \frac{P_k}{P}\left[1 - P'\right].$$

Similar to the case of P'_k, Π_{BTC} is a linear function with respect to R'_k as well. At this, the linear coefficient is defined as follows:

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R'_k} = \frac{\partial}{\partial R'_k}\left[\frac{1}{P}\sum_{k=1}^{n}(P'_k \cdot R_k + P_k \cdot R'_k)\right] = \frac{P_k}{P}.$$

With Respect to Sum

The case of derivative with respect to P is even simpler. Indeed,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P} = \frac{\partial}{\partial P}\left[\frac{P'}{P}\right] = -\frac{P'}{P^2}.$$

With Respect to Transactions

In this case,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} = \frac{\partial}{\partial T'_j}\left[\frac{\partial}{\partial t}\left[\ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right)\right]\right] = \frac{\partial}{\partial T'_j}\left[\frac{\sum_{j=1}^{m} T''_j}{\sum_{j=1}^{m} T'_j}\right] = -\frac{1}{\left(T'_j\right)^2}\cdot\sum_{l=1}^{m} T''_l.$$

With Respect to Output Parameters

We easily evaluate the following partial derivatives:

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b} = -\frac{\partial}{\partial b}\left[\frac{\partial \ln b}{\partial t}\right] = -\frac{\partial}{\partial b}\left(\frac{b'}{b}\right) = \frac{b'}{b^2},$$

$$\frac{\partial \Pi_{\text{BTC}}}{\partial h} = -\frac{\partial}{\partial h}\left[\frac{\partial \ln h}{\partial t}\right] = \frac{h'}{h^2},$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial d} = \frac{\partial}{\partial d}\left[\frac{\partial \ln d}{\partial t}\right] = -\frac{d'}{d^2}.$$

And for the derivatives we have

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b'} = -\frac{\partial}{\partial b'}\left(\frac{b'}{b}\right) = -\frac{1}{b},$$

$$\frac{\partial \Pi_{\text{BTC}}}{\partial h'} = -\frac{\partial}{\partial h'}\left[\frac{h'}{h}\right] = -\frac{1}{h},$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial d'} = \frac{\partial}{\partial d'}\left[\frac{d'}{d}\right] = \frac{1}{d}.$$

Further Derivations

With Respect to Velocity

Here we present derivations of some partial derivatives required for sensitivity analysis carried out in previous sections.

In order to study the sensitivity of Π_{BTC} with respect to V_{BTC}, we compute

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V_{\mathrm{BTC}}} &= \frac{\partial}{\partial V_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) - \ln Q_{\mathrm{BTC}}\right]\right] = \\ &= \frac{\partial}{\partial V_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t}\right] = \frac{\partial}{\partial V_{\mathrm{BTC}}}\left[\frac{V'_{\mathrm{BTC}}}{V_{\mathrm{BTC}}}\right] = -\frac{V'_{\mathrm{BTC}}}{V^2_{\mathrm{BTC}}}.\end{aligned}$$

On the other hand, the sensitivity of Π_{BTC} with respect to V'_{BTC} is carried out on the basis of the derivative

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V'_{\mathrm{BTC}}} &= \frac{\partial}{\partial V'_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) - \ln Q_{\mathrm{BTC}}\right]\right] = \\ &= \frac{\partial}{\partial V'_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t}\right] = \frac{\partial}{\partial V'_{\mathrm{BTC}}}\left[\frac{V'_{\mathrm{BTC}}}{V_{\mathrm{BTC}}}\right] = \frac{1}{V_{\mathrm{BTC}}}.\end{aligned}$$

With Respect to Output

In this case, we have

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q_{\mathrm{BTC}}} &= \frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[-\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}}\right]\right] = \\ &= -\frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t}\right] = -\frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[\frac{Q'_{\mathrm{BTC}}}{Q_{\mathrm{BTC}}}\right] = \frac{Q'_{\mathrm{BTC}}}{Q^2_{\mathrm{BTC}}}.\end{aligned}$$

In the same way, we compute

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q'_{\mathrm{BTC}}} &= \frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[-\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}}\right]\right] = \\ &= -\frac{\partial}{\partial Q'_{\mathrm{BTC}}}\left[\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t}\right] = -\frac{\partial}{\partial Q'_{\mathrm{BTC}}}\left[\frac{Q'_{\mathrm{BTC}}}{Q_{\mathrm{BTC}}}\right] = -\frac{1}{Q_{\mathrm{BTC}}}.\end{aligned}$$

With Respect to Asset Price

We start with simplifying the corresponding term in the expression of Π_{BTC},

$$\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] = \frac{\sum\limits_{k=1}^{n} (P_k \cdot U_{s_k})'}{\sum\limits_{k=1}^{n} P_k \cdot U_{s_k}} = \frac{1}{S} \cdot \sum_{k=1}^{n} \left(P_k' \cdot U_{s_k} + P_k \cdot U_{s_k}'\right) =$$
$$= \frac{1}{S} \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}',$$

in which

$$S = \sum_{k=1}^{n} P_k \cdot U_{s_k}.$$

As a matter of fact, Π_{BTC} is linear in P_k'. Therefore, the derivative of Π_{BTC} with respect to that variable is easy to compute. Indeed,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k'} = \frac{\partial}{\partial P_k'}\left[\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] + \frac{\partial}{\partial t}\left[\ln V_{\text{BTC}} - \ln Q_{\text{BTC}}\right]\right] =$$
$$= \frac{\partial}{\partial P_k'}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}'\right] = \frac{U_{s_k}}{S}.$$

On the other hand,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k} = \frac{\partial}{\partial P_k}\left[\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] + \frac{\partial}{\partial t}\left[\ln V_{\text{BTC}} - \ln Q_{\text{BTC}}\right]\right] =$$
$$= \frac{\partial}{\partial P_k}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}'\right] =$$
$$= \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k' \cdot U_{s_k}\right] +$$
$$+ \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}' + \frac{1}{S} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k \cdot U_{s_k}'\right].$$

Let us compute the derivatives above one by one:

$$\frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] = -\frac{1}{S^2} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k \cdot U_{s_k}\right] = -\frac{U_{s_k}}{S^2},$$

$$\frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k' \cdot U_{s_k}\right] = 0,$$

$$\frac{\partial}{\partial M_k}\left[\sum_{k=1}^{n} P_k \cdot U'_{s_k}\right] = U'_{s_k}.$$

Substituting these expressions into $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k}$, we obtain

$$\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} &= \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P'_k \cdot U_{s_k}\right] + \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} = \\
&= -\frac{U_{s_k}}{S^2} \cdot \sum_{l=1}^{n} P'_l \cdot U_{s_l} + \frac{U'_{s_k}}{S} - \frac{U_{s_k}}{S^2} \cdot \sum_{l=1}^{n} P_l \cdot U'_{s_l} = \\
&= \frac{U_{s_k}}{S} \cdot \left[\frac{U'_{s_k}}{U_{s_k}} \cdot S - \frac{\partial}{\partial t}\left(\sum_{l=1}^{n} P_l \cdot U_{s_l}\right)\right] = \\
&= \frac{U_{s_k}}{S^2} \cdot S \cdot \left[\frac{U'_{s_k}}{U_{s_k}} - \frac{\partial \ln S}{\partial t}\right] = \\
&= \frac{U_{s_k}}{S} \cdot \frac{\partial}{\partial t}\left[\ln U_{s_k} - \ln S\right]
\end{aligned}$$

or

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} = \frac{U_{s_k}}{S} \cdot \frac{\partial}{\partial t}\left[\ln\left(\frac{U_{s_k}}{S}\right)\right].$$

With Respect to Absorption

In this case as well, we are going to use the expression

$$\Pi_{\mathrm{BTC}} = \frac{1}{S} \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{\partial}{\partial t}\left[\ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].$$

Then, obviously,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U'_{s_k}} = \frac{\partial}{\partial U'_{s_k}}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k}\right] = \frac{P_k}{S}.$$

On the other hand,

$$\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U_{s_k}} &= \frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{\partial}{\partial t}\left[\ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]\right] = \\
&= \frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n} P'_k \cdot U_{s_k}\right] + \\
&+ \frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n} P_k \cdot U'_{s_k}\right].
\end{aligned}$$

Apparently,

$$\frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S}\right] = -\frac{P_k}{S^2},$$

$$\frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n} P'_k \cdot U_{s_k}\right] = P'_k,$$

$$\frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n} P_k \cdot U'_{s_k}\right] = 0.$$

Therefore,

$$\begin{aligned}
\frac{\partial \Pi_{\text{BTC}}}{\partial U_{s_k}} &= -\frac{P_k}{S^2}\cdot\sum_{k=1}^{n} P'_k \cdot U_{s_k} - \frac{P_k}{S^2}\cdot\sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{P'_k}{S} = \\
&= \frac{P_k}{S}\cdot\left[-\frac{1}{S}\cdot\sum_{k=1}^{n} P'_k \cdot U_{s_k} - \frac{1}{S}\cdot\sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{P'_k}{P_k}\right] = \\
&= \frac{P_k}{S}\cdot\left[-\frac{1}{S}\cdot\left(\sum_{k=1}^{n} P'_k \cdot U_{s_k} + \sum_{k=1}^{n} P_k \cdot U'_{s_k}\right) + \frac{P'_k}{P_k}\right] = \\
&= \frac{P_k}{S}\cdot\left[\frac{P'_k}{P_k} - \frac{1}{S}\cdot\frac{\partial}{\partial t}\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] = \\
&= \frac{P_k}{S}\cdot\left[\frac{P'_k}{P_k} - \frac{P'_k}{S}\right] = \frac{P_k}{S}\cdot\frac{\partial}{\partial t}\left[\ln P_k - \ln S\right]
\end{aligned}$$

or

$$\frac{\partial \Pi_{\text{BTC}}}{\partial U_{s_k}} = \frac{P_k}{S}\cdot\frac{\partial}{\partial t}\left[\ln\left(\frac{P_k}{S}\right)\right].$$

With Respect to Volume

In this case,

$$\begin{aligned}
\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} &= \frac{\partial}{\partial R_k}\left[\frac{\partial}{\partial t}\ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right)\right] = \frac{\partial}{\partial R_k}\left[\frac{\left(\sum_{k=1}^{n} W_k \cdot R_k\right)'}{\sum_{k=1}^{n} W_k \cdot R_k}\right] = \\
&= \frac{\partial}{\partial R_k}\left[\frac{1}{R}\cdot\sum_{k=1}^{n}\left(W'_k \cdot R_k + W_k \cdot R'_k\right)\right],
\end{aligned}$$

where

$$W_k = P_k \cdot \alpha_k, \quad R = \sum_{k=1}^{n} P_k \cdot R_k.$$

Therefore,

$$\begin{aligned}\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} &= \frac{\partial}{\partial R_k}\left[\frac{1}{R}\cdot\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k')\right] = \\ &= \frac{\partial}{\partial R_k}\left[\frac{1}{R}\right]\cdot\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k') + \frac{1}{R}\cdot\frac{\partial}{\partial R_k}\left[\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k')\right].\end{aligned}$$

Taking into account the chain rule, we obtain

$$\frac{\partial}{\partial R_k}\left[\frac{1}{R}\right] = -\frac{1}{R^2}\cdot\frac{\partial R}{\partial R_k} = -\frac{W_k}{R^2}.$$

On the other hand,

$$\frac{\partial}{\partial R_k}\left[\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k')\right] = W_k',$$

providing us with this final form:

$$\begin{aligned}\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} &= -\frac{W_k}{R^2}\cdot\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k') + \frac{P_k'}{R} = \\ &= \frac{W_k}{R}\cdot\left[-\frac{1}{R}\cdot\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k') + \frac{W_k'}{W_k}\right] = \\ &= \frac{W_k}{R}\cdot\left[-\frac{R'}{R} + \frac{W_k'}{W_k}\right] = \\ &= \frac{W_k}{R}\cdot\frac{\partial}{\partial t}\left[\ln W_k - \ln R\right]\end{aligned}$$

or

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{W_k}{R}\cdot\frac{\partial}{\partial t}\left[\ln\frac{W_k}{R}\right].$$

It is much easier to establish that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k'} = \frac{\partial}{\partial R_k'}\left[\frac{1}{R}\cdot\sum_{k=1}^{n}(W_k'\cdot R_k + W_k\cdot R_k')\right] = \frac{W_k}{R},$$

Bass Model for Alpha

For the absorption rate α_k we consider the following model based on the well-known Bass Diffusion

$$\alpha_k = \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{1 + \dfrac{p_k}{q_k} \cdot \exp\left[-\left(p_k + q_k\right)t\right]},$$

where p_k and q_k are time-dependent coefficients to be estimated.

Then, Π_{BTC} will obtain the following form:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{1 + \dfrac{p_k}{q_k} \cdot \exp\left[-\left(p_k + q_k\right)t\right]} \cdot R_k\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d\right].$$

Bass Derivatives

It is easy to see that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\frac{X'}{X}\right] = \frac{1}{X} \cdot \frac{\partial X'}{\partial p_k} + X' \cdot \frac{\partial}{\partial p_k}\left[\frac{1}{X}\right] = \frac{1}{X} \cdot \frac{\partial X'}{\partial p_k} - \frac{X'}{X^2} \cdot \frac{\partial X}{\partial p_k},$$

and, similarly,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_k} = \frac{1}{X} \cdot \frac{\partial X'}{\partial q_k} - \frac{X'}{X^2} \cdot \frac{\partial X}{\partial q_k},$$

where

$$X = \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}.$$

Therefore,

$$\begin{aligned} X' = \frac{\partial X}{\partial t} &= \sum_{k=1}^{n} \left(P_k \cdot R_k\right)' \cdot q_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} + \\ &+ \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k' \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} + \\ &+ \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \left(\frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}\right)'. \end{aligned}$$

Since p_k and q_k depend on t, we compute

$$\left(\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right)' = \frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)'}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} - \\ -\frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)'}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.$$

Now, making use of the chain rule, it is easy to evaluate

$$\begin{aligned}\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)' &= -\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left[-\left(p_k+q_k\right)t\right]' = \\ &= \left[p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right],\end{aligned}$$

and

$$\begin{aligned}\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)' &= q'_k+p'_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]+ \\ &+ p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left[-\left(p_k+q_k\right)\cdot t\right]' = \\ &= q'_k+p'_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]+ \\ &- p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left[p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right] = \\ &= q'_k+\left[p'_k-p_k\cdot\left(p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right].\end{aligned}$$

Hence,

$$\left(\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right)' = \frac{\left(p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} - \\ -\frac{q'_k+\left[p'_k-p_k\cdot\left(p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.$$

Thus,

$$\begin{aligned}X' = \frac{\partial X}{\partial t} &= \sum_{k=1}^{n}\left(P_k\cdot R_k\right)'\cdot q_k\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}+ \\ &+\sum_{k=1}^{n}P_k\cdot R_k\cdot q'_k\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}+ \\ &+\sum_{k=1}^{n}P_k\cdot R_k\cdot q_k\cdot\frac{\left(p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}- \\ &-\sum_{k=1}^{n}P_k\cdot R_k\cdot q_k\cdot\frac{q'_k+\left[p'_k-p_k\cdot\left(p_k+q_k+\left(p'_k+q'_k\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.\end{aligned}$$

Now, in order to continue the sensitivity analysis of Π_{BTC} with respect to p_k, we need to evaluate the derivatives $\dfrac{\partial X}{\partial p_k}$ and $\dfrac{\partial X'}{\partial p_k}$. First,

$$\frac{\partial X}{\partial p_k} = \frac{\partial}{\partial p_k}\left[P_k \cdot R_k \cdot q_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] =$$
$$= P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right],$$

in which

$$\frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] = \frac{1}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\frac{\partial}{\partial p_k}\left[1-\exp\left[-\left(p_k+q_k\right)t\right]\right]+$$
$$+\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\frac{\partial}{\partial p_k}\left[\frac{1}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] =$$
$$= -\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\cdot\frac{\partial}{\partial p_k}\left[-\left(p_k+q_k\right)t\right]-$$
$$-\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\frac{\partial}{\partial p_k}\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right) =$$
$$= \frac{t\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}-$$
$$-\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\left(\exp\left[-\left(p_k+q_k\right)t\right]+p_k\cdot\frac{\partial\exp\left[-\left(p_k+q_k\right)t\right]}{\partial p_k}\right) =$$
$$= \frac{t\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}-$$
$$-\frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\left(1+p_k\cdot\frac{\partial}{\partial p_k}\left[-\left(p_k+q_k\right)t\right]\right) =$$
$$= \frac{t\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}-\frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-p_k\cdot t\right)}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} =$$
$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\left[t-\frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\right)}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] =$$
$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$
$$\times\left[t\cdot q_k+t\cdot p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]-\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\right)\right] =$$
$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$
$$\times\left[t\cdot q_k+t\cdot p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]-\left(1-p_k\cdot t-\exp\left[-\left(p_k+q_k\right)t\right]+p_k\cdot t\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)\right] =$$
$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right],$$

or

$$\frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] = \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$
$$\times\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right],$$

leading to

$$\frac{\partial X}{\partial p_k} = P_k \cdot R_k \cdot q_k \cdot \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2} \cdot \left[\left(p_k+q_k\right)\cdot t + \exp\left[-\left(p_k+q_k\right)t\right] - 1\right],$$

Now, we evaluate the derivative $\dfrac{\partial X'}{\partial p_k}$:

$$\begin{aligned}
\frac{\partial X'}{\partial p_k} &= \left(P_k \cdot R_k \cdot q_k\right)' \cdot \frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\
&+ P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\
&- P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{q_k' + \left[p_k' - p_k \cdot \left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right]\right) \cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right] = \\
&= \left(P_k \cdot R_k \cdot q_k\right)' \cdot \frac{\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left[\left(p_k+q_k\right)\cdot t + \exp\left[-\left(p_k+q_k\right)t\right] - 1\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2} + \\
&+ P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\
&- P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{q_k' + \left[p_k' - p_k \cdot \left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right]\right) \cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right].
\end{aligned}$$

Next, we evaluate

$$\begin{aligned}
&\frac{\partial}{\partial p_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] = \\
&= \frac{\exp\left[-\left(p_k+q_k\right)t\right] + \left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot(-t)}{q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} - \\
&- \frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2} \cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-p_k t\right) = \\
&= \frac{\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot t\right)}{q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} - \\
&- \frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-2\left(p_k+q_k\right)t\right]\cdot\left(1-p_k t\right)}{\left(q_k + p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.
\end{aligned}$$

and

$$\frac{\partial}{\partial p_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right]=$$

$$=\frac{\left[-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}+$$

$$+\frac{\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]\cdot(-t)}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}-$$

$$-\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$

$$\times\frac{\partial}{\partial p_k}\left[\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2\right]=$$

$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$

$$\times\cdot\left(\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot t-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right)-$$

$$-\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$

$$\times 2\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\frac{\partial}{\partial p_k}\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)=$$

$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$

$$\times\cdot\left(\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot t-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right)-$$

$$-\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$

$$\times 2\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)\left(1-p_k\cdot t\right).$$

Thus, eventually,

$$\frac{\partial X'}{\partial p_k}=\left(P_k\cdot R_k\cdot q_k\right)'\cdot\frac{\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}+$$

$$+P_k\cdot R_k\cdot q_k\cdot\frac{\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot t\right)}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}-$$

$$-P_k\cdot R_k\cdot q_k\cdot\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-2\left(p_k+q_k\right)t\right]\cdot\left(1-p_kt\right)}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}-$$

$$-P_k\cdot R_k\cdot q_k\cdot\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$

$$\times\cdot\left(\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot t-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right)+$$

$$+P_k\cdot R_k\cdot q_k\cdot\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$

$$\times 2\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)\left(1-p_k\cdot t\right).$$

Similar steps as above will lead us to the following expressions:

$$\frac{\partial X}{\partial q_k} = \frac{\partial}{\partial q_k}\left[P_k \cdot R_k \cdot q_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] =$$
$$= P_k \cdot R_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} + P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right],$$

in which

$$\frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] = \frac{1}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\frac{\partial}{\partial q_k}\left[1-\exp\left[-\left(p_k+q_k\right)t\right]\right]+$$
$$+\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right) \cdot \frac{\partial}{\partial q_k}\left[\frac{1}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right] =$$
$$= -\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} \cdot \frac{\partial}{\partial q_k}\left[-\left(p_k+q_k\right)t\right] -$$
$$- \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2} \cdot \frac{\partial}{\partial q_k}\left(q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right) =$$
$$= \frac{t \cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} -$$
$$- \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2} \cdot \left(1+p_k \cdot \frac{\partial \exp\left[-\left(p_k+q_k\right)t\right]}{\partial q_k}\right) =$$
$$= \frac{t \cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} -$$
$$- \frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2} \cdot \left(1+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right] \cdot \frac{\partial}{\partial q_k}\left[-\left(p_k+q_k\right)t\right]\right) =$$
$$= \frac{t \cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} - \frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right) \cdot \left(1-p_k \cdot t \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2},$$

leading to

$$\frac{\partial X}{\partial q_k} = P_k \cdot R_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} + P_k \cdot R_k \cdot q_k \cdot \frac{t \cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]} -$$
$$- P_k \cdot R_k \cdot q_k \cdot \frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right) \cdot \left(1-p_k \cdot t \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k \cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.$$

Now, we evaluate the derivative $\dfrac{\partial X'}{\partial q_k}$:

$$\begin{aligned}
\frac{\partial X'}{\partial q_k} &= (P_k \cdot R_k)' \cdot \frac{\partial}{\partial q_k}\left[q_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\
&+ P_k \cdot R_k \cdot q_k' \cdot \frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\
&+ P_k \cdot R_k \cdot \frac{\partial}{\partial q_k}\left[q_k \cdot \frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\
&- P_k \cdot R_k \cdot \frac{\partial}{\partial q_k}\left[q_k \cdot \frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right] = \\
&= (P_k \cdot R_k)' \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + (P_k \cdot R_k)' \cdot q_k \cdot \frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\
&+ P_k \cdot R_k \cdot q_k' \cdot \frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\
&+ P_k \cdot R_k \cdot \frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \\
&+ P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial q_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\
&- P_k \cdot R_k \cdot \frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} - \\
&- P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial q_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right]
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial X'}{\partial q_k} &= (P_k \cdot R_k)' \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \\
&+ (P_k \cdot R_k \cdot q_k)' \cdot \frac{t\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} - \\
&- (P_k \cdot R_k \cdot q_k)' \cdot \frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} + \\
&+ P_k \cdot R_k \cdot \frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \\
&+ P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial q_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\
&- P_k \cdot R_k \cdot \frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} - \\
&- P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial q_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right].
\end{aligned}$$

Furthermore,

$$\frac{\partial}{\partial q_k}\left[\frac{(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right]=$$

$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]-(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot t}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}-$$

$$-\frac{(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\frac{\partial}{\partial q_k}\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)=$$

$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]-(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot t}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}-$$

$$-\frac{(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-p_k\cdot t\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2},$$

and

$$\frac{\partial}{\partial q_k}\left[\frac{q'_k+\left[p'_k-p_k\cdot(p_k+q_k+(p'_k+q'_k)\cdot t)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right]=$$

$$=\frac{-p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]+\left[p'_k-p_k\cdot(p_k+q_k+(p'_k+q'_k)\cdot t)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot(-t)}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}-$$

$$-\frac{q'_k+\left[p'_k-p_k\cdot(p_k+q_k+(p'_k+q'_k)\cdot t)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\cdot\frac{\partial}{\partial q_k}\left[\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2\right]=$$

$$=-\frac{\left(p_k+t\cdot\left[p'_k-p_k\cdot(p_k+q_k+(p'_k+q'_k)\cdot t)\right]\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}-$$

$$-\frac{q'_k+\left[p'_k-p_k\cdot(p_k+q_k+(p'_k+q'_k)\cdot t)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\cdot\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)\times$$

$$\times 2\cdot\left(1-p_k\cdot t\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right).$$

Thence,

$$\frac{\partial X'}{\partial q_k}=(P_k\cdot R_k)'\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}+$$

$$+(P_k\cdot R_k\cdot q_k)'\cdot\frac{t\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}-$$

$$-(P_k\cdot R_k\cdot q_k)'\cdot\frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}+$$

$$+P_k\cdot R_k\cdot\frac{(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}+$$

$$+P_k\cdot R_k\cdot q_k\cdot\frac{\exp\left[-\left(p_k+q_k\right)t\right]-(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot t}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}-$$

$$-P_k\cdot R_k\cdot q_k\cdot\frac{(p_k+q_k+(p'_k+q'_k)\cdot t)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-p_k\cdot t\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}-$$

$$
\begin{aligned}
&- P_k \cdot R_k \cdot \frac{q_k' + [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)] \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} - \\
&+ P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + t \cdot [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)]) \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} - \\
&+ P_k \cdot R_k \cdot q_k \cdot \frac{q_k' + [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)] \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^4} \times \\
&\times 2\,(q_k + p_k \cdot \exp[-(p_k + q_k)\, t]) \cdot (1 - p_k \cdot t \cdot \exp[-(p_k + q_k)\, t])\,.
\end{aligned}
$$

Apparently,

$$
\frac{\partial X}{\partial p_k'} = \frac{\partial X}{\partial q_k'} = 0,
$$

and

$$
\begin{aligned}
\frac{\partial X'}{\partial p_k'} &= \frac{\partial}{\partial p_k'} \left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} \right] - \\
&- \frac{\partial}{\partial p_k'} \left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{q_k' + [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)] \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} \right] = \\
&= P_k \cdot R_k \cdot q_k \cdot \frac{t \cdot \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} - P_k \cdot R_k \cdot q_k \cdot \frac{(1 - p_k \cdot t) \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} = \\
&= \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} \left[t \cdot (q_k + p_k \cdot \exp[-(p_k + q_k)\, t]) - 1 + p_k \cdot t \right] = \\
&= \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} \left[(q_k + p_k) \cdot t + p_k \cdot t \cdot \exp[-(p_k + q_k)\, t] - 1 \right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k'} &= \frac{1}{X} \cdot \frac{\partial X}{\partial p_k'} = \\
&= \frac{1}{X} \cdot \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} \left[(q_k + p_k) \cdot t + p_k \cdot t \cdot \exp[-(p_k + q_k)\, t] - 1 \right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial X'}{\partial q_k'} &= \frac{\partial}{\partial p_k'} \left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} \right] - \\
&- \frac{\partial}{\partial p_k'} \left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{q_k' + [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)] \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} \right] = \\
&= P_k \cdot R_k \cdot q_k \cdot \frac{t \cdot \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} - P_k \cdot R_k \cdot q_k \cdot \frac{(1 - p_k \cdot t) \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} = \\
&= \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2} \left[(q_k + p_k) \cdot t + p_k \cdot t \cdot \exp[-(p_k + q_k)\, t] - 1 \right],
\end{aligned}
$$

i.e.,

$$
\frac{\partial X'}{\partial p_k'} = \frac{\partial X'}{\partial q_k'}.
$$

Frechet Model for Alpha

For the absorption rate α_k we consider the following model based on the well-known Frechet distribution:

$$\alpha_k = p_k \cdot q_k \cdot t^{-p_k - 1},$$

where p_k and q_k are to be estimated.

Then, Π_{BTC} and $\ln P_{\text{BTC}}$ will obtain the following form:

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right],$$

$$\ln P_{\text{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d.$$

Frechet Derivations

Here we calculate the sensitivity derivatives of

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right],$$

and

$$\ln P_{\text{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d.$$

with respect to p_k and q_k.

Apparently,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial p_k} = \frac{\partial^2}{\partial t \partial p_k}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right)\right]$$

and

$$\frac{\partial \ln P_{\text{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k} \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right),$$

since other terms do not depend on p_k and q_k.

Taking into account that

$$\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k - 1}\right)\right] = \frac{X'}{X},$$

where

$$X=\sum_{k=1}^{n} P_k\cdot R_k\cdot p_k\cdot q_k\cdot t^{-p_k-1},\quad X'=\frac{\partial X}{\partial t},$$

we have

$$X'=\frac{\partial}{\partial t}\left(\sum_{k=1}^{n} P_k\cdot R_k\cdot p_k\cdot q_k\cdot t^{-p_k-1}\right)=-\sum_{k=1}^{n} P_k\cdot R_k\cdot p_k\cdot (p_k+1)\cdot q_k\cdot t^{-p_k-2}.$$

Let us now calculate the sensitivity derivative

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k}=\frac{\partial}{\partial p_k}\left[\frac{X'}{X}\right]=\frac{\partial X'}{\partial p_k}\cdot\frac{1}{X}+X'\cdot\frac{\partial}{\partial p_k}\left[\frac{1}{X}\right]=\frac{\partial X'}{\partial p_k}\cdot\frac{1}{X}-\frac{X'}{X^2}\cdot\frac{\partial X}{\partial p_k}.$$

Evidently,

$$\frac{\partial X'}{\partial p_k}=-P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial p_k}\left(p_k\cdot(p_k+1)\cdot t^{-p_k-2}\right).$$

Applying the multiplication derivative rule, we get

$$\frac{\partial}{\partial p_k}\left(p_k\cdot(p_k+1)\cdot t^{-p_k-2}\right)=(2p_k+1)\cdot t^{-p_k-2}-p_k\cdot(p_k+1)\cdot t^{-p_k-2}\cdot\ln t=$$

$$=t^{-p_k-2}\cdot\left[2p_k+1-p_k\cdot(p_k+1)\cdot\ln t\right].$$

Thus,

$$\frac{\partial X'}{\partial p_k}=-P_k\cdot R_k\cdot q_k\cdot t^{-p_k-2}\cdot\left[2p_k+1-p_k\cdot(p_k+1)\cdot\ln t\right].$$

On the other hand,

$$\frac{\partial X}{\partial p_k}=P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial p_k}\left(p_k\cdot t^{-p_k-1}\right)=$$

$$=P_k\cdot R_k\cdot q_k\cdot\left[t^{-p_k-1}-p_k\cdot t^{-p_k-1}\cdot\ln t\right]=$$

$$=P_k\cdot R_k\cdot q_k\cdot t^{-p_k-1}\cdot\left[1-p_k\cdot\ln t\right].$$

Substituting the above expressions into the sensitivity derivative, we finally obtain

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k}=-\frac{P_k\cdot R_k\cdot q_k}{X}\cdot t^{-p_k-2}\cdot\left[2p_k+1-p_k\cdot(p_k+1)\cdot\ln t\right]-$$

$$-\frac{X'}{X^2}\cdot P_k\cdot R_k\cdot q_k\cdot t^{-p_k-1}\cdot\left[1-p_k\cdot\ln t\right].$$

Let us now proceed with calculating the derivative of Π_{BTC} with respect to q_k. To that aim, notice that X and X' are linear in q_k leading to

$$\frac{\partial X}{\partial q_k}=P_k\cdot R_k\cdot p_k\cdot t^{-p_k-1}\cdot\frac{\partial}{\partial q_k}(q_k)=P_k\cdot R_k\cdot p_k\cdot t^{-p_k-1},$$

and

$$\frac{\partial X'}{\partial q_k} = -P_k \cdot R_k \cdot t^{-p_k-2} \cdot \left[2p_k + 1 - p_k \cdot (p_k + 1) \cdot \ln t\right].$$

Similarly, in the case of q_k, we have

$$\begin{aligned}\frac{\partial \Pi_{\text{BTC}}}{\partial q_k} = -\frac{P_k \cdot R_k}{X} \cdot t^{-p_k-2} \cdot \left[2p_k + 1 - p_k \cdot (p_k + 1) \cdot \ln t\right] - \\ - \frac{X'}{X^2} \cdot P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1}.\end{aligned}$$

Let us now proceed to the sensitivity derivatives calculations for $\ln P_{\text{BTC}}$. Apparently,

$$\frac{\partial \ln P_{\text{BTC}}}{\partial p_k} = \frac{\partial \ln X}{\partial p_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial p_k} = \frac{1}{X} \cdot P_k \cdot R_k \cdot q_k \cdot t^{-p_k-1} \cdot \left[1 - p_k \cdot \ln t\right].$$

Similarly,

$$\frac{\partial \ln P_{\text{BTC}}}{\partial q_k} = \frac{\partial \ln X}{\partial q_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial q_k} = \frac{1}{X} \cdot P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1}.$$

Weibull Model for Alpha

For the absorption rate α_k we consider the following model based on the well-known Weibull distribution:

$$\alpha_k = \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right],$$

where p_k and q_k are to be estimated.

Then, Π_{BTC} and $\ln P_{\mathrm{BTC}}$ will obtain the following form:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) + \right.$$
$$\left. + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right],$$

$$\ln P_{\mathrm{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) +$$
$$+ \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d.$$

Weibull Derivations

Here we calculate the sensitivity derivatives of

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) + \right.$$
$$\left. + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right] \quad and$$

$$\ln P_{\mathrm{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) +$$
$$+ \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d.$$

with respect to p_k and q_k.

Let us start with the sensitivity with respect to p_k. Apparently, only the first term in both expressions above depends on p_k. Therefore, we may dismiss other terms when calculating the partial derivatives. In other words,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial^2}{\partial t \partial p_k}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right)\right]$$

and

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k}\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right).$$

Taking into account that

$$\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right)\right] = \frac{X'}{X},$$

where

$$X = \sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]$$

and

$$X' = \frac{\partial X}{\partial t},$$

we have

$$\begin{aligned} X' &= \frac{\partial}{\partial t}\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) = \\ &= \sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \frac{\partial}{\partial t}\left(\left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) = \frac{\partial}{\partial t}\left[\left(\frac{t}{q_k}\right)^{p_k-1}\right] \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] + \\ &\quad + \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \frac{\partial}{\partial t}\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] = \frac{p_k-1}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] + \\ &\quad + \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left(-\frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1}\right) = \\ &\quad = \frac{p_k-1}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] - \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{2p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] = \\ &\quad = \left(\frac{t}{q_k}\right)^{p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[\frac{p_k-1}{q_k} - \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k}\right]. \end{aligned}$$

Thus, finally,

$$X' = \sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[\frac{p_k-1}{q_k} - \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k}\right].$$

Coming back to the sensitivity problem, let us now calculate the sensitivity derivative

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\frac{X'}{X}\right] = \frac{\partial X'}{\partial p_k}\cdot\frac{1}{X} + X'\cdot\frac{\partial}{\partial p_k}\left[\frac{1}{X}\right] = \frac{\partial X'}{\partial p_k}\cdot\frac{1}{X} - \frac{X'}{X^2}\cdot\frac{\partial X}{\partial p_k}.$$

Apparently,

$$\frac{\partial X'}{\partial p_k} = P_k\cdot R_k\cdot\frac{\partial}{\partial p_k}\left(\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]\right).$$

Applying the multiplication derivative rule, we get

$$\begin{aligned}
&\frac{\partial}{\partial p_k}\left(\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]\right) = \\
&\quad = \frac{\partial}{\partial p_k}\left(\frac{p_k}{q_k}\right)\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right] + \\
&\quad + \frac{p_k}{q_k}\cdot\frac{\partial}{\partial p_k}\left(\left(\frac{t}{q_k}\right)^{p_k-2}\right)\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right] + \\
&\quad + \frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\frac{\partial}{\partial p_k}\left(\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right)\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right] + \\
&\quad + \frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\frac{\partial}{\partial p_k}\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right].
\end{aligned}$$

Since

$$\frac{\partial}{\partial p_k}\left(\frac{p_k}{q_k}\right) = \frac{1}{q_k},\quad \frac{\partial}{\partial p_k}\left(\left(\frac{t}{q_k}\right)^{p_k-2}\right) = \left(\frac{t}{q_k}\right)^{p_k-2}\cdot\ln\left(\frac{t}{q_k}\right),$$

$$\begin{aligned}
\frac{\partial}{\partial p_k}\left(\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) &= \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[-\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right)\right] = \\
&= -\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right)\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial p_k}\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right] &= \frac{1}{q_k}-\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right) = \\
&= \frac{1}{q_k}-\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[1-p_k\cdot\ln\left(\frac{t}{q_k}\right)\right] = \\
&= \frac{1}{q_k}-\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[1-\ln\left(\frac{t}{q_k}\right)^{p_k}\right].
\end{aligned}$$

Combining all separate derivatives, we will obtain

$$
\begin{aligned}
&\frac{\partial}{\partial p_k}\left(\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]\right)=\\
&\qquad=\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]+\\
&\qquad+\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\ln\left(\frac{t}{q_k}\right)\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]-\\
&\qquad-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right)\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]+\\
&\qquad+\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left(\frac{1}{q_k}-\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[1-\ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right)=\\
&\qquad=\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{1}{q_k}\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]+\right.\\
&\qquad+\frac{1}{q_k}\cdot\ln\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]-\\
&\qquad-\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[\frac{p_k-1}{q_k}\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]+\\
&\qquad\left.+\frac{p_k}{q_k}\cdot\frac{1}{q_k}\cdot\left(1-\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[1-\ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right)\right]=\\
&\qquad=\frac{1}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k}{q_k}\cdot\left(1-\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[1-\ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right)+\right.\\
&\qquad\left.+\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[1+\ln\left(\frac{t}{q_k}\right)^{p_k}-\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial X'}{\partial p_k}=\frac{P_k\cdot R_k}{q_k}&\cdot\left(\frac{t}{q_k}\right)^{p_k-2}\cdot\exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[\frac{p_k}{q_k}\cdot\left(1-\left(\frac{t}{q_k}\right)^{p_k}\cdot\left[1-\ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right)+\right.\\
&\left.+\left[\frac{p_k-1}{q_k}-\frac{p_k}{q_k}\cdot\left(\frac{t}{q_k}\right)^{p_k}\right]\cdot\left[1+\ln\left(\frac{t}{q_k}\right)^{p_k}-\left(\frac{t}{q_k}\right)^{p_k}\cdot\ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right].
\end{aligned}
$$

Now, compute the following derivative:

$$\frac{\partial X}{\partial p_k} = P_k \cdot R_k \cdot \frac{\partial}{\partial p_k}\left(\frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right]\right) =$$

$$= P_k \cdot R_k \cdot \left[\frac{1}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] + \right.$$

$$+ \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \ln\left(\frac{t}{q_k}\right) \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] +$$

$$\left. + \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left(-\left(\frac{t}{q_k}\right)^{p_k} \cdot \ln\left(\frac{t}{q_k}\right)\right)\right] =$$

$$= \frac{P_k \cdot R_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \ln\left(\frac{t}{q_k}\right)^{p_k} \cdot \left(1 - \left(\frac{t}{q_k}\right)^{p_k}\right)\right].$$

Substituting the above expressions into the sensitivity derivative, we finally obtain

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{1}{X} \cdot \frac{P_k \cdot R_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[\frac{p_k}{q_k} \cdot \left(1 - \left(\frac{t}{q_k}\right)^{p_k} \cdot \left[1 - \ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right) + \right.$$

$$\left. + \left[\frac{p_k - 1}{q_k} - \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \ln\left(\frac{t}{q_k}\right)^{p_k} - \left(\frac{t}{q_k}\right)^{p_k} \cdot \ln\left(\frac{t}{q_k}\right)^{p_k}\right]\right] -$$

$$- \frac{X'}{X^2} \cdot \frac{P_k \cdot R_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \ln\left(\frac{t}{q_k}\right)^{p_k} \cdot \left(1 - \left(\frac{t}{q_k}\right)^{p_k}\right)\right].$$

Before proceeding to computing of derivatives with respect to q_k, let us simplify the expressions for X and X'. Namely,

$$X = \sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] =$$

$$= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k^{-1} \cdot t^{p_k-1} \cdot q_k^{-p_k+1} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] =$$

$$= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot t^{p_k-1} \cdot q_k^{-p_k} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right],$$

$$X' = \sum_{k=1}^{n} P_k \cdot R_k \cdot \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[\frac{p_k - 1}{q_k} - \frac{p_k}{q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k}\right] =$$

$$= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k^{-1} \cdot t^{p_k-2} \cdot q_k^{-p_k+2} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] \cdot q_k^{-1} \cdot \left[p_k - 1 - p_k \cdot t^{p_k} \cdot q_k^{-p_k}\right] =$$

$$= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot t^{p_k-2} \cdot q_k^{-p_k} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] \cdot \left[p_k - 1 - p_k \cdot t^{p_k} \cdot q_k^{-p_k}\right].$$

Thus,

$$\frac{\partial X}{\partial q_k} = P_k \cdot R_k \cdot p_k \cdot t^{p_k-1} \cdot \frac{\partial}{\partial q_k}\left(q_k^{-p_k} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right]\right) =$$

$$= P_k \cdot R_k \cdot p_k \cdot t^{p_k-1} \cdot \left[\frac{\partial q_k^{-p_k}}{\partial q_k} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] + q_k^{-p_k} \cdot \frac{\partial}{\partial q_k}\left(\exp\left[-t^{p_k} \cdot q_k^{-p_k}\right]\right)\right] =$$

$$= P_k \cdot R_k \cdot p_k \cdot t^{p_k-1} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] \cdot \left[-p_k q_k^{-p_k-1} + q_k^{-p_k} \cdot \left(-p_k \cdot t^{p_k} \cdot q_k^{-p_k-1}\right)\right] =$$

$$= -P_k \cdot R_k \cdot p_k^2 \cdot t^{p_k-1} \cdot q_k^{-p_k-1} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] \cdot \left[1 + q_k^{-p_k} \cdot t^{p_k}\right] =$$

$$= -P_k \cdot R_k \cdot p_k^2 \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \left(\frac{t}{q_k}\right)^{p_k}\right],$$

and

$$\frac{\partial X'}{\partial q_k} = P_k \cdot R_k \cdot p_k \cdot t^{p_k-2} \cdot \frac{\partial}{\partial q_k}\left(q_k^{-p_k} \cdot \exp\left[-t^{p_k} \cdot q_k^{-p_k}\right] \cdot \left[p_k - 1 - p_k \cdot t^{p_k} \cdot q_k^{-p_k}\right]\right) =$$

$$= -P_k \cdot R_k \cdot p_k^2 \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[p_k \cdot \left(1 - \left(\frac{t}{q_k}\right)^{p_k}\right) - 1\right] +$$

$$+ P_k \cdot R_k \cdot p_k^3 \cdot q_k \cdot \left(\frac{t}{q_k}\right)^{2p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right].$$

Similarly, in the case of q_k, we have

$$\frac{\partial \Pi_{\text{BTC}}}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{X'}{X}\right] = \frac{\partial X'}{\partial q_k} \cdot \frac{1}{X} - \frac{X'}{X^2} \cdot \frac{\partial X}{\partial q_k} =$$

$$= -\frac{P_k \cdot R_k \cdot p_k^2}{X} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[p_k \cdot \left(1 - \left(\frac{t}{q_k}\right)^{p_k}\right) - 1\right] +$$

$$+ \frac{P_k \cdot R_k \cdot p_k^3 \cdot q_k}{X} \cdot \left(\frac{t}{q_k}\right)^{2p_k-2} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] +$$

$$+ \frac{X' \cdot P_k \cdot R_k \cdot p_k^2}{X^2} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \left(\frac{t}{q_k}\right)^{p_k}\right].$$

Let us now proceed to the sensitivity derivatives calculations for $\ln P_{\text{BTC}}$. Apparently,

$$\frac{\partial \ln P_{\text{BTC}}}{\partial p_k} = \frac{\partial \ln X}{\partial p_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial p_k} =$$

$$= \frac{P_k \cdot R_k}{X \cdot q_k} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \ln\left(\frac{t}{q_k}\right)^{p_k} \cdot \left(1 - \left(\frac{t}{q_k}\right)^{p_k}\right)\right].$$

Similarly,

$$\frac{\partial \ln P_{\text{BTC}}}{\partial q_k} = \frac{\partial \ln X}{\partial q_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial q_k} =$$

$$= -\frac{P_k \cdot R_k \cdot p_k^2}{X} \cdot \left(\frac{t}{q_k}\right)^{p_k-1} \cdot \exp\left[-\left(\frac{t}{q_k}\right)^{p_k}\right] \cdot \left[1 + \left(\frac{t}{q_k}\right)^{p_k}\right].$$

Gumbel Model for Alpha

For the absorption rate α_k we consider the following model based on the well-known Gumbel distribution:

$$\alpha_k = p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right],$$

where p_k and q_k are to be estimated.

Then, Π_{BTC} and $\ln P_{\mathrm{BTC}}$ will obtain the following form:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]\right) + \right.$$
$$\left. + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d\right],$$

$$\ln P_{\mathrm{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]\right) +$$
$$+ \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d.$$

Gumbel Derivations

Here we calculate the sensitivity derivatives of

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]\right) + \right.$$
$$\left. + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d\right],$$

and

$$\ln P_{\mathrm{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]\right) +$$
$$+ \ln\left(\frac{1}{m}\sum_{j=1}^{m} T_j'\right) - \ln b - \ln h + \ln d.$$

with respect to p_k and q_k.

Apparently,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial^2}{\partial t \partial p_k}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]\right)\right]$$

and

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k} \ln \left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \right),$$

since other terms do not depend on p_k and q_k.

Taking into account that

$$\frac{\partial}{\partial t} \left[\ln \left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \right) \right] = \frac{X'}{X},$$

where

$$X = \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]$$

and

$$X' = \frac{\partial X}{\partial t},$$

we have

$$\begin{aligned}
X' &= \frac{\partial}{\partial t} \left(\sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \right) = \\
&= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot \left(\frac{\partial t^{-p_k-1}}{\partial t} \cdot \exp\left[-q_k t^{-p_k}\right] + t^{-p_k-1} \cdot \frac{\partial}{\partial t} \exp\left[-q_k t^{-p_k}\right] \right) = \\
&= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot \left((-p_k - 1) \cdot t^{-p_k-2} + t^{-p_k-1} \cdot \left[p_k \cdot q_k \cdot t^{-p_k-1}\right] \right) \cdot \exp\left[-q_k t^{-p_k}\right] = \\
&= \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right].
\end{aligned}$$

Thus, finally,

$$X' = \sum_{k=1}^{n} P_k \cdot R_k \cdot p_k \cdot q_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right].$$

Coming back to the sensitivity problem, let us now calculate the sensitivity derivative

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k} \left[\frac{X'}{X}\right] = \frac{\partial X'}{\partial p_k} \cdot \frac{1}{X} + X' \cdot \frac{\partial}{\partial p_k} \left[\frac{1}{X}\right] = \frac{\partial X'}{\partial p_k} \cdot \frac{1}{X} - \frac{X'}{X^2} \cdot \frac{\partial X}{\partial p_k}.$$

Evidently,

$$\frac{\partial X'}{\partial p_k} = P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k} \left(p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right]\right).$$

Applying the multiplication derivative rule, we get

$$\frac{\partial}{\partial p_k}\left(p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right]\right) =$$

$$= \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] +$$

$$+ p_k \cdot \frac{\partial}{\partial p_k}\left[\left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right)\right] \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] +$$

$$+ p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \frac{\partial t^{-p_k-2}}{\partial p_k} \cdot \exp\left[-q_k t^{-p_k}\right] +$$

$$+ p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \frac{\partial \exp\left[-q_k t^{-p_k}\right]}{\partial p_k} =$$

$$= \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] +$$

$$+ p_k \cdot \left(-1 + q_k \cdot t^{-p_k} - p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] -$$

$$- p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \ln t \cdot \exp\left[-q_k t^{-p_k}\right] +$$

$$+ p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot q_k \cdot t^{-p_k} \cdot \ln t$$

or

$$\frac{\partial}{\partial p_k}\left(p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right]\right) =$$

$$= t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Bigg[\left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) +$$

$$+ p_k \cdot \left(-1 + q_k \cdot t^{-p_k} - p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right) -$$

$$- p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) +$$

$$+ p_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot q_k \cdot t^{-p_k} \cdot \ln t\Bigg] =$$

$$= t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Bigg[p_k \cdot \left(-1 + q_k \cdot t^{-p_k} - p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right) +$$

$$+ \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \left(1 - p_k + p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right)\Bigg].$$

Thus, combining all separate derivatives, we will obtain

$$\frac{\partial X'}{\partial p_k} = P_k \cdot R_k \cdot q_k \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Bigg[p_k \cdot \left(-1 + q_k \cdot t^{-p_k} - p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right) +$$

$$+ \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \left(1 - p_k + p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right)\Bigg].$$

Now, compute the following derivative:

$$\begin{aligned}\frac{\partial X}{\partial p_k} &= P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left(p_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right]\right) = \\ &= P_k \cdot R_k \cdot q_k \cdot \Big(t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] - p_k \cdot t^{-p_k-1} \cdot \ln t \cdot \exp\left[-q_k t^{-p_k}\right] + \\ &+ p_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot q_k \cdot t^{-p_k} \cdot \ln t\Big) = \\ &= P_k \cdot R_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Big(1 - p_k \cdot \ln t + p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\Big) = \\ &= P_k \cdot R_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Big(1 - p_k \cdot \ln t\left[1 - q_k \cdot t^{-p_k}\right]\Big).\end{aligned}$$

Substituting the above expressions into the sensitivity derivative, we finally obtain

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} &= \frac{P_k \cdot R_k \cdot q_k}{X} \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Bigg[p_k \cdot \left(-1 + q_k \cdot t^{-p_k} - p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right) + \\ &+ \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \left(1 - p_k + p_k \cdot q_k \cdot t^{-p_k} \cdot \ln t\right)\Bigg] - \\ &- \frac{X'}{X^2} \cdot P_k \cdot R_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \Big(1 - p_k \cdot \ln t\left[1 - q_k \cdot t^{-p_k}\right]\Big).\end{aligned}$$

Let us now proceed with calculating the derivative of Π_{BTC} with respect to q_k. To that aim, let us first compute the analogous derivatives

$$\begin{aligned}\frac{\partial X}{\partial q_k} &= P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1} \cdot \frac{\partial}{\partial q_k}\left(q_k \cdot \exp\left[-q_k t^{-p_k}\right]\right) = \\ &= P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1} \cdot \left(\exp\left[-q_k t^{-p_k}\right] + \exp\left[-q_k t^{-p_k}\right] \cdot \left[-t^{-p_k}\right]\right) = \\ &= P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \left(1 - t^{-p_k}\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial X'}{\partial q_k} &= P_k \cdot R_k \cdot p_k \cdot t^{-p_k-2} \cdot \frac{\partial}{\partial q_k}\left(q_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \exp\left[-q_k t^{-p_k}\right]\right) = \\ &= P_k \cdot R_k \cdot p_k \cdot t^{-p_k-2} \cdot \Big(\left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \exp\left[-q_k t^{-p_k}\right] + \\ &+ q_k \cdot p_k \cdot t^{-p_k} \cdot \exp\left[-q_k t^{-p_k}\right] - q_k \cdot \left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \exp\left[-q_k t^{-p_k}\right] \cdot t^{-p_k}\Big) = \\ &= P_k \cdot R_k \cdot p_k \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \Big(\left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \left(1 - t^{-p_k}\right) + \\ &+ q_k \cdot p_k \cdot t^{-p_k}\Big).\end{aligned}$$

Similarly, in the case of q_k, we have

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{X'}{X}\right] = \frac{\partial X'}{\partial q_k} \cdot \frac{1}{X} - \frac{X'}{X^2} \cdot \frac{\partial X}{\partial q_k} = \\ &= \frac{P_k \cdot R_k \cdot p_k}{X} \cdot t^{-p_k-2} \cdot \exp\left[-q_k t^{-p_k}\right] \Big(\left(-p_k - 1 + p_k \cdot q_k \cdot t^{-p_k}\right) \cdot \left(1 - t^{-p_k}\right) + \\ &+ q_k \cdot p_k \cdot t^{-p_k}\Big) - \frac{X'}{X^2} \cdot P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \left(1 - t^{-p_k}\right).\end{aligned}$$

Let us now proceed to the sensitivity derivatives calculations for $\ln P_{\mathrm{BTC}}$. Apparently,

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial \ln X}{\partial p_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial p_k} = \\ = \frac{1}{X} \cdot P_k \cdot R_k \cdot q_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \left(1 - p_k \cdot \ln t \left[1 - q_k \cdot t^{-p_k}\right]\right).$$

Similarly,

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial q_k} = \frac{\partial \ln X}{\partial q_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial q_k} = \\ = \frac{1}{X} \cdot P_k \cdot R_k \cdot p_k \cdot t^{-p_k-1} \cdot \exp\left[-q_k t^{-p_k}\right] \cdot \left(1 - t^{-p_k}\right).$$

Shifted Gompertz Model for Alpha

Consider now the following particular model for the absorption rate α_k corresponding to the well-known shifted Gompertz distribution:

$$\alpha_k = p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right),$$

where p_k and q_k are to be estimated.

Then, Π_{BTC} and $\ln P_{\mathrm{BTC}}$ will obtain the following form:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right) + \right.$$
$$\left. + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right],$$

$$\ln P_{\mathrm{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right) +$$
$$+ \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d.$$

Shifted Gompertz Derivations

In this appendix, we will consider the sensitivity of

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right) + \right.$$
$$\left. + \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d\right]$$

and

$$\ln P_{\mathrm{BTC}} = \ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right) +$$
$$+ \ln\left(\frac{1}{m}\sum_{j=1}^{m} T'_j\right) - \ln b - \ln h + \ln d$$

with respect to p_k and q_k.

Let us start with the sensitivity with respect to p_k. Apparently, only the first term in both expressions above depends on p_k. Therefore, we may dismiss other terms when calculating the partial derivatives. In other words,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial^2}{\partial t \partial p_k}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right)\right]$$

and

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k} \ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right)$$

Taking into account that

$$\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right)\right] = \frac{X'}{X},$$

where

$$X = \sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k$$

and

$$X' = \frac{\partial X}{\partial t},$$

we have

$$X' = \frac{\partial}{\partial t}\left[\sum_{k=1}^{n} P_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \cdot R_k\right] =$$
$$= \sum_{k=1}^{n} P_k \cdot p_k \cdot R_k \cdot \frac{\partial}{\partial t}\Big[\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right)\Big].$$

On the other hand,

$$\frac{\partial}{\partial t}\Big[\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right)\Big] =$$

$$\begin{aligned}
&= \frac{\partial}{\partial t}\Big[\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\Big]\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right) + \\
&+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\frac{\partial}{\partial t}\Big[\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right)\Big] = \\
&= \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\frac{\partial}{\partial t}\Big(-p_k t - q_k \exp\left(-p_k t\right)\Big)\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right) + \\
&+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\left(-q_k\cdot\frac{\partial}{\partial t}\Big[\exp\left(-p_k t\right)\Big]\right) = \\
&= \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\Big(-p_k - q_k \exp\left(-p_k t\right)\cdot\left(-p_k\right)\Big)\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right) + \\
&+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\left(p_k\cdot q_k\cdot\exp\left(-p_k t\right)\right) = \\
&= -p_k\cdot\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\Big(1 - q_k \exp\left(-p_k t\right)\Big)\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right) + \\
&+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot p_k\cdot q_k\cdot\exp\left(-p_k t\right) = \\
&= p_k\cdot\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\cdot\Bigg[q_k\cdot\exp\left(-p_k t\right) - \\
&- \Big(1 - q_k \exp\left(-p_k t\right)\Big)\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right)\Bigg].
\end{aligned}$$

Simplifying the expression in the last bracket, we derive

$$\begin{aligned}
&q_k\cdot\exp\left(-p_k t\right) - \Big(1 - q_k \exp\left(-p_k t\right)\Big)\cdot\left(1+q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\right) = \\
&\qquad = q_k\cdot\exp\left(-p_k t\right) - 1 - q_k\cdot\left[1-\exp\left(-p_k t\right)\right] + q_k\cdot\exp\left(-p_k t\right) + \\
&\qquad + q_k^2\cdot\exp\left(-p_k t\right)\cdot\left[1-\exp\left(-p_k t\right)\right] = \\
&\qquad = 2q_k\cdot\exp\left(-p_k t\right) - q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\cdot\left(1-q_k\cdot\exp\left(-p_k t\right)\right) - 1.
\end{aligned}$$

Therefore, finally,

$$\begin{aligned}
X' = -\sum_{k=1}^{n} P_k\cdot R_k\cdot p_k^2\cdot\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]\times \\
\times\big[2q_k\cdot\exp\left(-p_k t\right) - q_k\cdot\left[1-\exp\left(-p_k t\right)\right]\cdot\left(1-q_k\cdot\exp\left(-p_k t\right)\right) - 1\big].
\end{aligned}$$

Coming back to the sensitivity problem, let us now calculate the sensitivity derivative

$$\frac{\partial\Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\frac{X'}{X}\right] = \frac{\partial X'}{\partial p_k}\cdot\frac{1}{X} + X'\cdot\frac{\partial}{\partial p_k}\left[\frac{1}{X}\right] = \frac{\partial X'}{\partial p_k}\cdot\frac{1}{X} - \frac{X'}{X^2}\cdot\frac{\partial X}{\partial p_k}.$$

Apparently,

$$\frac{\partial X'}{\partial p_k} = -P_k \cdot R_k \cdot \frac{\partial}{\partial p_k}\Big(p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big]\Big).$$

Applying the multiplication derivative rule, we get

$$\frac{\partial}{\partial p_k}\Big(p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big]\Big) =$$
$$= \frac{\partial p_k^2}{\partial p_k} \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big] +$$
$$+ p_k^2 \cdot \frac{\partial \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{\partial p_k} \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big] +$$
$$+ p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$
$$\times \frac{\partial}{\partial p_k}\Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big] =$$
$$= 2p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big] +$$
$$+ p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[-t - q_k \exp\left(-p_k t\right) \cdot (-t)\right] \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\Big] +$$
$$+ p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$
$$\times \Big[2q_k \cdot \exp\left(-p_k t\right) \cdot (-t) - q_k \cdot \left[-\exp\left(-p_k t\right) \cdot (-t)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) -$$
$$- q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(-q_k \cdot \exp\left(-p_k t\right) \cdot (-t)\right)\Big] =$$

$$
\begin{aligned}
&= 2p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] - \\
&- p_k^2 t \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] + \\
&+ p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[- 2q_k t \cdot \exp\left(-p_k t\right) - q_k t \cdot \exp\left(-p_k t\right) \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - \\
&- q_k^2 t \cdot \exp\left(-p_k t\right) \cdot \left[1 - \exp\left(-p_k t\right)\right] \big] = \\
&= 2p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] - \\
&- p_k^2 t \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] + \\
&+ p_k^2 \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times q_k t \cdot \exp\left(-p_k t\right) \cdot \big[- 2 - \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \big] = \\
&= 2p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] - \\
&- p_k^2 t \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] - \\
&- p_k^2 \cdot q_k t \cdot \exp\left(-p_k t\right) \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2 + \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial X'}{\partial p_k} &= -P_k \cdot R_k \cdot 2p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] - \\
&- p_k^2 t \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] - \\
&- p_k^2 \cdot q_k t \cdot \exp\left(-p_k t\right) \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \big[2 + \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \big].
\end{aligned}
$$

Now, compute the following derivative:

$$
\begin{aligned}
\frac{\partial X}{\partial p_k} &= P_k \cdot R_k \cdot \frac{\partial}{\partial p_k} \left[p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right)\right] = \\
&= P_k \cdot R_k \cdot \Big[\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + \\
&+ p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[-t - q_k \exp\left(-p_k t\right) \cdot (-t)\right] \times \\
&\times \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot q_k \cdot \left[-\exp\left(-p_k t\right) \cdot (-t)\right] \Big] = \\
&= P_k \cdot R_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \Big[1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] - \\
&- p_k t \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + p_k \cdot q_k t \cdot \exp\left(-p_k t\right) \Big].
\end{aligned}
$$

Substituting the above expressions into the sensitivity derivative, we finally obtain

$$
\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} &= -\frac{P_k \cdot R_k \cdot 2p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{X} \times \\
&\times \left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right] - \\
&- p_k^2 t \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right] - \\
&- p_k^2 \cdot q_k t \cdot \exp\left(-p_k t\right) \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times \\
&\times \left[2 + \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right] - \\
&- \frac{X' \cdot P_k \cdot R_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{X^2} \cdot \Big[1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] - \\
&- p_k t \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + p_k \cdot q_k t \cdot \exp\left(-p_k t\right) \Big].
\end{aligned}
$$

Similarly, in the case of q_k, we have

$$
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{X'}{X}\right] = \frac{\partial X'}{\partial q_k} \cdot \frac{1}{X} - \frac{X'}{X^2} \cdot \frac{\partial X}{\partial q_k}.
$$

Here,

$$\frac{\partial X'}{\partial q_k} = -P_k \cdot R_k \cdot p_k^2 \cdot \frac{\partial}{\partial q_k}\Big(\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$

$$\times \left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right]\Big) =$$

$$= -P_k \cdot R_k \cdot p_k^2 \cdot \Bigg[\frac{\partial}{\partial q_k}\Big(\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \Big) \times$$

$$\times \left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right] +$$

$$+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$

$$\times \frac{\partial}{\partial q_k}\left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right] \Bigg] =$$

$$= -P_k \cdot R_k \cdot p_k^2 \cdot \Bigg[\exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \left(-\exp\left(-p_k t\right)\right) \times$$

$$\times \left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right] +$$

$$+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$

$$\times \Big[2 \cdot \exp\left(-p_k t\right) - \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) -$$

$$- q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(-\exp\left(-p_k t\right)\right) \Big] \Bigg]$$

$$= P_k \cdot R_k \cdot p_k^2 \cdot \exp\left(-p_k t\right) \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$

$$\times \Bigg[\left[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\right] +$$

$$+ \left[\exp\left(p_k t\right) - 1\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] - 2 \Bigg].$$

On the other hand,

$$\frac{\partial X}{\partial q_k} = P_k \cdot R_k \cdot p_k \cdot \Bigg[\frac{\partial \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{\partial q_k} \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) +$$

$$+ \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \cdot \frac{\partial}{\partial q_k}\left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) \Bigg] =$$

$$= P_k \cdot R_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right] \times$$

$$\times \left[-\exp\left(-p_k t\right) \cdot \left(2 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + 1\right].$$

Therefore,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_k} = \frac{P_k \cdot R_k \cdot p_k^2 \cdot \exp\left(-p_k t\right) \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{X} \times$$
$$\times \Bigg[\big[2q_k \cdot \exp\left(-p_k t\right) - q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) - 1\big] +$$
$$+ \left[\exp\left(p_k t\right) - 1\right] \cdot \left(1 - q_k \cdot \exp\left(-p_k t\right)\right) + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] - 2 \Bigg] -$$
$$- \frac{X' \cdot P_k \cdot R_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{X^2} \times$$
$$\times \left[-\exp\left(-p_k t\right) \cdot \left(2 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + 1\right]$$

Let us now proceed to the sensitivity derivatives calculations for $\ln P_{\mathrm{BTC}}$. Apparently,

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial p_k} = \frac{\partial \ln X}{\partial p_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial p_k} =$$
$$= \frac{P_k \cdot R_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{X} \cdot \Big[1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right] -$$
$$- p_k t \cdot \left[1 - q_k \exp\left(-p_k t\right)\right] \cdot \left(1 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + p_k \cdot q_k t \cdot \exp\left(-p_k t\right) \Big].$$

Similarly,

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial q_k} = \frac{\partial \ln X}{\partial q_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial q_k} =$$
$$= \frac{P_k \cdot R_k \cdot p_k \cdot \exp\left[-p_k t - q_k \exp\left(-p_k t\right)\right]}{X} \times$$
$$\times \left[-\exp\left(-p_k t\right) \cdot \left(2 + q_k \cdot \left[1 - \exp\left(-p_k t\right)\right]\right) + 1\right].$$