MATH 212 CAT 2 SOLUTIONS

MATH 212 – LINEAR ALGEBRA 1 CAT 2 TIME: 1 HOUR 21/11/2022

a) State the steps followed in showing that a given set of vectors is a vector space. Hence show that the set of all polynomials of degree $\leq n$ is a vector space. [5 marks]

b) Given a linear transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

find the basis for $R(T)$ [5 marks]

c) Determine if $S = \{(1,2), (-1,1)\}$ is a basis for $\mathbb{R}^2$. [5 marks]

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a) A vector space is a set of vectors in which addition is defined as scalar multiplication such that the set is closed under vector addition and scalar mutiplication. Thus given any set / collection of vectors say S, to find out if it's a vector space, check that

i) $u+v \in S$, for all $u, v \in S$

ii) $ku \in S$, for all $u \in S$, and $k \in \mathbb{R}$

Now let P_n denote the set of all polynomials of degrees $\leq n$. Then each of such polynomial is a vector of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Let $p(x)$ and $q(x)$ be such polynomials

Say $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$

Then

$$p(x) + q(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n$$

which is a polynomial in P_n

Also
$$k\,p(x) = k(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$
$$= ka_0 + (ka_1)x + (ka_2)x^2 + \cdots + (ka_n)x^n$$

which is also a polynomial in P_n.

Thus P_n is closed with respect to both vector addition and scalar multiplication and is thus a vector space.

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b) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T(x_1, x_2, x_3) = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We have $R(T) = \{ v \in V : \exists\ u \in U \text{ for which } T(u) = v \}$

Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Thus if v is in the range of T, we have

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

i.e. $$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & a \\ 4 & 0 & -2 & b \\ 0 & 0 & 0 & c \end{array}\right) \begin{array}{c} \xrightarrow{R_1} \\ \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3} \end{array} \left(\begin{array}{ccc|c} 2 & 0 & -1 & a \\ 0 & 0 & 0 & b-2a \\ 0 & 0 & 0 & c \end{array}\right)$$

For consistency, we need $c = 0$ and $b - 2a = 0 \Rightarrow b = 2a$

If $a = t$, then $b = 2t$ where $t \in \mathbb{R}$

$$\text{so } R(T) = \left\{ \begin{pmatrix} t \\ 2t \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\} = \left\{ t \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}, t \in \mathbb{R} \right\}$$

The basis for $\mathbb{R}(T)$ is thus $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$.

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c) $S = \{(1,2), (-1,1)\}$

First we check if S is linearly independent. We have,

$k_1 v_1 + k_2 v_2 = 0$ or $k_1(1,2) + k_2(-1,1) = (0,0)$

Thus $k_1 - k_2 = 0$

$2k_1 + k_2 = 0$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right)$$

$3k_2 = 0 \Rightarrow k_2 = 0$

also $k_1 - k_2 = 0$ or $k_2 = 0 \Rightarrow k_1 = 0$

Thus $k_1 = k_2 = 0$ and so

S is linearly independent.

Next we check if S spans $\mathbb{R}^2$

Let $(a,b) \in \mathbb{R}^2$. We have

$c_1 v_1 + c_2 v_2 = (a,b)$

ie. $c_1(1,2) + c_2(-1,1) = (a,b)$

ie. $c_1 - c_2 = a$

$2c_1 + c_2 = b$

$$\left(\begin{array}{cc|c} 1 & -1 & a \\ 2 & 1 & b \end{array}\right) \begin{array}{c} \xrightarrow{R_1} \\ \xrightarrow{R_2 - 2R_1} \end{array} \left(\begin{array}{cc|c} 1 & -1 & a \\ 0 & 3 & b-2a \end{array}\right)$$

It follows that there is a unique solution for c_1, c_2 and so

S spans $\mathbb{R}^2$

We therefore conclude that S is a basis for $\mathbb{R}^2$.