# 3-ADIC IDENTITIES ON $\sum_{k=0}^{n-1}\binom{2k}{k}$

PRESTON FU

**Problem 1** (**6625 [1990, 252]**, *Proposed by Nicholas Strauss, Pontificia Universidade Católica do Rio de Janeiro, Brazil, and Jeffrey Shallit, Dartmouth College*)**.**

If $k$ is a positive integer, let $3^{v_3(k)}$ be the highest power of 3 dividing $k$. Let $r(n) = \sum_{i=0}^{n-1}\binom{2i}{i}$ for all positive integers $n$. Prove that

(i) $v_3(r(n)) \geq 2v_3(n)$,
(ii) $v_3(r(n)) = v_3\left(\binom{2n}{n}\right) + 2v_3(n)$.

*Solution:* (*by Don Zagier, University of Maryland, College Park, and Max-Planck-Insitut für Mathematik, Bonn, Germany*) If we can prove (ii), (i) immediately follows since $v_3\left(\binom{2n}{n}\right) \geq 0$.

The problem statement can be rewritten as follows:

$$v_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right) = v_3\left(n^2\binom{2n}{n}\right)\ \forall\, n \in \mathbb{N}. \tag{1}$$

We provide a proof of (1) and of various other 3-adic identities related to it.

Let us set

$$f(n) = \frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}.$$

I claim that $f(n) \equiv -1 \pmod 3\ \forall\, n \in \mathbb{N}$, and a few calculations suggest the congruences

$$n \equiv m \pmod{3^j} \implies f(n) \equiv f(m) \pmod{3^{j+1}}.$$

This means that the function $f : \mathbb{N} \to \mathbb{Q} \subset \mathbb{Q}_3$ extends to a 3-adic continuous map $\mathbb{Z}_3 \to -1+3\mathbb{Z}_3$. The range studied by computer ($n \leq 2200$) lets one check these congruences for $j \leq 7 = \lfloor \log_3 2200 \rfloor$ and therefore to interpolate $f(n)$ with accuracy $O(3^8)$. In fact, Zagier interpolated values for negative integers and half-integers, calculating the following:

$$f(-1) = -1, f(-2) = -\frac{7}{4}, f(-3) = -4, \ldots, f\left(-\frac{1}{2}\right) = -4, f\left(-\frac{3}{2}\right) = -4, f\left(-\frac{5}{2}\right) = -\frac{196}{25}, \ldots$$

Below is a result that captures all of his experimental observations:

**Theorem 2.** *The function $f$ extends to a 3-adic analytic function from $\mathbb{Z}_3$ to $-1+3\mathbb{Z}_3$. For $n \in \mathbb{N}$, we have*

$$f(-n) = -\frac{(2n-1)!}{(n!)^2}\sum_{k=0}^{n-1}\frac{(k!)^2}{(k-1)!}, \tag{2}$$

*and for $n \in \mathbb{N} \cup \{0\}$ we have*

$$f\left(-n-\frac{1}{2}\right) = -\frac{2^{4n+2}}{(2n+1)^2\binom{2n}{n}}\sum_{k=0}^{n}2^{-4k}\binom{2k}{k}. \tag{3}$$

*Date*: October 5, 2021.

*Proof.* It can be checked that $f(n)$ satisfies the following recurrence relation:

$$(2n+1)(2n+2)f(n+1) = 1 + n^2 f(n)\,\forall\, n \in \mathbb{N}. \tag{4}$$

The left hand side is zero at $n = -1$ and $n = -\frac{1}{2}$, so we can plug in to find $f(-1) = -1$, $f\left(-\frac{1}{2}\right) = -4$. ((2) and (3) can be proven via induction on $n$ using (4), but we won't go into detail about that.) It remains to show the first statement.

Let $g(n) = 2nf(n)$; we show that $g$ extends to a 3-adic analytic function of $n$, then that $x \mid g(x)$. For $g$, (4) becomes

$$2(2n+1)g(n+1) = 2 + ng(n). \tag{5}$$

We can define rational numbers $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that

$$g(n) = \sum_{k=0}^{\infty} a_k \binom{n-1}{k}. \tag{6}$$

If we can show that $\lim_{k\to\infty} v_3(a_k) = \infty$, then (6) will converge 3-adically for all $n \in \mathbb{Z}_3$, and the desired result will follow. Substituting (6) into (5), we have

$$2 + \sum_{k=0}^{n-1}(k+1)a_k\binom{n}{k+1} = \sum_{k=0}^{n}\left(2(2k+1)\binom{n}{k} + 4(k+1)\binom{n}{k+1}\right)a_k.$$

Comparing coefficients of $\binom{n}{k}$ for each $k$, we get $2(2k+1)a_k = -3ka_{k-1}$, and thus $a_k = \dfrac{(-3)^k(k!)^2}{(2k+1)!}$ (this can be proven by induction). Indeed, the 3-adic valuation does grow to infinity with $k$, so (6) gives the analytic continuation of $g$.

**Lemma 3.** *The series* $\sum_{k=0}^{\infty}\frac{3^k(k!)^2}{(2k+1)!}$ *converges* 3*-adically to* 0*.*

Assuming the lemma to be true (we won't prove it here since it uses beta integrals), we see that

$$\begin{aligned} g(n) &= \sum_{k=0}^{n-1}(-3)^k\frac{k!}{(2k+1)!}(n-1)(n-2)\cdots(n-k) \\ &= \sum_{k=0}^{n-1}\frac{3^k(k!)^2}{(2k+1)!} - \frac{n}{2} \\ &\quad + \sum_{k=2}^{n-1}(-3)^k\frac{k!}{(2k+1)!}\left((n-1)(n-2)\cdots(n-k) - (-1)^k k!\right). \end{aligned} \tag{7}$$

By the lemma, the first term in (7) has valuation

$$v_3\left(\sum_{k=0}^{n-1}\frac{3^k(k!)^2}{(2k+1)!}\right) = v_3\left(\sum_{r=n}^{\infty}\frac{3^k(k!)^2}{(2k+1)!}\right) \geq 2\cdot\frac{n-2}{3} \geq v_3(n) + 1\,\forall\, n \geq 4,$$

since $v_3\left(\frac{3^k(k!)^2}{(2k+1)!}\right) \geq 2v_3(k!) \geq 2\cdot\frac{k-2}{3}$ for all $k$. Also,

$$n \mid (n-1)(n-2)\cdots(n-k) - (-1)^k k!$$

and

$$3 \mid \frac{(-3)^k k!}{(2k+1)!}\,\forall\, k \geq 2,$$

so we know by (7) that

$$g(n) = -\frac{n}{2} \pmod{3^{v_3(n)+1}}.$$

Thus, $f(n) = \frac{g(n)}{2n} \equiv -1 \pmod 3$, as desired. $\square$

Therefore, we know that $f(n) = \frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}$ is a 3-adic unit $\forall\, n \in \mathbb{N}$, which implies $v_3(f(n)) = 0$. Thus,

$$\begin{aligned} v_3\left(\frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}\right) &= v_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right) - v_3\left(n^2\binom{2n}{n}\right) = 0 \\ \implies v_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right) &= v_3\left(n^2\binom{2n}{n}\right) \\ \implies v_3(r(n)) &= v_3\left(\binom{2n}{n}\right) + 2v_3(n), \end{aligned}$$

as desired. $\square$

The calculations to $n = 2200$ suggested the further congruence

$$n \equiv m \equiv 0 \pmod{3^j} \implies f(n) \equiv f(m) \pmod{3^{2j+1}},$$

and with a bit of work with Taylor series, the following (a bit stronger than our lemma from above), is equivalent to the following statement:

**Conjecture 4.** The series

$$\sum_{k=0}^{\infty} \frac{3^k (k!)^2}{(2k+1)!}\sigma_2\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right)$$

converges 3-adically to 0, where $\sigma_2$ denotes the second elementary symmetric sum.

## References


[1] Nicholas Strauss, Jeffrey Shallit, and Don Zagier. *The American Mathematical Monthly, Vol. 99, No. 1.* Mathematical Association of America, `https://people.mpim-bonn.mpg.de/zagier/files/amm/99/fulltext.pdf`, January 1992.