

SDS 383D Sol 1

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A

Sol:

$$\begin{aligned} p(w|X_1, \dots, X_N) &\propto p(w)p(X_1, \dots, X_N|w) \\ p(X_1, \dots, X_N|w) &= \prod_{i=1}^N w^{X_i}(1-w)^{1-X_i} \\ p(w) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1} \\ \implies p(w|X_1, \dots, X_N) &\propto w^{a-1}(1-w)^{b-1} \prod_{i=1}^N w^{X_i}(1-w)^{1-X_i} \\ &= w^{a-1}(1-w)^{b-1} w^{\sum_{i=1}^N X_i} (1-w)^{n-\sum_{i=1}^N X_i} \\ &= w^{a+n\bar{X}-1} (1-w)^{b+n(1-\bar{X})-1} \\ \implies w|X_{1:N} &\sim \text{Beta}(a+n\bar{X}, b+n(1-\bar{X})) \end{aligned}$$

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B

Sol:

$$\begin{aligned}
y_1 &= \frac{x_1}{x_1 + x_2} \\
y_2 &= x_1 + x_2 \\
y_1 &= \frac{x_1}{y_2} \\
\implies x_1 &= y_1 y_2 \\
y_2 &= y_1 y_2 + x_2 \\
\implies x_2 &= y_2(1 - y_1) \\
f_{Y_1, Y_2}(y_1, y_2) &= f_{X_1, X_2}(y_1 y_2, y_2(1 - y_1)) |J| \\
|J| &= \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} - \frac{\partial x_2}{\partial y_1} \frac{\partial x_1}{\partial y_2} \\
&= y_2(1 - y_1) + y_2 y_1 = y_2 \\
\implies f_{Y_1, Y_2}(y_1, y_2) &= f_{X_1, X_2}(y_1 y_2, y_2(1 - y_1)) |J| \\
&= f_{X_1}(y_1 y_2) f_{X_2}(y_2(1 - y_1)) y_2 \\
&= \frac{(y_1 y_2)^{a_1 - 1}}{\Gamma(a_1)} e^{-y_1 y_2} \frac{(y_2(1 - y_1))^{a_2 - 1}}{\Gamma(a_2)} e^{-y_2(1 - y_1)} y_2 \\
f_{Y_1, Y_2}(y_1, y_2) &= \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1} y_2^{a_1 + a_2 - 1}}{\Gamma(a_1) \Gamma(a_2)} e^{-y_2} \\
p(y_1) &= \int f_{Y_1, Y_2}(y_1, y_2) dy_2 \\
&= \int \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1} y_2^{a_1 + a_2 - 1}}{\Gamma(a_1) \Gamma(a_2)} e^{-y_2} dy_2 \\
&= \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1} \Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)} \int \frac{y_2^{a_1 + a_2 - 1}}{\Gamma(a_1 + a_2)} e^{-y_2} dy_2 \\
&= \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1}}{\text{Beta}(a_1, a_2)} \int Ga(a_1 + a_2, 1) dy_2 \\
p(y_1) &= \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1}}{\text{Beta}(a_1, a_2)} \\
\implies Y_1 &\sim \text{Beta}(p = y_1, a_1, a_2) \\
p(y_2) &= \int f_{Y_1, Y_2}(y_1, y_2) dy_1 \\
&= \int \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1} y_2^{a_1 + a_2 - 1}}{\Gamma(a_1) \Gamma(a_2)} e^{-y_2} dy_1 \\
&= \frac{y_2^{a_1 + a_2 - 1} e^{-y_2}}{\Gamma(a_1 + a_2 - 1)} \int \frac{y_1^{a_1 - 1} (1 - y_1)^{a_2 - 1}}{\text{Beta}(a_1, a_2)} dy_1 \\
p(y_2) &= \frac{y_2^{a_1 + a_2 - 1} e^{-y_2}}{\Gamma(a_1 + a_2 - 1)} \\
\implies Y_2 &\sim Ga(a_1 + a_2, 1)
\end{aligned}$$

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C

Sol:

$$\begin{aligned}
\theta &\sim N(m, v) \\
x_i &\sim N(\theta, \sigma^2) \\
p(\theta|x_1, \dots, x_N) &\propto p(\theta)p(x_1, \dots, x_N|\theta) \\
&\propto e^{-\frac{(\theta-m)^2}{2v}} \prod_{i=1}^N e^{-\frac{(x_i-\theta)^2}{2\sigma^2}} \\
&= e^{-\frac{(\theta-m)^2}{2v} - \sum_{i=1}^N \frac{(x_i-\theta)^2}{2\sigma^2}} \\
&= e^{-\frac{1}{2}(\frac{(\theta-m)^2}{v} + n\frac{(\theta-\bar{x})^2}{\sigma^2} + \sum_{i=1}^N \frac{(x_i-\bar{x})^2}{\sigma^2})} \\
&\propto e^{-\frac{1}{2}(\frac{\theta^2-2m\theta}{v} + \frac{n\theta^2-2n\theta\bar{x}}{\sigma^2})} \\
p(\theta|x_1, \dots, x_N) &\propto e^{-\frac{(\frac{1}{v} + \frac{n}{\sigma^2})}{2}(\theta - (\frac{m}{v} + \frac{n\bar{x}}{\sigma^2})(\frac{1}{v} + \frac{n}{\sigma^2})^{-1})^2} \\
\theta|x_{1:N} &\sim N((\frac{m}{v} + \frac{n\bar{x}}{\sigma^2})(\frac{1}{v} + \frac{n}{\sigma^2})^{-1}, (\frac{1}{v} + \frac{n}{\sigma^2})^{-1})
\end{aligned}$$

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D

Sol:

$$\begin{aligned}
w &\sim Ga(a, b) \\
p(w|x_1, \dots, x_N) &\propto p(w)p(x_1, \dots, x_N|w) \\
&\propto w^{a-1}e^{-bw} \prod_{i=1}^N (w)^{1/2}e^{-\frac{w}{2}(x_i-\theta)^2} \\
&= w^{a-1}e^{-bw}w^{N/2}e^{-\frac{w}{2}\sum(x_i-\theta)^2} \\
p(w|x_1, \dots, x_N) &\propto w^{\frac{N}{2}+a-1}e^{-w(b+\frac{\sum(x_i-\theta)^2}{2})} \\
w|x_{1:N} &\sim Ga(\frac{N}{2} + a, b + \frac{\sum(x_i-\theta)^2}{2}) \\
p(\sigma^2|x_1, \dots, x_N) &\propto (\frac{1}{\sigma^2})^{\frac{N}{2}+a-1}e^{-(\frac{1}{\sigma^2})(b+\frac{\sum(x_i-\theta)^2}{2})} \\
\sigma^2|x_{1:N} &\sim IG(\frac{N}{2} + a, b + \frac{\sum(x_i-\theta)^2}{2})
\end{aligned}$$

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E

Sol:

$$x_i \sim N(\theta, \sigma_i^2)$$

$$\theta \sim N(m, v)$$

$$p(\theta|x_1, \dots, x_N) \propto p(\theta)p(x_{1:N}|\theta)$$

$$= \frac{1}{\sqrt{2\pi v}} e^{-\frac{(\theta-m)^2}{2v}} \prod_{i=1}^N \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(x_i-\theta)^2}{2\sigma_i^2}}$$

$$= \frac{1}{\sqrt{2\pi v}} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\theta-m)^2}{2v} - \sum_{i=1}^N \frac{(x_i-\theta)^2}{2\sigma_i^2}}$$

$$\propto e^{-\frac{1}{2}(\frac{\theta^2 - 2\theta m + m^2}{v} + \sum \frac{\theta^2 - 2\theta x_i + x_i^2}{\sigma_i^2})}$$

$$\propto e^{-\frac{1}{2}\theta^2(\frac{1}{v} + \sum \frac{1}{\sigma_i^2}) - 2\theta(\frac{m}{v} + \sum \frac{x_i}{\sigma_i^2})}$$

$$\propto e^{-\frac{1}{2(\frac{1}{v} + \sum \frac{1}{\sigma_i^2})^{-1}}(\theta - (\frac{m}{v} + \sum \frac{x_i}{\sigma_i^2})(\frac{1}{v} + \sum \frac{1}{\sigma_i^2})^{-1})^2}$$

$$\theta|x \sim N((\frac{m}{v} + \sum \frac{x_i}{\sigma_i^2})(\frac{1}{v} + \sum \frac{1}{\sigma_i^2})^{-1}, (\frac{1}{v} + \sum \frac{1}{\sigma_i^2})^{-1})$$

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F

Sol:

$$\begin{aligned}
p(x, w) &= p(x|w)p(w) \\
&= \frac{w^{1/2}}{\sqrt{2\pi}} e^{-\frac{(x-m)^2 w}{2}} \frac{(b/2)^{a/2}}{\Gamma(a/2)} w^{a/2-1} e^{-bw/2} \\
&= \frac{(b/2)^{a/2}}{\sqrt{2\pi}\Gamma(a/2)} w^{\frac{a+1}{2}-1} e^{\frac{w}{2}(b+(x-m)^2)} \\
p(x) &= \int p(x, w)dw \\
&= \int_0^\infty \frac{(b/2)^{a/2}}{\sqrt{2\pi}\Gamma(a/2)} w^{\frac{a+1}{2}-1} e^{\frac{w}{2}(b+(x-m)^2)} dw \\
&= \frac{(b/2)^{a/2}}{\sqrt{2\pi}\Gamma(a/2)} \int_0^\infty w^{\frac{a+1}{2}-1} e^{\frac{w}{2}(b+(x-m)^2)} dw \\
&= \frac{(b/2)^{a/2}}{\sqrt{2\pi}\Gamma(a/2)} \frac{\Gamma(\frac{a+1}{2})}{(\frac{b+(x-m)^2}{2})^{\frac{a+1}{2}}} \int_0^\infty \frac{(\frac{b+(x-m)^2}{2})^{\frac{a+1}{2}}}{\Gamma(\frac{a+1}{2})} w^{\frac{a+1}{2}-1} e^{\frac{w}{2}(b+(x-m)^2)} dw \\
&= \frac{(b/2)^{a/2}}{\sqrt{2\pi}\Gamma(a/2)} \frac{\Gamma(\frac{a+1}{2})}{(\frac{b+(x-m)^2}{2})^{\frac{a+1}{2}}} \int_0^\infty Ga(\frac{a+1}{2}, \frac{b+(x-m)^2}{2}) dw \\
&= \frac{(b/2)^{a/2}}{\sqrt{2\pi}\Gamma(a/2)} \frac{\Gamma(\frac{a+1}{2})}{(\frac{b+(x-m)^2}{2})^{\frac{a+1}{2}}} \\
&= \frac{(\frac{b}{2})^{\frac{a}{2}} \Gamma(\frac{a+1}{2})}{\sqrt{2\pi}\Gamma(\frac{a}{2})} (\frac{b+(x-m)^2}{2})^{-\frac{a+1}{2}} \\
&= \frac{(\frac{b}{2})^{\frac{a}{2}} \Gamma(\frac{a+1}{2})}{\sqrt{2\pi}\Gamma(\frac{a}{2})} (1 + \frac{(x-m)^2}{b})^{-\frac{a+1}{2}} (\frac{b}{2})^{-\frac{a+1}{2}} \\
&= \frac{\sqrt{a}}{\sqrt{a}} \frac{\Gamma(\frac{a+1}{2})}{\sqrt{b\pi}\Gamma(\frac{a}{2})} (\frac{a + \frac{(x-m)^2}{b}}{a})^{-\frac{a+1}{2}} \\
&= \frac{\Gamma(\frac{a+1}{2})}{\sqrt{a\pi}\sqrt{\frac{b}{a}}\Gamma(\frac{a}{2})} (\frac{a + \frac{(x-m)^2}{\sqrt{\frac{b}{a}}}}{a})^{-\frac{a+1}{2}}
\end{aligned}$$

Which is by definition, a Student t distribution with scale $\sqrt{\frac{b}{a}}$, mean m, and d = a degrees of freedom. ■