Rutgers University

Symplectic Summer School 2024

Gary Hu

Acknowledgements

From August 19 to August 23, Rutgers University ran a summer school on symplectic geometry that aimed to provide graduate students and advanced undergraduate students tutorials in various advanced topics in symplectic geometry and introductions to recent developments. This year was focused on, but was not restricted to, the foundational aspects, including the theory of global Kuranishi charts, integer-valued curve-counting invariants, Hamiltonian dynamics, and contact topology.

These notes were scribed by Gary Hu, who is responsible for all mistakes. If you do find any errors, please report them to: gh7@williams.edu

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Contents

Part I

Erkao Bao: Introduction to Contact Homology

There were three lectures:

1. Day 1: Moduli Spaces of J-holomorphic Curves and Compactness

   In this lecture, we begin with an introduction to basic contact geometry. We then introduce J-holomorphic curves as the gradient of the action functional. The focus will be on the moduli space of J-holomorphic curves, with a discussion on compactness. We will provide heuristic definitions of cylindrical contact homology and full contact homology.

2. Day 2: Cylindrical Contact Homology in Dimension Three via Obstruction Bundle Gluing

   This lecture addresses the transversality issues associated with the moduli space of J-holomorphic curves. We specifically focus on cylindrical contact homology in the 3-dimensional case. The lecture will cover the resolution of transversality issues using obstruction bundle gluing techniques.

3. Day 3: Semi-Global Kuranishi Structure and Full Contact Homology

   In this lecture, we introduce the semi-global Kuranishi structure. We explore its application in relation to obstruction bundle gluing, including computations of simple examples. The discussion will culminate in the rigorous definition of full contact homology, facilitated by the semi-global Kuranishi structure.

Chapter 1

Moduli Spaces of J-Holomorphic Curves and Compactness

Abstract In this lecture, we begin with an introduction to basic contact geometry. We then introduce J-holomorphic curves as the gradient of the action functional. The focus will be on the moduli space of J-holomorphic curves, with a discussion on compactness. We will provide heuristic definitions of cylindrical contact homology and full contact homology.

1.1 Introduction

Definition 1.1 Let $M^{2n+1}, \xi^{2n} \subset TM$. We say ξ is a **contact structure** if there exists 1-form α on M, called the **contact form**, such that

- $\xi = \ker \alpha$
- $\alpha \wedge (d\alpha)^n \neq 0$

Together, these two conditions imply that ξ is non-integrable.

Example 1.1 $(\mathbb{R}^{2n+1}, \xi_{\text{std}} = \ker \alpha_{\text{std}})$ where

$$\alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i \, dx_i .$$

When $n = 1$, at the origin, we have the contact plane.

Theorem 1.1 (Darboux Theorem)

All contact structures in $\dim 2n+1$ *are locally isomorphic.*

Theorem 1.2 (Gray Stability Theorem)

Consider $\{\xi_t\}_{0\leq t\leq 1}$ contact structures on M. If M is closed, then there exists a 1-parameter family of diffeomorphisms ϕ^t of M such that $(\phi^t)_*\xi_t = \xi_0$ where $\phi_0 = id$.

1.2 Contact Homology

Contact homologies are some invariants of contact structures. Before we discuss further, lets look at some applications of contact homologies:

- [Ustilovsky, 1999]: S^{4m+1} admits ∞ many contact structures in each homotopy class of **almost contact structures** $(\xi, J, \xi \to \xi, J^2 = -\mathrm{id})$ where ξ is a hyperplane.
- [Bourgeois, 2004]: T^5 and $T^2 \times S^3$ have ∞ contact structures in some homotopy class of almost contact structures.
- [Giroux, 1994; Eliashberg, Hofer, Givental, 2000]: On T^3, $\alpha = \cos 2\pi nz\, dx + \sin 2\pi nz\, dy$, $\xi_n = \ker \alpha_n$, where ξ are pairwise non-isomorphic.

Definition 1.2 Given a contact form α, R_α is called a **Reeb vector field** if

- $\alpha(R_\alpha) = 1$ (positively transverses to ξ)
- $d\alpha(R_\alpha, \cdot) = 0$ (the flow of R_α preserves ξ)

Definition 1.3 Periodic orbits of R_α are called **Reeb orbits**.

Conjecture 1.1 (Weinstein Conjecture)

If M^{2n+1} is closed, then for any contact form α, there exists at least one Reeb orbit.

Theorem 1.3 (Taubes, 2007)

The Weinstein conjecture holds for $n = 1$.

Definition 1.4 Let γ be a Reeb orbit of period T, φ^t be a time t flow of R_α We say γ is non-degenerate if $d\varphi^T : \xi_{\gamma(0)} \to \xi_{\gamma(T)}$ does not have 1 as an eigenvalue.

Definition 1.5 Trivialize $(\xi|_\gamma, d\alpha)$ symplectically. Then $d\varphi^t$ gives a path of symplectic matrices starting at id. For such a path, we can define an integer, called the **Conley-Zehnder index** of γ.

The general definition is complicated so we will only present simple examples:

Example 1.2 Take $n = 1, \dim M = 3$.

- Positive hyperbolic: If the eigenvalues of $d\varphi^T$ are positive real numbers and $d\varphi^t(v)$ winds around the origin k times (where v is an eigenvector), then $\mu_{\text{cz}} = 2k$ are negative real numbers.
- Negative hyperbolic: If $d\varphi^t(v)$ winds around $k + \frac{1}{2}$ times, then $\mu_{\text{cz}}(\gamma) = 2k + 1$.
- Elliptic: If the eigenvalues are not real, $d\varphi^t(w)$ where $w \in \xi_{\gamma(0)}$ winds between k and $k + 1$ times then $\mu_{\text{cz}}(\gamma) = 2k + 1$.

Consider the actional functional $\mathcal{A} : C^\infty(s^t, M) \to \mathbb{R}, \gamma \mapsto \int_{S^1} \gamma^*(\alpha)$ with Reeb orbits crit A, with a complex structure $J : \xi$ acting on itself with $T^2 = -\text{id}$ and $\langle u, v \rangle = d\alpha(u, Tv)$ for any $u, v \in \xi$. Then $\langle \cdot, \cdot \rangle$ defines an inner product which implies

- $d\alpha(u, v) = d\alpha(Ju, Jv)$
- $d\alpha(u, Ju) > 0$ for any $u \neq 0 \in \xi$.

Take $\eta_1, \eta_2 \in T_\gamma C^\infty(S^1, M)$ with

$$\langle \eta_1, \eta_2 \rangle = \int_{S^1} \langle \eta_1, \eta_2 \rangle + \alpha(\eta_1)\alpha(\eta_2)\, dt.$$

Take $u : \mathbb{R} \to C^\infty(S^1, M)$ with $s \in \mathbb{R}$ and $t \in C^\infty(S^1, M)$ with $\dim u(s)$ are Reeb orbits as $s \to \pm\infty$. Then

$$\frac{du}{ds} = -\text{grad}\, \mathcal{A}$$

with $u : \mathbb{R} \times S^1 \to M$. This gives

$$d(u^*\alpha \cdot j) = 0\pi_\xi u_s + J\pi_\xi u_t = 0$$

where j is a complex structure on $\mathbb{R} \times S^1$.

Let's require $u^* d\alpha \cdot j = da$ where $a : \mathbb{R} \times S^1 \to \mathbb{R}$. Let $\tilde{u} = (a, u) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$. Now, we extend J: $T(\mathbb{R} \times M) = \mathbb{R}(\partial a) \oplus \mathbb{R}(R_\alpha) \oplus \xi$ where $a \in \mathbb{R}$, and extend $J : \mathbb{R}(\partial_a) \to \mathbb{R}(R_\alpha)$. $\tilde{u}$ is J-holomorphic, i.e.

$$\overline{\partial}\tilde{u} = \frac{1}{2}(d\tilde{u} + J(\tilde{u})d\tilde{u} \cdot j) = 0$$

or

$$\tilde{u}_s + J(\tilde{u})\tilde{u}_t = 0$$

Next time, we will study the compactification of moduli spaces of J-holomorphic cylinders.

Chapter 2

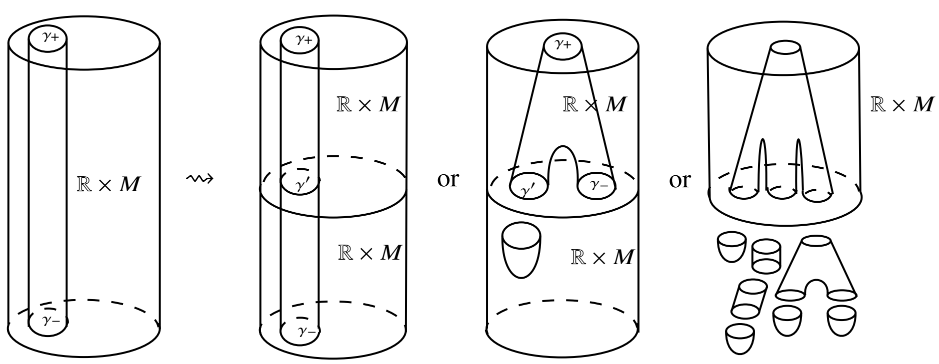
Cylindrical Contact Homology in Dimension Three via Obstruction Bundle Gluing

Abstract This lecture addresses the transversality issues associated with the moduli space of J-holomorphic curves. We specifically focus on cylindrical contact homology in the 3-dimensional case. The lecture will cover the resolution of transversality issues using obstruction bundle gluing techniques.

2.1 Compactification

Let $\tilde{\mathcal{M}}(\gamma_+, \gamma_-)$: $\left\{J\text{-holomorphic curves } \mathbb{R} \times S^1 \to \mathbb{R} \times M\right\}$ /out of domain. There are asymptotic markers γ_+ and γ_- on $\mathbb{R} \times M$, and at the points at ∞ of $\mathbb{R} \times S^1$ are markers in a way such that markers are mapped to markers. At each embedded Reeb orbit, choose a starting point. Define $\mathcal{M} = \tilde{\mathcal{M}}/\mathbb{R}$, which acts on $\mathbb{R} \times M$ by translation.

A sequence of J-holomorphic cylinder can converge to broken ones: we can break a cylinder $\mathbb{R} \times M$ into two cylinders $\mathbb{R} \times M$:



where for the rightmost diagram, each of the bottom components is a map to $\mathbb{R}^n$.

Definition 2.1 Let (F, j) be a Riemann surface, $\dot{F} = F$ a finite subset, and $u : (\dot{F}, j) \overset{J\text{-holomorphic}}{\longrightarrow} (\mathbb{R} \times M, J)$. Then the **Hofer energy** is

$$E(u) = \sup_{\phi \in C} \int_{\dot{F}} u^* d(\phi, \alpha)$$

where

$$C = \left\{ \phi : \mathbb{R} \to [1, 2] \mid \begin{cases} \phi(s) = 1 & \text{if } s \ll 0, \\ \phi(s) = 2 & \text{if } s \gg 0, \\ \phi'(s) \geq 0 & \text{for all } s \in \mathbb{R} \end{cases} \right\}$$

Proposition 2.1

$$\begin{aligned} E(u) &= 2 \sum_{i=1}^{k_+} \mathcal{A}(\gamma_{+,i}) - \sum_{i=1}^{k_-} \mathcal{A}(\gamma_{-,i}) \\ &\geq 0. \end{aligned}$$

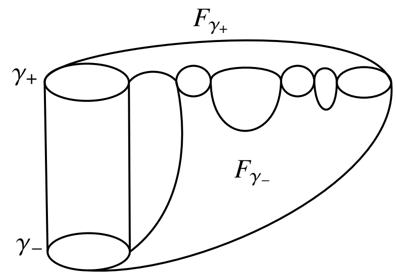
2.2 Grading

Theorem 2.1 *If $u : (\dot{E}, j) \to \mathbb{R} \times M$ is J-holomorphic, the following are equivalent:*

- $E(u) < \infty$
- *For any puncture p of $\dot{F}$, either p is removable or u converges to some Reeb orbit along P.*

Assume the $H_1(M; \mathbb{Z})$ is torsion free. If $H_1(M) = 0$, then for each Reeb orbit γ, fix a surface $F_\gamma \subset M$ spanning γ. If $H_1(M; \mathbb{Z}) \neq 0$, choose a basis of $H_1(M)$ and represent it by oriented curves $c_1, \ldots c_k$. Choose a trivialization of ξ / c_1. For any Reeb orbit γ, choose F_γ such that $[\partial F_\gamma] = [\gamma] - \sum n_i [c_i]$. Choose a trivialization of $\xi \mid_\gamma$ so that it extends over F_γ and agrees with the trivialization of ξ / c_i. This defines a Conley-Zehder index for each γ.

For each J-holomorphic cylinder u, we can attach F_{γ_+} and F_{γ_-}:



which gives an element in $H_2(M)$.

Assume does not exist a contractible Reeb orbits. We define a chain complex (C_*, ∂) where C_* is a free $\mathbb{Q}[H_2(M)]$ module generated by all good Reeb orbits. For each $A \in H_2(M)$, $|A| = -2\langle c_1(\xi), A\rangle$, define $\partial_{\gamma_+} = \sum_{\gamma_-, A} \#\mathcal{M}(\gamma_+, \gamma_-, A) e^A \frac{1}{m(\gamma_-)} \gamma_-$ where $m(\gamma_-)$ is the multiplcity of γ_-. This gives $|\gamma_+| - |\gamma_-| - |A| = 1$. Define $|\gamma| = \mu_{c_2}(\gamma) + n - 3$.

Definition 2.2 γ is **bad** if γ is multiple cover of embedded γ' and $\mu_{\mathrm{cz}}(\gamma) - \mu_{\mathrm{cz}}(\gamma')$ is odd.

Example 2.1 For dim $\mathcal{M} = 3$, γ is bad if and only if γ is an even multiple cover of a negative hyperbolic one.

If there exists a contractible Reeb orbits:

$$\partial\gamma_+ = \#(\quad) + \sum_{\gamma_-} \#(\overset{\gamma_-}{\underset{\gamma_+}{\quad}}) \frac{1}{m(\gamma_-)}\gamma_- + \sum_{\gamma_-} \#(\overset{\gamma_+}{\underset{\gamma_{-1}\ \gamma_{-2}}{\quad}}) \frac{1}{m(\gamma_-)m(\gamma_2)}\gamma_{-1}\gamma_{-2} + \ldots$$

This gives the contact homology. One important variant of the contact homology is the (rational) symplectic field theory.

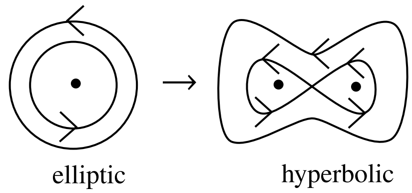
Example 2.2 In dim = 3, if γ is hyperbolic, then

$$\mu_{\mathrm{cz}}(\gamma, k) = k\mu_{\mathrm{cz}}(\gamma).$$

Let u be J-holomorphic. If ind $u = \mu(\gamma_+) - \mu(\gamma) - |A|$. If u is k-fold cover of u' and there are no elliptic orbits, then ind $u = k \cdot$ ind u'.

If u' transverses ind and ind $u' \geq 1$, then ind $u \geq k$. In the definition of ∂, ind $(u) = 1 \implies k = 1$.

We can always eliminate elliptic orbits up to any action:



These two images have the same contact structure, but different contact forms.

Chapter 3

Semi-Global Kuranishi Structure and Full Contact Homology

Abstract In this lecture, we introduce the semi-global Kuranishi structure. We explore its application in relation to obstruction bundle gluing, including computations of simple examples. The discussion will culminate in the rigorous definition of full contact homology, facilitated by the semi-global Kuranishi structure.

3.1 Introduction

Suppose $(W,\ d\alpha)$ is an exact symplectic manifold with $\partial W = W_+ + W_-$, $\alpha|_{W_\pm} = W_\pm$, and J an almost complex surface on $\hat{W}$. To count ind $= 0$ J-holomorphic cylinders, we obtain a chain map

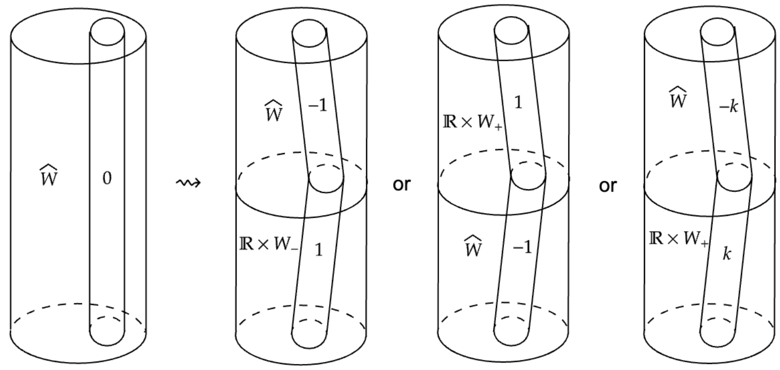
$$\Phi : c_*(W_+, \alpha_+, J_+) \to c_*(w_-, \alpha_-, J_-)$$

Given a 1-parameter family of this data $(W^2,\ d\alpha^2, J^2)$ we want $\Phi_0 < \Phi_1$ to induce the same map on homology:

$$\mathcal{M}(\gamma_+, \gamma_-) = \bigsqcup_{\iota} \{\iota\} \times \mathcal{M}^{\mathrm{ind}=0}_{J^2}(\gamma_+, \gamma_-).$$

We want to show that there exists a linear map $K : c_+(W_+, \alpha_+, J_+) \to c_*(W_-, \alpha_-, J_-)$ such that $\Phi_0 - \Phi_1 = K\partial + \partial K$.

We have:



For the right most cylinder, we can attach $\mathcal{U}$ to the top half and $\mathcal{M}$ to the bottom half.

For each $\mathcal{U}$, we have a map $\mathcal{S}^{-1} : O \to [R, \infty) \times M$ and a vector space attached to O defined by $\ker D_u^* / \mathbb{R}\langle Y \rangle$ where Y comes from variation of J^2 and $\mathcal{S}^{-1}(0)$ is the set of curves in M that glue with $\mathcal{U}$.

3.2 Semi-Global Kuranishi Structure

We start with Morse homology case. Let X be a closed manifold, $f : X \to \mathbb{R}$ be a Morse function, g a Riemannian metric, ϕ^t the time-t flow of $-\mathrm{grad}_f$, $p \in \mathrm{Crit}(f)$, $\mathcal{D}_p$ be the descending manifold of p, and $\mathcal{A}_p$ be the ascending manifold of p.

For any $p, q \in \mathrm{Crit}(f)$,

$$\mathcal{M}(p, q) = \left(\mathcal{A} + q \cap \mathcal{D}_p\right) / \mathbb{R}.$$

Definition 3.1 An **interior semi-global Kuranishi chart** is a quadruple $(K, \pi_v : E \to V, \mathcal{L} : V \to E, \psi)$ where

1. $K \subset M$ is a capacity.
2. $\pi_V : E \to V$ is a finite rank vector bundle of a finite dimensional manifold.
3. $\mathcal{L}$ is a section.
4. $\psi : \mathcal{L}^{-1}(0) \to M$ is homeomorphic onto an open sbuset of M containing K.
5. $\dim V - \mathrm{rank} E = \mathrm{vir}\dim \mathcal{M}$

We order the moduli space $\mathcal{M}_1, \mathcal{M}_2, \ldots$ such that the energy increase. Let

$$\mathcal{M}_i = \mathcal{M}(p_i, q_i), \qquad E(\mathcal{M}_i) = f(p_i) - f(q_i).$$

and suppose we have an index tuple $I = (i_1, ..., i_n)$ such that $p_{i_m} = q_{i_{m+1}}$. Suppose $S \subset I$ is a subindex tuple. Then I/S is the index tuple obtained by replacing S by an integer which is the index of $\mathcal{M}(p_{s_1}, q_{s_k})$ if $S = (s_1, ..., s_k)$, and $\mathbf{S} \subset I$ is a disjoint union of subindex tuples. We say $I < J$ if there exists $\mathbf{S} \subset J$ such that $I = J/\mathbf{S}$.

Definition 3.2 A **semi-global Kuranishi structure for** $\mathcal{M}_1, ..., \mathcal{M}_p$ for any $1 \leq i \leq p$ such that

1. For each I, such that $I/I = i$, $C_I = (\pi_I : E_I \to V_I, \mathcal{L}_{:} V_I \to E_I, \psi_I : \mathcal{L}_I^{-1}(0) \to M_i)$.
2. For each $I' < I$, we have the restriction inclusion: for $V_{I'I} \subset V_{I'}$ open, we have

$$\begin{tikzcd} & E_{I'}|_{V_{I'I}} \arrow[r, "\phi^{\#}_{I'I}"] \arrow[d] & E_I \arrow[d] \\ V_{I'} \arrow[r, phantom, "\supset"] & V_{I'I} \arrow[u, bend left, "\mathcal{L}_{I'}"] \arrow[r, hook, "\phi_{II'}"] & V_I \arrow[u, bend right, "\mathcal{L}_I"'] \end{tikzcd}$$

   where the top map is injective and the bottom map is an embedding
3. $\mathcal{L}_I \circ \phi_{I'I} = \phi^{\#}_{I'I}\mathcal{L}_{I'}|_{V_{II'}}$
4. $(\mathcal{L}_I)_*$ descends to an isomorphism:

$$(\mathcal{L})_I : TV_I/TV_{I\cdot I} \xrightarrow{\cong} E_I/E_I$$

5. Composition of $(\phi_{I'I}, \phi^{\#})$ is associative
6. $\mathcal{M}_i = \bigcup_{I, I/I=i} \psi_I(\mathcal{L}_I^{-1}(0))$

3.3 Strata Compatibility

Definition 3.3 For $I = (i_1, ..., i_m)$ define

$$\mathbb{V}_I = V_{i_1} \times ... \times V_{i_m} \times [R, \infty)^{m-1},$$
$$\mathbb{E}_I = E_{i_1} \oplus ... \oplus E_{i_m}.$$

Suppose that G_I a diffeomorphism onto its image, $G_I^{\#}$ a bundle isomorphism, and $\mathcal{L}_I$ are C'-close as $T_1, ..., T_{m-1} \to \infty$.

For all i, a bundle map satisfies the **strata compatibility** conditions if the following map commutes

$$\begin{tikzcd} \mathbb{E}_I \arrow[r, "G_I^{\#}"] \arrow[d] & E_I \arrow[d] \\ \mathbb{V}_I \arrow[r, "G_I"'] \arrow[u, bend left, "(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m)"] & V_I \arrow[u, bend right, "\mathcal{L}_I"'] \end{tikzcd}$$

Here, the perturbation section is

$$\sigma = \{G_I : V_I \to F_I \mid I/I = i\}$$

where σ_I transverses $\mathcal{L}_I$, σ_I is small, the section is compatible with $(\phi_{I'I}, \phi^{\#}_{I'I})$, and both σ_I and $(\sigma_{i_1, i_2, \ldots, i_n})$ are C^1-close as $T_1, \ldots, T_{n-1} \to \infty$.

The perturbed moduli space is

$$Z_i = \coprod_{I/I=i} \mathcal{L}_I^{-1}(\sigma_I)/\sim$$

3.4 Construction of Semi-Global Kuranishi Structures for Cylindrical Contact Homology

Suppose we have an interior chart $\mathcal{E} \to \mathcal{B}$ and $\overline{\partial}$ on the inverse.

Theorem 3.1 *Given any compact $K \subset \mathcal{M}$, there exists an integer and a neighborhood $N(K) \subset B$ and a rank ℓ sub-bundle $E \subset \mathcal{E}$ defined over $N(K)$ such that $\overline{\partial}$ transverses E over $N(K)$.*

Proof Pick $\epsilon \gg 0$ small. Find $s_0 \in \mathbb{R}$ such that $\mathcal{A}(u(s - s_0)) = \mathcal{A}(\gamma_+) - \epsilon$. Choose J such that $\overline{\partial}_J u = \partial_S u + Au = 0$ (where A is a linear self-adjoint operator) is a linear near ∞. □

Let f_i, λ_i be eigenvectors and eigenvalues of A ordered such that $\lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and consider

$$\tilde{f}_j(s,t) = \beta(s) f_k(t) \otimes (\,ds - i\,dt).$$

Then let $E = \mathrm{span}\{\tilde{f}_1, \ldots, \tilde{f}_\ell\}$. The claim is that E transverses $\overline{\partial}$. we have $V = \overline{\partial}^{-1}(E)$.

This involves proving the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_+ \oplus \mathcal{O}_- \arrow[r, "G_\#"'] \arrow[d] & \mathcal{O}_{+-} \arrow[d] \\
\mathcal{M}_+ \times \mathcal{M}_- \times [\mathbb{R}, \infty) \arrow[r, "G"'] \arrow[u, bend left, "{(0,0)}"] & V \arrow[u, bend right, "\mathcal{L}"']
\end{tikzcd}$$

Part II

Mike Usher: Quantitative Symplectic Geometry

There were three lectures:

1. Day 1: Symplectic Embedding Obstructions and Constructions

   The question of which subsets of $\mathbb{R}^{2n}$ embed symplectically into which others has turned out to be quite rich and has led to the development of many techniques over the past 40 years. In my first lecture, I will explain proofs of classic results of Gromov that give obstructions to symplectic squeezing and packing, and will contrast this with cases where an explicit construction allows one to give a non-obvious positive answer to a symplectic embedding question.

2. Day 2: Capacities and Symplectic Homology

   The second lecture will formally introduce the notion of a symplectic capacity, and will discuss two examples of these: the Hofer-Zehnder capacity based on periodic orbits of Hamiltonian systems, and the Floer-Hofer-Wysocki capacity based on symplectic homology.

3. Day 3: Obstructing Embeddings Using Equivariant Symplectic Homology

   The third lecture will explain how S^1-equivariant symplectic homology supplies additional restrictions on symplectic embeddings, both via a sequence of capacities coming from spectral invariants associated to various homology classes, and via chain-level information that vanishes in homology but can in some cases be used to show that two known embeddings are not symplectically isotopic.

Chapter 4

Symplectic Embedding Obstructions and Constructions

Abstract The question of which subsets of $\mathbb{R}^{2n}$ embed symplectically into which others has turned out to be quite rich and has led to the development of many techniques over the past 40 years. In my first lecture, I will explain proofs of classic results of Gromov that give obstructions to symplectic squeezing and packing, and will contrast this with cases where an explicit construction allows one to give a non-obvious positive answer to a symplectic embedding question.

4.1 Introduction

Here are some examples of questions that symplectic geometers care about:

1. Suppose $X(\mathbf{r})$ and $Y(\mathbf{s})$ are symplectic manifolds depending on parameters $\mathbf{r}$ and $\mathbf{s}$. For what values of $\mathbf{r}$ and $\mathbf{s}$ do there exist symplectic embeddings $X(\mathbf{r}) \hookrightarrow Y(\mathbf{s})$? For example, do there exist symplectic embeddings $X(\mathbf{r}) \hookrightarrow Y(\mathbf{s})$ where
$$X(\mathbf{r}) = X(a) = B^{2n}(a) = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid \pi \sum (x_j^2 + y_j^2) \leq a\right\}$$
$$Y(\mathbf{s}) = E^{2n}(s_1, \ldots, s_n) = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid \pi \sum \frac{(x_j^2 + y_j^2)}{s_j} \leq 1\right\}$$
with $a > 0$?
2. If $X \subset \mathbb{R}^{2n}$ is a domain with contact type boundary, what can be said about the action of closed characteristics on ∂X? What is the connection between this question and the previous one?
3. For $\phi : M \to M$ a Hamiltonian diffeomorphism, what happens to the actions of fixed points of ϕ^k, denoted #Fix (ϕ^k), and the Hofer norm $|| \ \phi^k \ ||_{\text{Hofer}}$ as $k \to \infty$?

For the rest of this talk, we will mostly focus on the first question, partially because the first result that got mathematicians interested in studying quantitative symplectic geometry arose from this problem.

Theorem 4.1 (Gromov's Non-Squeezing Theorem, 1985)

Let

$$B^{2n}(a) = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid \pi \sum (x_j^2 + y_j^2) \leq a\right\}$$

and

$$Z^{2n}(A) = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid \pi (x_j^2 + y_j^2) \leq A\right\} = B^2(A) \times \mathbb{R}^{2n-2}.$$

Then there exists a symplectic embedding $B^{2n}(a) \hookrightarrow Z^{2n}(A)$ only if $a \leq A$.

Basically, this result tells us that we cannot squeeze a ball into a cylinder of smaller radius while preserving the symplectic structure.

4.2 Proof of Gromov's Non-Squeezing Theorem

We present a proof sketch.

Suppose there exists a symplectic embedding $\phi : B^{2n}(a) \hookrightarrow Z^{2n}(A)$. Let $\epsilon > 0$. We want to show that $a \leq A$, so it suffices to show that $a - \epsilon < A + \epsilon$.

Choose L such that $\mathrm{Im}(\phi) \subset B^2(A) \times (-L, L)^{2n-2}$. Regard $B^2(A)$ as a subset of $S^2(A + \epsilon)$, the 2-sphere with area $A + \epsilon$. Then ϕ can be regarded as having image in $(M, \omega) = (S^2(A + \epsilon) \times \mathbb{R}^{2n}/2L\mathbb{Z}^{2n-2})$ with symplectic form $\omega_{\text{std}} \oplus \omega_{\text{std}}$.

There are 2 key facts:

1. For any ω-compatible almost complex structure J on M, there exists a J-holomorphic map $u : S^2 \to M$ such that $\phi(\mathbf{0}) \subset \mathrm{Im}(u)$ and $u_*[S^2] = [S^2 \times \{\text{pts}\}] \in H_c(M)$. In particular, this applies to J's agreeing on $\phi(B^{2n}(a - \frac{\epsilon}{2}))$ with $\phi_* J_0$, where J_0 is the standard complex structure on $B^{2n}(u - \frac{\epsilon}{2})$.
2. For a J_0-holomorphic map $v : \sum \to B^{2n}(c)$ where $\sum$ is a compact surface with boundary and $c \in (a - \epsilon, a - \frac{\epsilon}{2})$ such that $v(\partial \sum) \subset \partial B^{2n}(c)$ and $\mathbf{0} \subset \mathrm{Im}(v)$, then $\mathrm{Area}(v) \geq c$.

Together, these two facts prove the desired claim: assuming we have both key facts, for generic $c \in (a - \frac{\epsilon}{2}, a - \epsilon)$, take $\sum u^{-1}(\phi(B^{2n}(c)))$ and take $v : \sum \to B^{2n}(c)$ with $v = \phi^{-1} \cdot u \mid_{\Sigma}$. Then

$$\begin{aligned} c &\leq a - \frac{\epsilon}{2} \\ &\leq \text{Area}(v) = \int_\Sigma v^* \omega_0 = \int_\Sigma u^* \omega = \text{Area}(u \mid_\Sigma) \\ &\leq \text{Area}(u) = A + \epsilon \end{aligned}$$

The idea of the proof of (1) is as follows:

Proof For any ω-compatible J, consider

$$M_J = \left\{ u : S^2 \to M \mid u_*[S^2] = S^2 \times \{\text{pt}\}, U(0,0,1) = \phi\left(u\left(\mathbf{0}\right)\right) \right\}.$$

If J is the standard complex structure J_0, M_{J_0} has one element. For contradiction, suppose $M_{J_1} = \phi$. For a generic path $\{J_t\}_{0 \leq t \leq 1}$,

$$\bigcup_{t \in [0,1]} \{t\} \times M_{J_t}$$

would be a compact 1-manifold with boundary equal to M_{J_0}. But compact 1-manifolds have an even number of boundary points, contradiction. □

We will not prove (2).

4.3 4-Dimensional Packing Problem

Problem 4.1 (4-Dimensional Packing Problem)

Given $k \in \mathbb{N}, a > 0$, does there exist a symplectic embedding $\amalg_k B^4(a) \hookrightarrow B^4(1)$?

But we have the following relation:

Proposition 4.1 (McDuff-Polterovich)

$$B^4(1) = \mathbb{CP}^1$$

So we can also study symplectic embeddings $\amalg_k B^4(a) \hookrightarrow \mathbb{CP}^1$ instead.

We have $\text{Vol}(\amalg_k B^4(a)) = k\frac{a^2}{2}$ and $\text{Vol}(B^4(1)) = \frac{1}{2}$ so the fraction of volume filled is ka^2 - can this a be close to 1?

Theorem 4.2 (2-Ball Theorem)

If $k = 2$, this embedding exists only if $a \leq \frac{1}{2}$.

Proof We present two proofs:

1. If $\varphi_1, \varphi_2 : B^4(a) \hookrightarrow \mathbb{CP}^2(1)$ are symplectic embeddings with disjoint images, find a holomorphic curve for a J agreeing with $\phi_{1*}J_0, \phi_{2*}J_0$ on images, passing through $\phi_j(\mathbf{0})$ and representing $\mathbb{CP}^1$ in H_2. Then

$$1 = \text{Area}(u) \geq a + a \implies a \leq \frac{1}{2}.$$

2. Blowup the images from the balls, giving a symplectic form on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ such that $[\mathbb{CP}^1]$ has area 1 and $\#2\overline{\mathbb{CP}^2}$ are exceptional divisors on E_1, E_2, and E_1, E_2 have area a. But $[\mathbb{CP}^1] - E_1 - E_2$ is represented by holomorphic spheres which have area ≥ 0, which implies $1 - 2a \geq 0 \implies a \leq \frac{1}{2}$. □

Chapter 5
Capacities and Symplectic Homology

Abstract The second lecture will formally introduce the notion of a symplectic capacity, and will discuss two examples of these: the Hofer-Zehnder capacity based on periodic orbits of Hamiltonian systems, and the Floer-Hofer-Wysocki capacity based on symplectic homology.

5.1 Symplectic Capacities

Let $\mathcal{C}$ be some collection of $2n$-dimensional symplectic manifolds such that for all $(M, \omega) \in \mathcal{C}$ and for all $a > 0$, then $(M, a\omega) \in \mathcal{C}$.

Definition 5.1 A **symplectic capacity on** $\mathcal{C}$ is a function $c : \mathcal{C} \to [0, \infty]$ satisfying:

1. **Monotonicity**: if there exists a symplectic embedding $(M, \omega) \hookrightarrow (M', \omega')$, then $c(M, \omega) \leq c(M', \omega')$.
2. **Comformality**: $c(M, a\omega) = ac(M, \omega)$.
3. **Nontriviality**: $c(B^{2n}(1)) > 0$ and $c(Z^{2n}(1)) < \infty$.

Remark 5.1 A stronger version of (3) is **normalization**: $c(B^{2n}(1)) = c(Z^{2n}(1)) = 1$.

Often, we have $\mathcal{C} \subset \left\{\text{subsets of } \mathbb{R}^{2n}\right\}$. Then, instead of scaling via $(U, \omega_0) \to (U, a\omega)$, we'll keep the same ω_0, and replace U by $aU := \left\{\sqrt{a}\mathbf{z} \mid \mathbf{z} \in U\right\}$.

Let's look at some examples of capacities:

Example 5.1 (Gromov Capacity)

The **Gromov capacity** is defined as

$$c_B(M,\omega) := \sup\left\{a \mid \exists \text{ symplectic embedding } B^{2n}(a) \hookrightarrow (M,\omega)\right\}.$$

One major source of symplectic capacities are Hamiltonian systems, and we can study symplectic capacities via filtered Floer homology or symplectic homology.

Here is how symplectic capacities arise from Hamiltonian systems: suppose (M,ω) is a symplectic manifold and $F : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ is smooth with $(t,m) \mapsto H_t(m)$. Then the time-dependent Hamiltonian X_{H_t}, given by $\omega(\cdot, X_{H_t}) = dH_t$, gives a function $\varphi_H^t : M \to M$ satisfying

$$\frac{d\varphi_H^t(m)}{dt} = X_{H_t}(\varphi_H^t(m)).$$

Here is another example of a capacity:

Example 5.2 (Hofer-Zehnder Capacity)

The **Hofer-Zehnder capacity** is defined as

$$c_{\mathrm{HZ}}(M,\omega) := \sup\{\max H \mid \text{condition}\}$$

where the condition is that $H : M \to \mathbb{R}$, $H = 0$ on some open set compactly supported, and the only periodic orbits of X_H having period < 1 are constants.

5.2 (Filtered) Floer Homology

For the rest of today, assume $\omega = d\lambda$. For example, take the Euler vector field $E_X, M \subset \mathbb{R}^{2n}$, and $\lambda = \lambda_0 := \frac{1}{2}\sum_j (x_j dy_j - y_j dx_j)$.

For H as above, let $\mathrm{HF}(H)$ be the Morse homology for the action functional $\mathcal{A}_H : C^\infty(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$ defined by

$$\mathcal{A}_H(\gamma) = -\int_{S^1} \gamma^*\lambda + \int_0^1 H(t,\gamma(t))dt.$$

Additionally, define the critical points

$$\mathrm{Crit}(\mathcal{A}_H) := \left\{\gamma : \mathbb{R}/\mathbb{Z} \to M \mid \gamma'(t) = X_{H_t}(\gamma(t))\right\}.$$

There is a 1-to-1 correspondence between $\mathrm{Crit}(\mathcal{A}_H)$ and $\mathrm{Fix}(\varphi_H')$ given by $\gamma \mapsto \gamma(0)$.

The negative gradient flowlines of $\mathcal{A}_H$ are described by the function $u : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ such that

$$\frac{\partial u}{\partial s} + J_t(u(s,t))\left(\frac{\partial u}{\partial t} - X_{H_t}\right) = 0.$$

Additionally, we have $\mathrm{CF} = \mathrm{span}\,\{\mathrm{Crit}(\mathcal{A}_H)\}$, graded by μ_{CZ}, with the boundary operator $\partial : \mathrm{CF}_k(H) \to \mathrm{CF}_{k-1}(H)$ defined by $\partial\gamma_- = \sum \#(\text{flowlines from } \gamma_- \text{ to } \gamma_+)\gamma_+$, satisfying $\mathrm{ind}(\gamma_-) - \mathrm{ind}(\gamma_+) = 1$.

Proposition 5.1 *Assuming the conditions at infinity guarantee spaces are compact:*

- $\partial^2 = 0$*, which gives the homology* $HF(H)$.
- *If* H *and* H' *satisfy the same conditions at* ∞*, then there exists continuation chain maps* $CF(H) \rightleftarrows CF(H')$ *inducing isomorphisms on homology.*

The filtered version can be given as follows: given $t \in \mathbb{R}$, define $\mathrm{CF}^t(H) = \mathrm{span}\{\gamma \in \mathrm{Crit}(\mathcal{A}_H) \mid \mathcal{A}_H(\gamma) = t\}$. $\mathcal{A}_H$ decreases along its negative gradient flowlines, so $\partial(\mathrm{CF}^t(H)) \subset \mathrm{CF}^t(H)$. This gives a homology $\mathrm{HF}^t(H)$ for all $t \in \mathbb{R}$, with inclusion-induced $\mathrm{HF}^s(H) \to \mathrm{HF}^t(H)$ for $s \leq t$.

If $H \geq H'$, we have continuation chain maps $\mathrm{CF}^t(H) \to \mathrm{CF}^t(H')$, so for $s \leq t, H \geq H'$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{HF}^s(H) & \longrightarrow & \mathrm{HF}^s(H') \\ \downarrow & & \downarrow \\ \mathrm{HF}^t(H) & \longrightarrow & \mathrm{HF}^t(H') \end{array}$$

5.3 Liouville Embeddings

Definition 5.2 A **Liouville domain** (W, ω) is a compact manifold with ∂, W, together with a $\lambda \in \Omega^1(W)$ such that $d\lambda$ is symplectic and the Liouville vector field V_λ, defined by $d\lambda(V_\lambda, \cdot) = \lambda$, points outward along ∂W.

Example 5.3 For $\lambda_0 = \frac{1}{2}\sum(x_j\,dy_j - y_j\,dx_j)$, we have $V_{\lambda_0} = \frac{1}{2}\sum(x_j\partial x_j + y_j\partial_{y_j})$, so we can take W to be the strongly star-shaped around the region in $\mathbb{R}^{2n}$.

In this case, $\alpha := \lambda\mid_{\partial W}$ for $\alpha \in \Omega^1(\partial W)$ is a contact form on ∂W, and there exists a collar neighborhood U of ∂W such that $(U, \lambda) \approx ([1-\epsilon, 1] \times \partial W, r\alpha)$. From the completion, we have

$$(\hat{W}, \hat{\lambda}) = (W, \lambda) \bigcup_{\partial W} ([1, \infty) \times \partial W, r\alpha).$$

So in this case, $\hat{W} \simeq \mathbb{R}^{2n}$ is the **Liouville isomorphism**.

Definition 5.3 Given a Liouville domain (W, λ), a W**-admissible Hamiltonian** is a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \hat{W} \to \mathbb{R}$ such that $H(t, w) \geq 0$ and for all $w \in W$, there exist

- $a > 0$ that is not the period of any closed orbit R_α
- $b \in \mathbb{R}$ such that for (r, y) where $r \in [1, \infty)$ and $y \in \partial W$
- $H(t(r, y)) = -ar + b$.

The condition that a is not the period of any closed orbit R_α implies that $\mathrm{Fix}(\varphi_H^t) \subset W$. The condition that $H(t, (r, y)) = -ar+b$ implies that $X_H = -aR_\alpha$ on $(1, \infty)\times\partial W$.

Let $H_W = \{W\text{-admissible Hamiltonians}\}$. Then we can associate H with $\mathrm{HF}^t(H)$ and if $H \geq H'$, we have a map $\mathrm{HF}^t(H) \to \mathrm{HF}^t(H')$.

Definition 5.4 For all $t \in \mathbb{R}$, define

$$\mathrm{SH}^t(W) := \varinjlim_{H \in \mathcal{H}_W} \mathrm{HF}^t(H)$$

via the map we just mentioned.

So we have $\mathrm{SH}^s(W) \to \mathrm{SH}^t(W)$ for all $s \leq t$.

Proposition 5.2 *$SH^\infty(W)$ depends only on the Liouville isomorphism type of $\hat{W}$. For $W \subset \mathbb{R}^{2n}$ strongly star-shaped, then $SH^\infty(W) = 0$.*

Proposition 5.3 *If $W' \subset W^0$, we get continuation maps from W-admissible Hamiltonians to W'-admissible Hamiltonians. This gives transfer maps*

$$SH^t(W) \to SH^t(W').$$

More generally:

Proposition 5.4 *For a Liouville embedding $\varphi : W' \hookrightarrow W^0$ (meaning $\varphi^*\lambda - \lambda' = d \cdot f$ for some function f), we get $\varphi' : SH^t(W) \to SH^t(W')$.*

5.4 Floer-Hofer-Wysocki Capacity

Let's discuss Floer-Hofer-Wysocki Capacity for $W \subset \mathbb{R}^{2n}$ strongly star-shaped. We have the following result:

Proposition 5.5 *For $\epsilon > 0$,*

$$SH^t_n(W) = \mathbb{Q}.$$

with

$$SH^{\epsilon'}_n(W) \xrightarrow{\sim} SH^{\epsilon}(W)$$

for all $\epsilon' < \epsilon$.

This motivates the following definition:

Definition 5.5 The **Floer-Hofer-Wysocki capacity** is

$$c_{\mathrm{FHW}}(W) := \inf\left\{t \mid \mathrm{SH}^{\epsilon}_n(W) \to \mathrm{SH}^t(W) = 0 \text{ for small } \epsilon\right\}$$

Theorem 5.1 *This is a capacity.*

Proof We prove the three conditions:

- Monotonicity: follows from transfer maps:

$$\begin{tikzcd} \mathrm{SH}^{\epsilon}(W) \arrow[r, "\cong", "\varphi'"'] \arrow[d] & \mathrm{SH}^{\epsilon}(W') \arrow[d] \\ \mathrm{SH}^{t}(W) \arrow[r, "\varphi"'] & \mathrm{SH}^{t}(W') \end{tikzcd}$$

  where the left map if 0 if $t > c_{\mathrm{FHW}}(W)$ and the right map is 0 if $t > c_{\mathrm{FHW}}(W')$.
- Conformality: follows from behavior of action under scaling ω.
- Nontriviality: holds by construction. □

Example 5.4 For $W = E(a_1, ..., a_n) = \left\{\sum \frac{\pi(x_j^2+y_j^2)}{a_j} \leq 1\right\}$ with $a_1 \leq ... \leq a_n$, we have $c_{\mathrm{FHW}(W)} = a_1$.

Chapter 6

Obstructing Embeddings Using Equivariant Symplectic Homology

Abstract The third lecture will explain how S^1-equivariant symplectic homology supplies additional restrictions on symplectic embeddings, both via a sequence of capacities coming from spectral invariants associated to various homology classes, and via chain-level information that vanishes in homology but can in some cases be used to show that two known embeddings are not symplectically isotopic.

6.1 Symplectic Homology

Suppose we have (W, λ) a Liouville domain, $\eta = \partial W$, a contact form $\alpha = \lambda_Y$, and C a scalar of Y such that $(C\lambda) \cong ((1-\epsilon, 1] \times Y, r\alpha)$. The completion is

$$\hat{W} := W \bigcap_{Y} ([1, \infty] \times Y, r\alpha).$$

Define

$$\mathcal{H}_W := \{W\text{-admissible Hamiltonians}\}$$

where $H : \mathbb{R}/\mathbb{Z} \times \hat{W} \to \mathbb{R}$ such that $H\,|_{\mathbb{R}/\mathbb{Z}\times H} \geq 0$ and for $r \geq 1$, $y \in Y$, $H(t, (r, y)) = -ar + b$ for $a > 0$, $b \in \mathbb{R}$ such that a is not the closed period of any Reeb orbit for α.

Suppose $H, H \in \mathcal{H}_W$ and $H \geq H'$. Then for $t \in \mathbb{R}$ autonomous, we have construction maps $\mathrm{HF}^t(H) \to \mathrm{HF}^t(H')$.

Definition 6.1

$$\mathrm{SH}^t(W) = \varinjlim_{H \in \mathcal{H}_W} \mathrm{HF}^t(H).$$

A family of H's approaching the limit might look like the following: take $H\mid_W$ to be a small Morse function $W \to [0, \infty)$ that is C^2 small in the complement of the collar, with $H\mid_{\partial W} = 0$.

- On $C \cong (1-\epsilon, 1] \times Y$, we have $H(r, y) = -h(r)$, where h' increases rapidly from $\delta \approx 0$ to $a \gg 0$.
- On $\hat{W} \backslash W = (1, \infty) \times Y$, we have $H(r, y) = -a(r-1)$.

Note that the Hamiltonian vector field given by $H(r, y) = -h(r)$ is $X_H = -h'(r)R_\alpha$.

Recall that for $\gamma : \mathbb{R}/\mathbb{Z} \to M$,

$$\mathcal{A}_H(\gamma) = -\int_{\mathbb{R}/\mathbb{Z}} \gamma^*\lambda + \int_0^1 H(t, \gamma(t))\, dt.$$

For these H, Crit $\mathcal{A}_H$ consist of

- Constants at critical points of $H_{W\backslash C}$ with $\mathcal{A}_H = H(p) \approx 0$.
- In $C = [1-\epsilon, 1] \times Y$, the critical points are given by reparameterizations of closed orbits of R_α with period $h'(r)$, with $\mathcal{A}_H \approx h'(r)$.

If $t_0 <$ the minimal action of a Reeb orbit,

$$\begin{aligned} \mathrm{SH}_*^t(W) &= H_{*+n}(\text{Morse complex of } H\mid_W) \\ &= H_{*+n}(W, Y). \end{aligned}$$

Once t is bigger than the minimal period of a Reeb orbit, SH^t is affected by the Reeb orbits, any of which gives an S^1 family in Crit($\mathcal{A}_H$). Morse-Bott perturbation splits these into two orbits different in index by 1, both with action approximately the period of the original orbit.

Example 6.1 (Ellipsoid)

Consider the ellipsoid

$$E(a_1, \ldots, a_n) = \left\{ \sum_j \frac{\pi(x_j^2 + y_j^2)}{a_j} \leq 1 \right\}.$$

with $a_1 < \ldots < a_n$. Assume that a_j are linearly independent over $\mathbb{Q}$. The Reeb orbits are circles in $x_j y_j$ planes (and their m-fold covers) with action ma_j. After Morse-Bott perturbation, there exists 1 orbit in each index starting at $n+1$ arranged in increasing order of action.

6.2 Positive Symplectic Homology

The small variation on SH^t is the

$$\text{positive SH} = \mathrm{SH}^{+,t} = \varinjlim_{H} H_* \left(\frac{\mathrm{HF}^t(H)}{\mathrm{HF}^\epsilon(H)} \right)$$

for $\epsilon \ll 1$.

Example 6.2 For $W \subset \mathbb{R}^{2n}$ star-shaped,

$$\mathrm{SH}_*^{+,\infty}(W) = \begin{cases} \mathbb{Q} & * = n+1 \\ 0 & \text{otherwise} \end{cases}$$

The larger filtration on SH^t is the filtered positive S^1-equivariant symplectic homology. Formally, do filtered Morse theory on

$$\frac{C^\infty(S^1, \hat{W}) \times S^\infty}{S^1}.$$

With $\mathbb{Q}$ coefficients, we obtain the filtered homology of a complex with 1 generator per Reeb orbit (instead of 2).

Example 6.3 Abbreviate $\mathrm{CH}^t = \mathrm{SH}_*^{S^1,+,t}$. For the ellipsoid case,

$$\mathrm{CH}_*^\infty(\text{any strongly star-shaped domain in } \mathbb{R}^{2n}) = \begin{cases} \mathbb{Q} & * = n-1+2k \\ 0 & \text{otherwise} \end{cases}$$

where $k \geq 1$.

6.3 The Gutt-Hutchings Capacity

Definition 6.2 (Gutt-Hutchings Capacity)

For $k \geq 1$ and W strongly star-shaped $\subset \mathbb{R}^{2n}$, the **Gutt-Hutchings capacity** is

$$c_k^{\mathrm{GH}}(W) := \inf\{t \mid \mathrm{CH}_{n-1+2k}^t(W) \to \mathrm{CH}_{n-1+2k}^\infty(W) \text{ is nonzero}\}.$$

Example 6.4

$$c_k^{\mathrm{GH}}(E(a_1, ..., a_k)) = k^{\text{th}} \text{ smallest number among } ma_j$$

for $m \geq 1$, $j = 1, \ldots, n$.

Example 6.5

$$c_k^{\mathrm{GH}}(B^2(a_1) \times \ldots \times B^2(a_n)) = k \min a_j.$$

6.4 Symplectic Banach-Mazur Distances

We quickly introduce a different problem for the remainder of the class.

Definition 6.3 The **symplectic Banach-Mazur distance** is given by

$$\delta(X, Y) := \left\{\lambda \mid \exists \text{ Liouville embedding } \varphi : X \hookrightarrow \text{ such that } \lambda^{-1}(Y) \subset \varphi(X) \subset \lambda Y\right\}$$

A high-level overview to approach problems related to the symplectic Banach-Mazur distance is the following trick:

$$\mathrm{CH}_k^t(\lambda Y) \to \mathrm{CH}_k^t(X) \to \mathrm{CH}_k^t\left(\lambda^{-1} Y\right)$$

is equivalent to

$$\mathrm{CH}_k^{\lambda^{-1} t}(Y) \to \mathrm{CH}_k^t(X) \to \mathrm{CH}_k^{\lambda t}(Y).$$

If no such factorization exists, $\delta(X, Y) > \lambda$.

Part III

Mohan Swaminathan: Global Kuranishi Charts

There were three lectures:

1. Day 1: Local Structure of Holomorphic Curve Moduli Spaces

   We discuss how the implicit function theorem and gluing analysis give rise to 'local Kuranishi charts' for holomorphic curve moduli spaces. We also explain what it means for two or more such local Kuranishi charts to be 'compatible' on their overlap and briefly discuss how an atlas of Kuranishi charts allows us to 'virtually count' points in a compact moduli space of expected dimension 0.

2. Day 2: Global Kuranishi Charts: Definitions and Preliminaries

   We introduce the notion of a 'global Kuranishi chart' and explain how having one of these substantially simplifies the previous discussion. We also explain what it means for two global Kuranishi charts to be 'equivalent', which is analogous to the notion of compatibility for local Kuranishi charts. For the remainder, we discuss some geometric preliminaries necessary to understand the construction of global Kuranishi charts for moduli spaces of closed holomorphic curves of genus 0.

3. Day 3: Global Kuranishi Charts: Construction

   Following Abouzaid–McLean–Smith 2021, we explain the construction of global Kuranishi charts for genus 0 Gromov–Witten moduli spaces and show that the outcome of the construction is unique up to equivalence. Time permitting, we will also briefly discuss how one can extend this construction to settings beyond genus 0 GW theory.

Chapter 7

Local Structure of Holomorphic Curve Moduli Spaces

Abstract We discuss how the implicit function theorem and gluing analysis give rise to 'local Kuranishi charts' for holomorphic curve moduli spaces. We also explain what it means for two or more such local Kuranishi charts to be 'compatible' on their overlap and briefly discuss how an atlas of Kuranishi charts allows us to 'virtually count' points in a compact moduli space of expected dimension 0.

7.1 The Moduli Space

Suppose (X^{2n}, ω) is a closed symplectic manifold, $A \in H_2(X;\mathbb{Z})$ with $m \geq 0$, and J is an almost complex structure X tamed by ω, i.e. $\omega(v, Jv) > 0$ where $0 \neq v \in T_X X$. Let U be a J-holomorphic map $\mathbb{CP}^1 \to X$. Here, du is linear and $u_*[\mathbb{CP}^1] = A$. This gives the moduli space $\mathcal{M}_{0,m}(X, A, J)$.

Instead of just consider $\mathbb{CP}^1$, we can also consider more complicated objects where we glue together several $\mathbb{CP}^1$ components at nodes, giving a tree of $\mathbb{CP}^1$. The curves arising from trees of $\mathbb{CP}^1 \to X$ are called **nodal genus 0 curves**.

These nodal genus 0 curves allow us to enrich our moduli space. In particular, we can now study

$$\overline{\mathcal{M}}_{0,m}(X, A, J) = \left\{ \left(\sum x_1, ..., x_m, u : \Sigma \to X \right) |\text{condition} \right\} /\sim$$

where the condition is that Σ is a nodal genus 0 curve with m marked points $x_1, ..., x_m$, u is J-holomorphic, $u_*[\Sigma] = A$ and $\#\mathrm{Aut}(\Sigma, x_1, ..., x_m, u) < \infty$. This condition is equivalent to the condition that on any component of Σ where u is constant, there are 3 special points.

We present two basic properties of this moduli space:

Theorem 7.1 (Gromov Compactness Theorem)

$\overline{\mathcal{M}}_{0,m}(X, A, J)$ is compact and Hausdorff.

Theorem 7.2 *$\overline{\mathcal{M}}_{0,m}(X, A, J)$ has an expected/virtual dimension*

$$d = 2(m - 3 + n + c_1(TX) \cdot A).$$

Here is the motivating problem:

Problem 7.1 Can we count the number of points in $\overline{\mathcal{M}}_{0,m}(X, A, J)$ after cutting down the virtual dimension to 0?

The issue is that in general, the moduli space is not a manifold of the expected dimension.

7.2 The Transverse Case

Problem 7.2 Suppose that we are given $(\Sigma, x_1, \ldots, x_m, u : \Sigma \to X) \in \overline{\mathcal{M}}_{0,m}(X, A, J)$? What is the local structure of $\overline{\mathcal{M}}_{0,m}(X, A, J)$ near this point?

We have two cases: when Σ is smooth and when Σ is nodal.

7.2.1 Σ is Smooth

Here, smooth means $\Sigma \cong \mathbb{CP}^1$. Recall that

- $\mathcal{B} = C^\infty(\Sigma, X)_A = \{v : \Sigma \to X | v \text{ is } C^\infty, v_*[\Sigma] = A\}$, which we should think of an infinite dimensional manifold.
- $\mathcal{E}$ is an infinite rank vector bundle on $\mathcal{B}$ whose fiber is $\Omega^{0,1}(\Sigma, v^*TX) = \overline{\mathrm{Hom}_{\mathbb{C}}}(T\Sigma, v^*TX)$ over any $v \in g\mathcal{B}$.
- σ is a section of $\mathcal{E}$ over $\mathcal{B}$ given by $\left(v \mapsto \overline{\partial}_J v\right)$ where $\overline{\partial}_K = \frac{1}{2}(dv + J(v) \cdot dv \cdot j_\Sigma)$

Note that $\sigma^{-1}(0) = \mathrm{Hol}(\Sigma, X, A, J)$.

We know that $u \in \sigma^{-1}(0)$, which means σ has a well-defined linearization of u, namely

$$D_u\sigma : T_u\mathcal{B} \to \mathcal{E}_u.$$

More explicitly, this map is given by

$$D(\overline{\partial}_J)_u\sigma : \Omega^0(\Sigma, u^*TX) \to \Omega^{0,1}(\Sigma, u^*TX).$$

In holomorphic local coordinates, $z = s + it$ on Σ,

$$\overline{\partial}_J u = \frac{1}{2}\left(\frac{\partial U}{\partial s} + J(u)\frac{\partial u}{\partial t}\right) \otimes (ds - idt).$$

Given $\xi \in \Omega^0(\Sigma, u^*TX)$, we want to look at $\frac{d}{d\epsilon}|_{\epsilon=0}(u + \epsilon\xi)$. We have

$$D\left(\overline{\partial}_J\right)_u \xi = \frac{1}{2}\left(\frac{\partial \xi}{\partial s} + J(u)\frac{\partial \xi}{\partial t} + \left(\partial_\xi J\right)(u)\frac{\partial u}{\partial t}\right) \otimes (ds - idt).$$

We can check that $\dfrac{\partial \xi}{\partial s} + J(u)\dfrac{\partial \xi}{\partial t}$ is 1st order and $\left(\partial_\xi J\right)(u)\dfrac{\partial u}{\partial t}$ is 0th order.

For notation purposes, write $D_u := D(\overline{\partial}_J)_u$. We have D_u is Fredholm, i.e. ker, coker are finite dimensional, and

$$\text{ind}(D_u) = \dim(\ker D_u) - \dim(\text{coker} D_u) = 2(n + c_1(TX) \cdot A)$$

where the second equality comes from Riemann-Roch.

The implicit function theorem tells us that if D_u is surjective, then $\mathcal{M}_{0,m}(X, A, J)$ is a orbifold of expected dimension near $(\Sigma, x_1, ..., x_m, u)$.

We conclude that

$$\mathcal{M}_{0,m}(X, A, J) = \text{Hol}(\mathbb{CP}^1, X, A, J) \times \left(\left(\mathbb{CP}^1\right)^m \setminus \Delta\right) / \text{PSL}_2(\mathbb{C})$$

where $\Delta = \left\{(x_1, ..., x_m) | x_i = x_j \text{ for some } i \neq j\right\}$.

7.2.2 Σ is Nodal

Here, nodal means $\Sigma \cong$ trees of $\mathbb{CP}^1$'s. Consider $\tilde{\Sigma}$, the normalization of Σ where we disjoint the spheres. The map

$$\tilde{u} : \tilde{\Sigma} \xrightarrow{\text{normalization}} \Sigma \xrightarrow{u} \Sigma$$

gives

$$D_{\tilde{u}} : \Omega^0\left(\tilde{\Sigma}, \tilde{u}^*TX\right) \to \Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX)$$

and

$$D_u : \Omega^0\left(\Sigma, u^*TX\right) \to \Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX)$$

where $\Omega^0(\Sigma, u^*TX) \subseteq \Omega^0\left(\tilde{\Sigma}, \tilde{u}^*TX\right)$.

Exercise 7.1 Check that $\mathrm{ind}(D_u) = 2(n + c_1(TX) \cdot A)$

Theorem 7.3 (Gluing Theorem)

If D_u is surjective, then $\overline{\mathcal{M}}_{0,m}(X, A, J)$ has a local chart near $(\Sigma, x_1, ..., x_m, u)$ of the form V/Γ where V is a vector space of the expected dimension and Γ is a finite group acting linearly on the vector space.

7.3 Local Kuranishi Charts

Let $\mathcal{B}$ be a Banach manifold, $\mathcal{E}$ a Banach vector bundle, and J a smooth section with Fredholm linearizations. We want to study $\overline{\mathcal{M}} = \left(D\overline{\partial}\right)^{-1}(0) \subset \mathcal{B}$.

Suppose $u \in \overline{\mathcal{M}}$ is given. If $\left(D\overline{\partial}\right)_u$ is surjective, then we are done, i.e. $\overline{\mathcal{M}}$ is a manifold near u and $T_u\overline{\mathcal{M}} = \ker\left(D\overline{\partial}\right)_u$.

So let's assume that $D_u : T_u\mathcal{B} \to \mathcal{E}_u$ is not surjective. Choose a finite dimensional vector space V and a linear map $\lambda : E \to \mathcal{E}_u$ such that $E \twoheadrightarrow \mathrm{coker} D_u$, i.e. $D_u \oplus \lambda : T_u\mathcal{B} \oplus E \twoheadrightarrow \mathcal{E}_u$. Choose a neighborhood $u \in \mathcal{U} \subset \mathcal{B}$ and an extension $\lambda : \mathcal{U} \times v \to \mathcal{E}_{\mathcal{U}}$.

Now, consider

$$\mathcal{M}_{\mathcal{U},E,\lambda} = \left\{v \in \mathcal{U}, e \in E | \overline{\partial}v + \lambda(v, e) = 0\right\} \hookrightarrow \overline{\mathcal{M}} \cap \mathcal{U}.$$

There is a projection $s : \overline{\mathcal{M}}_{\mathcal{U},E,\lambda} \to E$.

Let's consider the linearized operators at $(u, 0) \in \overline{\mathcal{M}}_{\mathcal{U},E,\lambda}$ given by

$$T_u\mathcal{B} \to \mathcal{E}_u(\xi, e) \mapsto D_u\xi + \lambda(u, e)$$

which is surjective.

Definition 7.1 Suppose $\overline{\mathcal{M}}$ is a compact Hausdorff space. Then a **local Kuranishi chart of virtual dimension** d for $\overline{\mathcal{M}}$ is a quintuple $(\overline{\mathcal{M}}, E_\alpha, \Gamma_\alpha, s_\alpha, \psi_\alpha)$ where we have

- A finite dimensional topological manifold $\overline{\mathcal{M}}_\alpha$
- A finite dimensional vector space E_α such that $\dim \overline{\mathcal{M}}_\alpha = d + \dim E_\alpha$

- A finite group Γ_α which acts on $\overline{\mathcal{M}}_\alpha$ and E_α
- A Γ_α-equivariant function $s_\alpha : \mathcal{M}_\alpha \to E_\alpha$
- A homeomorphism $\psi_\alpha : s_\alpha^{-1}(0)/\Gamma_\alpha \xrightarrow{\cong} U_\alpha \subset \overline{\mathcal{M}}$ where the subset is open.

The upshot is that $\overline{\mathcal{M}}_{0,m}(X, A, J)$ is covered by local Kuranishi charts.

A local Kuranishi chart $(\overline{\mathcal{M}}, E_\alpha, \Gamma_\alpha, s_\alpha, \psi_\alpha)$ for $\overline{\mathcal{M}}$ induces a local virtual fundamental class on U_α via the following map:

$$
\begin{aligned}
\check{H}_c^d(U_\alpha;\mathbb{Q}) &\xrightarrow[\cong]{\frac{1}{|\Gamma_a|}\psi_\alpha^*} \check{H}_c^d\left(s_a^{-1}(0);\mathbb{Q}\right)^{\Gamma_\alpha} \\
&\xrightarrow[\cong]{\text{Pardon}} H_{\dim E_\alpha}\left(\overline{\mathcal{M}}_\alpha, \overline{\mathcal{M}}\backslash s_\alpha^{-1};\mathbb{Q}\right)^{\Gamma_\alpha} \\
&\xrightarrow{(s_\alpha)_*} H_{\dim E_\alpha}\left(E_\alpha, E_\alpha\backslash s_\alpha^{-1};\mathbb{Q}\right)^{\Gamma_\alpha} \\
&\xrightarrow[\cong]{\text{orientation}} \mathbb{Q}
\end{aligned}
$$

where Pardon is the map found in [Pardon, 2016, Appendix A].

Definition 7.2 The **local virtual fundamental class** is this entire map

$$[\nu_\alpha]^{\text{vir}}_{\text{local}} : \check{H}_c^d(U_\alpha;\mathbb{Q}) \to \mathbb{Q}.$$

Example 7.1 For $\overline{\mathcal{M}}_\alpha = \mathbb{C}, E_\alpha = \mathbb{C}, \Gamma_\alpha = (1), s_\alpha(z) = z^n$, then

$$[\nu_\alpha]^{\text{vir}}_{\text{local}} = n[\text{pt}].$$

Chapter 8

Global Kuranishi Charts: Definitions and Preliminaries

Abstract We introduce the notion of a 'global Kuranishi chart' and explain how having one of these substantially simplifies the previous discussion. We also explain what it means for two global Kuranishi charts to be 'equivalent', which is analogous to the notion of compatibility for local Kuranishi charts. For the remainder, we discuss some geometric preliminaries necessary to understand the construction of global Kuranishi charts for moduli spaces of closed holomorphic curves of genus 0.

8.1 Global Kuranishi Charts and Equivalence

Definition 8.1 Given $\overline{\mathcal{M}}$ a compact Hausdorff space, a **global Kuranishi chart of virtual dimension** d for $\overline{\mathcal{M}}$ consists of

- A finite dimensional topological manifold $\mathcal{T}$, called the **thickening**
- A finite rank vector bundle on $\mathcal{E} \to \mathcal{T}$, called the **obstruction bundle**
- A section $S : \mathcal{T} \to \mathcal{E}$, called the **obstruction section**
- A compact Lie group G which acts on $s^{-1} : \mathcal{E} \to \mathcal{T}$ such that the action of G on $\mathcal{T}$ has finite stabilizers (and these stabilizers act linearly in suitable local coordinates), satisfying $\dim \mathcal{T} = d + \operatorname{rank}\mathcal{E} - \dim G$, called the **symmetry group**
- A homeomorphism $s^{-1}(0)/G \xrightarrow{\sim} \overline{\mathcal{M}}$, called the **footprint map**

Why do we allow infinite G? It turns out allowing finite groups is not flexible enough. For example, if we take $\mathbb{CP}^1$ and consider a disk $\mathbb{D}$ at the origin with a $\mathbb{Z}/2$ action, we need infinite G.

On the other hand, our current condition is sufficient: Every effective orbifold is a global quotient $\mathcal{M}/G$ for some orbifold M quotiented by a Lie group G.

We want to study orientations on

- $\mathcal{T}, \mathcal{E}$ preserved by G
- $\mathfrak{g} = \mathrm{Lie}(G)$

which together induce a virtual fundamental class for $\overline{\mathcal{M}}$:

$$\begin{aligned}\check{H}^*(\overline{\mathcal{M}}, \mathbb{Q}) &\xrightarrow{\text{Poincaré Duality}} H_{\mathrm{rank}\ \mathcal{E}}(\mathcal{T}/G; \mathcal{T}/g - \overline{\mathcal{M}}, \mathbb{Q}) = H^G_{\mathrm{rank}\ \mathcal{E}}(\mathcal{T} - \mathcal{T} - s^{-1}(0); \mathbb{Q})\\ &\xrightarrow{s_*} H^G_{\mathrm{rank}\ \mathcal{E}}(\mathcal{E}, \mathcal{E} - 0_{\mathcal{E}}; \mathbb{Q})\\ &\xrightarrow{\tau^G_{\mathcal{E}}} H^G_0(\mathrm{pt}, \mathbb{Q}) \cong \mathbb{Q}\end{aligned}$$

where the first $\longrightarrow$ follows from Poincaré duality.

Problem 8.1 When are two global Kuranishi charts equivalent?

Proposition 8.1 *Two global Kuranishi charts are equivalent if we can reach one from the other using a sequence consisting of the following moves:*

1. *Germ equivalence: choose a G-invariant neighborhood $s^{-1}(0) \subset \mathcal{U} \subset \mathcal{T}$ where the second subset is open, and take $(G, \mathcal{U}, \mathcal{E}|_{\mathcal{U}}, s|_{\mathcal{U}})$.*
2. *Group enlargement: choose another compact Lie group H and a G-equivariant principle H-bundle $p : P \to \mathcal{T}$ and take $(G \times H, P, p^*\mathcal{E}, p^*s)$*
3. *Stabilization: choose a G-equivariant vector bundle $\pi : \mathcal{W} \to \mathcal{T}$ and take $(G, \mathcal{W}, \pi^*(\mathcal{E} \oplus \mathcal{W}), \pi^*s \oplus \Delta_{\mathcal{W}})$.*

This should be thought of as an analog of the Reidemeister moves.

8.2 Complex Geometry Background

On $\mathbb{CP}^1$, we have the tautological line bundle $\mathcal{O}(-1) \hookrightarrow \mathbb{CP}^n \times \mathbb{C}^{n+1}$, which has a dual $\mathcal{O}(1)$. Define $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$.

8.2.1 Holomorphic Line Bundles on Curves (Riemann Surfaces)

Lemma 8.1 *Suppose Σ is a Riemann surface, $L \to \Sigma$ is a C^∞ complex line bundle, and ∇ is a $\mathbb{C}$-linear connection on it. Then $\nabla^{0,1}$ defines an unique holomorphic structure on L.*

Proof Given $p \in \Sigma$, choose a C^∞ section τ of L defined near p such that $\tau(p) \neq 0$. Then

$$\nabla^{0,1}(\tau) = g \otimes \tau$$

where $g \in \Omega^{0,1}(\Sigma)$.

The $\overline{\partial}$-Poincaré lemma states that we can find a C^∞-function such that $g = \overline{\partial} f$ near p. Define $\sigma = e^{-f}\tau$, and we have

$$\begin{aligned}\Delta^{0,1}\sigma &= e^{-f}\nabla^{0,1}\tau + \overline{\partial}(e^{-f}) \otimes \tau \\ &= e^{-f}(g \otimes \tau - \overline{\partial} f \otimes \tau) \\ &= 0\end{aligned}$$

as desired. □

Lemma 8.2 *Suppose Σ is a nodal genus 0 curve. Then the isomorphism class of a holomorphic line bundle L on Σ is determined by the degree of L on each component of Σ.*

Corollary 8.1 *Consider $L \to \Sigma$ as above. Suppose L has total degree d and has degree ≥ 0 on each component. Then* $\dim H^0(\Sigma; L) = d + 1$ *and* $\dim_{\mathbb{C}} H^1(\Sigma; L) = 0$.

8.2.2 Genus 0 Curves in $\mathbb{CP}^n$

Suppose X is a complex manifold with

1. a holomorphic map $f : X \to \mathbb{CP}^n$
2. a holomorphic line bundle $\mathcal{L} \to X$ and holomorphic sections $s_0, \ldots, s_n$ such that they have no common zero in X.

To go between these two, we can do the following:

$$(X \xrightarrow{f} \mathbb{P}^n_{[x_0:\ldots:x_n]}) \mapsto (f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)(\mathcal{L}, s_0, \ldots, s_n) \mapsto (X \overset{[s_0:\ldots:s_n]}{\longrightarrow} \mathbb{CP}^n).$$

Consider $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n, d)$ for some $n, d \geq 1, m \geq 0$.

Lemma 8.3 *This space is a complex orbifold of the expected dimension.*

Proof Take $f : \Sigma \to \mathbb{P}^n$ where Σ is a genus 0 nodal curve. Recall $D_f : \Omega^0(\Sigma, f^*T\mathbb{CP}^n) \to \Omega^{0,1}(\Sigma)$. We want to show that this is surjective, which is equivalent to $\mathrm{coker} D_f = H^1(\Sigma; f^*T\mathbb{CP}^n)$. We have the Euler exact sequence:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus n+1} \longrightarrow T\mathbb{CP}^n \longrightarrow$$

If we pullback along $f : \Sigma \to \mathbb{CP}^n$ and take long exact sequence

$$\longrightarrow H^1(\Sigma; f^*\mathcal{O})^{\oplus n+1} \longrightarrow H^1(\Sigma; f^*T\mathbb{CP}^n) \to 0$$

and we are done. $\square$

Chapter 9
Global Kuranishi Charts: Construction

Abstract Following Abouzaid–McLean–Smith 2021, we explain the construction of global Kuranishi charts for genus 0 Gromov–Witten moduli spaces and show that the outcome of the construction is unique up to equivalence. Time permitting, we will also briefly discuss how one can extend this construction to settings beyond genus 0 GW theory.

9.1 The AMS Trick

Take (X^{2n}, ω), $A \in H_2(X; \mathbb{Z})$, and J an ω-tame almost complex structure on X. This gives $\overline{\mathcal{M}}_0(X, A, J)$ which consists of J-holomorphic maps u from trees of spheres Σ to X with $u_k[\Sigma] = A$.

Theorem 9.1 (Abouzaid, McLean, Smith, 2021)

$\overline{\mathcal{M}}_0(X, A, J)$ has a global Kuranishi chart, constructed often making some choices, but the resulting chart is unique up to equivalence.

Proposition 9.1

$$\overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d) = \left\{ f : \Sigma \to \mathbb{CP}^d \mid f \textit{ is a degree } d \textit{ genus } 0 \textit{ stable map and } f \textit{ is non-degenerate} \right\}$$

where non-degenerate means the image of f is not contained in any hyperplane on $\mathbb{CP}^n$. Then there is a smooth quasi-projective variety of the expected dimension.

Here is an example of a non-degenerate function:

Example 9.1

$$\mathbb{CP}^1 \to \mathbb{CP}^d$$
$$[u, v] \mapsto [u^d; u^{d-1}v, ..., v^d]$$

The proposition gives rise to a nontrivial family

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{CP}^d \\ \downarrow & & \\ \overline{\mathcal{M}}_0^*(\mathbb{CP}^n, d) & & \end{array}$$

where C is also a smooth, quasiprojective variety.

Every map $\mathcal{L} \to Z$ is induced with an embedding $Z \overset{[s_0:...]}{\hookrightarrow} \mathbb{CP}^n$ where $s_0, ..., s_p \in H^0(Z; \mathcal{L})$ have no common zero.

Let's recall some basic definitions about ample line bundles.

Definition 9.1 $\mathcal{L}$ is **very ample** if $\exists s_0, ..., s_n \in H^0(Z; \mathcal{L})$ such that we get an embedding $Z \overset{[s_0:...:s_n]}{\hookrightarrow} \mathbb{CP}^n$

Definition 9.2 $\mathcal{L}$ is **ample** if there exists $m > 1$ such that $L^{\otimes m}$ is very ample.

Proposition 9.2 *If Z is a cone, then* $\mathcal{L}$ *is ample if and only if* $\deg \mathcal{L} > 0$ *on each component of Z.*

We can finally state the AMS trick:

Lemma 9.1 (AMS Trick)

Suppose Z is a compact complex manifold $\mathcal{L} \to Z$ *is an ample line bundle and* $\mathcal{E} \to Z$ *is a holomorphic vector bundle, and endow everything with the Hermitian action. For* $k \gg 1$*, define*

$$W_k := Im\left(H^0(Z; \mathcal{E} \otimes \mathcal{L}^{\otimes k}) \otimes_{\mathbb{C}} \overline{H^0(Z; \mathcal{L}^{\otimes k})} \xrightarrow{\langle \cdot, \cdot \rangle} \Omega^0(Z, \mathcal{E})\right).$$

As $k \to \infty$*,* W_k *provides an* L^2 *extension of* $\Omega^0(Z, \mathcal{E})$*, i.e. for all* $\xi \in \Omega^0(Z, \mathcal{E})$*,* $\exists k \gg 1$ *and* $\eta \in W_k$ *such that* $\langle \xi, \eta \rangle_{L_2} \neq 0$.

9.2 Construction

9.2.1 Line Bundles on X

Approximate Ω by a symplectic form Ω which tames J and satisfies $[\Omega] \in H^2(X;\mathbb{Q})$. Clear the denominator to get $[\Omega] \in H^2(X;\mathbb{Z})/\text{torsion}$ which implies that there exists a C^∞ complex line bundle $L_\Omega \to X$ such that the first Chern class $c_1(L_\Omega) = [\Omega]$.

From Chern-Weil theory, we have the following lemma:

Lemma 9.2 *There exists a Hermitian metric and a Hermitian connection ∇ on L_Ω such that its curvature is $-2\pi i\Omega$.*

Notation: $[\Omega] \cdot A = d$.

9.2.2 Framed Genus 0 Curves

Let's start with a (genus 0) J-holomorphic stable $u : \Sigma \to X$. Then $u^* \mid_\Sigma \to \Sigma$ has a holomorphic structured generated by $(u^*\nabla)^{0,1}$. We know that $\int u^*\Omega \geq 0$ on each component of Σ where u is J-holomorphic and Ω tames J. We also know that $\int u^*\Omega > 0$ on each unstable component. This implies that the line bundle u^*L_Ω has non-negative degree on each component Σ and positive degree on each unstable component of Σ.

From last time, we know that

$$\dim_{\mathbb{C}} H^0(\Sigma; u^*L_\Omega) = d+1, H^1(\Sigma; u^*L_\Omega) = 0.$$

Choose a basis $F = (f_0, ..., f_d)$ of $H^0(\Sigma; u^*L_\Omega)$, called a **framing**. Consider the degree d genus 0 stable map

$$\Sigma \xrightarrow{\Phi_F = [f_0 : ... : f_d]} \mathbb{CP}^d$$

Then $(\Sigma, \Phi_F) \in \overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d)$. So we have

$$\begin{tikzcd} \Sigma \arrow[r, hook, "i_F"] \arrow[d] & C \arrow[r] \arrow[d] & \mathbb{CP}^d \\ (\Sigma, \Phi_F) & \in \quad \overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d) & \end{tikzcd}$$

where i_F is a embedding as a fiber.

We have

$$H(\Sigma, u, F) = \left(\int_\Sigma \langle f_i, f_j\rangle u^*\Omega\right)_{0\le i,j\le d}$$

is a Hermitian positive definite $(d+1)\times(d+1)$ matrix.

Definition 9.3 A **framed genus 0 curve in** X is a tuple (Σ, u, F) where

1. Σ is a nodal genus 0 curve.
2. $u : \Sigma \to X$ is a C^∞ map in class A such that $\int u^*\Omega \ge 0$ on each component of Σ and $\int u^*\Omega > 0$ on each unstable component.
3. $F : (f_0, ..., f_d)$ is a basis of $H^0(\Sigma; u^*L_\Omega)$ such that the matrix $H(\Sigma, u, F)$ is positive definite.

So $(\Sigma, u, F) \sim (\Sigma', u', F')$ if

$$\begin{tikzcd}
\Sigma \arrow[dr, "u"] \arrow[dd, "\varphi"', "\cong"] & \\
& X \\
\Sigma' \arrow[ur, "u'"'] &
\end{tikzcd}$$

commutes.

9.2.3 Achieving Transversality

Choose

1. A relatively ample line bundle $\mathcal{L}$ on $C \to \overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d)$ equipped with a Hermitian metric such that the natural $U(d+1)$ action on $\overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d)$ lifts to an action of $\mathcal{L}$ and preserves the metric.
2. A $\mathbb{C}$-linear connection on $T^{*\,0,1}C$ which is an invariant under the $U(d+1)$ action.
3. A $\mathbb{C}$-linear connection on TX (viewed as a $\mathbb{C}$-vector bundle on J)
4. A large integer $k \gg 1$.

For more details, see [Horschi, Swaminathan, 2021, Section 2.1].

Proposition 9.3 *The **thickening** $\mathcal{T}$ is the space of tuples (Σ, u, F, η) where*

1. (Σ, u, F) *is a framed genus* 0 *curve in* X
2. *We have*

$$\eta \in H^0(\Sigma; u^*TX \otimes i_F^*(T^{*\,0,1}C \otimes \mathcal{L}^{\otimes k})) \otimes_{\mathbb{C}} \overline{H^0(\Sigma; i_F^*\mathcal{L}^{\otimes k})}$$

satisfying the equation

$$\overline{\partial}_J u + \langle \eta \rangle \circ di_F = 0$$

Proposition 9.4 *The **obstruction bundle** $\mathcal{E} \to \mathcal{T}$ is a vector bundle whose fiber over (Σ, u, F, η) is*

$$E_{(\Sigma,u,F)} \oplus \mathcal{H}_{d+1}$$

where $\mathcal{H}_{d+1}$ is a space of $(d+1) \times (d+1)$ Hermitian matrices.

Proposition 9.5 *The **obstruction section** is*

$$\mathcal{S}(\Sigma, u, F, \eta) = (\eta, \log \mathcal{H}(\Sigma, u, F)).$$

Proposition 9.6 *The **symmetry group** is*

$$G = U(d+1).$$

The key point is that $\mathcal{T}$ is cut out transversally for $k \gg 1$ by the AMS trick.

Part IV

Talks

There were ten talks:

1. Alex Pieloch: Spectral Equivalence of Nearby Lagrangians

   Fix a commutative ring spectrum R. In this talk, we will show that any nearby Lagrangian in a cotangent bundle of a closed manifold is equivalent in the wrapped Fukaya category with R-coefficients to an R-brane supported on the zero section. As an application, we impose topological restrictions on the embeddings of exact Lagrangian fillings of the standard Legendrian unknot in sub-critical Stein domains. This is joint work with Johan Asplund and Yash Deshmukh.

2. Dan Cristofaro-Gardiner: Low-Action Holomorphic Curves and Invariant Sets

   I will discuss a new compactness theorem for sequences of low-action punctured holomorphic curves of controlled topology, in any dimension, without imposing the typical assumption of uniformly bounded Hofer energy; in the limit, we extract a family of closed Reeb-invariant subsets. I will also explain why such sequences exist in abundance in low-dimensional symplectic dynamics, via the theory of embedded contact homology. This has various applications: the one I want to focus on in my talk is a generalization to higher genus surfaces and three-manifolds of the celebrated Le Calvez–Yoccoz theorem. All of this is joint with Rohil Prasad.

3. Daniel Pomerleano: Homological Mirror Symmetry for Batyrev Mirror Pairs

   I will survey a recent proof of a version of Kontsevich's homological mirror symmetry conjecture for a large class of mirror pairs of Calabi–Yau hypersurfaces in toric varieties. These mirror pairs were constructed by Batyrev from dual reflexive polytopes. The theorem holds in characteristic zero and in all but finitely many positive characteristics. This is joint work with Ganatra, Hanlon, Hicks, and Sheridan.

4. Luya Wang: Deformation Inequivalent Symplectic Structures and Donaldson's Four-Six Question

   Studying symplectic structures up to deformation equivalences is a fundamental question in symplectic geometry. Donaldson asked: given two homeomorphic closed symplectic four-manifolds, are they diffeomorphic if and only if their stabilized symplectic six-manifolds, obtained by taking products with $\mathbb{CP}^1$ with the standard symplectic form, are deformation equivalent? I will discuss joint work with Amanda Hirschi on showing how deformation inequivalent symplectic forms remain deformation inequivalent when stabilized, under certain algebraic conditions. This gives the first counterexamples to one direction of Donaldson's "four-six" question and the related Stabilizing Conjecture by Ruan. In the other direction, I will also discuss more supporting evidence via Gromov–Witten invariants.

5. Kristen Hendricks: Symplectic Annular Khovanov Homology and Knot Symmetry

   Khovanov homology is a combinatorially-defined invariant which has proved to contain a wealth of geometric information. In 2006 Seidel and Smith introduced a candidate Lagrangian Floer analog of the theory, which has been shown by Abouzaid and Smith to be isomorphic to the original theory over fields of characteristic zero. The relationship between the theories is still unknown over other fields. In 2010 Seidel and Smith showed there is a spectral sequence relating the symplectic Khovanov homology of a two-periodic knot to the symplectic Khovanov homology of its quotient; in contrast, in 2018 Stoffregen and Zhang used the Khovanov homotopy type to show that there is a spectral sequence from the combinatorial Khovanov homology of a two-periodic knot to the annular Khovanov homology of its quotient. (An alternate proof of this result was subsequently given by Borodzik, Politarczyk, and Silvero.) These results necessarily use coefficients in the field of two elements. This inspired investigations of Mak and Seidel into an annular version of symplectic Khovanov homology, which they defined over characteristic zero. In this talk we introduce a new, conceptually straightforward, formulation of symplectic annular Khovanov homology, defined over any field. Using this theory, we show how to recover the Stoffregen-Zhang spectral sequence on the symplectic side. We further give an analog of recent results of Lipshitz and Sarkar for the Khovanov homology of strongly invertible knots. This is work in progress with Cheuk Yu Mak and Sriram Raghunath.

6. John Pardon: Derived Moduli Spaces of Pseudo-Holomorphic Curves

   I will present the derived representability approach to working with moduli spaces of pseudo-holomorphic curves.

7. Mark McLean: Symplectic Orbifold Gromov-Witten Invariants

   Chen and Ruan constructed symplectic orbifold Gromov-Witten invariants more than 20 years ago. In ongoing work with Alex Ritter, we show that moduli spaces of pseudo-holomorphic curves mapping to a symplectic orbifold admit global Kuranishi charts. This allows us to construct other types of Gromov-Witten invariants, such as K-theoretic counts. The construction relies on an orbifold embedding theorem of Ross and Thomas.

8. Rohil Prasad: High-Dimensional Families of Holomorphic Curves and Three-Dimensional Energy Surfaces

   Let H be any smooth function on $\mathbb{R}^4$. I'll discuss some recent dynamical theorems for the Hamiltonian flow on level sets of H ("energy surfaces"). The results are proved using holomorphic curves and neck stretching. One important tool is the compactness theorem from Dan's talk.

9. Thomas Massoni: Taut Foliations Through a Contact Lens

In the late '90s, Eliashberg and Thurston established a remarkable connection between foliations and contact structures in dimension three: any co-oriented, aspherical foliation on a closed, oriented 3-manifold can be approximated by both positive and negative contact structures. Additionally, if the foliation is taut then its contact approximations are tight. In this talk, I will present a converse result on constructing taut foliations from suitable pairs of contact structures. While taut foliations are rather rigid objects, this viewpoint reveals some degree of flexibility and offers a new perspective on the L-space conjecture.

10. Vardan Oganesyan: How to Construct Symplectic Homotopy Theory

    In 1968 Dold and Thom proved that singular homology groups of X are isomorphic to homotopy groups of infinite symmetric product of X. In 1990-2000 Morel, Suslin, and Voevodsky used a similar definition to define motivic cohomology groups of algebraic varieties. Moreover, they defined homotopy theory for algebraic varieties. Motivated by these results, we construct homotopy theory for symplectic manifolds. In particular, we define some new homology groups for symplectic manifolds and prove that these homology groups have all required properties. We will not discuss details, but we will show that these new homology groups appear in a very natural way. If time permits, we will also discuss some possible applications.

Chapter 10

Alex Pieloch: Spectral Equivalence of Nearby Lagrangians

Abstract Fix a commutative ring spectrum R. In this talk, we will show that any nearby Lagrangian in a cotangent bundle of a closed manifold is equivalent in the wrapped Fukaya category with R-coefficients to an R-brane supported on the zero section. As an application, we impose topological restrictions on the embeddings of exact Lagrangian fillings of the standard Legendrian unknot in sub-critical Stein domains. This is joint work with Johan Asplund and Yash Deshmukh.

10.1 Introduction

Theorem 10.1 (Abouzaid)

Let $L \subset T^*Q$ be exact, equipped with a choice of rank 1 local system L equivalent in the wrapped Fukaya category $\mathcal{W}(T^*Q, \mathbb{Z})$ to the zero section, with some choice of rank 1 local system.

Let R be a commutative ring spectrum. A spectrum is morally something that functions like a space. It's a bit more complicated, but this complication allows us to do more algebraic operations. There spectrum also allow us to define more homology theories. And each one of these, can be realized using the language of spectrum. For example:

Example 10.1 Take

$$\pi_*(M \wedge R) = H_*(M, \mathbb{R}).$$

If we let $R = HK$, then we have $H_*(M; K)$. If we let $R = \mathrm{MO}$, then we get $\Omega_*^{\mathrm{MO}}(M)$

Theorem 10.2 *Let $L \subset T^*Q$ be a nearby Lagrangian with an R-brane. In $\mathcal{W}(T^*Q, \mathbb{R})$, L is equivalent to an R-brane on the zero section.*

Here is one application of this theorem. We will prove this application later.

Theorem 10.3 *Let X be a subcritical Weinstein domain with $c_1(X) = 0 = c_2(X)$. Let $\Lambda \subseteq \partial X$ be a Legendrian unknot, with standard filling C. Fix a Lagrangian $L \subseteq X$ that is an exact filling of Λ. Then L is homotopic to C with Λ fixed in $X/X_{n-2} = \bigvee S^{n-1}$.*

Definition 10.1

1. $\mathcal{W}(X, \mathbb{Q})$ is a category where the objects are the exact canonical Maslov Lagrangians with rank 1 local systems and the morphisms are chain complexes built from $M(\emptyset)$.
2. $\mathcal{W}(X, \mathbb{R})$ is a category where the objects are the exact canonical Maslov Lagrangians with rank 1 local systems R-branes and the morphisms are R-module spectra built from $M(\emptyset)$.

Definition 10.2 A vector bundle $\mathcal{E} \to \mathcal{B}$ is R**-orientable** if

$$\mathcal{B} \xrightarrow{\mathcal{E}} \mathrm{BO} \longrightarrow \mathrm{BGL}() \longrightarrow \mathrm{BGL}(\mathbb{R}).$$

is null.

Example 10.2 Let $\mathbb{R} = H\mathbb{Z}$. Then

$$\mathcal{B} \to \mathrm{BO} \to \mathrm{BGL}(H\mathbb{Z}) = B\mathrm{Aut}(\mathbb{Z}) = B\mathbb{Z}/2.$$

10.2 *R*-Branes and Properties

Assume that X be a symplectically trivializable space, meaning $TX = \mathbb{C} \otimes \mathbb{R}^n$. Given $L \subset X$ a Lagrangian, denote $\mathrm{GL} : L \to \mathcal{U}/O$ as the Lagrangian Grassmannian.

Definition 10.3 An R**-brane** is a choice of null-homotopy of

$$L \xrightarrow{\mathrm{GL}} \mathcal{U}/O \xrightarrow{\mathrm{Bott}} B^2(0) \longrightarrow B^2\mathrm{GL}_1(R).$$

Remark 10.1

1. R-branes correspond to $[L, B\mathrm{GL}_1(R)]$ which are rank 1 local systems.
2. $R = H\mathbb{Z}$, $[L, B\mathrm{GL}_1(H\mathbb{Z})] = [L, B\mathbb{Z}/2]$ are rank 1 local systems.

3. Let $M_L = \{(D, \partial D) \to (X, L) | \overline{\partial} u = 0, \text{ based}\}$. We want

$$\begin{tikzcd} \mathcal{M}_L \arrow[r, "T\mathcal{M}_L"] \arrow[d] & \mathrm{BO} \arrow[r] & \mathrm{BGL}_1(R) \\ \Omega_L \arrow[r, "\Omega\mathrm{GL}"] \arrow[urr, bend right, "*"'] & \mathrm{BGL}_1(R) \arrow[u, "\mathrm{Bott}"] & \end{tikzcd}$$

where the left square commutes.

Proposition 10.1

1. We have

$$Mor(L, L) = HW(L, L, \mathbb{R}) = L \wedge R$$

2. For L compact, we have

$$\pi_*(HW(L, K, R)) \in Ab\pi_*(HW(L, L, R)) = H_*(L, R)$$

3. Change of coefficients: consider S a module over R. Then we have

$$\mathcal{W}(X, S) = \mathcal{W}(X, R) \wedge_R S$$

From now on, assume that R is connective, $\pi_0(R) = K$ is discrete, and the Hurwitz map Hw : $R \to$ HK is $\mathbb{1}$ on π_0.

Proposition 10.2 *Let M, M' be connected R-module spectra.*

1. Let

$$\pi_*(M \wedge_R HK) = \begin{cases} K & * = 0 \\ 0 & else \end{cases}$$

Then $M = R$.

2. If

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "H_W"'] & M' \arrow[d, "H_W"] \\ M \wedge_R HK \arrow[r, "H_n(f)"'] & M' \wedge_R HK \end{tikzcd}$$

then $Hw(f)$ are equivalent implies f is equivalent.

Proof We have $\mathrm{HW}(\mathrm{Fib} = F, F, R) = \Omega Q \wedge R$ and $\mathrm{HW}(F, L, R) = R$. The following commutative diagram

$$\begin{tikzcd}
\mathrm{HW}(F,F) \wedge_R \mathrm{HW}(F,L) \arrow[r, "\mu^2"] \arrow[d] & \mathrm{HW}(F,L) \arrow[d] \\
\Omega Q \wedge R \arrow[r] & R
\end{tikzcd}$$

implies that $\mathrm{Hw}(F, L) \leftrightarrow Q \to \mathrm{BGL}_1(R) \leftrightarrow$ rank 1 local system on Q. Every such local system is realized by $\mathrm{HW}(G, Q^\#)$ where $Q^\#$ is some R-brane. We have

$$\begin{tikzcd}
\mathrm{HW}(L,Q^\#) \arrow[rr, "\cong"] \arrow[d, "\mathrm{H}_W"'] & & F_{\Omega Q \wedge R}(R,R) \arrow[ddd, "\mathrm{H}_W"] \\
\mathrm{HW}(L,Q) \wedge_R \mathrm{HK} \arrow[d, phantom, "="] & & \\
& & \\
\mathrm{HW}(L,Q,K) \arrow[rr, "\cong"'] \arrow[uuurr, phantom, "\Uparrow"] & & F_{\Omega Q \wedge HK}(\mathrm{HK},\mathrm{HK})
\end{tikzcd}$$

which concludes the proof. □

10.3 Proof

Let's prove the application of the theorem from earlier. Recall that X is a subcritical Weinstein domain if $c_1(X) = 0 = a(X)$, $\dim_{\mathbb{C}} X \geq 4$, $\Lambda \subset \partial X$ the unknown with standard filling C, and $L \subset X$ is an exact filling of L (where $L = \mathbb{D}^n$).

Proposition 10.3 *It suffices to show that* $[L \cup_\Lambda C] = 0 \in \tilde{\Omega}_n^{spin}(X) = \tilde{H}_X(X, \mathcal{M}Spin)$

Proof It suffices to show that $L \cup_\Lambda C \cong S^n \implies X/X_{n-?} \simeq V$ where S^{n-1} is based null. We have

$$\mathbb{Z}/2 \cong \pi_n(S^{n-1}) \to \tilde{\Omega}_n^{\mathrm{spin}}(S^{n-1}) \xrightarrow{\cong} \Omega_1^{\mathrm{spin}}(\mathrm{pt}) \subset \Omega_1^{\mathrm{spin}}\mathbb{Z}/2$$

which is an isomorphism by Pontryagin-Thorn. If $X/X_{n-2} \simeq S^{n-1}$, then the claim holds. If $X/X_{n-2} = \vee S^{n-2}$, then the claim holds by the Milnor-Hilton argument. □

Let $\hat{X} = X \cup_\Lambda H^n$, $\hat{L} = L$, and $\hat{C} = C \subset U_n$ be the the core of H^n.

Proposition 10.4

$$[\hat{L}] = [\hat{C}] \in \tilde{\Omega}^{spin}(\hat{X}).$$

Proof Take $f : \hat{X} \to X$ such that $\hat{f}(\hat{L}) = [L \cup_\Lambda C]$. We have $0 = f(\hat{C}) \in \mathrm{Im}(B^{2n})$.

The obstruction to MSpin-brane is cohomology. But we can take $\pi_1(L) = 0, w_2(L) = 0, H_3(L, \mathbb{Z}/2) = 0$. Additionally, we have

$$\hat{X} = \text{subcritical handles} \cup T^*S^n \implies \mathcal{W}(\hat{X}, R) \cong \mathcal{W}(T^*S^n, R).$$

Similarly, $\hat{L} \cong \hat{C}$ in $\mathcal{W}(\hat{W}, R)$. We conclude that

$$\mathrm{HW}(L, L, R) \to H_n(\hat{X}, R) = \Omega_n^{\mathrm{spin}}(\hat{X})$$

and we are done. □

Chapter 11

Dan Cristofaro-Gardiner: Low-Action Holomorphic Curves and Invariant Sets

Abstract I will discuss a new compactness theorem for sequences of low-action punctured holomorphic curves of controlled topology, in any dimension, without imposing the typical assumption of uniformly bounded Hofer energy; in the limit, we extract a family of closed Reeb-invariant subsets. I will also explain why such sequences exist in abundance in low-dimensional symplectic dynamics, via the theory of embedded contact homology. This has various applications: the one I want to focus on in my talk is a generalization to higher genus surfaces and three-manifolds of the celebrated Le Calvez–Yoccoz theorem. All of this is joint with Rohil Prasad.

11.1 Introduction

11.1.1 A New Compactness Theorem

Consider a closed oriented $2n+1$ manifold Y. We are interested in framed Hamiltonian structures:

Definition 11.1 A **framed Hamiltonian structure** is a pair (λ, ω) where λ is a 1-form, ω is a closed 2-form, and $\lambda \wedge \omega^n > 0$.

Example 11.1

- Let λ be a contact form, $\omega = d\lambda$. The mapping torus is a symplectic automorphism $\phi : (M^{2n}, \omega) \to (M^{2n}, \omega)$. We have

$$Y = M^{2n} \times [0, 1]/\sim$$

where $(x, 1) \sim (\phi(x), 0)$. The pair (dt, ω) is a framed Hamiltonian structure.

- Suppose we have a proper Hamiltonian $H : (M^{2n}, \omega) \to R$. Suppose we have a regular value c. Then $H^{-1}(c)$ is a framed Hamiltonian structure.
- Non-singular volume preserving flows on a closed 3-manifold are framed Hamiltonian structures.

Let (λ, ω) is a Hamiltonian vector field. Suppose we have R satisfying $\omega(R, \cdot) = 0, \lambda(R) = 1$.

Example 11.2 In the contact case, R is the Reeb vector field.

We want non-trivial (nonempty and proper) closed invariant sets of R.

Example 11.3

- A periodic orbit
- The invariant tori
- Orbit closure for f proper

Consider the symplectization $X = \mathbb{R} \times Y$ with an almost complex structure: $J : TX \to TX, J^2 = -1, J(\partial_S) = R$ that preserves $\ker(\lambda)$ compatibly with ω. We are interested in sequences of holomorphic curves

$$u_k : C_k \to \mathbb{R} \times Y$$

where u_k is proper and J-holomorphic, C_k is a closed Riemann surface minus a finite number of punctures. For u_k, define the limit set

$$L(u_k) = \{\overline{K} \subset (-1, 1) \times Y | \text{condition}\}$$

where the condition is that $\overline{k}$ is closed and there exists a subsequence $u_k(C_k) \cap (s_k - 1, s_k + 1) \times Y$ converging (after shifting) to k. We should think of this as subsequential limits of height 2 slices.

Proposition 11.1 *$L(u_*)$ is a connected.*

We have the following classical quantities associated to u:

- The action
$$\mathcal{A}(u) = \int_C u^* \omega$$
- The Hofer energy
$$\mathcal{E}(u) = \sup_{s \in \mathbb{R}} \int_{C \cap u^{-1}(\{s\} \times Y)} u^* \lambda$$

Theorem 11.1 *Let $u_k : C_k \to R \times Y$ be such that $\lim_{k\to\infty} \mathcal{A}(u_k) = 0$ and $\inf_k x(C_k) > -\infty$. Then every $k \in L(u_*)$ satisfies $k = (-1, 1) \times \Lambda$ for some closed non-empty invariant set Λ.*

The upshot is that the sequence as in the theorem gives a connected family of invariant sets.

Remark 11.1 The main novelty is that there is no bound required on the Hofer energy $\mathcal{E}(u_k)$.

11.2 Dynamical Applications

The upshot is that the theorem widely applicable in low dimensions. In higher dimensions, problems are more open. In particular, they are very important in relation to the Le Calvez-Yoccoz Theorems

Definition 11.2 (Birkhoff)

A dynamical system Y is **minimal** if every orbit is dense.

One motivation is that if Y is not minimal, we can write $Y = k \cap (Y - k)$ where k is a non-trivial invariant set.

Problem 11.1 (Ulam)

Is there a minimal homeomorphism of $\mathbb{R}^n$ or $\mathbb{R}^n - \{p\}$?

Theorem 11.2 (Le Calvez, Yocroz, 1997)

A homeomorphism of $S^2 - \{p_1, ..., p_k\}$ is never minimal.

Definition 11.3 A system has the **(strong) Le Calvez-Yoccoz property** if the complement of any nontrivial invariant set is never minimal.

Theorem 11.3 (Cristofaro-Gardiner, Prasad)

The following have the strong Le Calvez-Yoccoz property:

1. *Any Hamiltonian diffeomorphism of a closed surface*
2. *Any Reeb flow on a rational homology sphere*
3. *Any geodesic flow on a closed surface (considered as a flow of its unit tangent bundle)*

Remark 11.2 There is no genericity required.

Corollary 11.1 $(1) - (3)$ *have the property that the non-trivial invariant sets are dense*

Corollary 11.2 *Any geodesic flow on a surface has the property that a dense set of points have n non-dense geodesic passing through them.*

11.3 Proof Ideas

11.3.1 Finding Low Action Curves of Controlled Topology

This part of the proof uses the embedded contact homology ECH. For (Y, λ) a closed 3-manifold, ECH(Y, λ) is a homology of a cochain complex that counts (mostly) embedded curves.

Theorem 11.4

$$ECH(X, \lambda) \cong HM(Y)$$

There exists a map u : ECH $\to$ ECH bounding curves through a marked point and the Weyl law allows us to use the u-map to produce low action curves

Problem 11.2 How do we bound $X_\pi(C_k)$?

There is no bound a priori on the genus.

Theorem 11.5 (Cristofaro, Gardiner, Prasad)

$$x_k(C_k) \geq -2.$$

11.3.2 Proving the Compactness Theorem

The main point is a new estimate that bounds of C in a small ball if C is low action in terms of $\chi(C)$.

Chapter 12

Daniel Pomerleano: Homological Mirror Symmetry for Batyrev Mirror Pairs

Abstract I will survey a recent proof of a version of Kontsevich's homological mirror symmetry conjecture for a large class of mirror pairs of Calabi–Yau hypersurfaces in toric varieties. These mirror pairs were constructed by Batyrev from dual reflexive polytopes. The theorem holds in characteristic zero and in all but finitely many positive characteristics. This is joint work with Ganatra, Hanlon, Hicks, and Sheridan.

12.1 Set Up

Let K be a field and $M_R = M \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a lattice $M \cong Z^n$ with $n \geq 4$ and a reflexive polytope Δ in $M_{\mathbb{R}}$. Let $\Delta_* \subset M^*_{\mathbb{R}}$ be the dual polytope and $\overline{\Sigma} \subset M^*_{\mathbb{R}}$ be the fan dual to Δ (rays of the fan point along the vertices of Δ^*)

We will assume that Σ^*, the fan dual to Δ^*, is a smooth fan. Additionally, assume that $\overline{\Sigma} \rightsquigarrow \overline{Y}$ is a toric variety where $\Sigma^* \rightsquigarrow Y^*$ smooth. Let P denote the integer lattice points $\Delta^* \cap M^*$ and let $P \subset P^0$ be the subset of lattice points which lie on a face of codimension ≥ 2.

12.2 B-Side

Consider $\mathcal{L}_{\Delta^*} \to Y^*$ and let

$$W_r = -z^0 + \sum_{p \in P} r_p z^p$$

Then $(r_p) \in \wedge_K^P$ where $\wedge_K := \sum_{i=0}^{\infty} a_i T^{b_i}$ satisfying $\lim_{i\to\infty} b_i = \infty, a_i \in K$ is the Novikov ring. We have

$$X_r^* = \{W_r = 0\} \subset Y.$$

12.3 A-Side

Assume $\overline{Y}$ is not assumed to be smooth, $A = \Delta \cap M$, and $z^\alpha \in \Gamma(\overline{Y}, \mathcal{L}_\Delta)$. Let Σ be a refinement of $\overline{\Sigma}$, where $\Sigma(1) = P$. Assume $Y = Y_\Sigma$ is smooth away from dimension

$$\overline{X}_t = \{-tz^0 + \sum_{\alpha \in A \backslash 0} z^\alpha = 0\} \subset \overline{Y}$$

The proper transform X_t is a smooth Calabi-Yau in Y.

Consider a Kähler class on Y of the form

$$[w] = \sum_{p\in P} \ell_p \mathrm{PD}([D_P^Y]), \qquad \ell_p \in \mathbb{R}^{>0}.$$

and restrict it to X_t.

On the A-side, we consider a variant of the Fukaya category

$$\mathrm{Fut}(X_t, D; \wedge)$$

where objects of this category are compact exact Lagrangian submanifolds in $X_t \backslash D$ and holomorphic curves u are weighted by $T^{\Sigma \ell_p u \cdot D_p}$.

Theorem 12.1 *Suppose that D_p are connected. Away from finitely many bad characteristics, there exist $b(\wedge) = (b(\wedge))_{p\in P} \in \wedge^P$ with $val(b(\wedge)_p) = \ell_p$ and an equivalence*

$$Fuk(X_t, D; A) \cong \mathcal{D}^b Coh(X_{b(\wedge)})^*$$

Remark 12.1 In characteristic 0, homological mirror symmetry implies Givental's Hodge-theoretic mirror symmetry.

This motivates the following problem:

Problem 12.1 Is there some kind of Gromov-Witten implication of homological mirror symmetry in a positive characteristic?

12.4 Strategy of Proof

This follows the groundbreaking worth of Seidel (in the case of quartic surface in $\mathbb{P}^3$):

1. Step 1: $\mathrm{Fuk}(X_t\backslash D) \cong \mathcal{D}^b\mathrm{Coh}(\partial Y^*)$ where ∂Y^* is the toric divisor which is cut out by z^0. For $\mathcal{A}_0 \subset \mathrm{Fuk}(X_t\backslash D)$, $\mathcal{B}_\gamma := \{\theta(i)\}_{i\in\mathbb{Z}}$.
2. Step 2: Employ a deformation theory argument.

We will only talk about Step 1.

Let $H = X_t\backslash D \subset (\mathbb{C}^\times)^n$. We can consider $\mathcal{W}H)$ where we allow certain non-compact Lagrangians and $\mathcal{W}(\mathbb{C}^\times)^n, H)$.

Theorem 12.2 (Gammage, Shende)

$$\begin{tikzcd}
\mathcal{W}H) \arrow[d, "\cup"] \arrow[r, phantom, "\cong"] & \mathcal{D}^b Coh(\partial Y^*) \arrow[d] \\
\mathcal{W}(\mathbb{C}^\times)^n, H) \arrow[r, phantom, "\cong"] & \mathcal{D}^b Coh(Y^*)
\end{tikzcd}$$

Abouzaid considered a different form of homological mirror symmetry for toric varieties where he considers certain Lagrangian actions with boundary on this hypersurface H:

$$\mathcal{F}_{\mathrm{trop}}((\mathbb{C}^\times)^n, H) \simeq \mathrm{Pic}^{dg} Y^*$$

Theorem 12.3

$$\begin{tikzcd}
\mathcal{W}(H) \arrow[r, "GS", "\cong"'] & \mathcal{D}Coh(\partial Y^\times) \\
\mathcal{W}\left((C^\times)^n, H\right) \arrow[u, "(\cup)^*"'] \arrow[r, "KGPS", "\cong"'] & \mathcal{D}^b Coh(Y^\times) \arrow[u, "C^\times"'] \\
\mathcal{F}_{trop}\left((\mathbb{C}^\times)^n, H\right) \arrow[uu, bend left, "\subset \textit{with the fiber}"] \arrow[r, phantom, "\overset{A}{\cong}"] & Pic^{dg}(Y^\times)
\end{tikzcd}$$

One thing that gets used that wasn't available to Seidel/Sheridan is that

$$\mathrm{CO} : \mathrm{SH}^*(X_t\backslash D) \longrightarrow \mathrm{HH}^*(\mathcal{W}X_t\backslash D)) \longrightarrow \mathrm{HH}^*(\mathrm{Fut}(X_t\backslash D))$$

are isomorphisms and $\mathrm{SH}^*(X_t\backslash D)$ can be computed in terms of the topology of the pair (X_t, D).

Chapter 13

Luya Wang: Deformation Inequivalent Symplectic Structures and Donaldson's Four-Six Question

Abstract Studying symplectic structures up to deformation equivalences is a fundamental question in symplectic geometry. Donaldson asked: given two homeomorphic closed symplectic four-manifolds, are they diffeomorphic if and only if their stabilized symplectic six-manifolds, obtained by taking products with $\mathbb{CP}^1$ with the standard symplectic form, are deformation equivalent? I will discuss joint work with Amanda Hirschi on showing how deformation inequivalent symplectic forms remain deformation inequivalent when stabilized, under certain algebraic conditions. This gives the first counterexamples to one direction of Donaldson's "four-six" question and the related Stabilizing Conjecture by Ruan. In the other direction, I will also discuss more supporting evidence via Gromov–Witten invariants.

13.1 Introduction

Definition 13.1 (X_1, ω_1) and (X_2, ω_2) are **deformation equivalent** if there exists a diffeomorphism $\varphi : X_1 \to X_2$ such that $\varphi^*\omega_2 \rightsquigarrow \omega_1$.

Problem 13.1 (Donaldson)

Given two closed simply-connected homeomorphic (X_1^4, ω_1) and (X_2^4, ω_2). Is X_1 diffeomorphic to X_2 equivalent to

$$(X_1 \times S^2 \omega_1 \oplus \omega_{\mathrm{std}}) \simeq (X_s \times S^2, \omega_2 \oplus \omega_{\mathrm{std}})?$$

Theorem 13.1 (Wall, 1964)

Two closed simply-connected homeomorphic 4-manifold are h-cobordant.

Theorem 13.2 (Smale, 1962)

Let $n \geq 5$. Then two closed simply connected n-manifolds are h-cobordant implies they are diffeomorphic.

Some history:

- [Ruan, 1994]: Homeomorphic but not diffeomorphic Kähler surfaces $\mathbb{C}^2\#\overline{\mathbb{CP}}^2$ and Barlow surface.
- [Ruan, Tian, 1997]: Stabilizing conjecture. For simply connected elliptic surfaces.
- [Ionel, Parker, 1999]: $E(n)$ using knot surgery.
- [Smith, 2000]: Given $n \geq 2$. Constructs n symplecitc structures on a fixed simply-connected Z^4 such that c, s are different, which implies the 4-6 question cannot be replaced by π^2.

Theorem 13.3 *[Hirschi, Wang, 2023]*

There exists infinitely many pairs $(X_1, \omega_1), (X_2, \omega_2)$ such that X_1, X_2 are diffeomorphic, but

$$(X_1 \times S^2, \omega_1 \oplus \omega_{std}) \not\simeq (X_2 \times S^2, \omega_2 \oplus \omega_{std}).$$

Here is another important theorem, which we will prove in the last section:

Theorem 13.4 *[Hirschi, Wang, 2023]*

Let (X_1, ω_1) and (X_2, ω_2) be closed simply-connected 4-manifolds such that

$$(X_1 \times S^2, \omega_1 \oplus \omega_{std}) \simeq (X_2 \times S^2, \omega_2 \oplus \omega_{std}).$$

Then $GW(X_1) = GW(X_2)$.

Corollary 13.1 *If (X_1, ω_1) and (X_2, ω_2) satisfy hypothesis of Theorem 13.4, and $b_2^+ \geq 2$, then $SW(X_1) = SW(X_2)$.*

The invariant is orbits under diffeomorphisms of c_1: given symplectic form ω, we can take a tame J and it's first chern class $c_1(TX, J)$. The goal is to show that $c_1(X \times S^2, \omega_1 \oplus \text{std})$.

Definition 13.2 Given X, Y let $G_{X,Y}$ be the set of cohomology equivalences ψ of $X \times Y$ such that ψ^* are maps $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ and $\text{pr}_G \psi(\cdot, Y)$ is cohomology equivalent.

13.2 Proof 1

Here are the steps toward a proof of Theorem 13.3:

1. Find a smooth manifold X^4 with symplectic forms ω_1 and ω_2 such that $c_1(\omega_1)$ and $c_1(\omega_2)$ lie in different orbits of cohomology equivalences.
2. Show that if $c_1(\omega_1)$ and $c_2(\omega_2)$ lie in different orbits of cohomology equivalence, then $c_1(\omega_1 \oplus \omega_{\text{std}})$ and $c_1(\omega_2 \oplus \omega_{\text{std}})$ lie in different orbits of G_{X,S^2}.
3. Show that any diffeomorphism of $X \times S^2$ lies in G_{X,S^2}.

Let's prove (2):

Proof Suppose there exists $\psi \in G_{X,S^2}$ such that $\psi^* c_1(\omega_2 \oplus \omega_{\text{std}}) = c_1(\omega_1 \oplus \omega_{\text{std}})$. Then $\psi^* h = h + \alpha$, where $\alpha \in H^2(x)$. Also, $\psi^*(h^2) = 0$ implies that $(h+\alpha)^2 = h^2 + 2\alpha h + \alpha^2 = 0$, which implies $2a = 0$ in $H^*(X \times \mathbb{CP}^1) \cong H^*(x)[h]/h^2$. Now we get

$$\begin{aligned} c_1(\omega_1) + 2h &= \psi^* c_1(\omega_2) + 2\psi^* h \\ &= \psi^* c_1(\omega_2) + 2h + 2\alpha \end{aligned}$$

as desired. □

Proposition 13.1 *$\hat{\psi}$ is a cohomology equivalence on X.*

Now, we present a counter example for (1) and (3) Let $Z := \mathbb{T}^4 \# 5E(1)$ where # is the fiber sum. Then we have $\langle x, t \rangle = T_X, T_Y, T_Z, 2T_W$ where $[T_W] = [T_x + T_y + T_Z]$. Then we have $T_X, 2T_W$ are symplectic and T_Y, T_Z are Lagrangian.

Theorem 13.5 (Smith)

$\xi \mid c_1(TZ, \omega)$ and $c_1(TZ, \omega)$ is prime.

13.3 Proof 2

We prove Theorem 13.4.

Consider (X_0, ω_0) and (X_1, ω_1) simply connected. Suppose

$$(X_0 \times S^2, \omega_0 \oplus \omega_{\text{std}}) \simeq (X_1 \times S^2, \omega \oplus \omega_{\text{std}}).$$

Then there exists a homeomorphism $\varphi : X_0 \to X_1$ such that for all $g, n \geq 0$ and $A \in H_2(X_0)$, we have

$$\mathrm{GW}^{X_0,\omega_0}_{g,n,A}(\varphi^*\alpha_1,...,\varphi^*\alpha_n) = \mathrm{GW}^{X_0,\omega_0}_{g,n,\varphi^* A}(\alpha_1,...,\alpha_n)$$

for any $\alpha \in H^*(X;\mathbb{Q})$.

Theorem 13.6 (Hirschi-Swaminathan Product Formula)

For (X_0,ω_0) and (X_1,ω_1) with torsion free $H_1(X;\mathbb{Z})$, we have

$$GW^{X\times Y,\omega_X\oplus\omega_Y}_{g,n,(B_X,B_Y)}(\alpha_1\times\beta_1,...,\alpha_n\times\beta_n) = GW^{X,\omega}_{g,n,B_X}(\alpha_1,...,\alpha_n)GW^{Y,\omega Y}_{g,nB_Y}(\beta_1,...,\beta_n)$$

in $H^*(\overline{\mathcal{M}}_{g.n};\mathbb{Q})$.

Problem 13.2 How do we find some $\beta_1,...,\beta_n$ in $H^*\left(\overline{M}_{g,n};\mathbb{Q}\right)$ and d such that

$$\mathrm{GW}^{S^2}_{g,n,d}\left(1^{\otimes n}\right)([\mathrm{pt}]) \neq 0$$

Lemma 13.1 *This is possible for g odd:*

$$GW^{S^2}_{g,n,d}\left(1^{\otimes n}\right)([pt]) = 2^g$$

We need one last lemma:

Lemma 13.2 *Suppose X_0, X_1 have nonvanishing signature. Then we can find a homeomorphism $\phi : X_0 \to X_1$ such that $\tilde{\phi}^*$ on $X_1\times S^2 \to X_0\times S^2$ agree with $\phi^*\otimes id$.*

Suppose $\sigma(X_0) = 0$. The simply-connected condition implies that X_0 is homeomorphic to the number of odd $(S^2\times S^2)$, or the number of odd $\left(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2\right)$. This concludes the proof.

Chapter 14

Kristen Hendricks: Symplectic Annular Khovanov Homology and Knot Symmetry

Abstract Khovanov homology is a combinatorially-defined invariant which has proved to contain a wealth of geometric information. In 2006 Seidel and Smith introduced a candidate Lagrangian Floer analog of the theory, which has been shown by Abouzaid and Smith to be isomorphic to the original theory over fields of characteristic zero. The relationship between the theories is still unknown over other fields. In 2010 Seidel and Smith showed there is a spectral sequence relating the symplectic Khovanov homology of a two-periodic knot to the symplectic Khovanov homology of its quotient; in contrast, in 2018 Stoffregen and Zhang used the Khovanov homotopy type to show that there is a spectral sequence from the combinatorial Khovanov homology of a two-periodic knot to the annular Khovanov homology of its quotient. (An alternate proof of this result was subsequently given by Borodzik, Politarczyk, and Silvero.) These results necessarily use coefficients in the field of two elements. This inspired investigations of Mak and Seidel into an annular version of symplectic Khovanov homology, which they defined over characteristic zero. In this talk we introduce a new, conceptually straightforward, formulation of symplectic annular Khovanov homology, defined over any field. Using this theory, we show how to recover the Stoffregen-Zhang spectral sequence on the symplectic side. We further give an analog of recent results of Lipshitz and Sarkar for the Khovanov homology of strongly invertible knots. This is work in progress with Cheuk Yu Mak and Sriram Raghunath.

14.1 (Symplectic) Khovanov Homology

Khovanov homology takes a link $L \subseteq S^3$ and gives back $\mathrm{Kh}(L)$, a bigraded vector space over some field with χ the Jones polynomial.

Theorem 14.1 (Ozswatch, Szabé)

There exists a spectral sequence

$$\widehat{Kh}(L;\mathbb{F}_2) \rightrightarrows \widehat{HF}(\Sigma(\overline{L}))$$

Here is the Floer theory analogy:

- (M,ω) are exact, $\omega = d\lambda$, and convex at ∞
- L_0, L_1 are exact compact Lagrangians satisfying $\lambda|_{L_i} = df_i$
- $(M, L_0, L_1) \rightsquigarrow \mathrm{CF}(M, L_0, L_1) = (\mathbb{F}\langle L_1 \text{ transverses } L_1\rangle, \partial)$.

Let's briefly discuss symplectic Khovanov homology. Take $p = \prod_{i=1}^{n}(z - k_i)$ and consider:

$$\left\{u^2 + v^2 + p(z) = 0 : (u,v,z) \in \mathbb{C}^3\right\}.$$

Define

$$\Sigma_{\beta_L} := \left\{(u,v,z) : z \in B_i, u, v \in \sqrt{-p(z)}\mathbb{R}\right\}.$$

Let $\mathcal{Y}_n \subseteq \mathrm{Hilb}^n(S) \xrightarrow{\text{Hilbert-Chow}} \mathrm{Sym}^n(S)$ be 1-to-1 away from the diagonal, where Hilb denotes the Hilbert scheme.

Define $\Sigma_A = \Sigma_{A_1} + \cdots + \Sigma_{A_n}$ and $\Sigma_B = \Sigma_{B_1} + \cdots + \Sigma_{B_n} \subseteq \mathrm{Sym}^n(S) \setminus \Delta \subseteq \mathcal{Y}_n \subseteq \mathrm{Hilb}^n(S)$, where

$$\mathcal{Y}_n = \mathrm{HC}^{-1}\left\{(u_1,v_1,z_1),\ldots,(u_n,v_n,z_n) : z_i = z_j \implies (u_i,v_i) = (u_j,v_j)\right\}.$$

Then, we have the following definition:

Definition 14.1

$$\mathrm{Kh}_{\mathrm{symp}}(L) = \mathrm{HF}(\Sigma_A, \Sigma_B).$$

Theorem 14.2 (Abouzaid, Smith)

Over characteristic 0, $Kh(K) = Kh_{symp}(K)$.

Since $O(2)$ acts on (u,v), we have an action on S and thus a symplectic action on $(\mathcal{Y}_n, \Sigma_A, \Sigma_B)$.

Let's briefly digress to discuss $\mathbb{F}_2$-actions:

Example 14.1 (Smith, Borel)

Consider τ acting on X satisfying $\tau^2 = \mathrm{Id}$, a fixed set X^{Fix}. Then there is a spectral sequence

$$H^*(x;\mathbb{F}_2)\otimes\mathbb{F}_2\left[\theta,\theta^{-1}\right]\rightrightarrows\theta^{-1}H_{\mathbb{Z}/2\mathbb{Z}}(X;\mathbb{F}_2)$$
$$\simeq H^*(X^{\mathrm{fix}};\mathbb{F}_2)\otimes\mathbb{F}\left[\theta,\theta^{-1}\right]$$

where
$$\mathbb{F}_2\left[\theta,\theta^{-1}\right]=\theta^{-1}H^*(\theta\mathbb{Z}_2;\mathbb{F}_2)$$
and
$$\theta^{-1}H_{\mathbb{Z}/2\mathbb{Z}}(X;\mathbb{F}_2)=\mathrm{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(C_*(X),\mathbb{F}_2)$$

Example 14.2 (Seidel, Smith, 2010)

Given τ acting on (M,L_0,L_1) with $\tau^2=\mathrm{Id},\tau^*\omega=\omega,\tau(L_i)=L_i$. Take $(M^{\mathrm{Fix}},L_0^{\mathrm{Fix}},L_1^{\mathrm{Fix}})$. Then

$$\mathrm{HF}(M,L_0,L_1)\otimes F_2[\theta,\theta^{-1}]\rightrightarrows\mathrm{HF}(M^{\mathrm{Fix}},L_0^{\mathrm{Fix}},L_1^{\mathrm{Fix}})\otimes\mathbb{F}_2[\theta,\theta^{-1}]$$

The intrinsic symmetry is $(u,v,z)\to(u,-v,z)$.

Theorem 14.3 (Manelescu)

Consider

$$\mathcal{Y}_n^{Fix}=Sym^n(\Sigma-\{\mathbf{z}\})\backslash\nabla$$
$$\Sigma_A^{Fix}=\alpha_1\times\ldots.\times\alpha_n$$
$$\Sigma_B^{Fix}=\beta_1\times\ldots.\times\beta_n$$

Then we have a spectral sequence

$$Kh_{symp}(L)\otimes\mathbb{F}_2[\theta,\theta^{-1}]\rightrightarrows g\widehat{HF}(\Sigma(\overline{L})\otimes H_*(S^1))\otimes\mathbb{F}_2[\theta,\theta^{-1}].$$

14.1.1 Extrinsic Symmetries

For a link L, we have $\mathcal{S},\mathcal{Y}_n$. Do the same for $\overline{L}$. $\tau:(u,v,z)\mapsto(u,v,-z)$. $S/\tau=\overline{S},z\mapsto z^2$.

Theorem 14.4 *The following commutative diagram commutes:*

$$\begin{tikzcd} Hilb^{n/2}\left(\overline{S}\right) \arrow[r, hook] \arrow[d, "HC"'] & Hilb^n(S) \arrow[d, "HC"] \\ Sym^{n/2}\left(\overline{S}\right) \arrow[r, hook] & Sym^n(S) \end{tikzcd}$$

$$(u, v, z) \longmapsto \left\{(u, v, -\sqrt{z}), (u, v, -\sqrt{z})\right\}$$

where τ acts on $Hilb^n(S)$ and $Sym^n(S)$.

Theorem 14.5 (Seidel, Smith)

$$Kh_{symp}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows Kh_{symp}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$$

Theorem 14.6 (Stoffregen, Zhang, 2018; Borodzik, Politarcyzk, Silvero)

$$Kh(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows AKh(\overline{L}) \otimes \mathbb{F}[\theta, \theta^{-1}].$$

14.2 Annular (Symplectic) Khovanov Homology

Following the works of [Asaeda; Przytycki, Sikora; Roberts].

AKh is an associated graded ring of an annular filtration.

Theorem 14.7 (Mak, Smith, 2019)

Over characteristic 0*,*

$$AKh^{HH}_{symp}(L) \simeq AKh(L)$$

where HH is the Hochschild homology.

Now, we present a new $\mathrm{AKh}_{\mathrm{symp}}$. Replace $\mathrm{Hilb}^n(S)$ with $\mathrm{Hilb}^n(S \backslash \pi^{-1}(0))$, deleting a divisor over 0.

Theorem 14.8 (Hendricks, Mak, Raghunath)

$AKh_{symp}(L)$ is a link invariant.

Conjecture 14.1 Over characteristic 0, $\mathrm{AKh}_{\mathrm{symp}}(L)$ is the same as $\mathrm{AKh}^{HH}_{\mathrm{symp}}(L)$.

We have the following intrisic cases:

- $(u, v, z) \mapsto (u, -v, z)$:

$$\begin{aligned}\mathrm{AKh}_{\mathrm{symp}}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] &\rightrightarrows g\widehat{CFK}(\Sigma(mL), \tilde{A}) \otimes H_*(S^1) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \\ &\rightrightarrows \widehat{CFK}(\Sigma(mL), \tilde{A}) \otimes H_*(S^1) \otimes \mathbb{F}_2[\theta, \theta^{-1}]\end{aligned}$$

- $(u, v, z) \mapsto (u, v, -z)$

$$\mathrm{AKh}_{\mathrm{symp}}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows \mathrm{AKh}_{\mathrm{symp}}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$$

- $(u, v, z) \mapsto (-u, -v, -z)$: start with the original theory and do not delete a divisor.
- $\{(u, v, z), (-u, -v, -z)\}$:

$$\mathrm{Kh}_{\mathrm{symp}}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows \mathrm{AKh}_{\mathrm{symp}}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

14.3 The Table

We can summarize the results in the following table:

Action	**Consequence**	**Analog**
$(u, v, z) \mapsto (u, -v, z)$	$\mathrm{Kh}_{\mathrm{symp}}(L)$ to $\mathrm{g}\widehat{\mathrm{HF}}(\Sigma(mL))$	[Ozsvath, Szabo]
$(u, v, z) \mapsto (u, v, -z)$	$\mathrm{Kh}_{\mathrm{symp}}(L)$ to $\mathrm{Kh}_{\mathrm{symp}}\left(\overline{L}\right)$	[None]
$(u, v, z) \mapsto (u, -v, z)$	$\mathrm{AKh}_{\mathrm{symp}}(L)$ to $\widehat{\mathrm{HF}}\left(\Sigma(mL), \hat{A}\right)$	[Roberts]
$(u, v, z) \mapsto (u, v, -z)$	$\mathrm{AKh}_{\mathrm{symp}}(L)$ to $\mathrm{AKh}_{\mathrm{symp}}\left(\overline{L}\right)$	[Zhang]
$(u, v, z) \mapsto (-u, -v, -z)$	$\mathrm{Kh}_{\mathrm{symp}}(L)$ to $\mathrm{AKh}_{\mathrm{symp}}\left(\overline{L}\right)$	[Szabó, Ozsváth; Borodzik, Politarczyk, Silvero]

Chapter 15

John Pardon: Derived Moduli Spaces of Pseudo-Holomorphic Curves

Abstract I will present the derived representability approach to working with moduli spaces of pseudo-holomorphic curves.

15.1 Introduction

We are interested in properties and structures of moduli spaces of solutions to elliptic partial differential equations, which lead to enumerative invariants:

1. **Global topological property**: (Uhlenbeck, Gromov) compactness
2. **Local structure**: Regularity, of which there are two types:
3. Classical regularity: $\mathcal{M}$ is locally $\cong \mathbb{R}^n$ (or more generally, sometimes $\mathbb{R} \times \mathbb{R}^m_{\geq 0}$, etc.). Classical regularity requires transversality (e.g. by choosing "generic" data for the partial differential equation).
4. Derived regularity: $\mathcal{M}$ is locally $\cong$ zero set for a smooth function on $\mathbb{R}^n$ (or more generally, sometimes $\mathbb{R} \times \mathbb{R}^m_{\geq 0}$, etc.). Derived regularity holds in wide generality, however, it becomes technical to say what the relevant structure on $\mathcal{M}$ encoding such chart actually is [Fukaya, Ono; Fukaya, Oh, Ohta, Ono; Li, Tian; Ruan, Seibert; etc.]

We want to associate with every moduli problem a moduli space, which is associated to an enumerative invariant. We will mostly focus on associating with every moduli problem a moduli space.

Problem 15.1 Why is derived regularity complicated (in comparison to the classical case)?

The answer is that derived regularity is a contact structure.

Example 15.1 Consider a proper submersion $Q \to B$, let $E, F/Q$ be vector bundles, and $L : C^\infty(Q, E) \to C^\infty(Q, F)$ be a vertical elliptic operator. What is the nature of $\pi_* L$, the 2-term complex finite vector bundle on B where pointwise cohomology is $\ker L_b$ and $\operatorname{coker} L_b$, where L_b is the restriction of L to Q_b?

Given $V \xrightarrow{d} W$ and $V \oplus \mathbb{R} \xrightarrow{d \oplus 1} W \oplus \mathbb{R}$, π_L is unique up to contractible choice in the 2-category of 2-term vector bundles on B, where contractible choice means

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & W \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] \\ V' \arrow[r, "d'"'] & W' \end{tikzcd}$$

where $d'h + hd = f - g$.

Let Sm be the category of smooth manifolds and DSm be the ∞-category of derived smooth manifolds. The functor Sm $\to$ DSm freely adjoints finite limits, modulo preserving finite products.

Concretely, a derived smooth manifold is a formal symbol $\lim_K p$ for some finite degree $p : K \to \mathrm{Sm}$, e.g.

$$\lim \left(\begin{tikzcd} & \mathbb{R} \arrow[d, "x \mapsto x^2"] \\ * \arrow[r, "0"] & \mathbb{R} \end{tikzcd} \right) \in \mathrm{DSm}$$

Additionally, we have

$$\mathrm{Hom}_{\mathrm{DSm}}(\lim_K p, \lim_L q) = \text{sheafification on } \lim_K |p| \left(\lim_L \operatorname{colim}_{K_\Delta} \mathrm{Hom}_{\mathrm{Sm}}(p_\Delta, q) \right)$$

Example 15.2 Let

$$\tau := \lim \left(\begin{tikzcd} & \mathbb{R} \arrow[d, "x \mapsto x^2"] \\ * \arrow[r, "0"] & \mathbb{R} \end{tikzcd} \right)$$

A map $\tau \to M \in \mathrm{Sm}$ is a a point $p \in M$ and a vector $v \in T_p M$.

Remark 15.1 D(topological manifolds) is full subcategory of Top.

Each $X \in \mathrm{DSm}$ has an associated $TX \in \mathrm{Perf}^{\geq 0}(X)$.

15.2 Representability

Definition 15.1 Let C be a Riemann surface, X an almost complex manifold. A **family** of ψ-holomorphic maps $C \to X$ parameterized by Z is a map $Z \times C \xrightarrow{u} X$ together with an isomorphism between 0 and the derivative

$$Z \times C \xrightarrow{(D_C u)^{0,1}} TX \otimes_{\mathbb{C}} \overline{T^*C}.$$

Theorem 15.1 (Pardon, Steffens)

There exists a derived smooth manifold $Hol(C, X)$ and a nontrivial bijection (for $Z \in DSm$)

$$\{\text{maps } Z \to Hol(C, X)\} \xrightarrow{\sim} \{\text{families of } \psi\text{-holomorphic maps } C \to X \text{ parameterized by } Z\}.$$

Remark 15.2 For $u : C \to X$ a part of $\mathrm{Hol}(C, X)$, we have a canonical isomorphism

$$H^*(T_n\mathrm{Hol}(C, X)) = \begin{cases} \ker D_n & * = 0 \\ \operatorname{coker} D_n & * = 1 \end{cases}$$

where

$$D_n : C^\infty(C, u^*TX) \to C^\infty(C, u^*TX \otimes_{\mathbb{C}} \overline{T^*C})$$

Example 15.3 If $\operatorname{coker} D_n = 0$, then $T_n\mathrm{Hol}(C, X)$ has H^* support in degree 0, which implies that $\mathrm{Hol}(C, X)$ is smooth near u.

$$\begin{array}{ccc} \{\text{compact smooth manifolds } \to A\}/\text{bordism} & = & \Omega_+(A) \\ & & \big\downarrow L \\ \{\text{compact derived smooth manifolds } \to A\}/\text{bordism} & = & \Omega_+^{\mathrm{der}}(A) \ni [\mathrm{Hol}(C, X)] \end{array}$$

The content in the proof is the following:

Proposition 15.1 *For $Z \in DSm$ and every map $Z \times C \to X$ (not necessarily ψ-holomorphic), there exists a map $Z \to Q \in Sm$ and an isomorphism $u = v|_Z$ with $v : Q \times C \to X$.*

An application of the above proposition is as follows:

Theorem 15.2 (Zung)

X a smooth stack with submersive atlas and a proper diagonal, then X is locally isomorphic to M/G for G a compact Lie group.

Using the proposition, we can show the same holds for any derived smooth stack.

Chapter 16

Mark McLean: Symplectic Orbifold Gromov-Witten Invariants

Abstract Chen and Ruan constructed symplectic orbifold Gromov-Witten invariants more than 20 years ago. In ongoing work with Alex Ritter, we show that moduli spaces of pseudo-holomorphic curves mapping to a symplectic orbifold admit global Kuranishi charts. This allows us to construct other types of Gromov-Witten invariants, such as K-theoretic counts. The construction relies on an orbifold embedding theorem of Ross and Thomas.

16.1 Introduction

The aim is to construct ∂W invariants for orbifolds. Over $\mathbb{Q}$, this was done by [Chen, Ruan]. We want to use global Kuranishi charts. This is part of a larger project proving a version of the crepant resolution conjecture. There is also work by [Mak, Seyfaddini, Smith] are working on the global quotient case. This talk is all joint work with [Ritter].

Informally, an orbifold is like a manifold, but the charts look like V/Γ where $V \subset \mathbb{R}^n$ is an open finite subset, $\Gamma \to \mathrm{GL}_n(\mathbb{R})$. For example, $\{\mathrm{pt}\}/\mathbb{Z}/2$ is an orbifold. We will not define an orbifold formally.

Suppose G is a compact Lie group acting on a smooth manifold M with finite stabilizers. Then the quotient $[M/G]$ is naturally an orbifold.

Theorem 16.1 (The Slice Theorem)

For each point $x \in M$, there exists a G_X-equivariant submanifold $S_X \subseteq M$ containing X and a G-equivariant neighborhood $U_X \subseteq M$ of X such that

$$G \times_{G_X} S_X \to U_X$$

is a diffeomorphism.

Definition 16.1 If S_X has a global G_X equivariant coordinate system, then (S_X, G_X) is an **orbifold chart** for $[M/G]$ centered at X.

Theorem 16.2 (Pardon)

Every compact orbifold is of the form $[M/G]$.

Definition 16.2 Let $[M_1, G_1], [M_2, G_2]$ be orbifolds. A **Hilsum-Skandalis morphism** between these orbifolds is a diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & M_2 \\ M_1 & \end{tikzcd}$$

where there is a $G_1 \times G_2$ action on P, π is a principal G_2-bundle that is G_1-equivariant, and f is G_2-equivariant.

We can define a symplectic orbifold, complex orbifold, etc.

Definition 16.3 If $X = [M/G]$ is an orbifold, then $\underline{X} = M/G$ is the **underlying coarse moduli space**.

16.2 Twisted Nodal Curves

Let's fix (X, ω, J) where X is an orbifold, ω is a symplectic form, and J is a J-taming almost complex structure. Take $\beta \in H_2(\underline{X}; \mathbb{Z})$.

Definition 16.4 A **twisted nodal domain** Σ is a space $\tilde{\Sigma}/\sim$ where $\tilde{\Sigma}$ is a 1d complex orbifold and the equivalence relation is identifies distinct pairs of points $p \sim q$ so that the following balancing condition holds:

- p admits an orbifold chart with coordinate z centered at p and where the stabilizer group $\mathbb{Z}/k\mathbb{Z}$ acts by $(m, z) \to e^{2\pi i m} z$
- q admits an orbifold chart with coordinate z centered at q and where the stabilizer group $\mathbb{Z}/k\mathbb{Z}$ acts by $(m, w) \to e^{-2\pi i m} w$

So, near a node, Σ looks like $\{XY = 0\}/\mathbb{Z}/k\mathbb{Z}$ with $(g, (x, y)) = (gx, g^{-1}x)$. We can extend this to $\{XY = t\}/\mathbb{Z}/k\mathbb{Z}$.

Definition 16.5 A **marking** on Σ is a collection of distinct points $p_1, ..., p_n$ disjoint from the nodes containing all smooth points with nontrivial stabilizer.

To learn more, see [Abramovich, Vistoli].

Definition 16.6 A **twisted nodal curve** $U : \Sigma \to X$ is a J-holomorphic Hilsum-Scandalis morphism $\Sigma \to X$ such that

- this descends to a continuous map $\Sigma \to \underline{X}$
- the induced map of stabilizer groups $G_\sigma \to G_{u(\sigma)}$ is injective for all $\sigma \in \tilde{\Sigma}$.

[Abramovich, Vistoli] studied the example $\{\mathrm{pt}\}/\,\mathbb{Z}/2 = X$.

For smooth $g = 0$ case, Sieburt considered (u, Σ, F) where $u : \Sigma \to X$, $L \to X$, and the framing is a basis of $H^0(u^*L)$. Then, we obtain $\mathcal{E} \to \mathbb{P}^d$ where $d = \dim H^0(u^*L) - 1$ and $\mathcal{F} = \mathcal{M}_{0,d}(\mathbb{P}^n)$.

16.3 Problems

There are many problems with generalizing this $g = 0$ case to the smooth orbifold case:

1. For higher genus curves, there is a moduli space of line bundles in each given degree
2. Twisted nodal curves with nontrivial stabilizer groups don't map to $\mathbb{CP}^d$

Let's deal with (2), using an idea by [Ross, Thomas]. Instead of looking at curves mapping to $\mathbb{CP}^d$, we'll look at curves mapping to the weighted projective space

$$P(w_0, ..., w_d) = \left(\mathbb{C}^{d+1} - 0\right)/\sim$$

with

$$(z_0, ..., z_d) \sim (t^{w_0} z_0, ..., t^{w_d} z_d)$$

for all $t \in \mathbb{C}^\times$.

Let Y be a compact complex orbifold with only cyclic stabilizer groups. We should think of $\tilde{\Sigma}$.

Definition 16.7 A line bundle L is **locally ample** if for all $y \in Y$, the stabilizer group of Y acts faithfully on the fiber $L|_Y$.

Definition 16.8 L is **globally positive** if L^N is a pullback of an ample line bundle from Y.

Definition 16.9 L is **orbi-ample** if it is locally ample and globally positive

Definition 16.10 Let $n_i = |H^0(L^i)|$ for i. A k**-framing** of L is a tuple

$$\left(f_{ij}\right)_{i=k,\ldots,2k,\, j=0,\ldots,n}$$

where $f_{ij}, J = 0, \ldots, n$ is a basis of $H^0(L^i)$.

Theorem 16.3 *Define $\mathbb{P}_k(L) = \mathbb{P}(k, \ldots, k, k+1, \ldots., k+1, \ldots, 2k, \ldots, 2k)$ and let n_i be the number of i's in this expression. Take $\phi_F : Y \to \mathbb{P}_k(L)$ to be the map that sends y to $[\tau f_{ij}(y)]_{i=k,\ldots,2k,\, j=0,\ldots,n_i} \subset \mathbb{P}_k(L)$. Then*

$$\tau : L|_y \to \mathbb{C}$$

is an isomorphism.

Chapter 17

Rohil Prasad: High-Dimensional Families of Holomorphic Curves and Three-Dimensional Energy Surfaces

Abstract Let H be any smooth function on $\mathbb{R}^4$. I'll discuss some recent dynamical theorems for the Hamiltonian flow on level sets of H ("energy surfaces"). The results are proved using holomorphic curves and neck stretching. One important tool is the compactness theorem from Dan's talk.

17.1 Basics

Consider $(\mathbb{R}^n, \Omega = \sum_{i=1}^n dx_i \wedge dy_i)$, a Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$, and the Hamiltonian vector field X_H.

Lemma 17.1 *The flow of X_H preserves H.*

Corollary 17.1 *The flow of X_H preserves the level sets of H.*

Remark 17.1 Historically, these H's were called energy surfaces.

Definition 17.1 Fix $s \in \text{Reg}(H)$. Then $H^{-1}(s)$ is a smooth $(2n-1)$-dimensional manifold. Energy surfaces that arise this way are called **regular**.

17.2 Invariant Sets

Problem 17.1 (Herman, 1998 ICM)

Fix $H : \mathbb{R}^{2n} \to \mathbb{R}$ and fix a compact regular energy surface. Does Y contain a proper closed X_H invariant subset?

Theorem 17.1 (Fish, Hofer, 2018)

Yes for $n = 2$.

Here is some more progress on the problem:

Theorem 17.2 (Weinstein, Rabinowitz, Vileda)

If Y is of contact type, it contains a closed orbit.

Theorem 17.3 *Examples exist where Y has no closed orbits:*

- *[Ginzburg, Kerman, Herman] for $n \geq 3$.*
- *[Ginzburg, Gurel] for $n \geq 2$, with H being C^2.*

Theorem 17.4 (Prasad, 2024; Theorem A)

Let $H : \mathbb{R}^n \to \mathbb{R}$, Y be a compact regular energy surface. There exists an infinite family of distinct, proper, closed invariant subsets with dense union in Y.

Notation: $\mathrm{Reg}_C(J) = \{s \in \mathrm{Reg}(H) | H^{-1}(s) \text{ compact}\}$.

Theorem 17.5 (Theorem B)

Let $H' : \mathbb{R}^{2n} \to \mathbb{R}$ for almost every $s \in Reg_c(H)$. $H'(s)$ has the following property: for any closed orbit $\Lambda \subset H^{-1}(s)$, $H^{-1}(s)/\Lambda$ is not minimal.

Remark 17.2 The Le Calvez-Yoccoz property implies dense existence on invariant sets.

17.3 Closed Orbits and Closed Holomorphic Curves

Theorem 17.6 (Hofer, Zehnder, 1987)

Fix $H : \mathbb{R}^{2n} \to \mathbb{R}$. For about every energy surface $s \in Reg_c(H), H^{-1}(c)$ contains a closed orbit.

Theorem 17.7 *Fix $H : \mathbb{R}^{2n} \to \mathbb{R}$. Almost every $s \in Reg_c H$, $H^{-1}(s)$ contains two closed orbits. This bound is sharp.*

Proof Consider

$$H = \frac{x_1^2 + y_1^2}{a} + \frac{x_2^2 + y_2^2}{b}$$

where $\frac{a}{b} \notin \mathbb{Q}$. Each regular energy surface has 2 closed orbits. $\square$

Theorem 17.8 *Fix $H : \mathbb{R}^4 \to \mathbb{R}$. Under C^∞-generic conditions on H, almost every compact regular energy surface has infinitely many closed orbits.*

Proof Follows from a strictly stronger version of Theorem B and known results about generic Hamiltonians. □

Theorem 17.9 (Taubes)

Fix $H : \mathbb{CP}^2 \to \mathbb{R}$. Fix base $J, J \geq 1, S \subset \mathbb{CP}^2$ such that $\#S \approx \lambda^2$. Then there exists a closed J-holomorphic curve $u : C \to \mathbb{CP}^2$ such that

1. $S = u(C)$
2. $\int_C u\Omega = d$
3. $\chi(C) \sim -d^2$

17.4 Theorem A Proof Idea

Take

$$W_k = \mathbb{CP}^2 \backslash Y \cup [-k, k] \times Y.$$

Consider the sequence $u_k : C_k \to W_k$.

Definition 17.2 The **stretched limit set**

$$\chi(u_k) \in \mathrm{Cl}((-1, 1) \times Y) \times (-1, 1).$$

We say $(\Xi, s) \in \chi(u_k)$ if there exists $\{s_k\}$ such that the following holds:

1. $u_k(C_k) \cap (s_k^{-1}, s_k + 1) \times Y \to \Xi$ after shifting
2. $k^{-1} s_k \to s$.

Define $u_{d,k}$ be the set of degree d curves as above. The main structural result is the following:

Proposition 17.1

1. *For almost every s and every $(\Xi, s) \in \chi(s_k)$,*

$$\Xi = (-1, 1) \times \Lambda$$

where Λ is X_H invariant.

2. *For all but* $\sim d^2$ *heights* s*, every* $(\Xi, s) \in \chi(u_{d,k})$ *is such that* Ξ *is* ϵ*-almost invariant where* $\epsilon \to 0$ *as* $d \to \infty$.

Chapter 18

Thomas Massoni: Taut Foliations Through a Contact Lens

Abstract In the late '90s, Eliashberg and Thurston established a remarkable connection between foliations and contact structures in dimension three: any co-oriented, aspherical foliation on a closed, oriented 3-manifold can be approximated by both positive and negative contact structures. Additionally, if the foliation is taut then its contact approximations are tight. In this talk, I will present a converse result on constructing taut foliations from suitable pairs of contact structures. While taut foliations are rather rigid objects, this viewpoint reveals some degree of flexibility and offers a new perspective on the L-space conjecture.

18.1 Introduction

Let M^2 be a closed, oriented, connected 3-manifold.

Informally, a foliation $\mathcal{F}$ is made by decomposing M into 2d leaves. If such a foliation is smooth, we can define $T\mathcal{F} = \text{Ker } \alpha$, where α is a 1-form.

Theorem 18.1 (Frobenius Integrability Theorme)

$$\alpha \wedge \, d\alpha = 0.$$

Definition 18.1 The **minimally integrable plane field** is

$$\xi = \text{Ker } (\alpha)$$

where α is a 1-form satisfying

$$\alpha \wedge d\alpha \neq 0$$

Locally, we have the following theorem:

Theorem 18.2 (Darboux)

$$\alpha = dz + x\,dy$$

Theorem 18.3 *Every homotopy class of a plane field is defined up to foliation.*

Proof Follows from the h-principal. □

Theorem 18.4 (Eliashberg)

The same holds for contact structures.

Definition 18.2 A **taut foliation** satisfies the following properties:

- for all $p \in M$, there exists γ cloosed loop passing through p, where γ transverses $\mathcal{F}$.
- there exists ω a closed 2-form satisfying $\omega_{T\mathcal{F}} > 0$.

We present the following (badly defined) definition.

Definition 18.3 A **tight contact structures** is a structure that isn't over-twisted.

Problem 18.1 Which M carry a taut foliation? Reebless foliation? Tight contact structure?

Theorem 18.5 *If the first Betti number $b_1(H) > 0$, there exists a taut foliation.*

Theorem 18.6 (Eliashberg, Thurson)

There exists tight contact structures.

Problem 18.2 What happens for $b = 1$, equivalently QHS^3?

Conjecture 18.1 (L-Space Conjecture)

Let M be QHS^3, irreducble, The following are equivalent:

1. M causes a taut foliation
2. M is not an L-space: $\mathrm{HF}^{\mathrm{red}} = 0$
3. $\pi_1(M)$ is left-orderable

We have the following progress so far:

Theorem 18.7 (Ozsvath, Szabo)

$$(1) \implies (2)$$

But in general, it's unsolved and people think it's wrong.

The punchline is taut foliations can be thought of as contact geometry objects.

18.2 From Foliations to Contact Structures

Theorem 18.8 (Eliashberg, Thurston, 1998)

1. *For a $\mathcal{F}$ foliation (C^2) with no spherical leaf (meaning no $\mathcal{F} = S^2 \times \{pt\}$ on $S^1 \times S^2$), then $T\mathcal{F}$ t_L are approximately contact structures.*
2. *If $\mathcal{F}$ taut, then the contact structures are tight.*

18.2.1 Proof of (2)

Proof Look at $[-1, 1] \times M$, choose ω closed in M, $w|_{T\mathcal{F}} > 0$, and choose α a 1-form such that $T\mathcal{F} = \ker \alpha$.

Suppose $\Omega = \omega(t\alpha)$. If $\xi_\pm$ weak symplectic filling of $(-M, \xi_-) \sqcup (M, \xi_+)$. If $\xi_\pm$ C^0-close to $T\mathcal{F}$, then Ω is symplectic and $\Omega|_{\xi_\pm} > 0$.

By [Gromov, Eilenberg], ξ_+ is tight. □

Definition 18.4 A C^0-foliation is a topological foliation where the leaves are smoothly immersed and $T\mathcal{F}$ is C^0

Theorem 18.9 (Bowden, Kasez, Robert)

Eliashberg-Thurson holds for C^0-foliations.

We should think of Eliashberg-Thurson as a machine that turns a foliation into a postiive contact pair (ξ_-, ξ_+).

18.3 From Positive Constant Pairs to Foliations

Definition 18.5 We say (ξ_-, ξ_+) is a **positive contact pair** if $\xi_\pm$ is a $\pm$ contact structure and there exists a vector field Z such that Z positively transverses $\xi_\pm$.

Fix (ξ_+, ξ_-) and Z.

Theorem 18.10 (Massoni, 2024)

Assume that at least one of $\xi_\pm$ is right, or ξ_- and ξ_+ are transverse. Then there exists a foliation $\mathcal{F}$ that transverses to Z.

Definition 18.6 If there exists a Z which is volume preserving, then we say (ξ_-, ξ_+) is **strongly right**.

A corollary of the theorem is as follows:

Corollary 18.1 *$M \neq S^1 \times S^2$ admits a taut foliation if and only if it admits a strongly tight contact pair.*

18.3.1 Sketch of Proof Idea

Consider a vector field $X \subset \xi_- \cap \xi_+$ which vanishes exactly along

$$\Delta = \{p \in M | \xi_-(p) = \xi_+(p)\}$$

Consider $\xi_\pm^t = (\phi_x^t) \times \xi_\pm$.

Proposition 18.1 *There exists η a continuous plane field such that $\xi_\pm^t$ converges to η.*

Proposition 18.2 *The map $(\xi_-, \xi_+) \mapsto \eta$ is continuous.*

Proposition 18.3 *For generic (ξ_-, ξ_+), η is locally integrable.*

Warning: η is only C^0, not always uniquely integrable tangent to the foliation.

There is also the useful notion of branching foliation, which we will not define right now.

Proposition 18.4 *Assume that there is no immersed $\overline{D} \to M$ tangent to η such that $\partial\overline{D}^2$ tangent to X. Then η is tangent to a branching foliation.*

Proposition 18.5 *η is approximatable by integrable plane fields.*

18.4 Speculation

Here is some speculation about the future of this area. Strongly tight pairs are hard to work with, but there are a few intruiging conjectures and problems to work on.

Conjecture 18.2 (Massoni)

If (ξ_-, ξ_+) is a positive pair and both $\xi_\pm$ are tight, then there exists a Reebless foliation.

We want an intrinsic result on the isotopy classes of ξ_- and ξ_+.

Problem 18.3 Consider a contact pair (ξ_-, ξ_+), not necessarily positive. Assume they are both tight, they are homotopic as plane fields, and $\langle c(\xi_-), c(\xi_+)\rangle = 1$. Then does there exist a Reeb-less foliation on M?

The third condition that we assumed is motivated by the following theorem:

Theorem 18.11 (Lin)

If $\mathcal{F}$ is a taut foliation, $\xi_\pm$ are contact approximations,

$$\langle c(\xi_-), c(\xi_+)\rangle = 1$$

where c is the contact invariant in HF.

Here is an application of this theorem:

Theorem 18.12 *Take $M^2 \neq S^1 \times S^2$, with $\mathcal{F}$ a taut foliation on M and K a transverse and framed knot. Then there exists s_0 such that for every $s \in \mathbb{Q}$ with $|s| \geq s_0$, there exists a a taut foliation on $M_k(s)$ transverse to K' (image of K).*

Chapter 19

Vardan Oganesyan: How to Construct Symplectic Homotopy Theory

Abstract In 1968 Dold and Thom proved that singular homology groups of X are isomorphic to homotopy groups of infinite symmetric product of X. In 1990-2000 Morel, Suslin, and Voevodsky used a similar definition to define motivic cohomology groups of algebraic varieties. Moreover, they defined homotopy theory for algebraic varieties. Motivated by these results, we construct homotopy theory for symplectic manifolds. In particular, we define some new homology groups for symplectic manifolds and prove that these homology groups have all required properties. We will not discuss details, but we will show that these new homology groups appear in a very natural way. If time permits, we will also discuss some possible applications.

19.1 Introduction

Let (X, e) where $e \in X$. Consider

$$\mathrm{SP}^n(X) = X^n/S_n$$

We have an embedding $\mathrm{SP}^n(X) \hookrightarrow \mathrm{SP}^{n+1}(X)$ through $\{p_1, ..., p_n\} \to \{p_1, ..., p_n, e\}$. So we have $\mathrm{SP}^0(X) \hookrightarrow \mathrm{SP}^1(X) \hookrightarrow ...$

Define

$$\mathrm{SP}(X) = \bigcup \mathrm{SP}^n(X)$$

This consists of elements $p = \{e, p_1, ..., p_n, e\}$. It turns out $\mathrm{SP}(X)$ is a semigroup under the following operation:

$$\{p_1, ..., p_n, e\} + \{q_1, ..., q_k, e\} = \{p_1, ..., p_n, q_1, ..., q_k, e\}$$

Let Δ^n be an n-simplex. We consider

$$\mathrm{Map}(\Delta^n, \mathrm{SP(X)})^+$$

which is also a semigroup. We can turn this into a gorup by considering the Grothendieck group

$$\mathrm{Map}(\Delta^n, \mathrm{SP(X)})^+ = C_n(X)$$

by adding $-f, -g$ satisfying $-(f+g) = -f - g$ and taking the restriction ∂_K : $C_n(X) \to C_{n-1} \to X$ with $d = \sum_{k=0}^{n} (-1)^k \partial_X$ satisfying $d^2 = 0$.

19.2 Dold-Thom Theorems

Theorem 19.1 (Dold, Thom, 1968)

$$H(C_*(X); d) = H_*^{sing}(X; \mathbb{Z}).$$

where $H(C_*(X); d) = \pi_*(SP(X))$

Theorem 19.2 (Dold, Thom, 1988-2000)

Let X *be a variety,* $SP^n(X)$, $\Delta^n_{alg} = \{z \in \mathbb{C}^{n-1} \mid z_1 + \ldots + z_{n+1} = 1\}$, *Consider* $C_n^{alg} = Map_{alg}(\Delta^n_{alg}, SP(X))^+$. *Define*

$$H(C_*^{alg,d}) = H_*^{sus}(X)$$

where sus means the Suslin homology.

We have

$$H_0^{sus}(T^2) = \mathbb{Z} \times T^2 H_0^{sus}(\Sigma_g) = \mathbb{Z} \times Jac(\Sigma_g)$$

19.3 Categories

Let Y be a variety, $\mathcal{U} \subset Y$ be open. We can define

$$C_n(\mathcal{U}, X) = \mathrm{Maps}_{\mathrm{alg}}(\Delta^n_{\mathrm{alg}}, \mathrm{SP}(X))$$

and $\mathcal{U} \to C_*(\mathcal{U}, X)$ is a sheaf on Y.

We can define the following categories:

1. Map are symplectic embeddings
2. Map are generalized Lagrangians corr

3. J-holomorphic maps

The second one is the most important but also the hardest.

Problem 19.1 What is Δ^n? What is $\mathrm{Map}(Y, \mathrm{SP}(X))$? $\mathrm{Map}(Y, \mathrm{SP}(X))$?

Consider $\mathcal{U} \overset{\text{symplectic}}{\hookrightarrow} X^n/S_n$. We are interested in $U \longrightarrow$ unordered maps $f_1, ..., f_n : U \to X$ such that $f_1^* w_X + ... + f_n^* w_X = w_Y$ which is a presheaf. When we perform sheafification, we obtain a global section $\int \mathrm{SCor}_n(Y, X)$. This is an analogue of maps $Y \to \mathrm{SP}^n(X)$.

We need to define a map $Y \to \mathrm{SP}(X)$. Consider $\mathrm{ICor}(Y, X)$ in the same way: $\mathcal{U} \overset{\text{isotropic}}{\longrightarrow} X^n$ where if $G \in \mathrm{ICor}_k(Y, X), F \in \mathrm{SCor}_n(T, X)$, then $F + G \in \mathrm{SCor}_{n+k}(T, X)$.

Definition 19.1 We say $F_1 \in \mathrm{SCor}_n(Y, X), F_2 \in \mathrm{Scor}_{n+k}(Y, X)$, then $F_1 \sim F_2$ if there exists $G \in \mathrm{ICor}(T, X)$ such that $F_2 = F_1 + G$.

We want to define $\mathrm{SCor}(Y, X) = (\bigsqcup \mathrm{SCor}_n(Y, X)) /\!\sim$ an analogue of $Y \to \mathrm{SP}(X)$. Fix a segment $M = (M, w_M, p_0, p_1)$.

Now, we define $\mathrm{SC}_n(Y, X) = \mathrm{Map}(Y \times M^n, X)$. We have an embedding $i_k^\epsilon : M^n \hookrightarrow M^{n+1}$ where $M^n \to M^k \times p_\epsilon \times M^{n-k}$ where $\epsilon = 0, 1$. We have

$$\partial_k : \mathrm{SC}_n(T, X) \to \mathrm{SC}_{n-1}(Y, X) d = \sum_{\epsilon=0}^{1} \sum_{k=0}^{n} \partial_k$$

where $d^2 = 0$.

We obtain

$$H(\mathrm{SC}_*(Y, X); d) = \mathrm{EH}_*(Y, X).$$

All standard symplectic embeddings belong to $\mathrm{SCor}(Y, X)$. If Y is contractible, then $\mathrm{SCor}(Y, X) \neq 0$.

19.4 Homotopy and Triangulated Persistence Categories

Definition 19.2 Let $F_0, F_1 \in \mathrm{SCor}(Y, X)$. F_0 is M**-homotopic** to F_1 if there exists $H \in \mathrm{SCor}(Y \times M, X)$ such that

$$H|_{Y \times p_0} = F_0, \qquad H|_{Y \times p_2} = F_2.$$

Proposition 19.1

1. *Homotopy equivalences defines equivalence relation* $SCor(Y, X)$.
2. *If* $\varphi_t : Y \to X$ *is a symplectic isotopy, then* φ_0 *is homotopic to* φ_1.
3. *Define* $H(SC_*(Y, X); d) = EH_*(Y, X)$. *Groups* $EH_*(Y, X)$ *are functorial in all nice ways.*
4. *These groups are homotopy invariant.*
5. *These groups have all of the required exact sequences.*

[Biran, Corea, Zhang] defined triangulated persistence category on $\mathcal{A}$, where $\mathrm{ob}(\mathcal{A})$ are symplectic manifolds and $\mathrm{Mor}(T, X) = \mathrm{SCor}(T, X)$. This category is additive, and we can consider a category of chain complexes to get a triangulated persistence category with metric $\mathrm{dist}(Y_1, Y_2) = \mathrm{dist}_*(Y_1, X) \cdot \mathrm{SC}_*(Y_2, X)$.

We can define $\mathrm{JH}_*(T, X)$. Let $M = \mathbb{CP}^{\times}$. If X is Kähler and $\mathrm{JH}_0(\mathrm{pt}, X) = 0$, then X is algebraic.