

# Fast Computation for Generalized Dedekind Sums

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# Motivation: Classical Dedekind Sum

## Definition

Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ .

$$s(a, c) = \sum_{n=0}^{c-1} B_1\left(\frac{n}{c}\right) B_1\left(\frac{an}{c}\right)$$

where

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

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## Example

$$\begin{aligned} s(5, 3) &= \sum_{n=0}^2 B_1\left(\frac{n}{3}\right) B_1\left(\frac{5n}{3}\right) = \\ &B_1\left(\frac{0}{3}\right) B_1\left(\frac{0}{3}\right) + B_1\left(\frac{1}{3}\right) B_1\left(\frac{5}{3}\right) + B_1\left(\frac{2}{3}\right) B_1\left(\frac{10}{3}\right) = -\frac{1}{18} \end{aligned}$$

# Motivation: Computing Classical Dedekind Sum

## Remark

It takes  $\mathcal{O}(c)$  time to compute  $s(h, k)$  from definition.

# Motivation: Computing Classical Dedekind Sum

## Properties

$$s(a, c) = -s(c, a) + \frac{1}{12} \left( \frac{a}{c} + \frac{1}{ac} + \frac{c}{a} \right) - \frac{1}{4}$$

$$s(a, c) = s(a \bmod c, c)$$

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## Example

$$\begin{aligned} s(99, 10) &= s(9, 10) \\ &= -s(10, 9) + R_1 \\ &= -s(1, 9) + R_1 \\ &= s(9, 1) + R_1 + R_2 \\ &= s(0, 1) + R_1 + R_2 \\ &= R_1 + R_2 \end{aligned}$$

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$$\mathcal{O}(c) \quad \longrightarrow \quad \mathcal{O}(\log(c))$$



# Research Question

Given  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ , is there an algorithm to compute the generalized Dedekind sum of  $\gamma$  faster than  $O(c)$ ?

# Definitions: Matrix Groups

## Definition

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}; ad - bc = 1 \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

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## Remark

For

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle.$$

# Definitions: Dirichlet Characters

## Definition

A *Dirichlet character*  $\chi \pmod q$  is a function  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  with the following properties:

1.  $\chi(n + ql) = \chi(n) \forall n, l \in \mathbb{Z}$
2.  $\chi(n) = 0$  if and only if  $\gcd(n, q) \neq 1$
3.  $\chi(mn) = \chi(m)\chi(n) \forall m, n \in \mathbb{Z}$ .

# Generalized Dedekind Sum

## Definition

Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$  with primitive Dirichlet characters  $\chi_1, \chi_2$  and respective conductors  $q_1, q_2$ . Let  $q_1, q_2 > 1$  and  $\chi_1 \chi_2(-1) = 1$ , then

$$S_{\chi_1, \chi_2}(\gamma) = \sum_{j=1}^c \sum_{i=1}^{q_1} \left( \overline{\chi_2(j) \chi_1(i)} B_1 \left( \frac{j}{c} \right) B_1 \left( \frac{n}{q_1} + \frac{aj}{c} \right) \right).$$

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## Crossed Homomorphism Property

Let  $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$ . Then

$$S_{\chi_1, \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1, \chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1, \chi_2}(\gamma_2).$$

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$$S_{\chi_1, \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1, \chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1, \chi_2}(\gamma_2).$$

Note that  $\psi$  is trivial in  $\Gamma_1(N)$ , so  $S_{\chi_1, \chi_2}$  is a homomorphism from  $\Gamma_1(N)$  to  $\mathbb{C}$  (more succinctly  $S_{\chi_1, \chi_2} \in \text{Hom}(\Gamma_1(N), \mathbb{C})$ ).

# Intuition

Given  $\Gamma_1(N) \leq \mathrm{SL}_2(\mathbb{Z})$ . Let  $\mathrm{SL}_2(\mathbb{Z}) = \langle S_i \rangle$  and  $\Gamma_1(N) = \langle \gamma_i \rangle$ . Given  $\gamma \in \Gamma_1(N)$ , we want to find  $S_{\chi_1, \chi_2}(\gamma)$ .



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$$\begin{aligned}\gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k.\end{aligned}$$

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$$\begin{aligned}\gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k.\end{aligned}$$

So

$$S_{\chi_1, \chi_2}(\gamma) = S_{\chi_1, \chi_2}(\gamma_1) + S_{\chi_1, \chi_2}(\gamma_2) + \cdots + S_{\chi_1, \chi_2}(\gamma_k)$$

# General Group Theoretic Preliminaries

## Definition

We say  $\mathcal{T}$  is a *right transversal* of  $H$  in  $G$  if each right coset of  $H$  in  $G$  contains exactly one element of  $\mathcal{T}$ . Moreover,  $\mathcal{T}$  must contain the identity.

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## Definition

Given a right transversal  $\mathcal{T}$  of  $H$  in  $G$ , a *right coset representative function* for  $\mathcal{T}$  is a mapping:  $G \rightarrow \mathcal{T}$  via  $g \mapsto \bar{g}$ , where  $\bar{g}$  is the unique element in  $\mathcal{T}$  such that  $Hg = H\bar{g}$ .

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## Example

Let  $G = \mathbb{Z}$ ,  $H = 5\mathbb{Z}$ , and  $G/H = \{0 + 5\mathbb{Z}, 1 + 5\mathbb{Z}, 2 + 5\mathbb{Z}, 3 + 5\mathbb{Z}, 4 + 5\mathbb{Z}\}$ .

Let  $\mathcal{T} = \{0, 6, 2, 18, -1\}$ .

Since  $23 \in 3 + 5\mathbb{Z}$ ,  $\overline{23} = 18$ .

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## Definition

Given a right transversal of  $H$  in  $G$  and  $a, b \in G$ , we define  $U(a, b) = ab(\overline{ab})^{-1}$ .

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## Lemma

*Given a right transversal of  $H$  in  $G$  and  $a, b \in G$ , then  $U(a, b) \in H$ .*

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## Lemma (Schreier's Lemma)

*Let  $\mathcal{S}$  be a set which finitely generates  $G$ , and let  $\mathcal{T}$  be a right transversal of  $H$  in  $G$ . The set*

$$\{U(t, s) : t \in \mathcal{T}, s \in \mathcal{S}\}$$

*generates  $H$ .*

This set is commonly referred to as the *Schreier generators* of  $H$ .



# General Group Theoretic Preliminaries

## Theorem (Reidemeister Rewriting Process)

Given a right transversal of  $H$  in  $G$ , let  $G = \langle g_1, \dots, g_n \rangle$ . Let  $h = g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_r}^{\epsilon_r} \in H$  (where  $\epsilon_k = \pm 1$ ) be a word in the  $g_i$ . Define the mapping  $\tau$  of the word  $h$  by

$$\tau(h) = U(p_1, g_{q_1})^{\epsilon_1} U(p_2, g_{q_2})^{\epsilon_2} \cdots U(p_r, g_{q_r})^{\epsilon_r},$$

where

$$p_k = \begin{cases} \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_{k-1}}^{\epsilon_{k-1}}} & \text{if } \epsilon_k = 1 \\ \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_k}^{\epsilon_k}} & \text{if } \epsilon_k = -1. \end{cases}$$

Then  $\tau(h) = h$ , for all  $h \in H$ .

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Then  $\tau(h) = h$ , for all  $h \in H$ .

## Remark

The Reidemeister rewriting process expresses a word in the generators of  $G$  as a word in the Schreier generators of  $H$ .

# Example: Reidemeister Rewriting Process

Let  $G = \langle g_i \rangle$ , let  $h \in G$  and  $h = g_1 g_1 g_1 g_2^{-1}$ , then

$$\begin{aligned}\tau(h) &= U(\bar{1}, g_1) U(\overline{g_1}, g_1) U(\overline{g_1^2}, g_1) U(\overline{g_1^3 g_2^{-1}}, g_2)^{-1} \\ &= \bar{1} g_1 (\overline{1 g_1})^{-1} \cdot (1 g_1) g_1 (\overline{g_1 g_1})^{-1} \cdot (\overline{g_1 g_1}) g_1 (\overline{g_1 g_1 g_1})^{-1} \cdot (\overline{g_1 g_1 g_1}) g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\ &= \bar{1} g_1 (\overline{1 g_1})^{-1} \cdot (1 g_1) g_1 (\overline{g_1 g_1})^{-1} \cdot (\overline{g_1 g_1}) g_1 (\overline{g_1 g_1 g_1})^{-1} \cdot (\overline{g_1 g_1 g_1}) g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\ &= \bar{1} g_1 g_1 g_1 g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\ &= g_1 g_1 g_1 g_2^{-1}\end{aligned}$$

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However, this takes a long time!

# Problem

## Lemma

$$SL_2(\mathbb{Z}) = \langle S, T \rangle.$$

*More specifically, one can decompose any matrix  $M \in SL_2(\mathbb{Z})$  into the following form:*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

*Note that  $-1 = S^2$ . Furthermore, we can precisely describe  $a_k$  via the Euclidean algorithm.*

## Remark

Note that  $2r$  grows as  $\log(c)$  and  $|a_1| + |a_2| + \dots + |a_r| + r$  grows as  $c$ .

# General Group Theoretic Preliminaries

## Theorem (Modified Reidemeister Rewriting Process)

*Given a right transversal of  $H$  in  $G$ , let  $G = \langle g_1, \dots, g_n \rangle$ . Let  $h = g_{q_1}^{a_1} g_{q_2}^{a_2} \cdots g_{q_r}^{a_r} \in H$  (where  $a_i \in \mathbb{Z}_{\neq 0}$ ) be a word in powers of the  $g_i$ . Define the mapping  $\tau$  of the word  $h$  by*

$$\tau(h) = U(p_1, g_{q_1}^{a_1}) U(p_2, g_{q_2}^{a_2}) \cdots U(p_r, g_{q_r}^{a_r}),$$

*where*

$$p_k = \overline{g_{q_1}^{a_1} g_{q_2}^{a_2} \cdots g_{q_{k-1}}^{a_{k-1}}}.$$

*Then  $\tau(h) = h$ , for all  $h \in H$ .*

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*Then  $\tau(h) = h$ , for all  $h \in H$ .*

## Remark

The modified Reidemeister rewriting process expresses a word in the generators of  $G$  as a word in specific elements of  $H$ .

## Example: Modified Reidemeister Rewriting Process

Let  $G = \langle g_i \rangle$ , let  $h \in G$  and  $h = g_1^3 g_2^{-1}$ . Reidemeister Rewriting Process:

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Modified Reidemeister Rewriting Process:

$$\tau(h) = U(\bar{1}, g_1^3)U(\overline{g_1^3}, g_2^{-1})$$

### Remark

$H$  now has an infinite alphabet.

# Specific Group Theoretic Preliminaries

## Lemma

*Let  $a = qN + r$  for  $0 \leq r < N$ . Let  $M \in SL_2(\mathbb{Z})$ , then given a right transversal of  $\Gamma_1(N)$  in  $SL_2(\mathbb{Z})$ :*

$$U(\overline{M}, T^a) = U^q(\overline{M}, T^N)U(\overline{M}, T^r).$$

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$$S_{\chi_1, \chi_2}(U(\overline{M}, T^a)) = qS_{\chi_1, \chi_2}(U(\overline{M}, T^N)) + S_{\chi_1, \chi_2}(U(\overline{M}, T^r)).$$

# Algorithm

Let  $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$  with primitive Dirichlet characters  $\chi_1, \chi_2$  and respective conductors  $q_1, q_2$ . Let  $q_1, q_2 > 1$  and  $\chi_1 \chi_2(-1) = 1$ . We present an algorithm to find  $S_{\chi_1, \chi_2}(\gamma_0)$ .

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## Group Theoretic Precomputation

- Find a right transversal  $\mathcal{T}_{\Gamma_0}$  of  $\Gamma_1(N)$  in  $\Gamma_0(N)$ .
- Find a right transversal  $\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}$  of  $\Gamma_1(N)$  in  $\mathrm{SL}_2(\mathbb{Z})$ .
- Find the set  $\mathcal{U} = \{U(t, T^i) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 1 \leq i \leq N\} \cup \{U(t, S^k) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 0 \leq k \leq 2\}$ .

# Algorithm

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## Group Theoretic Precomputation

- ▶ Find a right transversal  $\mathcal{T}_{\Gamma_0}$  of  $\Gamma_1(N)$  in  $\Gamma_0(N)$ .
- ▶ Find a right transversal  $\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}$  of  $\Gamma_1(N)$  in  $\mathrm{SL}_2(\mathbb{Z})$ .
- ▶ Find the set  $\mathcal{U} = \{U(t, T^i) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 1 \leq i \leq N\} \cup \{U(t, S^k) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 0 \leq k \leq 2\}$ .

## Dedekind Sum Precomputation

- ▶ Compute the Dedekind sums  $S_{\chi_1, \chi_2}(\mathcal{T}_{\Gamma_0})$ .
- ▶ Compute the Dedekind sums  $S_{\chi_1, \chi_2}(\mathcal{U})$ .

# Algorithm

## The Main Computation

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For each exponent of  $T$ , we write  $a_i = q_i N + r_i$  with  $0 \leq r_i < N$ . Then

$$U(\overline{p_i}, T^{a_i}) = U^{q_i}(\overline{p_i}, T^N) U(\overline{p_i}, T^{r_i}).$$

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$$S_{\chi_1, \chi_2} (U(\overline{p_i}, T^a)) = q_i S_{\chi_1, \chi_2} (U(\overline{p_i}, T^N)) + S_{\chi_1, \chi_2} (U(\overline{p_i}, T^{r_i})).$$

# Example Computation

Fix  $\Gamma_0(9)$ . Let  $\chi_1 = \chi_2$  be the primitive character modulo 3 with conductors  $q_1 = q_2 = 3$ . We want to compute  $S_{\chi_1, \chi_2}(\gamma_0)$  where

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$$\begin{aligned} \tau(\gamma_1) = & U\left(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, T^1\right) U\left(\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, S\right) U\left(\overline{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}, T^{-2}\right) U\left(\overline{\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}}, S\right) \cdots \\ & \cdots U\left(\overline{\begin{pmatrix} -15 & -13 \\ -8 & -7 \end{pmatrix}}, T^{-11}\right) U\left(\overline{\begin{pmatrix} -15 & 152 \\ -8 & 81 \end{pmatrix}}, S\right) U\left(\overline{\begin{pmatrix} 152 & 15 \\ 81 & 8 \end{pmatrix}}, T^{-1}\right) U\left(\overline{\begin{pmatrix} 152 & -137 \\ 81 & -73 \end{pmatrix}}, -I\right) = \gamma_1. \end{aligned}$$

# Example Computation

We expand and simplify the  $T$  terms.

$$U\left(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, T^1\right) = U^0\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^1\right)$$

$$U\left(\overline{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}, T^{-2}\right) = U^{-1}\left(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^7\right)$$

$$\vdots$$

$$U\left(\overline{\begin{pmatrix} -15 & -13 \\ -8 & -7 \end{pmatrix}}, T^{-11}\right) = U^{-2}\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^7\right)$$

$$U\left(\overline{\begin{pmatrix} 152 & 15 \\ 81 & 8 \end{pmatrix}}, T^{-1}\right) = U^{-1}\left(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^8\right)$$



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And we simplify the  $S$  terms.

$$\begin{aligned}U\left(\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, S\right) &= U\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S\right) \\U\left(\overline{\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}}, S\right) &= U\left(\begin{pmatrix} 1 & 6 \\ 1 & 7 \end{pmatrix}, S\right) \\&\vdots \\U\left(\overline{\begin{pmatrix} -15 & 152 \\ -8 & 81 \end{pmatrix}}, S\right) &= U\left(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, S\right) \\U\left(\overline{\begin{pmatrix} 152 & -137 \\ 81 & -73 \end{pmatrix}}, -I\right) &= U\left(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, S^2\right).\end{aligned}$$

Using this we can express our desired Dedekind sum as a linear combination of precomputed Dedekind sums.

$$\begin{aligned}
S_{\chi_1, \chi_2}(\gamma_1) = & 0 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^1)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S)) \\
& -1 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^7)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 6 \\ 1 & 7 \end{pmatrix}, S)) \\
& \vdots \\
& -2 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^7)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, S)) \\
& -1 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^8)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, S^2)).
\end{aligned}$$

Now, using the precomputed Dedekind sums, we arrive at the final result

$$S_{\chi_1, \chi_2}(\gamma_0) = S_{\chi_1, \chi_2}(\gamma_1) + S_{\chi_1, \chi_2}(g) = 0.$$

## Theorem


*Given primitive Dirichlet characters  $\chi_1, \chi_2$  and respective conductors  $q_1, q_2 > 1$  such that  $\chi_1 \chi_2(-1) = 1$ . Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$ . For fixed  $q_1, q_2$ , the time complexity of finding  $S_{\chi_1, \chi_2}(\gamma)$  as a function of  $\gamma$  is  $O(\log(c))$ .*

# Experimental Results

Consider  $\Gamma_0(28)$ . Let  $\chi_1$  be the primitive Dirichlet character with conductor  $q_1 = 4$ , and let  $\chi_2$  be the primitive Dirichlet character with conductor  $q_2 = 7$  such that  $\chi_2(3) = \exp(2\pi i(5/6))$ . We let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $c = 28k$ ,  $0 < a < c$ , and  $\gcd(a, c) = 1$ . We choose  $b$  and  $d$  such that the exponent  $a_r$  is 0. We graph the logarithm of the average time it takes to compute  $S_{\chi_1, \chi_2}(\gamma)$  for all  $\gamma$  as a function of  $k$ .

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images/graph.jpg

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images/graph.jpg

The performance of the algorithm always exceeds that of the definition.

# Experimental Results

Now we present an example for a large matrix. Consider  $\Gamma_0(35)$ . Let  $\chi_1$  be the primitive Dirichlet character with conductor  $q_1 = 5$  such that  $\chi_1(2) = -i$ , and let  $\chi_2$  be the primitive Dirichlet character with conductor  $q_2 = 7$  such that  $\chi_2(3) = \exp(2\pi i(1/3))$ .

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$$\gamma = \begin{pmatrix} 46741638 & 43234369 \\ 43234205 & 39990117 \end{pmatrix}.$$



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
Computing  $S_{\chi_1, \chi_2}(\gamma)$  by definition takes  $5.531 * 10^4$  seconds (around 15 hours), whereas it takes  $5.128 * 10^{-2}$  seconds using our algorithm.

# Conclusion

Thank you for listening!

<https://arxiv.org/abs/2210.01172>

<https://github.com/prestontranbarger/NFDSFastComputation>

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