3-ADIC IDENTITIES ON $\sum_{i=0}^{n-1} \binom{2i}{i}$

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Problem 1 (**6625 [1990, 252]**, *Proposed by Nicholas Strauss, Pontifícia Universidade Católica do Rio de Janeiro, Brazil, and Jeffrey Shallit, Dartmouth College*).

If k is a positive integer, let $3^{v_3(k)}$ be the highest power of 3 dividing k. Let $r(n) = \sum_{i=0}^{n-1} \binom{2i}{i}$ for all positive integers n. Prove that

    (i) $v_3(r(n)) \geq 2v_3(n)$,
    (ii) $v_3(r(n)) = v_3\left(\binom{2n}{n}\right) + 2v_3(n)$.

Solution: (*by Don Zagier, University of Maryland, College Park, and Max-Planck-Insitut für Mathematik, Bonn, Germany*) If we can prove (ii), (i) immediately follows since $v_3\left(\binom{2n}{n}\right) \geq 0$.

The problem statement can be rewritten as follows:

$$(1) \qquad v_3\left(\sum_{k=0}^{n-1} \binom{2k}{k}\right) = v_3\left(n^2 \binom{2n}{n}\right) \; \forall\, n \in \mathbb{N}.$$

We provide a proof of (1) and of various other 3-adic identities related to it.

Let us set

$$f(n) = \frac{\sum_{k=0}^{n-1} \binom{2k}{k}}{n^2 \binom{2n}{n}}.$$

I claim that $f(n) \equiv -1 \pmod 3 \; \forall\, n \in \mathbb{N}$, and a few calculations suggest the congruences

$$n \equiv m \pmod{3^j} \implies f(n) \equiv f(m) \pmod{3^{j+1}}.$$

This means that the function $f : \mathbb{N} \to \mathbb{Q} \subset \mathbb{Q}_3$ extends to a 3-adic continuous map $\mathbb{Z}_3 \to -1 + 3\mathbb{Z}_3$. The range studied by computer ($n \leq 2200$) lets one check these congruences for $j \leq 7 = \lfloor \log_3 2200 \rfloor$ and therefore to interpolate $f(n)$ with accuracy $O(3^8)$. In fact, Zagier interpolated values for negative integers and half-integers, calculating the following:

$$f(-1) = -1, f(-2) = -\frac{7}{4}, f(-3) = -4, \ldots, f\left(-\frac{1}{2}\right) = -4, f\left(-\frac{3}{2}\right) = -4, f\left(-\frac{5}{2}\right) = -\frac{196}{25}, \ldots$$

Below is a result that captures all of his experimental observations:

Theorem 2. *The function f extends to a 3-adic analytic function from $\mathbb{Z}_3$ to $-1 + 3\mathbb{Z}_3$. For $n \in \mathbb{N}$, we have*

$$(2) \qquad f(-n) = -\frac{(2n-1)!}{(n!)^2} \sum_{k=0}^{n-1} \frac{(k!)^2}{(k-1)!},$$

and for $n \in \mathbb{N} \cup \{0\}$ we have

$$(3) \qquad f\left(-n - \frac{1}{2}\right) = -\frac{2^{4n+2}}{(2n+1)^2 \binom{2n}{n}} \sum_{k=0}^{n} 2^{-4k} \binom{2k}{k}.$$

Proof. It can be checked that $f(n)$ satisfies the following recurrence relation:

$$(4) \qquad\qquad (2n+1)(2n+2)f(n+1) = 1 + n^2 f(n) \, \forall \, n \in \mathbb{N}.$$

The left hand side is zero at $n = -1$ and $n = -\frac{1}{2}$, so we can plug in to find $f(-1) = -1$, $f\left(-\frac{1}{2}\right) = -4$. ((2) and (3) can be proven via induction on n using (4), but we won't go into detail about that.) It remains to show the first statement.

Let $g(n) = 2nf(n)$; we show that g extends to a 3-adic analytic function of n, then that $x \mid g(x)$. For g, (4) becomes

$$(5) \qquad\qquad 2(2n+1)g(n+1) = 2 + ng(n).$$

We can define rational numbers $\{a_n\}_{n \in \mathbb{N} \cup \{0\}}$ such that

$$(6) \qquad\qquad g(n) = \sum_{k=0}^{\infty} a_k \binom{n-1}{k}.$$

If we can show that $\lim_{k \to \infty} v_3(a_k) = \infty$, then (6) will converge 3-adically for all $n \in \mathbb{Z}_3$, and the desired result will follow. Substituting (6) into (5), we have

$$2 + \sum_{k=0}^{n-1}(k+1)a_k\binom{n}{k+1} = \sum_{k=0}^{n}\left(2(2k+1)\binom{n}{k} + 4(k+1)\binom{n}{k+1}\right)a_k.$$

Comparing coefficients of $\binom{n}{k}$ for each k, we get $2(2k+1)a_k = -3ka_{k-1}$, and thus $a_k = \dfrac{(-3)^k(k!)^2}{(2k+1)!}$ (this can be proven by induction). Indeed, the 3-adic valuation does grow to infinity with k, so (6) gives the analytic continuation of g.

Lemma 3. *The series $\sum_{k=0}^{\infty} \frac{3^k(k!)^2}{(2k+1)!}$ converges 3-adically to 0.*

Assuming the lemma to be true (we won't prove it here since it uses beta integrals), we see that

$$g(n) = \sum_{k=0}^{n-1}(-3)^k \frac{k!}{(2k+1)!}(n-1)(n-2)\cdots(n-k)$$

$$= \sum_{k=0}^{n-1}\frac{3^k(k!)^2}{(2k+1)!} - \frac{n}{2}$$

$$(7) \qquad\qquad + \sum_{k=2}^{n-1}(-3)^k\frac{k!}{(2k+1)!}\left((n-1)(n-2)\cdots(n-k) - (-1)^k k!\right).$$

By the lemma, the first term in (7) has valuation

$$v_3\left(\sum_{k=0}^{n-1}\frac{3^k(k!)^2}{(2k+1)!}\right) = v_3\left(\sum_{r=n}^{\infty}\frac{3^k(k!)^2}{(2k+1)!}\right) \geq 2 \cdot \frac{n-2}{3} \geq v_3(n) + 1 \, \forall \, n \geq 4,$$

since $v_3\left(\frac{3^k(k!)^2}{(2k+1)!}\right) \geq 2v_3(k!) \geq 2 \cdot \frac{k-2}{3}$ for all k. Also,

$$n \mid (n-1)(n-2)\cdots(n-k) - (-1)^k k!$$

and

$$3 \mid \frac{(-3)^k k!}{(2k+1)!} \, \forall \, k \geq 2,$$

so we know by (7) that

$$g(n) = -\frac{n}{2} \quad (\text{mod } 3^{v_3(n)+1}).$$

Thus, $f(n) = \frac{g(n)}{2n} \equiv -1 \ (\text{mod } 3)$, as desired. $\qquad \square$

Therefore, we know that $f(n) = \frac{\sum_{k=0}^{n-1} \binom{2k}{k}}{n^2 \binom{2n}{n}}$ is a 3-adic unit $\forall\, n \in \mathbb{N}$, which implies $v_3(f(n)) = 0$. Thus,

$$v_3 \left(\frac{\sum_{k=0}^{n-1} \binom{2k}{k}}{n^2 \binom{2n}{n}} \right) = v_3 \left(\sum_{k=0}^{n-1} \binom{2k}{k} \right) - v_3 \left(n^2 \binom{2n}{n} \right) = 0$$

$$\implies v_3 \left(\sum_{k=0}^{n-1} \binom{2k}{k} \right) = v_3 \left(n^2 \binom{2n}{n} \right)$$

$$\implies v_3(r(n)) = v_3 \left(\binom{2n}{n} \right) + 2v_3(n),$$

as desired. $\qquad \square$

The calculations to $n = 2200$ suggested the further congruence

$$n \equiv m \equiv 0 \quad (\text{mod } 3^j) \implies f(n) \equiv f(m) \quad (\text{mod } 3^{2j+1}),$$

and with a bit of work with Taylor series, the following (a bit stronger than our lemma from above), is equivalent to the following statement:

Conjecture 4. The series

$$\sum_{k=0}^{\infty} \frac{3^k (k!)^2}{(2k+1)!} \sigma_2 \left(1, \frac{1}{2}, \ldots, \frac{1}{k} \right)$$

converges 3-adically to 0, where σ_2 denotes the second elementary symmetric sum.

References

[1] Nicholas Strauss, Jeffrey Shallit, and Don Zagier. *The American Mathematical Monthly, Vol. 99, No. 1.* Mathematical Association of America, `https://people.mpim-bonn.mpg.de/zagier/files/amm/99/fulltext.pdf`, January 1992.