Enumerative Geometry of Quiver Representations and Quantum Groups

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The main goal of this learning project is to explore this relationship.

$$\begin{tikzcd}
\begin{array}{c}\text{Gromov-Witten Theory}\\ \text{of quiver varieties}\end{array} \arrow[r] & \text{Maulik-Okounkov Yangians} \\
\begin{array}{c}\text{Enumerative geometry}\\ \text{of quiver representations}\end{array} \arrow[u] \arrow[d] & \text{Quantum groups} \arrow[u] \arrow[d] \\
\begin{array}{c}\text{Donaldson-Thomas Theory}\\ \text{of quivers with potential}\end{array} \arrow[r] & \text{Cohomological Hall algebras}
\end{tikzcd}$$

This is more or less an exploration of the intersection of three areas of math: enumerative geometry, quivers, and quantum groups.

Contents

1 Cartan Matrices

1.1 Motivation and Definition

The motivation for caring about Cartan matrices are Kac-Moody Lie algebras, where generalized Cartan matrices serve as the initial building blocks for their construction. Kac-Moody Lie algebras, particularly of the affine type, are a key ingredient in several fields of physics.

- **Conformal field theory**: Affine Kac-Moody algebras provide the symmetry structure necessary to construct conformal blocks and to implement the modular invariance of partition functions in conformal field theory.
- **String theory**: Affine Kac-Moody Lie algebras provide the algebraic structure for symmetry transformations and vertex operator algebras, which are essential for the formulation of consistent string interactions and compactifications.

Although these are the two main applications, Kac-Moody Lie algebras also appear in integrable systems, statistical mechanics, gauge theories, and more. See [1] for more.

Now that we know about why we should care about Cartan matrices, let's define a Cartan matrix.

Definition 1.1. *An $n \times n$ matrix $C = (c_{i,j})$ with integer entries $c_{i,j}$ is called a* ***Cartan matrix*** *if:*

- $c_{i,i} = 2$ *for all* i
- $c_{i,j} \leq 0$ *for all* $i \neq j$
- $c_{i,j} = 0$ *if and only if* $c_{j,i}$, *for all* i, j.

Most of the time, we will assume that C is symmetrizable: that is, there exists a diagonal matrix $D = \mathrm{diag}(d_1, ..., d_n)$ with d_i positive integers such that DC is symmetric.

Now, we present two different ways in which Cartan matrices naturally arise.

1.2 Finite Graphs with Automorphisms

The point of this realization is to motivate studying the representation theory of quivers with automorphisms.

We can encode all of the information of a Cartan matrix inside a valued graph, which can be then encoded in a graph with automorphism, such that there is no loss of information in this process.

Let's review some standard graph theory terminology: Consider a graph $\Sigma = (V, E, f)$ where $v \in V$ are the vertices, $e \in E$ are the edges, and a function $f(e) = \{u, v\}$. We allow loops as edges. An isomorphism $\Sigma \to \Sigma'$ consists of a pair of bijections $\sigma : V \to V', E \to E'$ such that if $f(e) = \{u, v\}$, then $f'(\sigma(e)) = \{\sigma(u), \sigma(v)\}$. An automorphism is an isomorphism $\Sigma \to \Sigma$.

Now, we define a valued graph Γ as a graph Σ with positive integers d_v or m_e assigned to each vertex or edge such that m_e is a common multiple of d_u and d_v. Two valued graphs are isomorphic if there is an underlying graph isomorphism such that the values attached to v and $\sigma(v)$ and to e and $\sigma(e)$ are the same for all v, e. The values d_u and m_u attached to u or v of Γ is the cardinality of the corresponding orbit.

Definition 1.2. *A graph with automorphism* (Σ, σ) *is a* ***graph realization*** *of* C *if* $\Gamma(\Sigma, \sigma) \cong \Gamma(C)$.

Theorem 1.3. *Every* C *has a graph realization.*

Proof. We perform the following procedure.

$$\begin{tikzcd}
(\Sigma, \sigma) \arrow[r, "1"] & \Gamma(\Sigma, \sigma) \arrow[d, dash, "4"] & \\
C \arrow[r, "2"] & \Gamma(C) \arrow[r, "3"] & (\Sigma(C), \sigma(C))
\end{tikzcd}$$

1. Given (Σ, σ), we construct $\Gamma(\Sigma, \sigma)$ through the folding procedure: Assign a graph with automorphism (Σ, σ) a valued graph $\Gamma(\Sigma, \sigma)$: the vertices and edges are the σ-orbits of the vertices and edges in Σ.

2. Given C, we construct $\Gamma(C)$ as follows.

   The underlying graph: the vertices are $1, ..., n$. If $i \neq j$, then there are $n_{i,j} = -\frac{d_i c_{i,j}}{d_{i,j}}$ parallel edges connecting i and j, where $d_{i,j} = \operatorname{lcm}(d_i, d_j)$.

   The weights: Each vertex i is assigned the weight d_i, and each edge is assigned the weight $m_e := d_{i,j}$.

3. Given $\Sigma(C)$, we construct $(\Sigma(C), \sigma(C))$.

   The underlying graph: There are $d_1 + ... + d_n$ vertices $v_{i,k}$ for $1 \leq i \leq n, k \in \mathbb{Z}/(d_i)$, and for fixed i, j there are $n_{i,j}$ parallel edges $e_{k,l}^{(a)}$ for $1 \leq a \leq n_{i,j}$ connecting $v_{i,k}$ with $v_{j,l}$, where (k, l) runs over the cyclic subgroup $\langle (1, 1) \rangle$ of the abelian group $\mathbb{Z}/(d_i) \times \mathbb{Z}/(d_j)$ generated by $(1, 1)$.

   The automorphism: for every i, σ cyclicly permutes the vertices $v_{i,1}, ..., v_{i,d_i}$. This is clearly an automorphism.

4. By construction, this is an isomorphism.

$\square$

To get a better sense of this construction, let's compute one example explicitly:

Example 1.4. *Consider*

$$C = \begin{pmatrix} 2 & -4 \\ -6 & 2 \end{pmatrix}, D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

- *Construction of* $\Gamma(C)$*:* C *is a* 2×2 *matrix, so* $n = 2$*. Thus, we have two vertices,* 1 *and* 2*. The weight on vertex* 1 *is* $d_1 = 3$*, and the weight on vertex* 2 *is* $d_2 = 2$*.*

  The number of edges between the two vertices is:

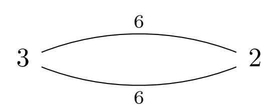
  $$n_{1,2} = -\frac{d_1 c_{1,2}}{lcm(d_1, d_2)} = -\frac{3 \cdot -4}{lcm(2,3)} = 2.$$

  We can check this:

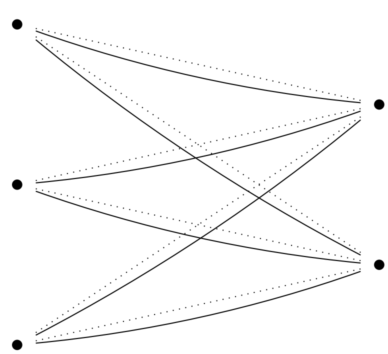
  $$n_{2,1} = -\frac{d_2 c_{2,1}}{lcm(d_2, d_1)} = -\frac{2 \cdot -6}{lcm(3,2)} = 2.$$

  The weights on the edges are $m_e := d_{i,j} = lcm(2,3) = 6$*.*

  This gives



- *Construction of* $(\Sigma(C), \sigma(C))$*: There are* $d_1 + d_2 = 5$ *vertices and each vertex coming from* d_1 *has* $n_{i,j} = 2$ *edges with each vertex coming from* d_2*:*



Definition 1.5. *A Cartan matrix* C *is called* ***indecomposable*** *if the underlying graph of* $\Gamma(C)$ *is a connected graph.*

Through lots of messy linear algebra, it can be shown that after renumbering the rows/columns, every Cartan matrix can be uniquely decomposed into block diagonal form such that each block is a Cartan matrix.

1.3 Root Datum Realization

The point of this realization is the Lie theory/quantum groups side of the story.

Definition 1.6. *Let $C = (c_{i,j})$ be an $n \times n$ Cartan matrix. A 4-tuple $\mathfrak{R} = (\Pi, X, \Pi^\vee, X^\vee)$ is called a* ***root datum realization*** *of $C = (c_{i,j})$ if*

- *$X^\vee$ is a free $\mathbb{Z}$-module of finite rank $n+m$, for some $m \in \mathbb{N}$, having an ordered basis $\{\alpha_1^\vee, ..., \alpha_n^\vee, b_1, ..., b_m\}$.*
- *$X := Hom(X^\vee, \mathbb{Z})$ is the $\mathbb{Z}$-linear dual of $X^\vee$, with the duality pairing*

$$X \times X^\vee \to \mathbb{Z}, \qquad (\alpha, h) \mapsto \langle \alpha, h\rangle = \alpha(h)$$

- *$\Pi^\vee := \{\alpha_1^\vee, ..., \alpha_n^\vee\}$ whose elements are called* ***simple coroots***
- *$\Pi = \{\alpha_1, ..., \alpha_n\}$ is a fixed subset of X satisfying*
  - *$\langle \alpha_i, \alpha_j^\vee\rangle = c_{j,i}$ for all i, j*
  - *$a_{j,i} = \langle \alpha_i, b_j\rangle$ for all i, j such that the combined $(n+m) \times n$ matrix $\begin{pmatrix} C \\ A \end{pmatrix}$, where $A = (a_{i,j})$, has rank n.*

  whose elements are linearly independent and are called ***simple roots****.*

By linear algebra, it is not hard to see that a realization exists if and only if $m \geq n - \text{rank}(C)$.

Definition 1.7. *The* ***root lattice*** *of $\mathfrak{R}$ is*

$$R(\Pi) := \mathbb{Z}\alpha_1 \oplus ... \oplus \mathbb{Z}\alpha_n \subseteq X$$

where X is a free $\mathbb{Z}$-module of rank $n+m$ called the ***weight lattice*** *and the* ***fundamental dominant weights*** *are the vectors $\overline{\omega_1}, ..., \overline{\omega_n} \in X$ defined by*

$$\langle \overline{\omega_i}, \alpha_j^\vee\rangle = \delta_{i,j}, \qquad \langle \overline{\omega_i}, b_t\rangle = 0 \qquad \text{for } 1 \leq i, j \leq n \text{ and } 1 \leq t \leq m.$$

Definition 1.8. *A weight $\lambda \in X$ is* ***dominant*** *if $\langle \lambda, \alpha_j^\vee\rangle \geq 0$ for $1 \leq j \leq n$.*

Definition 1.9. *The* ***Weyl group*** *$W(C)$ is the subgroup of $GL(\mathfrak{h}_\mathbb{R}^*)$ generated by s_i, where*

- *$\mathfrak{h}_\mathbb{R}^*$ is the $\mathbb{R}$-linear dual of $\mathfrak{h}_\mathbb{R} = X^\vee \otimes_\mathbb{Z} \mathbb{R}$*
- *$s_i \in GL(\mathfrak{h}_\mathbb{R}^*)$ are the linear transformations defined by*

$$s_i(\zeta) = \zeta - \langle \zeta, \alpha_i^\vee\rangle \alpha_i$$

  for $\zeta \in \mathfrak{h}_\mathbb{R}^*$.

It's not hard to see that $W(C)$ stabilizes X, and the action $W(C)$ on X induces a contragradient action of $W(C)$ on $X^\vee$ satisfying for $\zeta \in X, h \in X^\vee$,

$$\langle \zeta, w(h)\rangle = \langle w^{-1}(\zeta), h\rangle.$$

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References

[1] José Figueroa-O'Farrill. *Why Do Physicists Need Unitary Representation of Kac-Moody Algebra?* MathOverflow. Version: 2010-07-04. URL: `https://mathoverflow.net/q/30515`. URL: `https://mathoverflow.net/q/30515`.