Data, Math and Methods

Week 2, Gentle Math Recap

~ Vítek Růžička

Intro

- Last class you played chess. You tried thinking about strategies, then like a computer to win a game, predict the opponent. Abstract, logical, mathematical, systematic thinking.
- Listen: youtube.com/watch?v=cDIRT_NEMxo

AI playing games

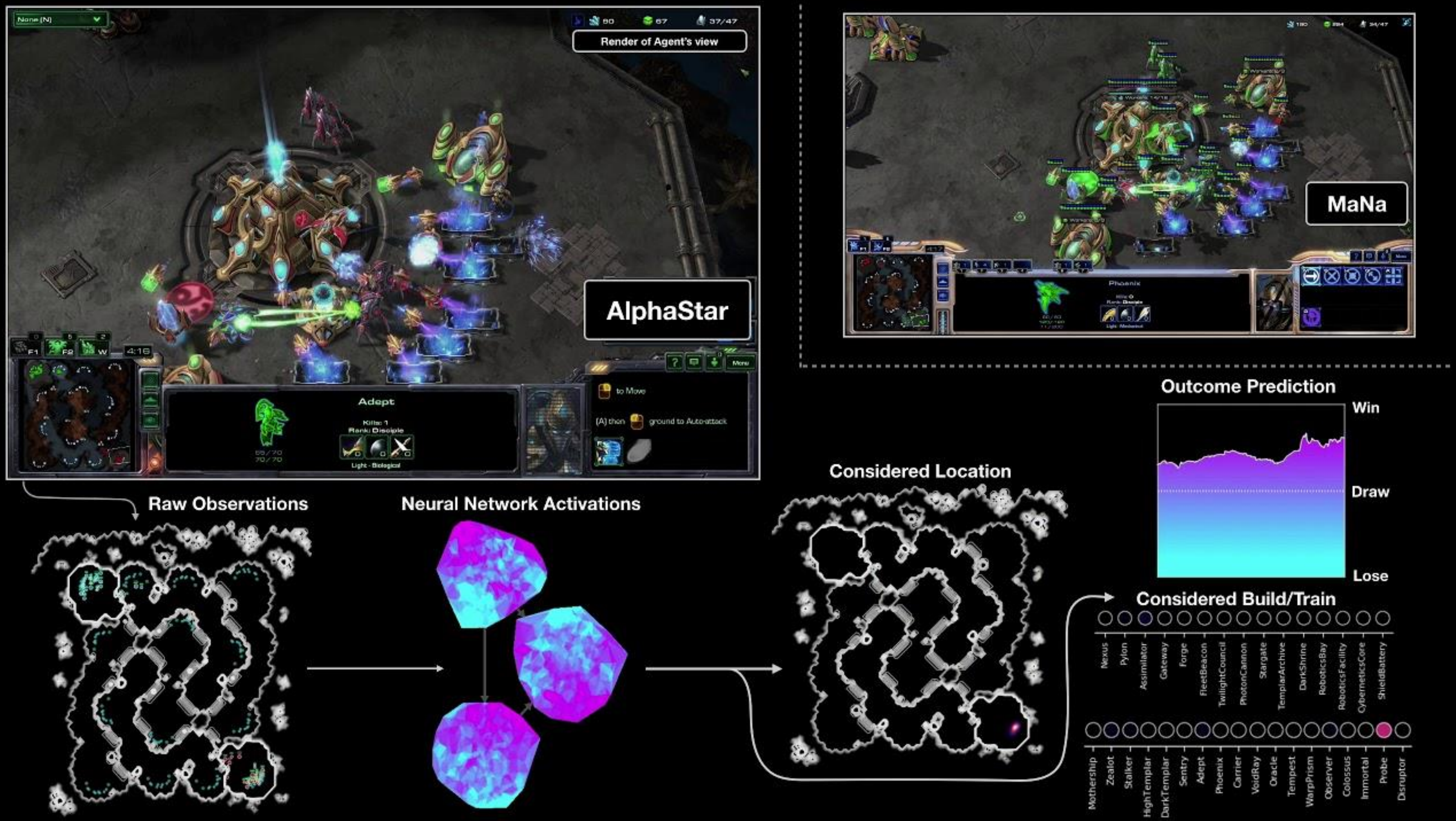
- Deep Blue (1996)

AI playing games

- Deep Blue (1996)
- Alpha GO (2015)
  - Self play

AI playing games

- Deep Blue (1996)
- Alpha GO (2015)
  - Self play
- Alpha Star (2019)
  - Neural Networks with Reinforcement Learning



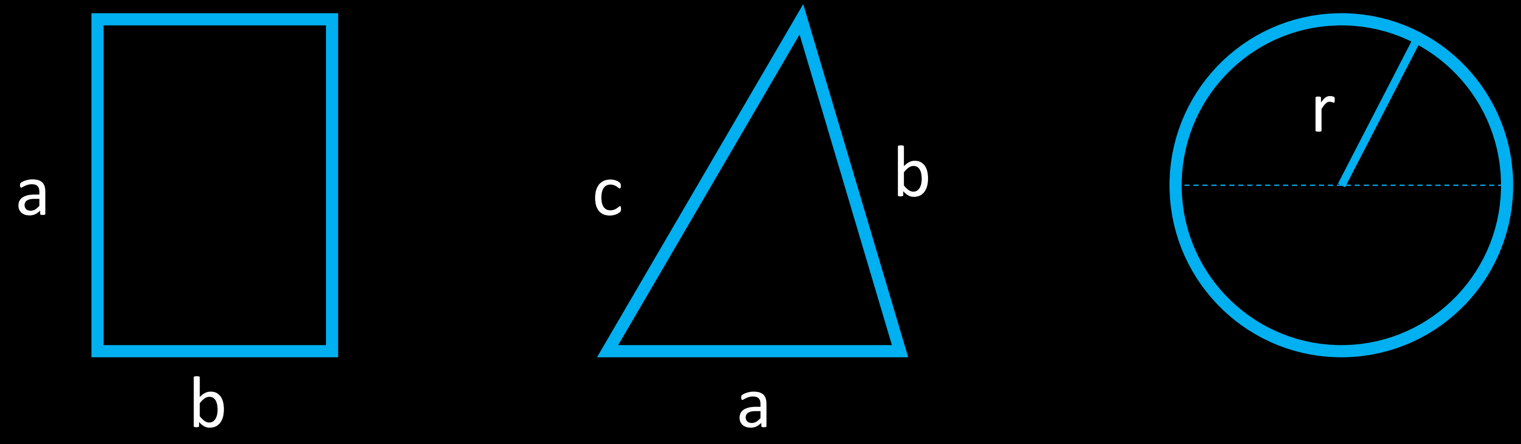
Motivation

- **Exciting times!**
- Examples where converting something into an abstract mathematical notation makes sense and helps the field.
- Real world problems are harder! Not possible to simulate reality (really exactly). We have only models which are being checked with real world experiments, compared.
  - Model of atmosphere => research of countering climate change effects. Then compare with real world experiments on smaller scale.

Algebra

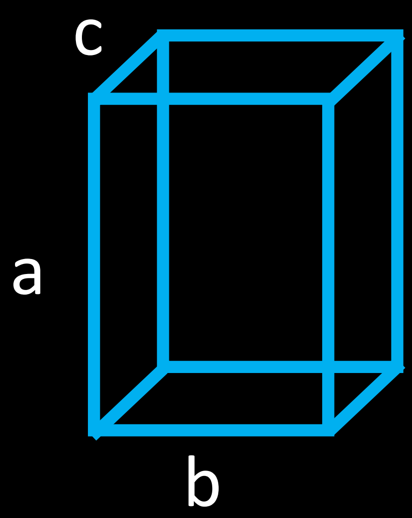
- … algebra is the study of mathematical symbols and the rules for manipulating these symbols …
- (abstractions and rephrasing them into better solvable forms)

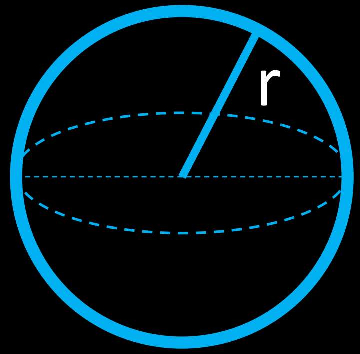
Volume of geometric shapes



• Area	$a*b$	$p*(p-a)*(p-b)*(p-c)$ $p = (a+b+c)/2$	πr^2
• Perimeter	$2a + 2b$	$a+b+c$	$2\pi r$ (circumference)

Volume of geometric shapes





• Volume	a*b*c	$4/3\ \pi r^3$
• Surface	2ab+2bc+2ac	$4\ \pi r^2$

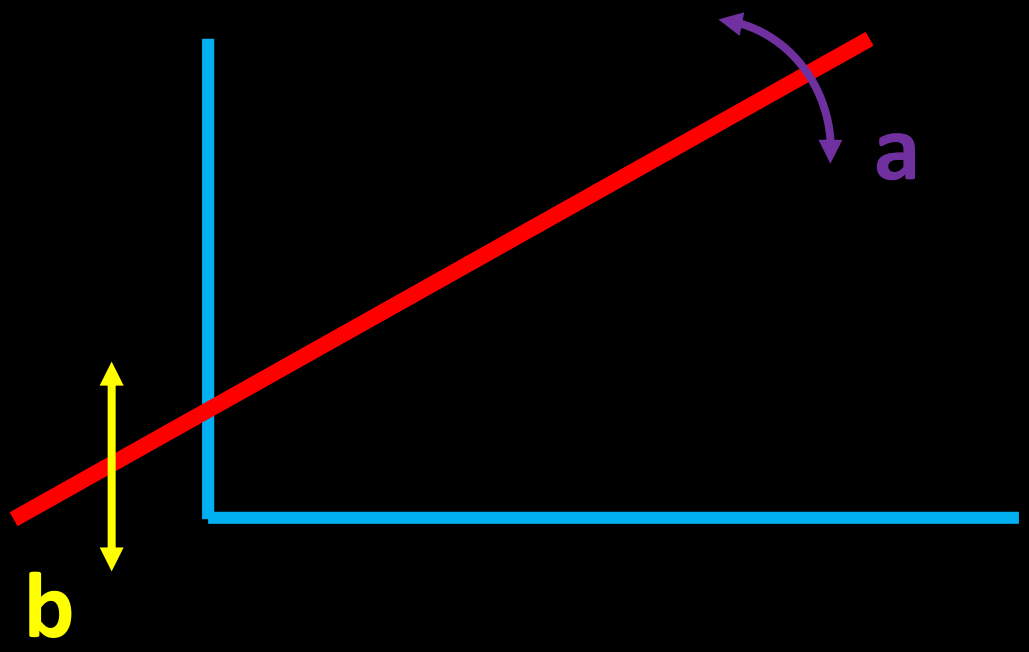
Equations

- Linear
- Quadratic
- Polynomial (wolfram)
- Multiple variables (wolfram)

Linear Functions

f(x): ax + b

y = ax + b



- Linear function is a line, parameters **a**, **b** influence it's location

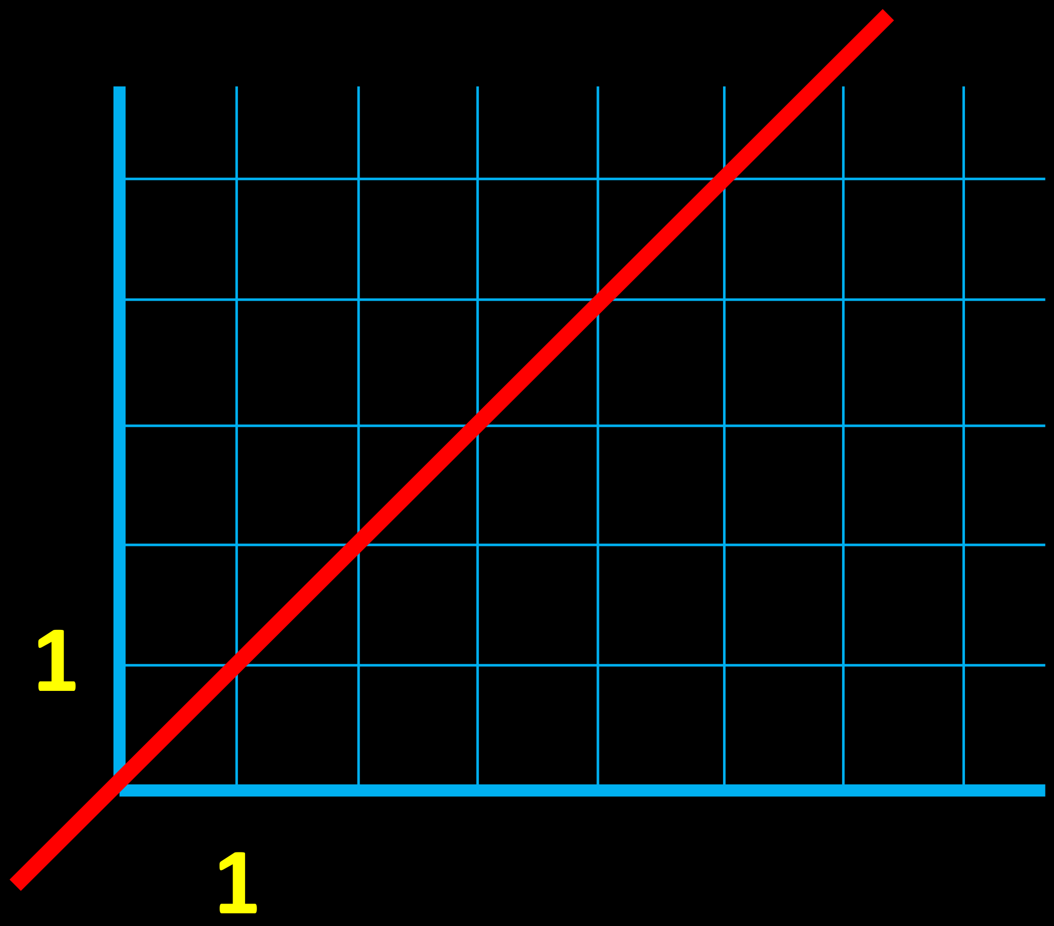
Linear Functions - example

f(x): ax + b

a = 1

b = 0

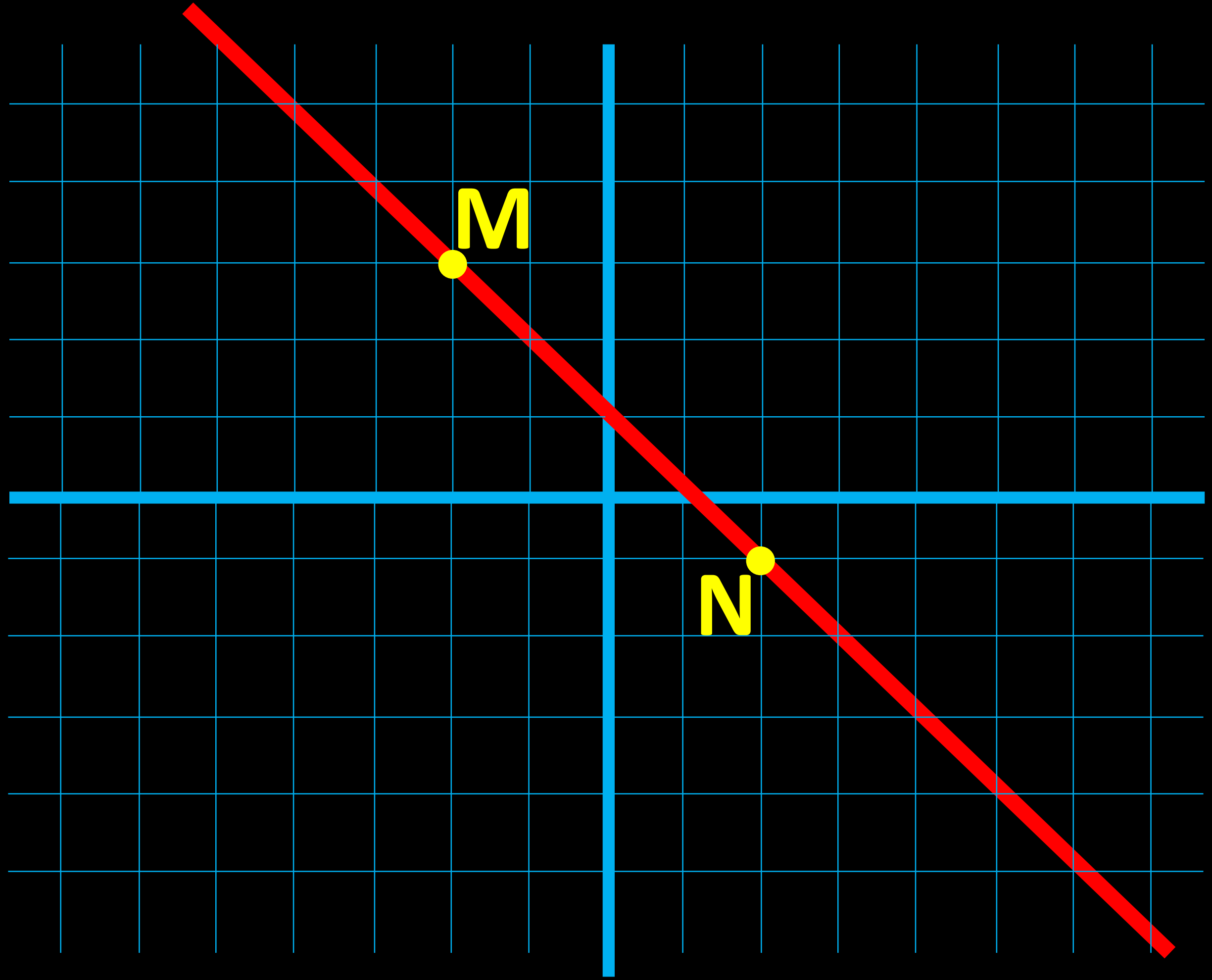
f(x): x



Linear Functions – given by points

f(x): ax + b

Find values for **a,b** so that the points **M** and **N** are on this line:



M = [-2; 3]

N = [2; -1]

Linear Functions – given by points

f(x): ax + b

Find values for **a,b** so that the points **M** and **N** are on this line:

M = [-2; 3]

N = [2; -1]

Points are on the line, which means:

M = [m_x, m_y] = [-2, 3]

N = [n_x, n_y] = [2,-1]

Are on:

ax + b = y

$a*m_x + b = m_y$

$a*n_x + b = n_y$

Linear Functions – given by points

f(x): ax + b

Find values for **a,b** so that the points **M** and **N** are on this line:

M = [-2; 3]

N = [2; -1]

```
Points are on the line, which means:
M = [mx, my] = [-2, 3]
N = [nx, ny] = [ 2,-1]

Are on:
       ax + b = y

a*mx + b = my          a*-2 + b = 3
a*nx + b = ny          a*2 + b = -1

… (solve from these two equations) …
```

Linear Functions – given by points

f(x): ax + b

Find values for **a,b** so that the points **M** and **N** are on this line:

M = [-2; 3]

N = [2; -1]

Points are on the line, which means:

M = [m_x, m_y] = [-2, 3]

N = [n_x, n_y] = [2,-1]

Are on:

ax + b = y

a*m_x + b = m_y	a*-2 + b = 3
a*n_x + b = n_y	a*2 + b = -1

… (solve from these two equations) …

a = -1

b = 1

Line: -x + 1 = y

Linear Functions

Super simple example, right?

f(x): ax + b

Wolfram Alpha command:

line points (-2,3) (2,-1)

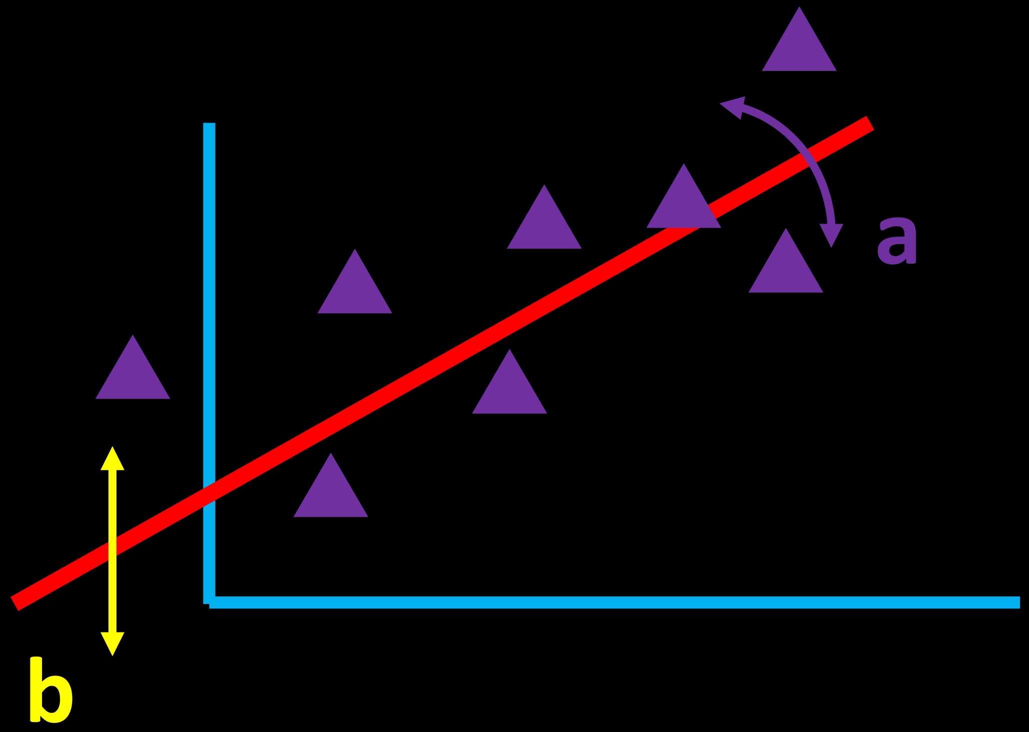
wolframalpha.com/input/?i=line points (-2,3) (2,-1)

Simplest Machine Learning algo.:

Linear regression:

f(x): ax + b

Find a, b, so that the line follows the data



Simplest Machine Learning algo.:

Linear regression:

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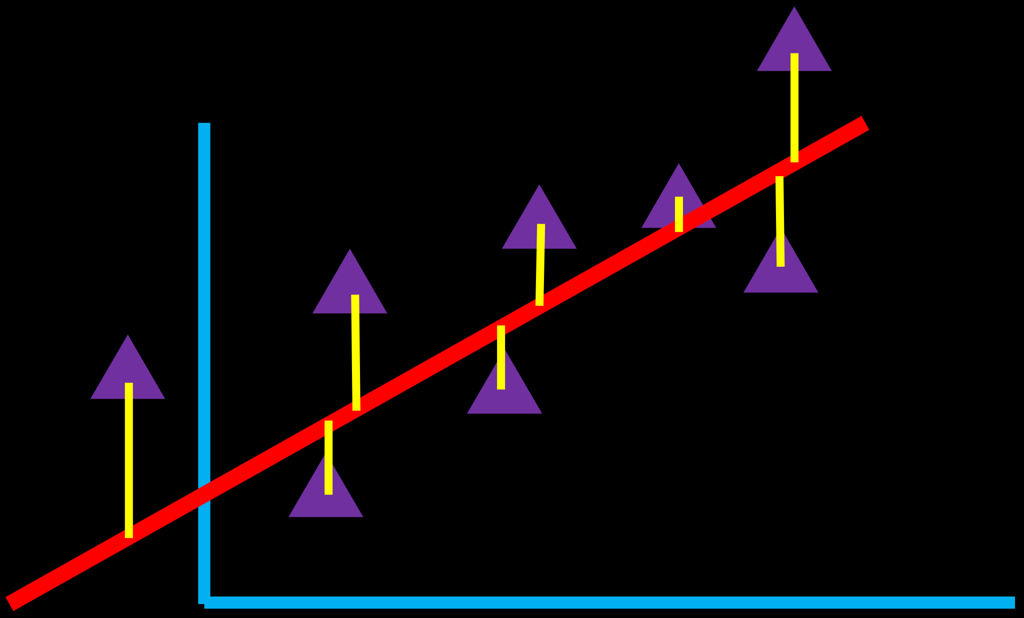
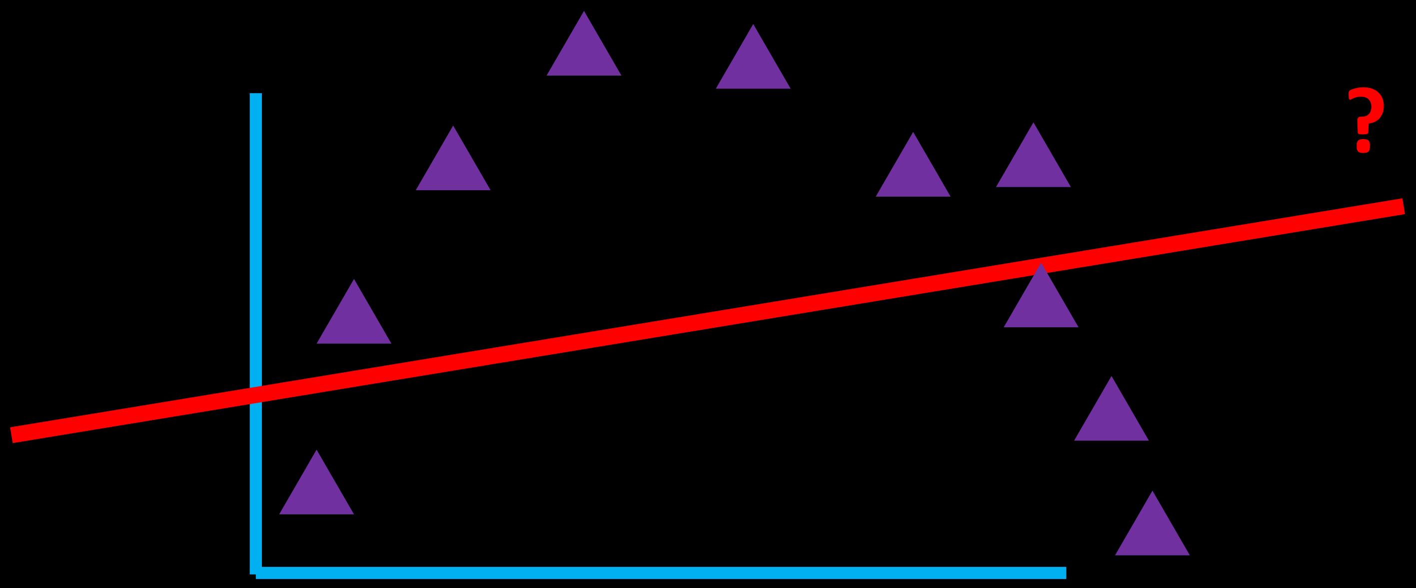


minimize the sum of the squared distances from the line

Simplest Machine Learning algo.:



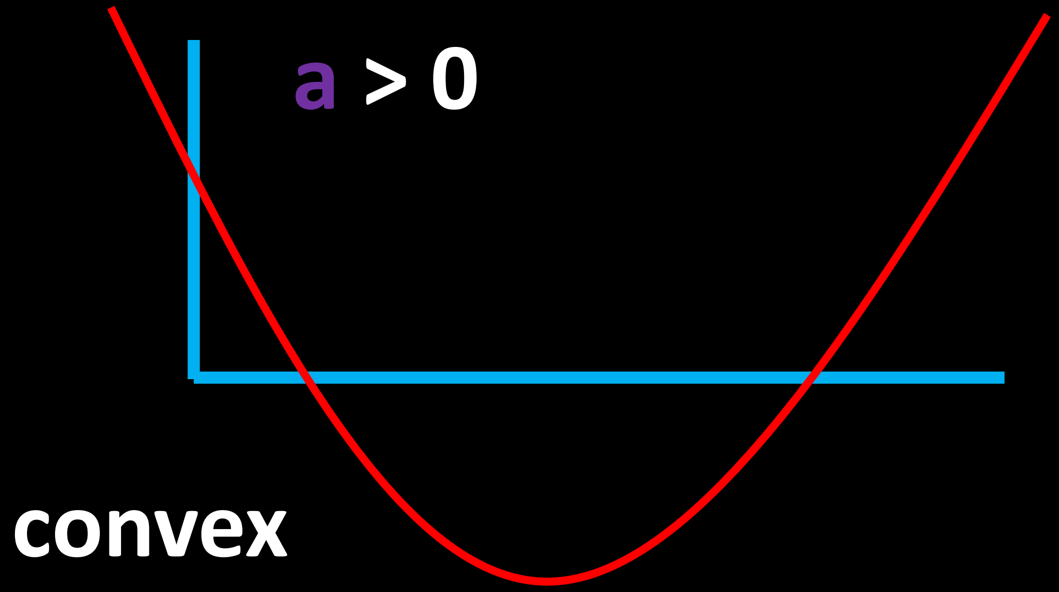
Depending on the points, might not be the most suitable …

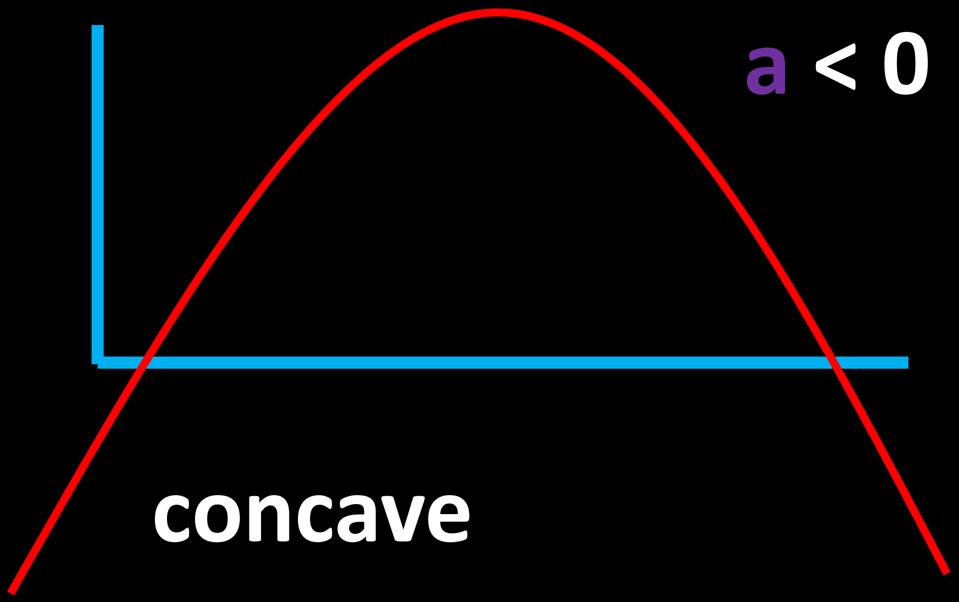
Simplest Machine Learning algo.:

See demo for
Least-Squares Regression

Quadratic Functions

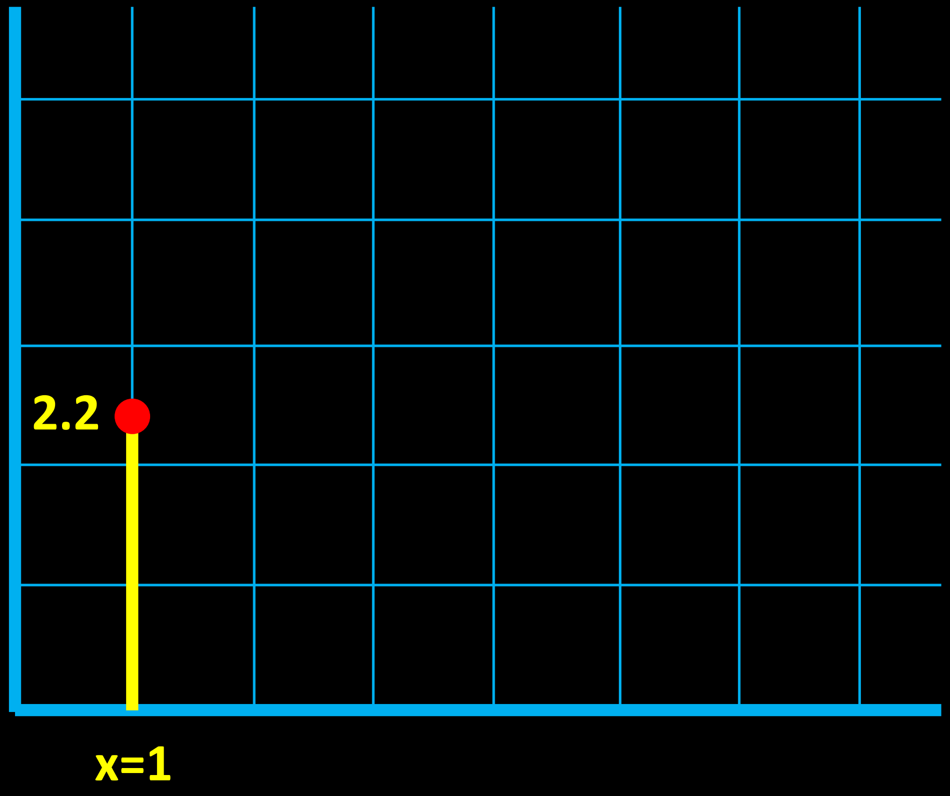
$f(x): ax^2 + bx + c$





Quadratic Functions – finding minimum

f(x): $ax^2 + bx + c$



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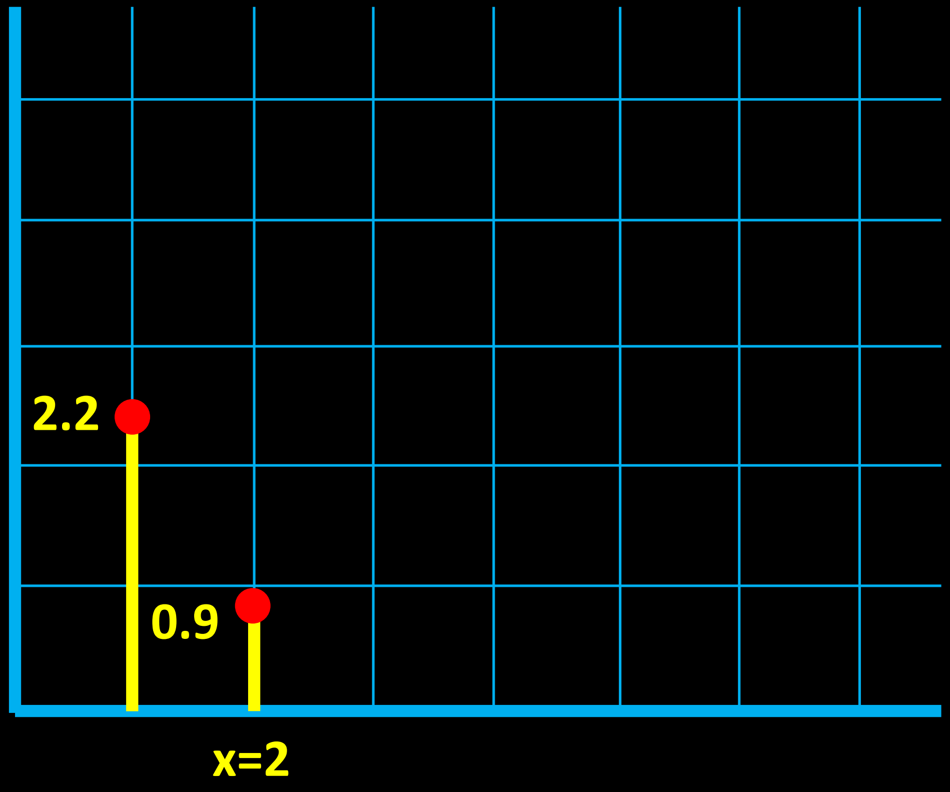

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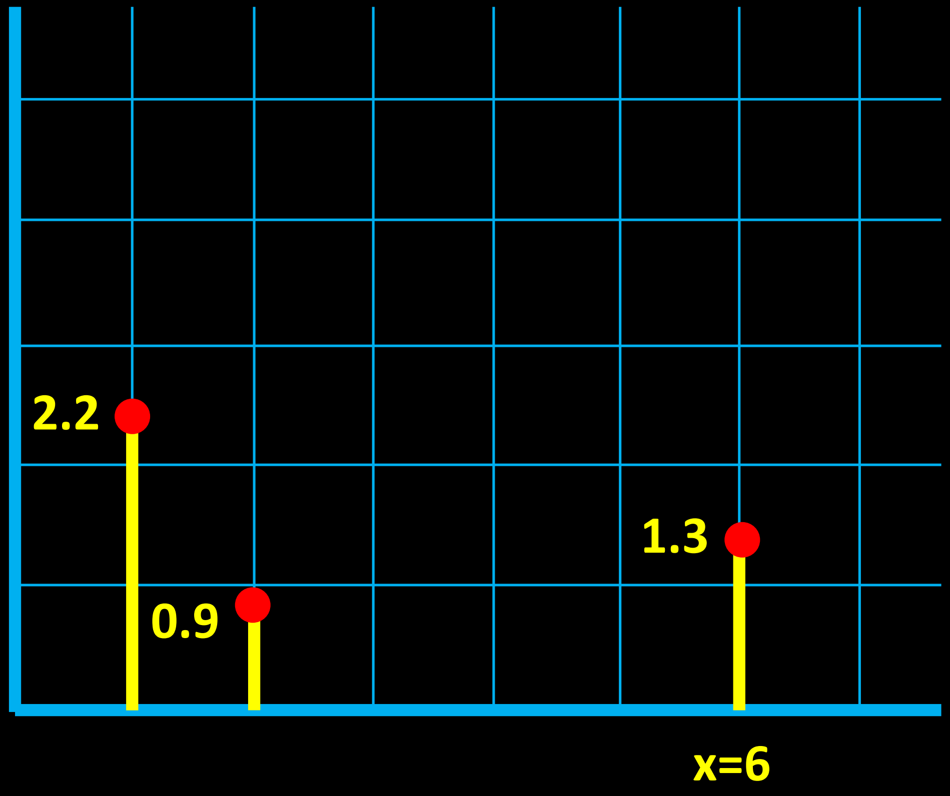

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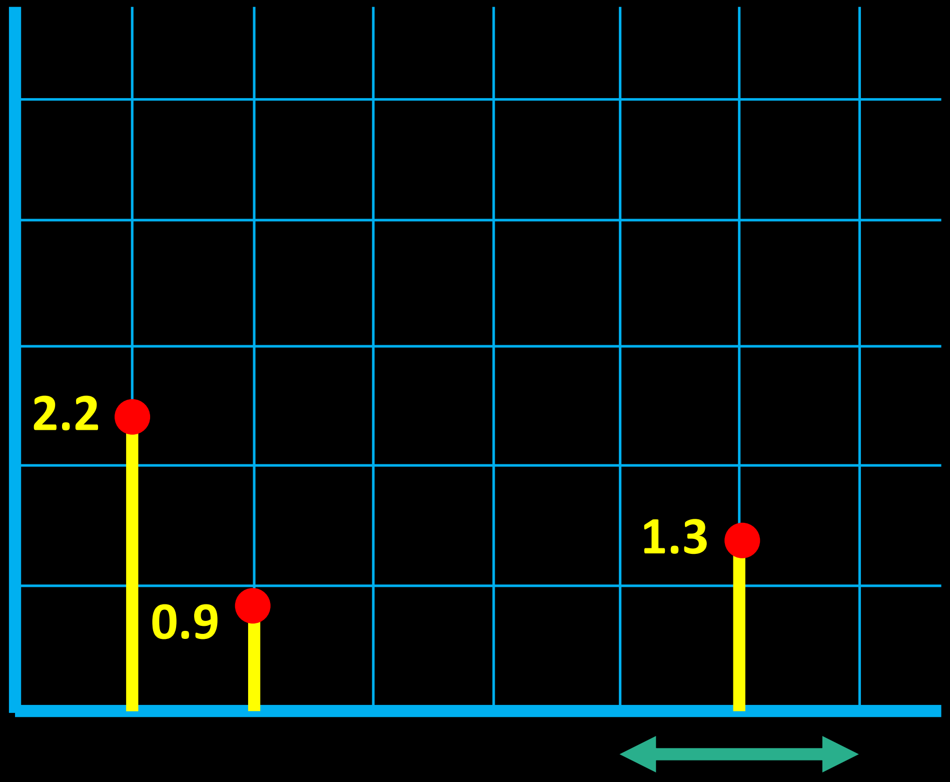
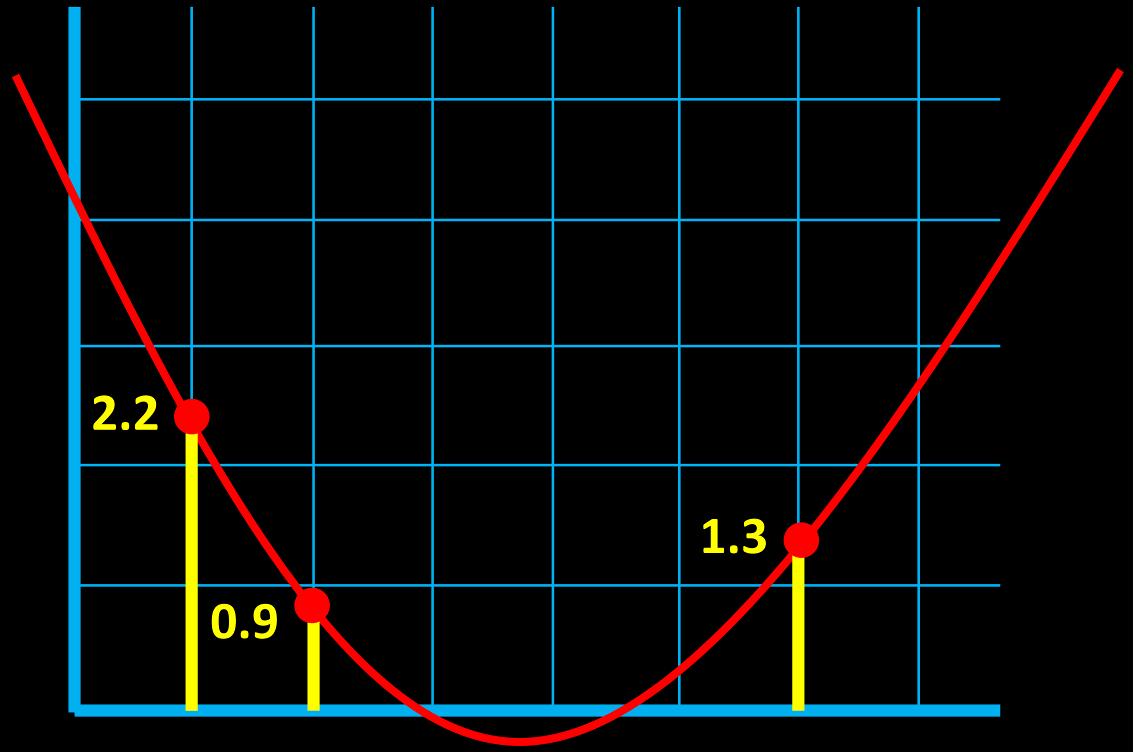


Where would you sample next?

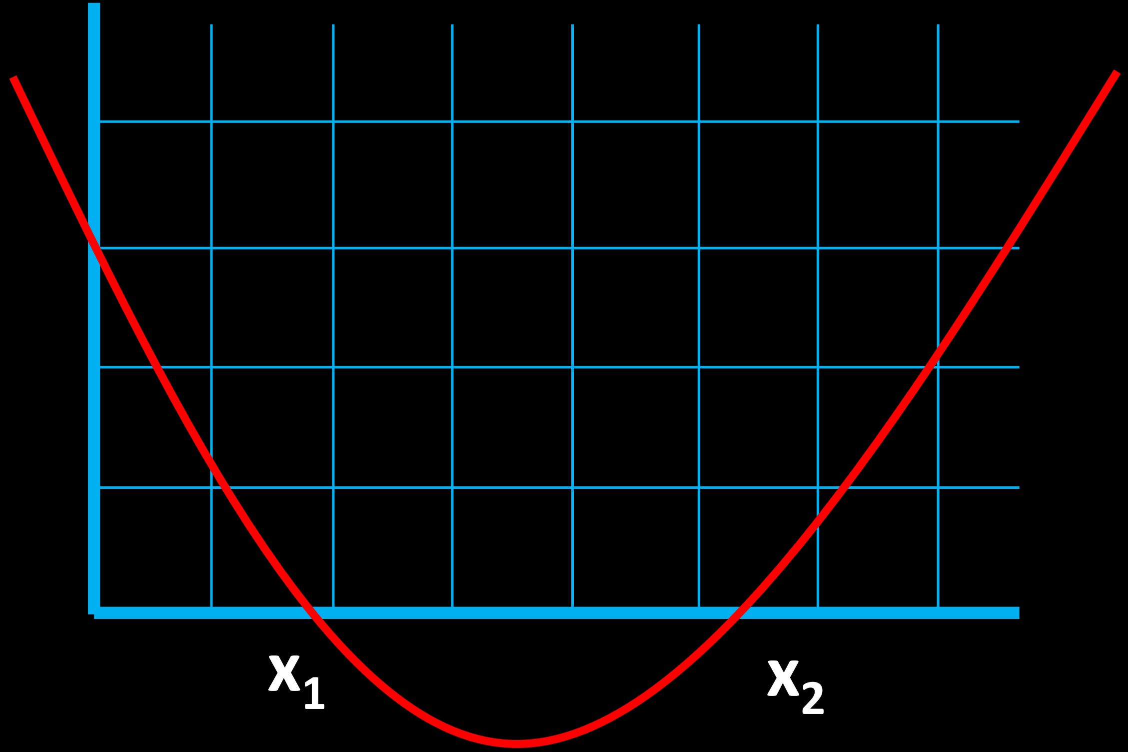
Quadratic Functions – finding minimum

f(x): $ax^2 + bx + c$



Quadratic Functions – intersections with x

f(x): ax² + bx + c



$a x^2 + b x + c = 0$	
x	indeterminate variable
a	quadratic coefficient
b	linear coefficient
c	constant coefficient
$\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$	

$$x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Functions – intersections with x

f(x): ax² + bx + c

$$x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Wolfram Alpha command:

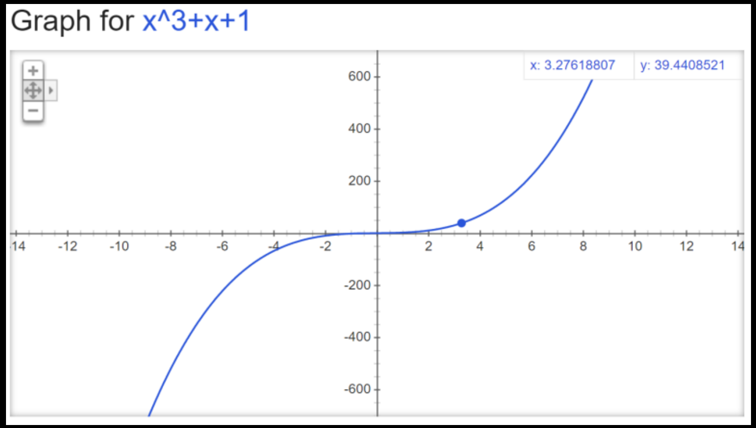
quadratic formula calculator

wolframalpha.com/input/?i=quadratic+formula+calculator

Polynomial Functions

For example:

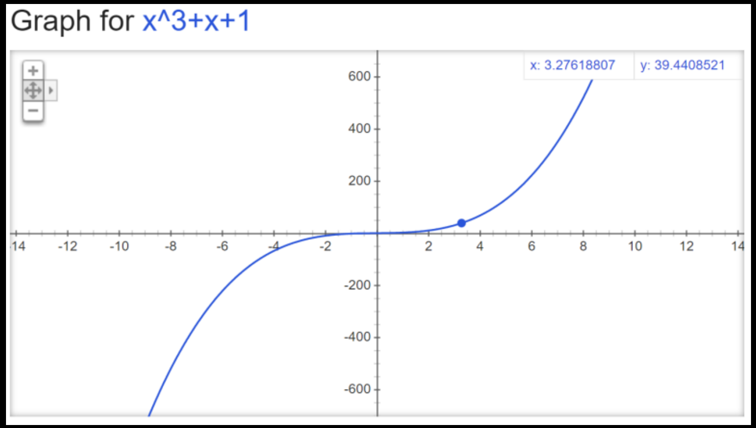
$$f(x): x^3 + x + 1$$



Polynomial Functions

For example:

f(x): $x^3 + x + 1$



More generally:

$$f(x): a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

ps: quadratic functions are also polynomials of 2 degree

Polynomial Functions

f(x): $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Wolfram alpha:

wolframalpha.com/examples/mathematics/algebra/polynomials/

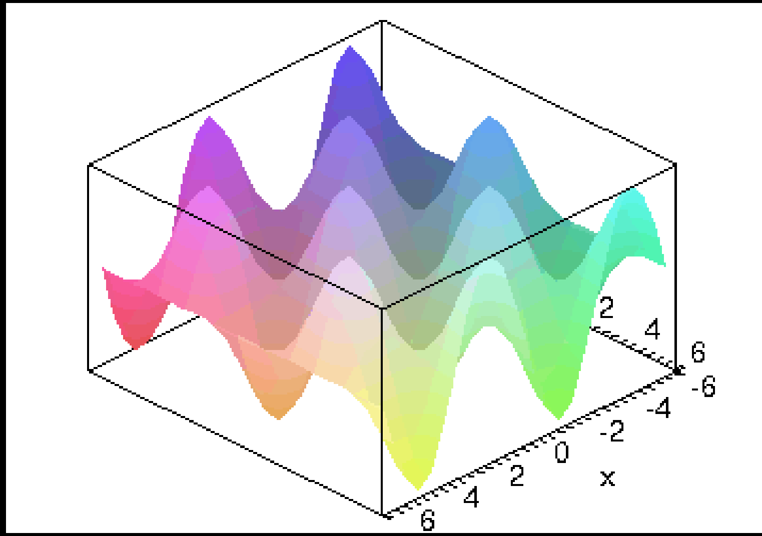
Multivariate Functions

So far…

$$f(x): x^3 + x + 1$$

Multiple variables – x,y,z, …

$$f(x,y): x^3 + y + 12$$



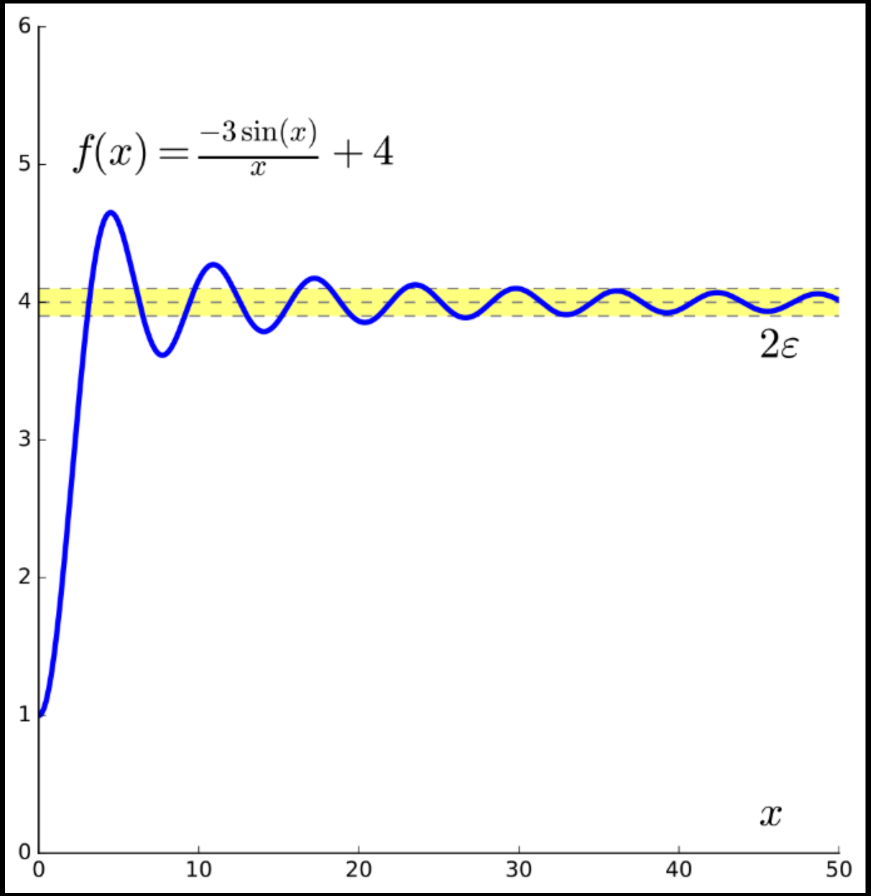
2 variables – visualize in 2D, 3 variables – vis in 3D, …

Functions

- Slope
- Min, Max, intersection with 0
- Visualization, inspection (wolfram)

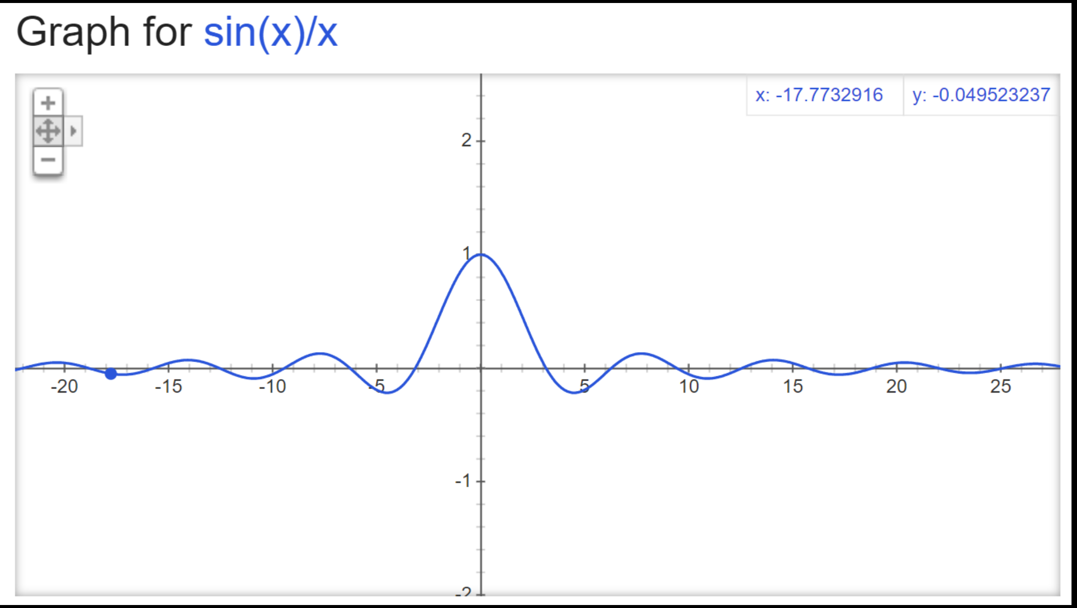
Limits

- In mathematics, a limit is the value that a function "approaches" as the input "approaches" some value.



Limits

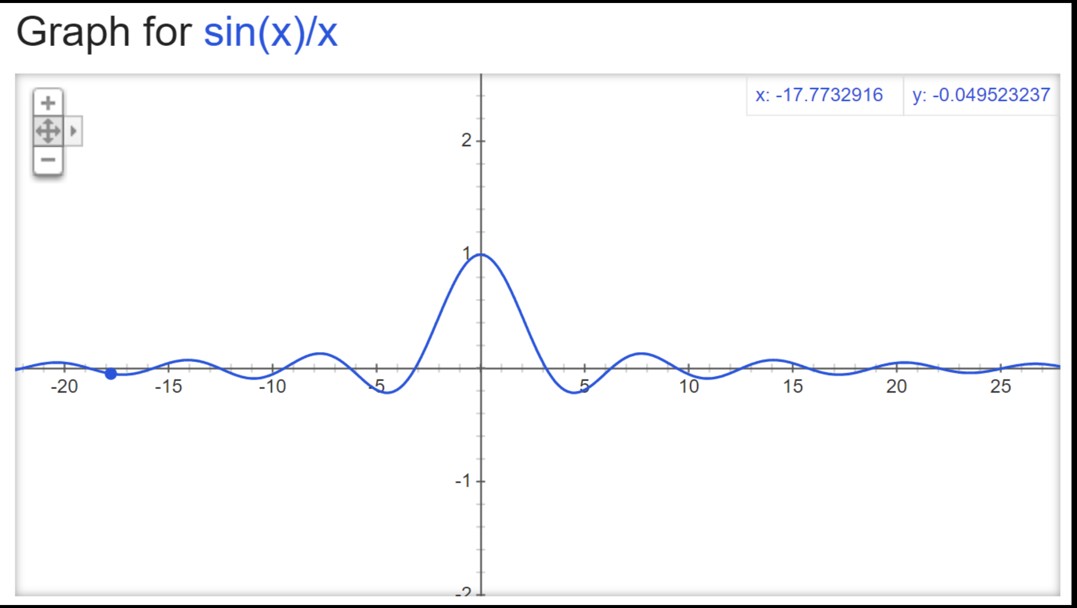
f(x): (sin x)/x



- f(0) = err, dividing by 0 = not defined

Limits

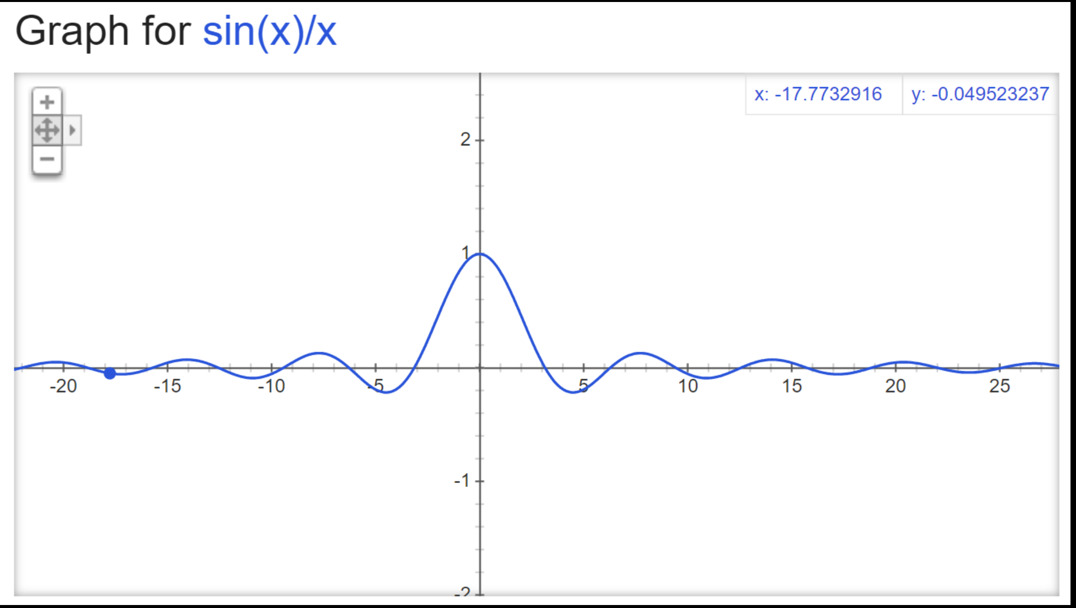
f(x): (sin x)/x



- f(0) = err, dividing by 0 = not defined
- f(1) = 0.841471…
- f(0.1) = 0.998334…
- f(0.01) = 0.999983…

Limits

f(x): (sin x)/x



- f(0) = err, dividing by 0 = not defined
- f(1) = 0.841471...
- f(0.1) = 0.998334...
- f(0.01) = 0.999983...

- Although the function **(sin *x*)/*x*** is **not defined at zero**, as *x* becomes closer and closer to zero, (sin *x*)/*x* becomes arbitrarily close to 1. In other words, the **limit of (sin *x*)/*x* as *x* approaches zero equals 1**.

Pause 1

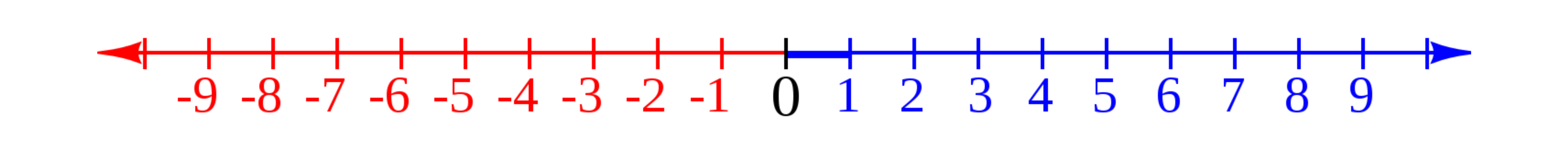
Number representations

- Integers
- Natural numbers
- Real numbers
- Complex numbers

- Int
- Float
- Double, …

Integers

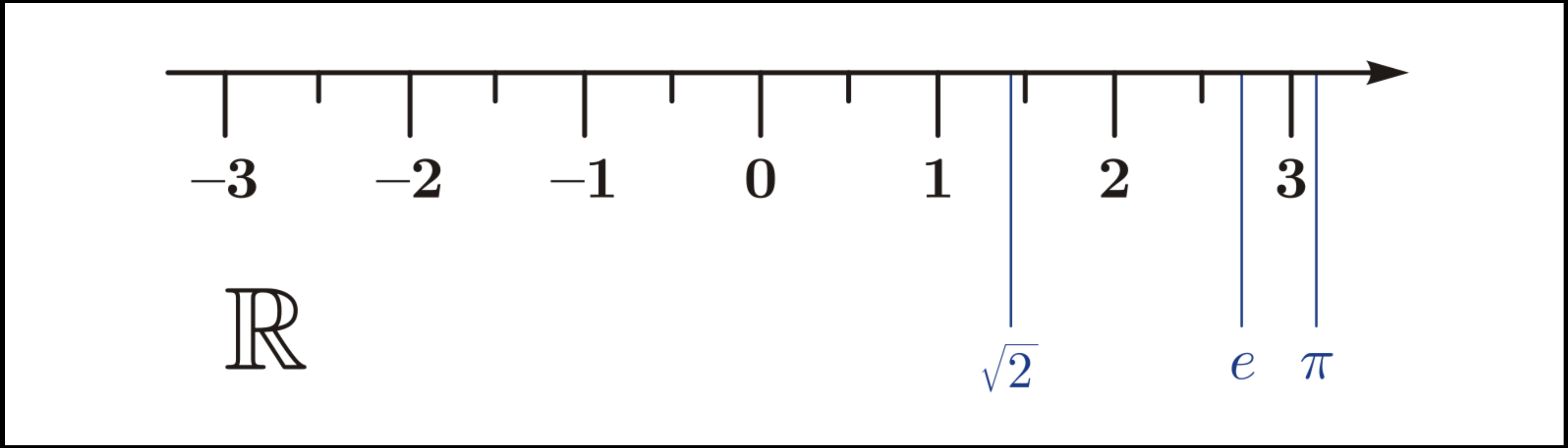
- $\mathbb{Z}$
- Integers can be thought of as discrete, equally spaced points on an infinitely long number line.



- $\mathbb{N}$
- Natural numbers are non-negative integers ($\mathbb{N}_0$ includes 0).

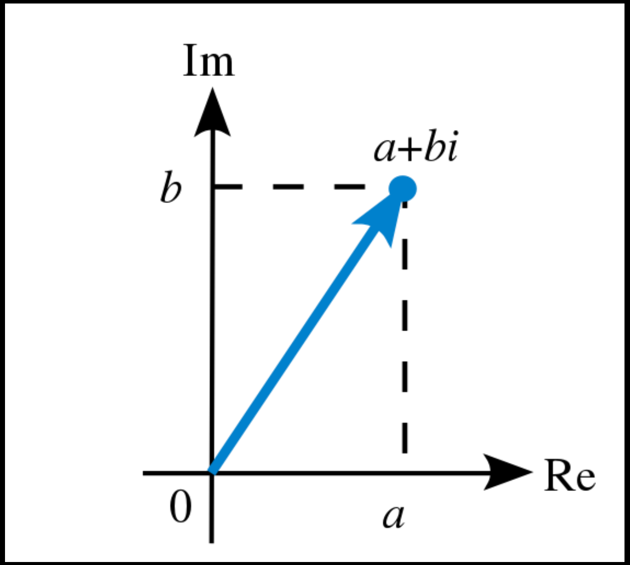
Real numbers

- $\mathbb{R}$
- In mathematics, a real number is a value of a continuous quantity that can represent a distance along a line.

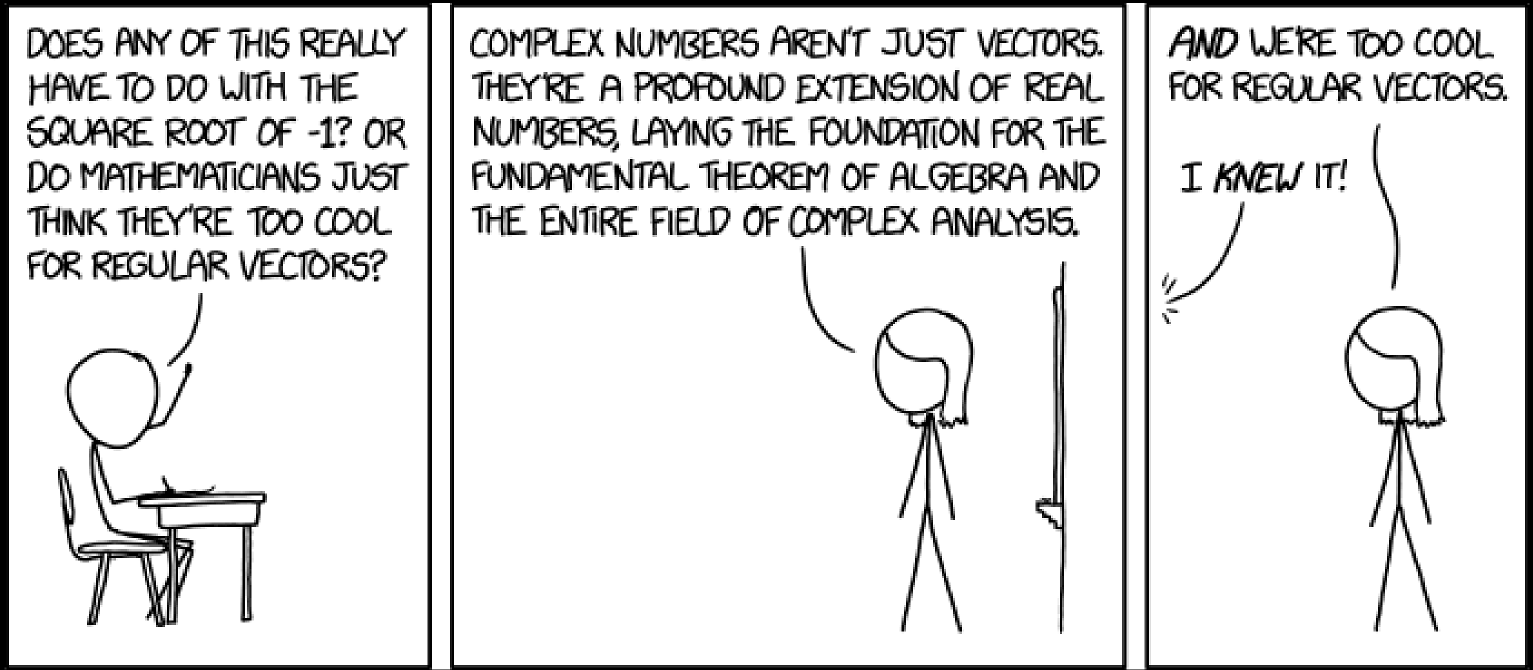


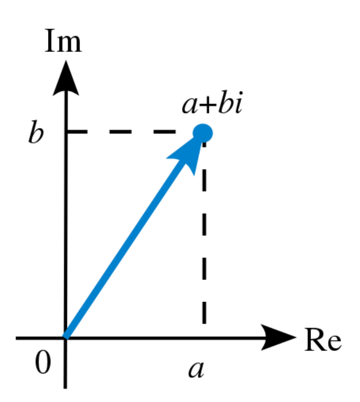
Complex numbers

- $\mathbb{C}$
- A complex number is a number that can be expressed in the form *a+bi*, where *a* and *b* are real numbers, and *i* is a solution of the equation $x^2=-1$. Because no real number satisfies this equation, *i* is called an imaginary number.



Complex numbers





Integers in programming

- Variable which holds integer number. It can be signed (allows negative values) or unsigned (doesn't).

- Number of bytes corresponds to what is the minimal and maximal number we can store in it:

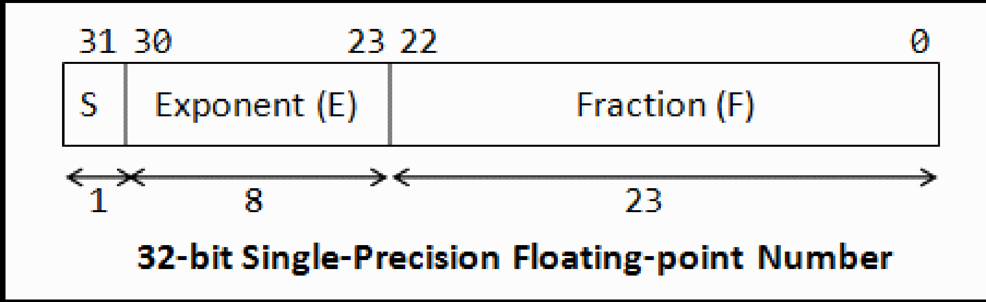
  https://en.wikipedia.org/wiki/Integer_(computer_science)

- In Python:

```python
import sys
print(sys.maxsize)
# 2147483647
```

Floats in programming

- Variables which hold real numbers.
- 32 bits used as Sign, Exponent, and Fraction



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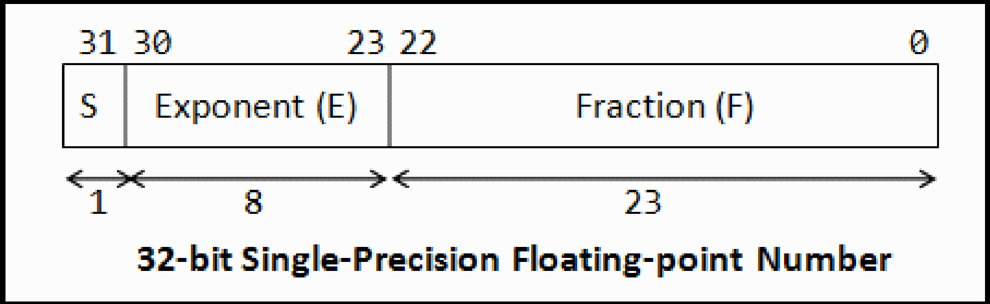
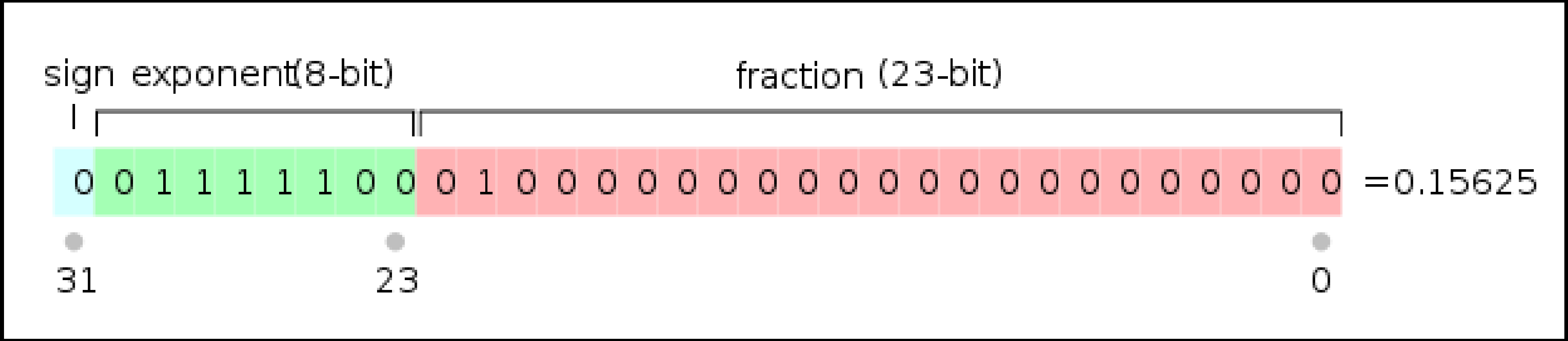
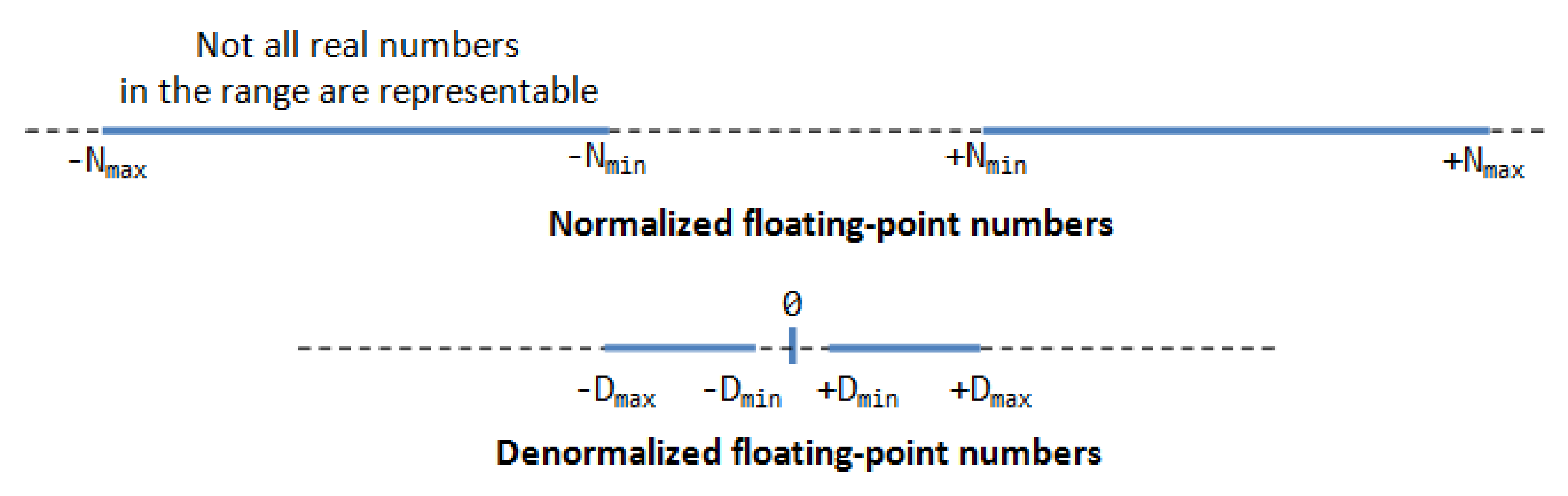




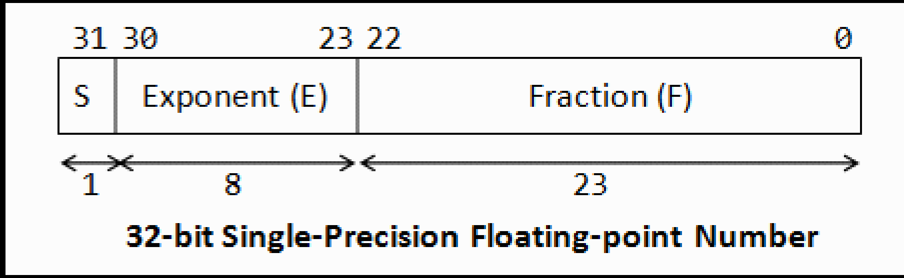
Floats in programming

- S = Sign, E = Exponent, F = Fraction
- For large numbers "normalized" (-1)^S * 1.F * 2^(E-127)
- For small numbers "denormalized" (E=0) (-1)^S * 0.F * 2^(-126)



Floats in programming

- In Python:




```python
import sys
print(sys.float_info)
# sys.float_info(max=1.7976931348623157e+308, max_exp=1024,
max_10_exp=308, min=2.2250738585072014e-308, min_exp=-1021, min_10_exp=-307,
dig=15, mant_dig=53, epsilon=2.220446049250313e-16, radix=2, rounds=1)
```

Pause 2

Task 1 - Python

- **Quadratic function solver**

$$x_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Ask the user for values of a,b,c in **f(x): ax² + bx + c**
- Then solve for the intersections

- *Hint*: depending on the value of the discriminant $\mathrm{D} = b^2 - 4ac$ we have
  - D > 0 … two solutions
  - D == 0 … one solution
  - D < 0 … no real solution

Task 2 - Python

- **Plot quadratic function**

```
import matplotlib.pyplot as plt

x_values=[]
y_values=[]

# TODO: sample the x and y values!

fig= plt.figure()
axes=fig.add_subplot(111)
axes.plot(x,y)
plt.show()
```

- Starter code: git / [colab]