

Fast Computation for Generalized Dedekind Sums

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Motivation: Classical Dedekind Sum

Definition

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

$$s(a, c) = \sum_{n=0}^{c-1} B_1\left(\frac{n}{c}\right) B_1\left(\frac{an}{c}\right)$$

where

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ x - [x] - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

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Example

$$\begin{aligned} s(5, 3) &= \sum_{n=0}^2 B_1\left(\frac{n}{3}\right) B_1\left(\frac{5n}{3}\right) = \\ &B_1\left(\frac{0}{3}\right) B_1\left(\frac{0}{3}\right) + B_1\left(\frac{1}{3}\right) B_1\left(\frac{5}{3}\right) + B_1\left(\frac{2}{3}\right) B_1\left(\frac{10}{3}\right) = -\frac{1}{18} \end{aligned}$$

Motivation: Computing Classical Dedekind Sum

Remark

It takes $\mathcal{O}(c)$ time to compute $s(h, k)$ from definition.

Motivation: Computing Classical Dedekind Sum

Properties

$$s(a, c) = -s(c, a) + \frac{1}{12} \left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a} \right) - \frac{1}{4}$$

$$s(a, c) = s(a \bmod c, c)$$

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Example

$$\begin{aligned} s(99, 10) &= s(9, 10) \\ &= -s(10, 9) + R_1 \\ &= -s(1, 9) + R_1 \\ &= s(9, 1) + R_1 + R_2 \\ &= s(0, 1) + R_1 + R_2 \\ &= R_1 + R_2 \end{aligned}$$

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$$\mathcal{O}(c) \quad \longrightarrow \quad \mathcal{O}(\log(c))$$

Research Question

Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, is there an algorithm to compute the generalized Dedekind sum of γ faster than $O(c)$?

Definitions: Matrix Groups

Definition

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}; ad - bc = 1 \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

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Remark

For

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle.$$

Definitions: Dirichlet Characters

Definition

A *Dirichlet character* $\chi \pmod q$ is a function $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ with the following properties:

1. $\chi(n + ql) = \chi(n) \forall n, l \in \mathbb{Z}$
2. $\chi(n) = 0$ if and only if $\gcd(n, q) \neq 1$
3. $\chi(mn) = \chi(m)\chi(n) \forall m, n \in \mathbb{Z}$.

Generalized Dedekind Sum

Definition

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$ with primitive Dirichlet characters χ_1, χ_2 and respective conductors q_1, q_2 . Let $q_1, q_2 > 1$ and $\chi_1 \chi_2(-1) = 1$, then

$$S_{\chi_1, \chi_2}(\gamma) = \sum_{j=1}^c \sum_{i=1}^{q_1} \left(\overline{\chi_2(j) \chi_1(i)} B_1 \left(\frac{j}{c} \right) B_1 \left(\frac{n}{q_1} + \frac{aj}{c} \right) \right).$$

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Crossed Homomorphism Property

Let $\gamma_1, \gamma_2 \in \Gamma_0(q_1 q_2)$. Then

$$S_{\chi_1, \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1, \chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1, \chi_2}(\gamma_2).$$

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$$S_{\chi_1, \chi_2}(\gamma_1 \gamma_2) = S_{\chi_1, \chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1, \chi_2}(\gamma_2).$$

Note that ψ is trivial in $\Gamma_1(N)$, so S_{χ_1, χ_2} is a homomorphism from $\Gamma_1(N)$ to \mathbb{C} (more succinctly $S_{\chi_1, \chi_2} \in \text{Hom}(\Gamma_1(N), \mathbb{C})$).

Intuition

Given $\Gamma_1(N) \leq \mathrm{SL}_2(\mathbb{Z})$. Let $\mathrm{SL}_2(\mathbb{Z}) = \langle S_i \rangle$ and $\Gamma_1(N) = \langle \gamma_i \rangle$. Given $\gamma \in \Gamma_1(N)$, we want to find $S_{\chi_1, \chi_2}(\gamma)$.

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$$\begin{aligned}\gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k.\end{aligned}$$

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$$\begin{aligned}\gamma &= S_1 S_2 \cdots S_m \\ &= \gamma_1 \gamma_2 \cdots \gamma_k.\end{aligned}$$

So

$$S_{\chi_1, \chi_2}(\gamma) = S_{\chi_1, \chi_2}(\gamma_1) + S_{\chi_1, \chi_2}(\gamma_2) + \cdots + S_{\chi_1, \chi_2}(\gamma_k)$$

General Group Theoretic Preliminaries

Definition

We say \mathcal{T} is a *right transversal* of H in G if each right coset of H in G contains exactly one element of \mathcal{T} . Moreover, \mathcal{T} must contain the identity.

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Given a right transversal \mathcal{T} of H in G , a *right coset representative function* for \mathcal{T} is a mapping: $G \rightarrow \mathcal{T}$ via $g \mapsto \bar{g}$, where \bar{g} is the unique element in \mathcal{T} such that $Hg = H\bar{g}$.

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Example

Let $G = \mathbb{Z}$, $H = 5\mathbb{Z}$, and $G/H = \{0 + 5\mathbb{Z}, 1 + 5\mathbb{Z}, 2 + 5\mathbb{Z}, 3 + 5\mathbb{Z}, 4 + 5\mathbb{Z}\}$.

Let $\mathcal{T} = \{0, 6, 2, 18, -1\}$.

Since $23 \in 3 + 5\mathbb{Z}$, $\overline{23} = 18$.

Definition

Given a right transversal of H in G and $a, b \in G$, we define $U(a, b) = ab(\overline{ab})^{-1}$.

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Lemma (Schreier's Lemma)

Let \mathcal{S} be a set which finitely generates G , and let \mathcal{T} be a right transversal of H in G . The set

$$\{U(t, s) : t \in \mathcal{T}, s \in \mathcal{S}\}$$

generates H .

This set is commonly referred to as the *Schreier generators* of H .

General Group Theoretic Preliminaries

Theorem (Reidemeister Rewriting Process)

Given a right transversal of H in G , let $G = \langle g_1, \dots, g_n \rangle$. Let $h = g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_r}^{\epsilon_r} \in H$ (where $\epsilon_k = \pm 1$) be a word in the g_i . Define the mapping τ of the word h by

$$\tau(h) = U(p_1, g_{q_1})^{\epsilon_1} U(p_2, g_{q_2})^{\epsilon_2} \cdots U(p_r, g_{q_r})^{\epsilon_r},$$

where

$$p_k = \begin{cases} \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_{k-1}}^{\epsilon_{k-1}}} & \text{if } \epsilon_k = 1 \\ \overline{g_{q_1}^{\epsilon_1} g_{q_2}^{\epsilon_2} \cdots g_{q_k}^{\epsilon_k}} & \text{if } \epsilon_k = -1. \end{cases}$$

Then $\tau(h) = h$, for all $h \in H$.

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Then $\tau(h) = h$, for all $h \in H$.

Remark

The Reidemeister rewriting process expresses a word in the generators of G as a word in the Schreier generators of H .

Example: Reidemeister Rewriting Process

Let $G = \langle g_i \rangle$, let $h \in G$ and $h = g_1 g_1 g_1 g_2^{-1}$, then

$$\begin{aligned}\tau(h) &= U(\bar{1}, g_1) U(\overline{g_1}, g_1) U(\overline{g_1^2}, g_1) U(\overline{g_1^3 g_2^{-1}}, g_2)^{-1} \\ &= \bar{1} g_1 (\overline{1 g_1})^{-1} \cdot (1 g_1) g_1 (\overline{g_1 g_1})^{-1} \cdot (\overline{g_1 g_1}) g_1 (\overline{g_1 g_1 g_1})^{-1} \cdot (\overline{g_1 g_1 g_1}) g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\ &= \bar{1} g_1 (\overline{1 g_1})^{-1} \cdot (1 g_1) g_1 (\overline{g_1 g_1})^{-1} \cdot (\overline{g_1 g_1}) g_1 (\overline{g_1 g_1 g_1})^{-1} \cdot (\overline{g_1 g_1 g_1}) g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\ &= \bar{1} g_1 g_1 g_1 g_2^{-1} (\overline{g_1 g_1 g_1 g_2^{-1}})^{-1} \\ &= g_1 g_1 g_1 g_2^{-1}\end{aligned}$$

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However, this takes a long time!

Problem

Lemma

$$SL_2(\mathbb{Z}) = \langle S, T \rangle.$$

More specifically, one can decompose any matrix $M \in SL_2(\mathbb{Z})$ into the following form:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

Note that $-1 = S^2$. Furthermore, we can precisely describe a_k via the Euclidean algorithm.

Remark

Note that $2r$ grows as $\log(c)$ and $|a_1| + |a_2| + \dots + |a_r| + r$ grows as c .

General Group Theoretic Preliminaries

Theorem (Modified Reidemeister Rewriting Process)

Given a right transversal of H in G , let $G = \langle g_1, \dots, g_n \rangle$. Let $h = g_{q_1}^{a_1} g_{q_2}^{a_2} \cdots g_{q_r}^{a_r} \in H$ (where $a_i \in \mathbb{Z}_{\neq 0}$) be a word in powers of the g_i . Define the mapping τ of the word h by

$$\tau(h) = U(p_1, g_{q_1}^{a_1}) U(p_2, g_{q_2}^{a_2}) \cdots U(p_r, g_{q_r}^{a_r}),$$

where

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Then $\tau(h) = h$, for all $h \in H$.

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Then $\tau(h) = h$, for all $h \in H$.

Remark

The modified Reidemeister rewriting process expresses a word in the generators of G as a word in specific elements of H .

Example: Modified Reidemeister Rewriting Process

Let $G = \langle g_i \rangle$, let $h \in G$ and $h = g_1^3 g_2^{-1}$. Reidemeister Rewriting Process:

$$\tau(h) = U(\overline{1}, g_1)U(\overline{g_1}, g_1)U(\overline{g_1^2}, g_1)U(\overline{g_1^3 g_2^{-1}}, g_2)^{-1}$$

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Modified Reidemeister Rewriting Process:

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Remark

H now has an infinite alphabet.

Specific Group Theoretic Preliminaries

Lemma

Let $a = qN + r$ for $0 \leq r < N$. Let $M \in SL_2(\mathbb{Z})$, then given a right transversal of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$:

$$U(\overline{M}, T^a) = U^q(\overline{M}, T^N)U(\overline{M}, T^r).$$

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Lemma

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$$S_{\chi_1, \chi_2}(U(\overline{M}, T^a)) = qS_{\chi_1, \chi_2}(U(\overline{M}, T^N)) + S_{\chi_1, \chi_2}(U(\overline{M}, T^r)).$$

Algorithm

Let $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$ with primitive Dirichlet characters χ_1, χ_2 and respective conductors q_1, q_2 . Let $q_1, q_2 > 1$ and $\chi_1 \chi_2(-1) = 1$. We present an algorithm to find $S_{\chi_1, \chi_2}(\gamma_0)$.

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Group Theoretic Precomputation

- Find a right transversal \mathcal{T}_{Γ_0} of $\Gamma_1(N)$ in $\Gamma_0(N)$.
- Find a right transversal $\mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}$ of $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$.
- Find the set $\mathcal{U} = \{U(t, T^i) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 1 \leq i \leq N\} \cup \{U(t, S^k) : t \in \mathcal{T}_{\mathrm{SL}_2(\mathbb{Z})}, 0 \leq k \leq 2\}$.

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Dedekind Sum Precomputation

- ▶ Compute the Dedekind sums $S_{\chi_1, \chi_2}(\mathcal{T}_{\Gamma_0})$.
- ▶ Compute the Dedekind sums $S_{\chi_1, \chi_2}(\mathcal{U})$.

Algorithm

The Main Computation

$$\gamma_0 = \gamma_1 g.$$

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$$\begin{aligned} \tau(\gamma_1) &= U(\overline{p_1}, T^{a_1}) U(\overline{p_1 T^{a_1}}, S) U(\overline{p_2}, T^{a_2}) U(\overline{p_2 T^{a_2}}, S) \dots \\ &\dots U(\overline{p_r}, T^{a_r}) U(\overline{p_r T^{a_r}}, \pm I) = \gamma_1, \end{aligned}$$

where

$$p_k = T^{a_1} S T^{a_2} S \dots T^{a_{k-1}} S.$$

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where

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For each exponent of T , we write $a_i = q_i N + r_i$ with $0 \leq r_i < N$. Then

$$U(\overline{p_i}, T^{a_i}) = U^{q_i}(\overline{p_i}, T^N) U(\overline{p_i}, T^{r_i}).$$

Algorithm

The Main Computation

$$\gamma_0 = \gamma_1 g.$$

$$\gamma_1 = \pm T^{a_1} S T^{a_2} S \dots T^{a_{r-1}} S T^{a_r}.$$

$$\begin{aligned} \tau(\gamma_1) &= U(\overline{p_1}, T^{a_1}) U(\overline{p_1 T^{a_1}}, S) U(\overline{p_2}, T^{a_2}) U(\overline{p_2 T^{a_2}}, S) \dots \\ &\dots U(\overline{p_r}, T^{a_r}) U(\overline{p_r T^{a_r}}, \pm I) = \gamma_1, \end{aligned}$$

where

$$p_k = T^{a_1} S T^{a_2} S \dots T^{a_{k-1}} S.$$

For each exponent of T , we write $a_i = q_i N + r_i$ with $0 \leq r_i < N$. Then

$$U(\overline{p_i}, T^{a_i}) = U^{q_i}(\overline{p_i}, T^N) U(\overline{p_i}, T^{r_i}).$$

$$S_{\chi_1, \chi_2} (U(\overline{p_i}, T^a)) = q_i S_{\chi_1, \chi_2} (U(\overline{p_i}, T^N)) + S_{\chi_1, \chi_2} (U(\overline{p_i}, T^{r_i})).$$

Example Computation

Fix $\Gamma_0(9)$. Let $\chi_1 = \chi_2$ be the primitive character modulo 3 with conductors $q_1 = q_2 = 3$. We want to compute $S_{\chi_1, \chi_2}(\gamma_0)$ where

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$$\begin{aligned} \tau(\gamma_1) = & U\left(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, T^1\right) U\left(\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, S\right) U\left(\overline{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}, T^{-2}\right) U\left(\overline{\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}}, S\right) \cdots \\ & \cdots U\left(\overline{\begin{pmatrix} -15 & -13 \\ -8 & -7 \end{pmatrix}}, T^{-11}\right) U\left(\overline{\begin{pmatrix} -15 & 152 \\ -8 & 81 \end{pmatrix}}, S\right) U\left(\overline{\begin{pmatrix} 152 & 15 \\ 81 & 8 \end{pmatrix}}, T^{-1}\right) U\left(\overline{\begin{pmatrix} 152 & -137 \\ 81 & -73 \end{pmatrix}}, -I\right) = \gamma_1. \end{aligned}$$

Example Computation

We expand and simplify the T terms.

$$U\left(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, T^1\right) = U^0\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^1\right)$$

$$U\left(\overline{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}, T^{-2}\right) = U^{-1}\left(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^7\right)$$

$$\vdots$$

$$U\left(\overline{\begin{pmatrix} -15 & -13 \\ -8 & -7 \end{pmatrix}}, T^{-11}\right) = U^{-2}\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^7\right)$$

$$U\left(\overline{\begin{pmatrix} 152 & 15 \\ 81 & 8 \end{pmatrix}}, T^{-1}\right) = U^{-1}\left(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^9\right)U\left(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^8\right)$$

Example Computation

We expand and simplify the T terms.

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And we simplify the S terms.

$$\begin{aligned}U\left(\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, S\right) &= U\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S\right) \\U\left(\overline{\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}}, S\right) &= U\left(\begin{pmatrix} 1 & 6 \\ 1 & 7 \end{pmatrix}, S\right) \\&\vdots \\U\left(\overline{\begin{pmatrix} -15 & 152 \\ -8 & 81 \end{pmatrix}}, S\right) &= U\left(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, S\right) \\U\left(\overline{\begin{pmatrix} 152 & -137 \\ 81 & -73 \end{pmatrix}}, -I\right) &= U\left(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, S^2\right).\end{aligned}$$

Using this we can express our desired Dedekind sum as a linear combination of precomputed Dedekind sums.

$$\begin{aligned}
S_{\chi_1, \chi_2}(\gamma_1) = & 0 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T^1)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S)) \\
& -1 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, T^7)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 6 \\ 1 & 7 \end{pmatrix}, S)) \\
& \vdots \\
& -2 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, T^7)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}, S)) \\
& -1 \cdot S_{\chi_1, \chi_2}(U(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^9)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, T^8)) + S_{\chi_1, \chi_2}(U(\begin{pmatrix} 8 & 7 \\ 9 & 8 \end{pmatrix}, S^2)).
\end{aligned}$$

Now, using the precomputed Dedekind sums, we arrive at the final result

$$S_{\chi_1, \chi_2}(\gamma_0) = S_{\chi_1, \chi_2}(\gamma_1) + S_{\chi_1, \chi_2}(g) = 0.$$

Theorem

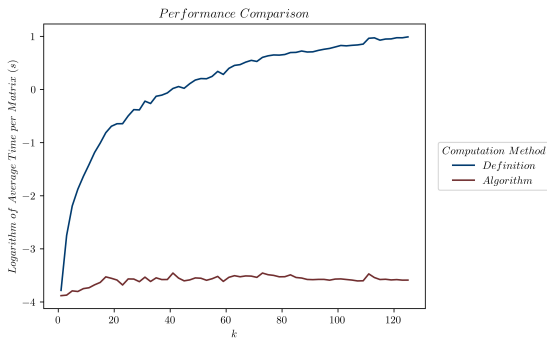
Given primitive Dirichlet characters χ_1, χ_2 and respective conductors $q_1, q_2 > 1$ such that $\chi_1 \chi_2(-1) = 1$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q_1 q_2)$. For fixed q_1, q_2 , the time complexity of finding $S_{\chi_1, \chi_2}(\gamma)$ as a function of γ is $O(\log(c))$.

Experimental Results

Consider $\Gamma_0(28)$. Let χ_1 be the primitive Dirichlet character with conductor $q_1 = 4$, and let χ_2 be the primitive Dirichlet character with conductor $q_2 = 7$ such that $\chi_2(3) = \exp(2\pi i(5/6))$. We let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c = 28k$, $0 < a < c$, and $\gcd(a, c) = 1$. We choose b and d such that the exponent a_r is 0. We graph the logarithm of the average time it takes to compute $S_{\chi_1, \chi_2}(\gamma)$ for all γ as a function of k .

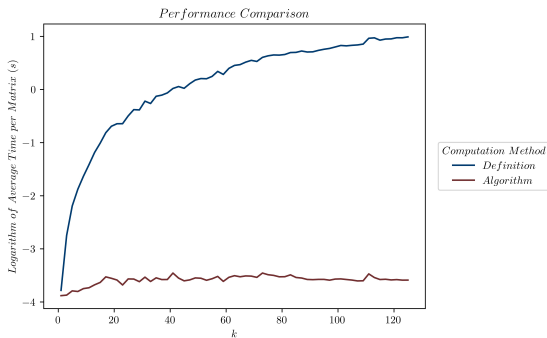
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The performance of the algorithm always exceeds that of the definition.

Experimental Results

Now we present an example for a large matrix. Consider $\Gamma_0(35)$. Let χ_1 be the primitive Dirichlet character with conductor $q_1 = 5$ such that $\chi_1(2) = -i$, and let χ_2 be the primitive Dirichlet character with conductor $q_2 = 7$ such that $\chi_2(3) = \exp(2\pi i(1/3))$.

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$$\gamma = \begin{pmatrix} 46741638 & 43234369 \\ 43234205 & 39990117 \end{pmatrix}.$$

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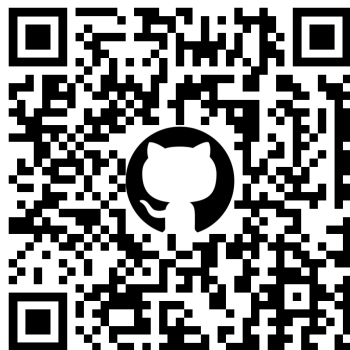
Computing $S_{\chi_1, \chi_2}(\gamma)$ by definition takes $5.531 * 10^4$ seconds (around 15 hours), whereas it takes $5.128 * 10^{-2}$ seconds using our algorithm.

Conclusion

Thank you for listening!

<https://arxiv.org/abs/2210.01172>

<https://github.com/prestontranbarger/NFDSFastComputation>



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