linear algebra group project

Characterizing quantum sensors with noise

report 1

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Assume that we have a single atom interacting with a single frequency of light within an optical cavity. We can write the interactions between the atom and cavity as[1]

$$\mathcal{H}_{\text{JC}} = \hbar\omega a^\dagger a + \frac{\hbar\Delta}{2}\sigma^z + g(a\sigma^- + a\sigma^+)\,, \tag{1}$$

where $a^\dagger$ the creation operator for a photon, σ^- is the raising operator for the atom, ω is the cavity frequency, Δ is the detuning of the cavity frequency from the atom's excited state, and g is the interaction strength between the atom and cavity. Next, assume that our atom has some Zeeman splitting,[2] we may then write our total Hamiltonian—neglecting the atom's own angular momentum—for this system as

$$\mathcal{H} = \mathcal{H}_{\text{JC}} - \frac{e_0}{m}\mathbf{S}\cdot\mathbf{B} + V(t)\,, \tag{2}$$

where e_0 is the electron charge, m is the atom's mass, $\mathbf{S}$ is the electron's total spin, and $\mathbf{B}$ is the input magnetic field, and $V(t)$ is a noise function such that $\langle V(t+dt)V(t)\rangle = 0$.

This Hamiltonian serves as a decent prototype for a quantum sensor,[3] since its energy levels change directly with an applied magnetic field. If we know the change in energy levels for an atom with zero noise, or at the very least *reduced* noise, then we can characterize the noise in a quantum sensor based off of the change in the sensor's sensitivity, which can be calculated by getting a least-squares fit of the energy of incident photons versus photons emitted from the atom.

If one is assuming an open quantum system—as quantum systems are in the real world—then it is prudent to use the Lindblad equation,[4]

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[\mathcal{H},\rho] + \sum_j\Big(L_j\rho L_j^\dagger - \frac{1}{2}\big\{L_j^\dagger L_j,\rho\big\}\Big), \tag{3}$$

where L_i is a "jump-operator" describing how a system interacts with its environment.

Least-squares fits can be calculated using the singular value decomposition (SVD),[5,6] where we assume that an $m\times n$ matrix with rank r, A, can be broken up into three components,[5] $A = U\Sigma V^{\text{T}}$, where U and V are orthogonal, and Σ is a matrix with the single values of A, usually expressed as

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{bmatrix},$$

and $\sigma_n = \sqrt{\lambda_n}$ are the singular values of A. U and V are constructed by the left and right singular vectors of A, respectively. A full treatment of least-squares will be covered in the next report, as I scrapped the entire report I worked on last week in favor of this, and I've spent all morning—and most of this afternoon—finishing *this* particular report. (I was doing some big power-grid optimization problem, and honestly it was just really bad.)

Calculations will be performed numerically, since finding solutions to this problem will be *extremely* nontrivial analytically even for the most talented physicists and mathematicians. In the next report, I will be showing my calculations, going into more detail with the singular value decomposition, and expanding my analysis to cover an extended, many-body interacting Hamiltonian. I would like to apologize for how brief and sparse this report is.

References

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3. Degen, C. L., Reinhard, F. & Cappellaro, P. Quantum sensing. *Rev. Mod. Phys.* **89**, 035002 (2017).
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5. Lay, D. C., Lay, S. R. & McDonald, J. J. *Linear algebra and its applications*. (Pearson Education Ltd, 2016).
6. Verschelde, J. Numerical analysis: Nonlinear least squares. (2022).