Generalized Dedekind Sums Arising from Specialized Eichler-Shimura Type Integrals

Preston Tranbarger
prestontranbarger@tamu.edu
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, U.S.A.

Abstract

Building upon the work of [SVY20] we derive generalized Dedekind sums arising from the Eichler-Shimura isomorphism applied to holomorphic weight $k \geq 3$ Eisenstein series attached to primitive non-trivial Dirichlet characters χ_1, χ_2. Furthermore, we demonstrate that these generalized Dedekind sums preserve certain important properties including a finite sum formula and a cohomological relation.

1 Introduction and Basic Definitions

The classical Dedekind sum is well studied due to its wide range of applications within mathematics and even specific subfields of physics. For more background on classical Dedekind sums we refer the reader to [RG72], and for more background on how the Dedekind sum appears in other areas of study we refer the reader to [Ati87].

Let h and k be coprime integers with $k > 0$. The classical Dedekind sum is defined as

$$s(h,k) = \sum_{n=1}^{k} B_1\left(\frac{n}{k}\right) B_1\left(\frac{hn}{k}\right),$$

where B_1 is the first periodic Bernoulli polynomial which is defined as follows.

Definition 1.1. *The periodic Bernoulli polynomials for integer $k \geq 1$ are given by the formula*

$$B_k(x) = \begin{cases} \displaystyle\sum_{m=0}^{k} \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(\{x\}+n)^k}{m+1} & x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

Recent work has further generalized the classical Dedekind sum. As stated in Theorem 1.2 of [SVY20], given primitive non-trivial Dirichlet characters χ_1 and χ_2 with conductors q_1 and q_2 respectively such that $\chi_1\chi_2(-1) = (-1)^k$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$, we have

$$S_{\chi_1,\chi_2,2}(\gamma) = \sum_{\substack{j \bmod c \\ n \bmod q_1}} \overline{\chi_1}(n)\overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) B_1\left(\frac{aj}{c} + \frac{n}{q_1}\right).$$

Conveniently, $S_{\chi_1,\chi_2,2}$ satisfies an elegant crossed homomorphism relation as shown in Lemma 2.2 of [SVY20]. For $\gamma_1, \gamma_2 \in \Gamma_0(q_1q_2)$

$$S_{\chi_1,\chi_2,2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2,2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2,2}(\gamma_2)$$

where $\psi(\gamma) = \chi_1\overline{\chi_2}(d_\gamma)$ is the central character of the weight $k = 2$ holomorphic Eisenstein series attached to characters.

Definition 1.2 ([DS05], Theorem 4.5.1)**.** *The Fourier expansion for holomorphic weight k Eisenstein series attached to primitive non-trivial Dirichlet characters χ_1 and χ_2 such that $\chi_1\chi_2(-1) = (-1)^k$ is given as*

$$E_{\chi_1,\chi_2,k}(z) = \sum_{1 \leq N} \sum_{A|N} \chi_1(A)\overline{\chi_2}(N/A)(N/A)^{k-1} e(Nz) \quad \textit{where} \quad e(z) = \exp(2\pi i z).$$

The central character of this series is $\psi = \chi_1\overline{\chi_2}$. Given a matrix $\gamma \in \Gamma_0(q_1q_2)$ recall that we have the relation under Mobius transformations given by $E_{\chi_1,\chi_2,k}(\gamma z) = \psi(\gamma)\, j(\gamma, z)^k\, E_{\chi_1,\chi_2,k}(z)$.

The focus of this paper builds upon a small but astute observation contained within Section 5 of [SVY20]: this generalization of the classical Dedekind sum is exactly an Eichler-Shimura type integral applied to the weight $k=2$ holomorphic Eisenstein series attached to characters. Building upon this, the main goal of this work is to evaluate this Eichler-Shimura type integral applied to weight $k \geq 3$ holomorphic Eisenstein series attached to characters. Using these calculations, we show that there are natural generalizations of $S_{\chi_1,\chi_2,2}$ to higher weight which preserve either its finite sum expression or its crossed homomorphism relation (see Theorem 1.5 and Lemma 4.3).

To adequately define our higher weight Eichler-Shimura type integrals we need to define some polynomials.

Definition 1.3. *Let us define the polynomial*

$$P_{k-2}(z) = (Xz+Y)^{k-2} = \left(\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}\right)^{k-2}.$$

Given a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *we have the relation under Mobius transformations given by*

$$P_{k-2}(\gamma z) = (X\gamma z+Y)^{k-2} = \left(j(\gamma,z)^{-1} \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}\right)^{k-2}.$$

For fixed z*, this polynomial* $P_{k-2}(z)$ *is a member of* $V_{k-2}(\mathbb{C})$ *which is the vector space of degree* $k-2$ *homogeneous polynomials in two variables having complex coefficients.*

Now we define our Eichler-Shimura type integral in terms of our Eisenstein series and polynomials.

Definition 1.4. *Given weight* k*, primitive non-trivial Dirichlet characters* χ_1 *and* χ_2 *with conductors* q_1 *and* q_2 *respectively such that* $\chi_1\chi_2(-1) = (-1)^k$*,* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$*, and* $z_0 = -d/c + i/(c^2u)$ *such that* $\gamma z_0 = a/c + iu$*; we define* $\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2})$ *as the Eichler-Shimura integral of* $E_{\chi_1,\chi_2,k}$ *against the polynomial* P_{k-2} *with base point at* ∞*. That is,*

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = \int_{\infty}^{\gamma\infty} E_{\chi_1,\chi_2,k}(z) P_{k-2}(z)\, dz = \lim_{u\to 0^+} \int_{z_0}^{\gamma z_0} E_{\chi_1,\chi_2,k}(z) P_{k-2}(z)\, dz.$$

This integral converges due to the exponential decay of $E_{\chi_1,\chi_2,k}$ *at the endpoints as seen in Definition 1.2.*

By specializing the values $X=1$ *and* $Y=-a/c$ *we have that* $P_{k-2}(z) = (z-a/c)^{k-2}$*. Using this we define our higher weight Dedekind sums in terms of this* ϕ *function,*

$$S_{\chi_1,\chi_2,k}(\gamma) = (-1)^k \tau(\overline{\chi_1})(k-1)\phi_{\chi_1,\chi_2,k}\left(\gamma, (z-a/c)^{k-2}\right). \tag{1}$$

This paper seeks to prove the following main theorem.

Theorem 1.5. *For* $k \geq 3$*, given* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1q_2)$ *and primitive non-trivial Dirichlet characters* χ_1 *and* χ_2 *with conductors* q_1 *and* q_2 *respectively such that* $\chi_1\chi_2(-1) = (-1)^k$*, we have*

$$S_{\chi_1,\chi_2,k}(\gamma) = \sum_{\substack{j \bmod c \\ n \bmod q_1}} \overline{\chi_1}(n)\overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) B_{k-1}\left(\frac{aj}{c} + \frac{n}{q_1}\right).$$

We will prove Theorem 1.5 in Section 3. This proof loosely follows the process of [SVY20]; however, we must first prove several analysis related results as there are significant changes to the necessary analysis required in this Eichler-Shimura type integral formulation of our generalized Dedekind sums.

2 Preliminaries for the Proof of Theorem 1.5

Before we provide our necessary analysis, we first collect some results from the literature which we will use in many of the following proofs. Note that in this section we impose the same restrictions on χ_1, χ_2, q_1, q_2, a, c, and u as in Definition 1.4.

2.1 Results from the Literature

Using the work of Berndt we define character analogues of Bernoulli polynomials.

Definition 2.1 ([Ber75], Definition 1)**.** *Given a primitive Dirichlet character χ with modulus m, for integer $k \geq 2$ we define $B_{k-1,\chi}(x)$ using the expression*

$$B_{k-1,\chi}(x) = \frac{(-i)^k \tau(\overline{\chi})(k-1)!}{im(2\pi/m)^{k-1}} \sum_{n \neq 0} \frac{\chi(n)}{n^{k-1}} e_m(nx).$$

Furthermore, we have an expression relating Bernoulli polynomials to character analogues of Bernoulli polynomials.

Lemma 2.2 ([Ber75], Theorem 3.1)**.** *Given a primitive Dirichlet character χ with modulus m, for integer $k \geq 1$ we have the following expression*

$$B_{k,\chi}(x) = m^{k-1} \sum_{n \bmod m} \overline{\chi}(n) B_k\left(\frac{x+n}{m}\right).$$

Berndt also provides us with the following character analogue of Poisson summation.

Theorem 2.3 ([Ber75], Theorem 2.3)**.** *Given a primitive Dirichlet character χ with conductor q and a function f of bounded variation on $[0, \infty)$ we have that*

$$\sum_{1 \leq n} \chi(n) f(n) = \frac{2\tau(\chi)}{q} \sum_{v \in \mathbb{Z}} \overline{\chi}(v) \int_0^\infty f(t)\, e^{2\pi i v t/q}\, dt.$$

Finally, we use the work of Stucker, Vennos, and Young to help simplify a particular sum in Section 2.2.

Lemma 2.4 ([SVY20], Lemma 3.2)**.** *We have that*

$$\sum_{1 \leq B} \overline{\chi_2}(B) e^{2\pi AB(ia/c-u)} = \sum_{j \bmod c} \overline{\chi_2}(j) e_c(Aaj) \left(\frac{e(Aiuj)-1}{1-e(Aiuc)}\right) \quad \textit{where} \quad e_c(z) = e(z/c).$$

Lemma 2.5 ([SVY20], Lemma 3.3)**.** *We have that*

$$\lim_{u \to 0^+} \sum_{1 \leq B} \overline{\chi_2}(B) e^{2\pi AB(ia/c-u)} = -\sum_{j \bmod c} \overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) e_c(Aaj).$$

2.2 Analysis Preliminaries

First we develop a twisted Poisson summation identity which will give us the proper convergence in future lemmas.

Lemma 2.6. *For an integer $K \geq 1$ and $z \in \mathcal{H}$ we have*

$$\sum_{1 \leq B} \overline{\chi_2}(B) B^K e^{2\pi ABiz} = 2q_2^K \left(\frac{\tau(\overline{\chi_2})\, K!}{(-2\pi i)^{K+1}}\right) \sum_{v \in \mathbb{Z}} \frac{\chi_2(v)}{(Aq_2 z + v)^{K+1}}.$$

Proof. Applying the formula given in Theorem 2.3 gives us

$$\sum_{1 \leq B} \overline{\chi_2}(B) B^K e^{2\pi ABiz} = \frac{2\tau(\overline{\chi_2})}{q_2} \sum_{v \in \mathbb{Z}} \chi_2(v) \int_0^\infty t^K e^{(2\pi i t/q_2)(Aq_2 z + v)}\, dt.$$

Evaluating this integral completes the proof. □

A similar sum will show up in the proof of a future lemma, so we show that it is finite.

Lemma 2.7. *For $K > 0$ and $(a, Q) = 1$, when $Q \nmid A$ we have that $\sum_{v \in \mathbb{Z}} |Aa/Q + v|^{-1-K} < \infty$.*

Proof. Note that when $Q \nmid A$ then

$$\begin{aligned} \sum_{v \in \mathbb{Z}} \frac{1}{|Aa/Q + v|^{K+1}} &= \sum_{v = \lceil Aa/Q \rceil}^\infty \frac{1}{(v - Aa/Q)^{K+1}} + \sum_{v = -\lfloor Aa/Q \rfloor}^\infty \frac{1}{(Aa/Q + v)^{K+1}} \\ &= \zeta(K+1, \lceil Aa/Q \rceil - Aa/Q) + \zeta(K+1, Aa/Q - \lfloor Aa/Q \rfloor) \leq 2\,\zeta(K+1, 1/Q) < \infty \end{aligned}$$

where $\zeta(s, a) = \sum_{0 \leq n} (n + a)^{-s}$ is the Hurwitz zeta function which converges absolutely for complex $\mathfrak{Re}(s) > 1$ and real $0 < a < 1$. □

We now provide a uniform bound on a particular summation.

Lemma 2.8. *For integers $k \geq 3$ and $0 \leq k-n-2$, given the following sum there exists a constant $C(k,n,c,q_2)$ depending solely on k, n, c, and q_2 such that we have the following uniform bound in terms of u*

$$\left|\sqrt{u}\sum_{1\leq A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\leq B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)}\right| < C(k,n,c,q_2).$$

Proof. We use Lemma 2.6 to give an alternate formula for the inner sum over B; thus we have

$$\begin{aligned}&\left|\sqrt{u}\sum_{1\leq A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\leq B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)}\right|\\ &\qquad\leq 2q_2^{k-n-2}\left(\frac{\sqrt{q_2}\,(k-n-2)!}{(2\pi)^{k-n-1}}\right)\sum_{1\leq A}\left|\frac{\chi_1(A)}{A^{n+1}}\sqrt{u}\sum_{v\in\mathbb{Z}}\frac{\chi_2(v)}{(Aq_2(a/c+iu)+v)^{k-n-1}}\right|.\end{aligned}$$

Now we will bound the summand independently of u. Let χ_0 be the trivial character modulo q_1. Note that

$$\left|\frac{\chi_1(A)}{A^{n+1}}\sqrt{u}\sum_{v\in\mathbb{Z}}\frac{\chi_2(v)}{(Aq_2(a/c+iu)+v)^{k-n-1}}\right| \leq \frac{\chi_0(A)}{A^{n+1}}\sqrt{u}\sum_{v\in\mathbb{Z}}\frac{1}{|Aq_2(a/c+iu)+v|^{k-n-1}}.$$

We borrow an $(Aq_2u)^{-1/2}$ from the sum by using the inequality

$$\frac{1}{|Aq_2(a/c+iu)+v|^{k-n-1}} \leq \frac{1}{(Aq_2u)^{1/2}|Aq_2a/c+v|^{k-n-3/2}}.$$

This gives

$$\frac{\chi_0(A)}{A^{n+1}}\sqrt{u}\sum_{v\in\mathbb{Z}}\frac{1}{|Aq_2(a/c+iu)+v|^{k-n-1}} \leq \frac{\chi_0(A)}{\sqrt{q_2}\,A^{n+3/2}}\sum_{v\in\mathbb{Z}}\frac{1}{\left|Aa/(cq_2^{-1})+v\right|^{k-n-3/2}}.$$

If $q_1 \nmid A$, then by Lemma 2.7 we have

$$\chi_0(A)\sum_{v\in\mathbb{Z}}\frac{1}{\left|Aa/(cq_2^{-1})+v\right|^{k-n-3/2}} \leq 2\,\zeta(k-n-3/2,q_2/c).$$

Otherwise if $q_1 \mid A$ then $\chi_0(A)=0$. Thus,

$$\frac{\chi_0(A)}{\sqrt{q_2}\,A^{n+3/2}}\sum_{v\in\mathbb{Z}}\frac{1}{|Aa/c+v|^{k-n-3/2}} \leq \frac{2\,\zeta(k-n-3/2,q_2/c)}{\sqrt{q_2}\,A^{n+3/2}}.$$

This bounds our summand independently of u. Hence

$$\left|\sqrt{u}\sum_{1\leq A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\leq B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)}\right| \leq 4q_2^{k-n-2}\left(\frac{(k-n-2)!}{(2\pi)^{k-n-1}}\right)\sum_{1\leq A}\frac{\zeta(k-n-3/2,q_2/c)}{A^{n+3/2}}.$$

Noting that this sum converges and is solely dependent on k, n, c, and q_2 completes our proof. □

From Lemma 2.8 it immediately follows that the following limit vanishes.

Corollary 2.9. *For integers $k \geq 3$ and $0 \leq n < k-2$ we have*

$$\lim_{u\to 0^+} u^{k-n-2}\sum_{1\leq A}\frac{\chi_1(A)}{A^{n+1}}\sum_{1\leq B}\overline{\chi_2}(B)B^{k-n-2}e^{2\pi AB(ia/c-u)} = 0.$$

And lastly we explicitly evaluate a particular limit.

Lemma 2.10. *For $k \geq 3$, we have the limit*

$$\lim_{u\to 0^+}\sum_{1\leq A}\frac{\chi_1(A)}{A^{k-1}}\sum_{1\leq B}\overline{\chi_2}(B)e^{2\pi AB(ia/c-u)} = -\sum_{1\leq A}\sum_{j \bmod c}\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}B_1\left(\frac{j}{c}\right)e_c(Aaj).$$

Proof. First we seek to interchange this limit and sum via Tannery's theorem, so we will bound the modulus of the summand independently of u. Applying Lemma 2.4 to the inner sum and noting $|(e(Aiuj)-1)/(1-e(Aiuc))| \leq 1$ gives us the bound

$$\left| \frac{\chi_1(A)}{A^{k-1}} \sum_{1\leq B} \overline{\chi_2}(B) e^{2\pi AB(ia/c-u)} \right| \leq \frac{1}{A^{k-1}} \sum_{j \bmod c} \left| \overline{\chi_2}(j) e_c(Aaj) \left(\frac{e(Aiuj)-1}{1-e(Aiuc)} \right) \right| \leq \frac{c}{A^{k-1}}.$$

Note that for $k \geq 3$ we have $\sum_{1\leq A} cA^{1-k} = c\,\zeta(k-1) < \infty$. Thus we can interchange limits via Tannery's theorem. So it follows that

$$\lim_{u\to 0^+} \sum_{1\leq A} \frac{\chi_1(A)}{A^{k-1}} \sum_{1\leq B} \overline{\chi_2}(B) e^{2\pi AB(ia/c-u)} = \sum_{1\leq A} \frac{\chi_1(A)}{A^{k-1}} \lim_{u\to 0^+} \sum_{1\leq B} \overline{\chi_2}(B) e^{2\pi AB(ia/c-u)}.$$

Applying Lemma 2.5 to the limit on the right hand side completes the proof. □

3 Proof of Theorem 1.5

Throughout this proof it is useful to recall that $z_0 = -d/c + i/(c^2u)$ and $\gamma z_0 = a/c + iu$. We begin by substituting Definition 1.2 into Definition 1.4; doing so gives

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2 \lim_{u\to 0^+} \int_{z_0}^{\gamma z_0} \left((Xz+Y)^{k-2} \sum_{1\leq N} \sum_{A|N} \chi_1(A)\overline{\chi_2}(N/A)(N/A)^{k-1} e(Nz) \right) dz.$$

We may interchange the sums and integral due to the rapid decay of $e(Nz)$ when $\mathfrak{Im}(z) > 0$; thus

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2 \lim_{u\to 0^+} \sum_{1\leq N} \sum_{A|N} \int_{z_0}^{\gamma z_0} (Xz+Y)^{k-2} \chi_1(A)\overline{\chi_2}(N/A)(N/A)^{k-1} e(Nz)\, dz.$$

Using repeated integration by parts (differentiating $(Xz+Y)^{k-2}$ and integrating $e(Nz)$), we have

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2 \lim_{u\to 0^+} \sum_{1\leq N} \sum_{A|N} \sum_{n=0}^{k-2} \left(\frac{\chi_1(A)\overline{\chi_2}(N/A) N^{k-n-2}}{-A^{k-1}(-2\pi i)^{n+1}} \left(\frac{d^n}{dz^n} (Xz+Y)^{k-2} \right) e(Nz) \right) \Bigg|_{z_0}^{\gamma z_0}.$$

We have chosen z_0 such that these terms evaluated at z_0 vanish in the limit, thus we are left with

$$\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2}) = 2 \lim_{u\to 0^+} \sum_{1\leq N} \sum_{A|N} \sum_{n=0}^{k-2} \left(\frac{\chi_1(A)\overline{\chi_2}(N/A) N^{k-n-2}}{-A^{k-1}(-2\pi i)^{n+1}} \right) \left(\frac{X^n (k-2)!}{(k-n-2)!} (X\gamma z_0 + Y)^{k-n-2} e(N\gamma z_0) \right).$$

Interchanging the summations we write $\phi_{\chi_1,\chi_2,k}(\gamma, P_{k-2})$ in the form

$$= \sum_{n=0}^{k-2} \left(-\frac{2X^n (k-2)!}{(-2\pi i)^{n+1}(k-n-2)!} \right) \left(\lim_{u\to 0^+} (X\gamma z_0 + Y)^{k-n-2} \sum_{1\leq A} \frac{\chi_1(A)}{A^{n+1}} \sum_{1\leq B} \overline{\chi_2}(B) B^{k-n-2} e(AB\gamma z_0) \right).$$

Recall from (1) that $S_{\chi_1,\chi_2,k}$ requires the specialization $X = 1$ and $Y = -a/c$ so that $P_{k-2}(z) = (z-a/c)^{k-2}$; doing so gives the formula

$$\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = \sum_{n=0}^{k-2} \left(-\frac{2 i^{k-n-2} (k-2)!}{(-2\pi i)^{n+1}(k-n-2)!} \right) \left(\lim_{u\to 0^+} u^{k-n-2} \sum_{1\leq A} \frac{\chi_1(A)}{A^{n+1}} \sum_{1\leq B} \overline{\chi_2}(B) B^{k-n-2} e^{2\pi AB(ia/c-u)} \right).$$

Corollary 2.9 implies that when $k \geq 3$ and $0 \leq n < k-2$, this limit vanishes. Thus,

$$\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = \frac{(k-2)!}{\pi i} \left(-\frac{1}{2\pi i} \right)^{k-2} \lim_{u\to 0^+} \sum_{1\leq A} \frac{\chi_1(A)}{A^{k-1}} \sum_{1\leq B} \overline{\chi_2}(B) e^{2\pi AB(ia/c-u)}.$$

In Lemma 2.10 we explicitly evaluated this limit; substituting this gives

$$\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = -\frac{(k-2)!}{\pi i}\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(Aaj). \tag{2}$$

Following the proof of Theorem 1.2 in [SVY20], we would like to use some type of symmetry relation to reindex our sum with A running over non-zero integers. To do this, we pause the proof of Theorem 1.5 to show that

$$S_{\chi_1,\chi_2,k}(\gamma) = \frac{1}{2}\left(S_{\chi_1,\chi_2,k}(\gamma) + \chi(-1)\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)\right)$$

First, note that

$$\chi_2(-1)\left(\frac{(-1)^k\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)}{\tau(\overline{\chi_1})(k-1)}\right) = \chi_2(-1)\left(\frac{(k-2)!}{\pi i}\right)\left(\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)\overline{e_c}(Aaj).$$

Observe that $\chi_2(-1) = \overline{\chi_2}(-1)$, $B_1(-x) = -B_1(x)$, and $\overline{e_c}(Aaj) = e_c(-Aaj)$. So,

$$\chi_2(-1)\left(\frac{(-1)^k\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)}{\tau(\overline{\chi_1})(k-1)}\right) = -\frac{(k-2)!}{\pi i}\left(\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(-j)}{A^{k-1}}\right)B_1\left(-\frac{j}{c}\right)e_c(-Aaj).$$

By sending $j \mapsto -j$, we re-index our summation

$$\chi_2(-1)\left(\frac{(-1)^k\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)}{\tau(\overline{\chi_1})(k-1)}\right) = -\frac{(k-2)!}{\pi i}\left(\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(Aaj).$$

Removing the scalars gives us that $(-1)^k S_{\chi_1,\chi_2,k}(\gamma) = \chi_2(-1)\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)$; thus we have the symmetric formula

$$S_{\chi_1,\chi_2,k}(\gamma) = \frac{1}{2}\left(S_{\chi_1,\chi_2,k}(\gamma) + (-1)^k\chi_2(-1)\overline{S}_{\overline{\chi_1},\overline{\chi_2},k}(\gamma)\right).$$

Recalling that $\chi_1\chi_2(-1) = (-1)^k$ gives us the desired symmetry relation.

Now we return to the proof of Theorem 1.5. Using our new symmetry relation we can rewrite (2) as

$$\begin{aligned}\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = &-\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(Aaj)\\ &+\chi_1(-1)\left(\frac{(k-2)!}{2\pi i}\right)\left(\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(-Aaj).\end{aligned}$$

Noting that $A^{-k+1} = (-1)^{-k+1}(-A)^{-k+1}$, we get

$$\begin{aligned}\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = &-\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(A)\overline{\chi_2}(j)}{A^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(Aaj)\\ &-\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{1\le A}\sum_{j \bmod c}\left(\frac{\chi_1(-A)\overline{\chi_2}(j)}{(-A)^{k-1}}\right)B_1\left(\frac{j}{c}\right)e_c(-Aaj)\end{aligned}$$

Re-indexing and swapping the order of summation we have

$$\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = -\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\sum_{j \bmod c}\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)\sum_{A\neq 0}\frac{\chi_1(A)}{A^{k-1}}e_c(Aaj).$$

Recognizing the inner sum from Definition 2.1 we rewrite our expression as

$$\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = -\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\left(\frac{iq_1(2\pi/q_1)^{k-1}}{(-i)^k\tau(\overline{\chi_1})(k-1)!}\right)\sum_{j \bmod c}\overline{\chi_2}(j)B_1\left(\frac{j}{c}\right)B_{k-1,\chi_1}\left(\frac{ajq_1}{c}\right).$$

Now applying Lemma 2.2 gives us the equivalent expression

$$= -\left(\frac{(k-2)!}{2\pi i}\right)\left(-\frac{1}{2\pi i}\right)^{k-2}\left(\frac{iq_1(2\pi/q_1)^{k-1}}{(-i)^k\tau(\overline{\chi_1})(k-1)!}\right) q_1^{k-2} \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_1}(n)\overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) B_{k-1}\left(\frac{aj}{c}+\frac{n}{q_1}\right).$$

Simplifying the leading coefficient gives

$$\frac{(-1)^k S_{\chi_1,\chi_2,k}(\gamma)}{\tau(\overline{\chi_1})(k-1)} = \frac{(-1)^k}{\tau(\overline{\chi_1})(k-1)} \sum_{\substack{j \bmod c \\ n \bmod q_1}} \overline{\chi_1}(n)\overline{\chi_2}(j) B_1\left(\frac{j}{c}\right) B_{k-1}\left(\frac{aj}{c}+\frac{n}{q_1}\right).$$

Removing the scalar factor from both sides proves the theorem.

4 Other Results

We seek to recover analogues of previous results involving $S_{\chi_1,\chi_2,2}$ in an effort to better understand how $\phi_{\chi_1,\chi_2,k}$ an $S_{\chi_1,\chi_2,k}$ behave when $k \geq 3$.

4.1 Cohomological Properties of $\phi_{\chi_1,\chi_2,k}$ and $S_{\chi_1,\chi_2,2}$

Recall the definition of the slash operator, as we will use this to demonstrate that $\phi_{\chi_1,\chi_2,k}$ exhibits a crossed homomorphism relation similar to Lemma 2.2 of [SVY20].

Definition 4.1. *Given a function $f : \mathcal{H} \to \mathbb{C}$ and a matrix $\gamma \in SL_2(\mathbb{Z})$, we define the weight k slash operator as*

$$f|_\gamma(z) = j(\gamma, z)^{-k} f(\gamma z),$$

such that weight k modular forms are invariant under the weight k slash operator.

As a formality we also endow $V_{k-2}(\mathbb{C})$ with an implied group action.

Definition 4.2. *Recall that $V_{k-2}(\mathbb{C})$ is the vector space of degree $k-2$ homogeneous polynomials in two variables having complex coefficients. Let $P(X,Y)$ be a polynomial in $V_{k-2}(\mathbb{C})$; given a central character ψ, we define $V_{k-2}^{\psi}(\mathbb{C})$ as the same vector space of polynomials with the right group action $V_{k-2}(\mathbb{C}) \times \Gamma_0(q_1q_2) \to V_{k-2}(\mathbb{C})$ via the map*

$$P(X,Y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi(d)\, P\left(\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

It is simple to verify that this indeed forms a right group action. Note that in the case of $P_{k-2}(z) \in V_{k-2}^{\psi}(\mathbb{C})$ one has the relation $P_{k-2}(z) \cdot \gamma = \psi(\gamma) P_{k-2}|_\gamma(z)$.

Now we prove that $\phi_{\chi_1,\chi_2,k}$ is a crossed homomorphism.

Lemma 4.3. *Suppose we have two matrices $\gamma_1, \gamma_2 \in \Gamma_0(q_1q_2)$. We have that $\phi_{\chi_1,\chi_2,k}$ is a crossed homomorphism, that is to say*

$$\phi_{\chi_1,\chi_2,k}(\gamma_1\gamma_2, P_{k-2}) = \phi_{\chi_1,\chi_2,k}(\gamma_1, P_{k-2}) + \phi_{\chi_1,\chi_2,k}(\gamma_2, P_{k-2} \cdot \gamma_1).$$

Proof. Noting that $E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)$ is holomorphic, by path independence we have that

$$\begin{aligned}\phi_{\chi_1,\chi_2,k}(\gamma_1\gamma_2, P_{k-2}) &= \int_\infty^{\gamma_1\gamma_2\infty} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\, dz \\ &= \int_\infty^{\gamma_1\infty} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\, dz + \int_{\gamma_1\infty}^{\gamma_1\gamma_2\infty} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\, dz.\end{aligned}$$

We can change variables on the second integral using $\gamma_1 u = z$ and $j(\gamma_1, u)^{-2}\, du = dz$; doing so gives

$$\begin{aligned}\int_{\gamma_1\infty}^{\gamma_1\gamma_2\infty} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\, dz &= \int_\infty^{\gamma_2\infty} j(\gamma_1,u)^k E_{\chi_1,\chi_2,k}|_{\gamma_1}(u)\left(\frac{P_{k-2}|_{\gamma_1}(u)}{j(\gamma_1,u)^{k-2}}\right)\left(\frac{du}{j(\gamma_1,u)^2}\right) \\ &= \int_\infty^{\gamma_2\infty} E_{\chi_1,\chi_2,k}|_{\gamma_1}(u)P_{k-2}|_{\gamma_1}(u)\, du \\ &= \int_\infty^{\gamma_2\infty} \psi(\gamma_1)E_{\chi_1,\chi_2,k}(u)P_{k-2}|_{\gamma_1}(u)\, du = \int_\infty^{\gamma_2\infty} E_{\chi_1,\chi_2,k}(u)(P_{k-2}\cdot\gamma_1)(u)\, du.\end{aligned}$$

Thus we have,

$$\begin{aligned}\phi_{\chi_1,\chi_2,k}(\gamma_1\gamma_2, P_{k-2}) &= \int_{\infty}^{\gamma_1\infty} E_{\chi_1,\chi_2,k}(z)P_{k-2}(z)\,dz + \int_{\infty}^{\gamma_2\infty} E_{\chi_1,\chi_2,k}(z)(P_{k-2}\cdot\gamma_1)(z)\,dz \\ &= \phi_{\chi_1,\chi_2,k}(\gamma_1, P_{k-2}) + \phi_{\chi_1,\chi_2,k}(\gamma_2, P_{k-2}\cdot\gamma_1)\end{aligned}$$

completing our proof as desired. □

Remark. *The above proof is standard in the literature for the Eichler-Shimura isomorphism. However, due to the lack of accessible discussion on the Eisenstein part of the Eichler-Shimura isomorphism, we opt to explicitly describe the crossed homomorpism as above. Given that there is little overlap between the literature on the Eichler-Shimura isomorphism and the literature on Dedekind sums, the inclusion of this proof may help those who are unfamiliar with the Eichler-Shimura isomorphism.*

Additionally, we can view $\phi_{\chi_1,\chi_2,k}$ as an element of the cohomology group $H^1(\Gamma_0(q_1q_2), V_{k-2}^{\psi}(\mathbb{C}))$.

Noting that $\phi_{\chi_1,\chi_2,2}$ is independent of our choice of X and Y allows us to recover the crossed homomorphism relation for $S_{\chi_1,\chi_2,2}$.

Corollary 4.4. *We recover the cohomological properties of $S_{\chi_1,\chi_2,2}$ as in [SVY20] Lemma 2.2. Specifically we can view $S_{\chi_1,\chi_2,2}$ as an element of the space Hom$(\Gamma_1(q_1q_2),\mathbb{C})$ and we have a crossed homomorphism relation for $S_{\chi_1,\chi_2,2}$ given by*

$$S_{\chi_1,\chi_2,2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2,2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1,\chi_2,2}(\gamma_2).$$

Proof. Note that the Eichler-Shimura type integral is independent of X and Y when $k=2$. So it follows if we are given $\gamma_1,\gamma_2 \in \Gamma_0(q_1q_2)$ then we have that

$$\phi_{\chi_1,\chi_2,2}(\gamma_1, P_0) = \int_{\infty}^{\gamma_1\infty} E_{\chi_1,\chi_2,2}(z)(Xz+Y)^0\,dz = \int_{\infty}^{\gamma_1\infty} E_{\chi_1,\chi_2,2}(z)(z - a_{\gamma_1}/c_{\gamma_1})^0\,dz = \frac{(-1)^k}{\tau(\overline{\chi_1})(k-1)}S_{\chi_1,\chi_2,2}(\gamma_1).$$

Using this independence in combination with Lemma 4.3 we have

$$\begin{aligned}\phi_{\chi_1,\chi_2,2}(\gamma_1\gamma_2, P_0) &= \phi_{\chi_1,\chi_2,2}(\gamma_1, P_0) + \int_{\infty}^{\gamma_2\infty} \psi(\gamma_1)E_{\chi_1,\chi_2,2}(z)P_0|_{\gamma_1}(z)\,dz \\ &= \phi_{\chi_1,\chi_2,2}(\gamma_1, P_0) + \psi(\gamma_1)\int_{\infty}^{\gamma_2\infty} E_{\chi_1,\chi_2,2}(z)P_0(z)\,dz = \phi_{\chi_1,\chi_2,2}(\gamma_1) + \psi(\gamma_1)\phi_{\chi_1,\chi_2,2}(\gamma_2, P_0).\end{aligned}$$

This explicitly gives a crossed homomorphism relation for $S_{\chi_1,\chi_2,2}$ up to scalar multiple. So we have

$$S_{\chi_1,\chi_2,2}(\gamma_1\gamma_2) = S_{\chi_1,\chi_2,2}(\gamma_1) + \psi(\gamma_1)S_{\chi_1,\chi_2,2}(\gamma_2).$$

Noting that $\psi(\gamma_1)$ is trivial on $\Gamma_1(q_1q_2)$ we also have that $S^2_{\chi_1,\chi_2} \in \mathrm{Hom}(\Gamma_1(q_1q_2),\mathbb{C})$. So we recover all of the results of Lemma 2.2 in [SVY20] as desired. □

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