

A Simple Gaussian Location Model

A

$$\begin{aligned} p(\theta) &= \int p(\theta|\omega)p(\omega)d\omega \\ &\propto \int \omega^{\frac{d+1}{2}-1} e^{-\omega \frac{\kappa(\theta-\mu)^2}{2}} e^{-\omega \frac{\eta}{2}} d\omega \\ &= \int \omega^{\frac{d+1}{2}-1} e^{-\omega \frac{\kappa(\theta-\mu)^2 + \eta}{2}} d\omega \\ &= \int \omega^{\alpha-1} e^{-\omega\beta} \end{aligned}$$

$$\begin{aligned} \text{Where } \alpha &= \frac{d+1}{2} \text{ and } \beta = \frac{\kappa(\theta-\mu)^2 + \eta}{2} \\ \implies &= \Gamma(\alpha)\beta^{-\alpha} \\ &\propto \left(\frac{\kappa(\theta-\mu)^2 + \eta}{2}\right)^{-\frac{d+1}{2}} \\ &= \left(\frac{\kappa(\theta-\mu)^2}{2} + \frac{\eta}{2}\right)^{-\frac{d+1}{2}} \\ &= \left(1 + \frac{1}{d} \frac{d\kappa(\theta-\mu)^2}{\eta}\right)^{-\frac{d+1}{2}} \\ p(\theta) &= \left(1 + \frac{1}{\nu} \frac{(\theta-m)^2}{s^2}\right)^{-\frac{\nu+1}{2}} \end{aligned}$$

With:

$$\begin{aligned} \nu &= d \\ m &= \mu \\ s^2 &= \frac{\eta}{d\kappa} \end{aligned}$$

B

$$\begin{aligned}
p(\theta, \omega | \mathbf{y}) &\propto p(\mathbf{y} | \theta, \omega) p(\theta, \omega) \\
p(\mathbf{y} | \theta, \omega) &= p(\mathbf{y} | \theta, \frac{1}{\sigma^2}) \\
p(\mathbf{y} | \theta, \sigma^2) &= \prod_{i=1}^n p(y_i | \theta, \sigma^2) \\
&= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\theta - y_i)^2}{2\sigma^2}} \\
&\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (\theta - y_i)^2} \\
&= (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} (\sum_{i=1}^n (\theta - \bar{y})^2 + \sum_{i=1}^n (\bar{y} - y_i)^2)} \\
p(\mathbf{y} | \theta, \sigma^2) &\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} (n(\theta - \bar{y})^2 + S_y)} \\
\text{Where } S_y &= \sum_{i=1}^n (y_i - \bar{y})^2 \\
\implies p(\mathbf{y} | \theta, \omega) &\propto \omega^{n/2} e^{-\frac{\omega}{2} (n(\theta - \bar{y})^2 + S_y)} \\
p(\theta, \omega | \mathbf{y}) &\propto p(\mathbf{y} | \theta, \omega) p(\theta, \omega) \\
&\propto \omega^{\frac{n}{2}} e^{-\frac{\omega}{2} (n(\theta - \bar{y})^2 + S_y)} \omega^{\frac{d+1}{2}-1} e^{-\frac{\omega}{2} \kappa (\theta - \mu)^2} e^{-\frac{\omega}{2} \eta} \\
&= \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2} (n(\theta - \bar{y})^2 + \kappa (\theta - \mu)^2)} e^{-\frac{\omega}{2} (S_y + \eta)}
\end{aligned}$$

Manipulating the exponent on the first exponential term:

$$\begin{aligned}
-\frac{\omega}{2} (n(\theta - \bar{y})^2 + \kappa (\theta - \mu)^2) &= -\frac{\omega}{2} (n\theta^2 - 2n\theta\bar{y} + n\bar{y}^2 + \kappa\theta^2 - 2\kappa\theta\mu + \kappa\mu^2) \\
&= -\frac{\omega}{2} (\theta^2(n + \kappa) - 2\theta(n\bar{y} + \kappa\mu) + n\bar{y}^2 + \kappa\mu^2) \\
&= -\frac{\omega}{2} ((n + \kappa)[\theta^2 - 2\theta \frac{(n\bar{y} + \kappa\mu)}{(n + \kappa)} + (\frac{n\bar{y} + \kappa\mu}{(n + \kappa)})^2] - (n + \kappa)(\frac{n\bar{y} + \kappa\mu}{(n + \kappa)})^2 + n\bar{y}^2 + \kappa\mu^2) \\
&= -\frac{\omega}{2} ((n + \kappa)(\theta - \frac{n\bar{y} + \kappa\mu}{n + \kappa})^2 - (n + \kappa)(\frac{n\bar{y} + \kappa\mu}{n + \kappa})^2 + n\bar{y}^2 + \kappa\mu^2)
\end{aligned}$$

We will now take these last three terms out of this exponential and group it with the following exponential term:

$$\begin{aligned}
p(\theta, \omega | \mathbf{y}) &\propto \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2} (n(\theta - \bar{y})^2 + \kappa (\theta - \mu)^2)} e^{-\frac{\omega}{2} (S_y + \eta)} \\
&= \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2} ((n + \kappa)(\theta - \frac{n\bar{y} + \kappa\mu}{n + \kappa})^2)} e^{-\frac{\omega}{2} (S_y + \eta - (n + \kappa)(\frac{n\bar{y} + \kappa\mu}{n + \kappa})^2 + n\bar{y}^2 + \kappa\mu^2)} \\
&= \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2} ((n + \kappa)(\theta - \frac{n\bar{y} + \kappa\mu}{n + \kappa})^2)} e^{-\frac{\omega}{2} (S_y + \eta + \frac{n\kappa}{n + \kappa} (\mu - \bar{y})^2)}
\end{aligned}$$

We can now see this is in the form of:

$$p(\theta, \omega | \mathbf{y}) \propto \omega^{\frac{d^*+1}{2}-1} \exp \left\{ -\omega \cdot \frac{\kappa^* (\theta - \mu^*)^2}{2} \right\} \cdot \exp \left\{ -\omega \cdot \frac{\eta^*}{2} \right\}$$

With

$$\begin{aligned}
d^* &= n + d \\
\kappa^* &= n + \kappa \\
\mu^* &= \frac{n\bar{y} + \kappa\mu}{n + \kappa} \\
\eta^* &= \eta + S_y + \frac{n\kappa}{n + \kappa}(\mu - \bar{y})^2
\end{aligned}$$

C

Since the posterior follows a Normal-Gamma model, the conditional posterior for θ should follow the same distribution as its prior but with the new parameters, thus:

$$\begin{aligned}
p(\theta|\omega, y) &\sim N(\mu^*, (\omega\kappa^*)^{-1}) \\
&\sim N\left(\frac{n\bar{y} + \kappa\mu}{n + \kappa}, \frac{\sigma^2}{n + \kappa}\right)
\end{aligned}$$

D

$$\begin{aligned}
p(\omega|y) &= \int p(\theta, \omega|y) d\theta \\
&= \int \omega^{\frac{d^*+1}{2}-1} \exp\left\{-\omega \cdot \frac{\kappa^*(\theta - \mu^*)^2}{2}\right\} \cdot \exp\left\{-\omega \cdot \frac{\eta^*}{2}\right\} d\theta \\
&= \omega^{\frac{d^*+1}{2}-1} e^{-\omega \frac{\eta^*}{2}} \int e^{-\omega \frac{\kappa^*(\theta - \mu^*)^2}{2}} d\theta \\
&= \omega^{\frac{d^*+1}{2}-1} e^{-\omega \frac{\eta^*}{2}} \int N(\mu^*, (\omega\kappa^*)^{-1}) d\theta \\
&\propto \omega^{\frac{d^*+1}{2}-1} e^{-\omega \frac{\eta^*}{2}} (\omega\kappa^*)^{-\frac{1}{2}} \\
p(\omega|y) &\sim Ga\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)
\end{aligned}$$

E

Since the posterior follows a Normal-Gamma model, the marginal posterior for θ should follow the same distribution as its prior but with the new parameters, thus:

$$\begin{aligned}
p(\theta|y) &\sim t_{\nu^*}(m^*, s^{2*}) \\
&= \left(1 + \frac{1}{\nu^*} \frac{(\theta - m^*)^2}{s^{2*}}\right)^{-\frac{\nu^*+1}{2}}
\end{aligned}$$

Where:

$$\begin{aligned}
\nu^* &= d^* \\
&= N + d \\
m^* &= \mu^* \\
&= \frac{n\bar{y} + \kappa\mu}{n + \kappa} \\
s^{2*} &= \frac{\eta^*}{d^* \kappa^*} \\
&= \frac{\eta + S_y + \kappa\mu^2 + n\bar{y}^2 - \frac{(n\bar{y} + \kappa\mu)^2}{n + \kappa}}{(n + d)(n + \kappa)}
\end{aligned}$$

F

False. In the limit of our parameters κ, η , and d approaching zero, our prior distributions are no longer valid probability distributions, as $p(\theta)$ would become 1 for all values of θ , and therefore would not integrate to a finite value, and $p(\omega)$ would become zero.

G

True. Even though the prior distributions would no longer hold in the limit, the posterior distribution would still be a valid probability distribution, as the parameters for the posterior κ^*, η^* , and d^* would be affected but still be nonzero values, as they are also influenced by the likelihood function.

H

True. As our prior parameters approach zero, our interval turns into the following:

$$\begin{aligned}
\theta &= m \pm t^* s \\
&= \frac{n\bar{y} + \kappa\mu}{n + \kappa} \pm t^* \sqrt{\frac{\eta + S_y + \kappa\mu^2 + n\bar{y}^2 - \frac{(n\bar{y} + \kappa\mu)^2}{n + \kappa}}{(n + d)(n + \kappa)}} \\
&= \frac{n\bar{y}}{n} \pm t^* \sqrt{\frac{S_y + n\bar{y}^2 - \frac{(n\bar{y})^2}{n}}{n^2}} \\
&= \bar{y} \pm t^* \sqrt{\frac{S_y}{n^2}} \\
&= \bar{y} \pm t^* s
\end{aligned}$$

Which is the exact formula for a confidence interval in the frequentist setting.