A Simple Gaussian Location Model

A

$$
\begin{aligned}
p(\theta) &= \int p(\theta|\omega)p(\omega)d\omega \\
&\propto \int \omega^{\frac{d+1}{2}-1} e^{-\omega\frac{\kappa(\theta-\mu)^2}{2}} e^{-\omega\frac{\eta}{2}} d\omega \\
&= \int \omega^{\frac{d+1}{2}-1} e^{-\omega\frac{\kappa(\theta-\mu)^2+\eta}{2}} d\omega \\
&= \int \omega^{\alpha-1} e^{-\omega\beta} \\
\text{Where } \alpha = \frac{d+1}{2} &\text{ and } \beta = \frac{\kappa(\theta-\mu)^2+\eta}{2} \\
\implies &= \Gamma(\alpha)\beta^{-\alpha} \\
&\propto (\frac{\kappa(\theta-\mu)^2+\eta}{2})^{-\frac{d+1}{2}} \\
&= (\frac{\kappa(\theta-\mu)^2}{2} + \frac{\eta}{2})^{-\frac{d+1}{2}} \\
&= (1 + \frac{1}{d}\frac{d\kappa(\theta-\mu)^2}{\eta})^{-\frac{d+1}{2}} \\
p(\theta) &= (1 + \frac{1}{\nu}\frac{(\theta-m)^2}{s^2})^{-\frac{\nu+1}{2}}
\end{aligned}
$$

With:

$$
\begin{aligned}
\nu &= d \\
m &= \mu \\
s^2 &= \frac{\eta}{d\kappa}
\end{aligned}
$$

B

$$
\begin{aligned}
p(\theta,\omega|y) &\propto p(y|\theta,\omega)p(\theta,\omega)\\
p(y|\theta,\omega) &= p(y|\theta,\frac{1}{\sigma^2})\\
p(y|\theta,\sigma^2) &= \prod_{i=1}^{n} p(y_i|\theta,\sigma^2)\\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(\theta-y_i)^2}{2\sigma^2}}\\
&\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\theta-y_i)^2}\\
&= (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}(\sum_{i=1}^{n}(\theta-\bar{y})^2+\sum_{i=1}^{n}(\bar{y}-y_i)^2)}\\
p(y|\theta,\sigma^2) &\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}(n(\theta-\bar{y})^2+S_y)}\\
\text{Where } S_y &= \sum_{i=1}^{n}(y_i-\bar{y})^2\\
\implies p(y|\theta,\omega) &\propto \omega^{n/2} e^{-\frac{\omega}{2}(n(\theta-\bar{y})^2+S_y)}\\
p(\theta,\omega|y) &\propto p(y|\theta,\omega)p(\theta,\omega)\\
&\propto \omega^{\frac{n}{2}} e^{-\frac{\omega}{2}(n(\theta-\bar{y})^2+S_y)} \omega^{\frac{d+1}{2}-1} e^{-\frac{\omega}{2}\kappa(\theta-\mu)^2} e^{-\frac{\omega}{2}\eta}\\
&= \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2}(n(\theta-\bar{y})^2+\kappa(\theta-\mu)^2)} e^{-\frac{\omega}{2}(S_y+\eta)}
\end{aligned}
$$

Manipulating the exponent on the first exponential term:

$$
\begin{aligned}
-\frac{\omega}{2}(n(\theta-\bar{y})^2+\kappa(\theta-\mu)^2) &= -\frac{\omega}{2}(n\theta^2-2n\theta\bar{y}+n\bar{y}^2+\kappa\theta^2-2\kappa\theta\mu+\kappa\mu^2)\\
&= -\frac{\omega}{2}(\theta^2(n+\kappa)-2\theta(n\bar{y}+\kappa\mu)+n\bar{y}^2+\kappa\mu^2)\\
&= -\frac{\omega}{2}((n+\kappa)[\theta^2-2\theta\frac{(n\bar{y}+\kappa\mu)}{(n+\kappa)}+(\frac{(n\bar{y}+\kappa\mu)}{(n+\kappa)})^2]-(n+\kappa)(\frac{(n\bar{y}+\kappa\mu)}{(n+\kappa)})^2+n\bar{y}^2+\kappa\mu^2)\\
&= -\frac{\omega}{2}((n+\kappa)(\theta-\frac{n\bar{y}+\kappa\mu}{n+\kappa})^2-(n+\kappa)(\frac{n\bar{y}+\kappa\mu}{n+\kappa})^2+n\bar{y}^2+\kappa\mu^2)
\end{aligned}
$$

We will now take these last three terms out of this exponential and group it with the following exponential term:

$$
\begin{aligned}
p(\theta,\omega|y) &\propto \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2}(n(\theta-\bar{y})^2+\kappa(\theta-\mu)^2)} e^{-\frac{\omega}{2}(S_y+\eta)}\\
&= \omega^{\frac{n+d+1}{2}-1} e^{-\frac{\omega}{2}((n+\kappa)(\theta-\frac{n\bar{y}+\kappa\mu}{n+\kappa})^2} e^{-\frac{\omega}{2}(S_y+\eta-(n+\kappa)(\frac{n\bar{y}+\kappa\mu}{n+\kappa})^2+n\bar{y}^2+\kappa\mu^2)}
\end{aligned}
$$

We can now see this is in the form of:

$$
p(\theta,\omega|y) \propto \omega^{\frac{d^\star+1}{2}-1} \exp\left\{-\omega\cdot\frac{\kappa^\star(\theta-\mu^\star)^2}{2}\right\}\cdot\exp\left\{-\omega\cdot\frac{\eta^\star}{2}\right\}
$$

With

$$
\begin{aligned}
d^\star &= n + d \\
\kappa^\star &= n + \kappa \\
\mu^\star &= \frac{n\bar{y} + \kappa\mu}{n + \kappa} \\
\eta^\star &= \eta + S_y + \kappa\mu^2 + n\bar{y}^2 - \frac{(n\bar{y} + \kappa\mu)^2}{n + \kappa}
\end{aligned}
$$

C

Since the posterior follows a Normal-Gamma model, the conditional posterior for θ should follow the same distribution as its prior but with the new parameters, thus:

$$
\begin{aligned}
p(\theta|\omega, y) &\sim N(\mu^\star, (\omega\kappa^\star)^{-1}) \\
&\sim N(\frac{n\bar{y} + \kappa\mu}{n + \kappa}, \frac{\sigma^2}{n + \kappa})
\end{aligned}
$$

D

$$
\begin{aligned}
p(\omega|y) &= \int p(\theta, \omega|y) d\theta \\
&= \int \omega^{\frac{d^\star+1}{2}-1} \exp\left\{-\omega \cdot \frac{\kappa^\star(\theta - \mu^\star)^2}{2}\right\} \cdot \exp\left\{-\omega \cdot \frac{\eta^\star}{2}\right\} d\theta \\
&= \omega^{\frac{d^\star+1}{2}-1} e^{-\omega\frac{\eta^\star}{2}} \int e^{-\omega\frac{\kappa^\star(\theta-\mu^\star)^2}{2}} d\theta \\
&= \omega^{\frac{d^\star+1}{2}-1} e^{-\omega\frac{\eta^\star}{2}} \int N(\mu^\star, (\omega\kappa^\star)^{-1}) d\theta \\
&\propto \omega^{\frac{d^\star+1}{2}-1} e^{-\omega\frac{\eta^\star}{2}} (\omega\kappa^\star)^{\frac{1}{2}} \\
p(\omega|y) &\sim Ga(\frac{d^\star}{2} + 1, \frac{\eta^\star}{2})
\end{aligned}
$$

E

Since the posterior follows a Normal-Gamma model, the marginal posterior for θ should follow the same distribution as its prior but with the new parameters, thus:

$$
\begin{aligned}
p(\theta|y) &\sim t_{v^\star}(m^\star, s^{2\star}) \\
&= (1 + \frac{1}{\nu^\star}\frac{(\theta - m^\star)^2}{s^{2\star}})^{-\frac{\nu^\star+1}{2}}
\end{aligned}
$$

Where:

$$\begin{aligned}
\nu^\star &= d^\star \\
&= N + d \\
m^\star &= \mu^\star \\
&= \frac{n\bar{y} + \kappa\mu}{n + \kappa} \\
s^{2\star} &= \frac{\eta^\star}{d^\star \kappa^\star} \\
&= \frac{\eta + S_y + \kappa\mu^2 + n\bar{y}^2 - \frac{(n\bar{y}+\kappa\mu)^2}{n+\kappa}}{(n+d)(n+\kappa)}
\end{aligned}$$

F

False. In the limit of our parameters κ, η, and d approaching zero, our prior distributions are no longer valid probability distributions, as $p(\theta)$ would become 1 for all values of θ, and therefore would not integrate to a finite value, and $p(\omega)$ would become zero.

G

True. Even though the prior distributions would no longer hold in the limit, the posterior distribution would still be a valid probability distribution, as the parameters for the posterior $\kappa^\star, \eta^\star$, and $d^\star$ would be affected but still be nonzero values, as they are also influenced by the likelihood function.

H

True. As our prior parameters approach zero, our interval turns into the following:

$$\begin{aligned}
\theta &= m \pm t^\star s \\
&= \frac{n\bar{y} + \kappa\mu}{n + \kappa} \pm t^\star \sqrt{\frac{\eta + S_y + \kappa\mu^2 + n\bar{y}^2 - \frac{(n\bar{y}+\kappa\mu)^2}{n+\kappa}}{(n+d)(n+\kappa)}} \\
&= \frac{n\bar{y}}{n} \pm t^\star \sqrt{\frac{S_y + n\bar{y}^2 - \frac{(n\bar{y})^2}{n}}{n^2}} \\
&= \bar{y} \pm t^\star \sqrt{\frac{S_y}{n^2}} \\
&= \bar{y} \pm t^\star s
\end{aligned}$$

Which is the exact formula for a confidence interval in the frequentist setting.