MATH 132

HOMEWORK – 6

V.3 EXERCISES.

1(c) $\sum_{k=1}^{\infty} k^2 z^k$

$a_k = k^2$

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1.$$

$\therefore$ $R = 1$.

2.(d) $\sum_{m=1}^{\infty} \frac{(z+i)^m}{m^2}$

$a_m = 1/m^2$

$$R = \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m\to\infty} \left| \frac{m^2}{(m+1)^2} \right| = 1.$$

Thus, the power series is convergent for $|z+i| \leq 1$ ~~ie~~

~~the circle of radius 1 centered at (0,1)~~

5. (a) $\sum_{k=1}^{\infty} k z^k$

We know,

$$\sum_{k=1}^{\infty} z^k = \frac{1}{1-z}$$

$$\frac{d}{dz}\left(\sum_{k=1}^{\infty} z^k\right) = \frac{d}{dz}\left(\frac{1}{1-z}\right)$$

$$\sum_{k=1}^{\infty} k z^{k-1} = \frac{1}{(1-z)^2}$$

multiplying both sides by z, we get

$$\sum_{k=1}^{\infty} k z^k = \frac{z}{(1-z)^2}$$

(b) $\sum_{k=1}^{\infty} k^2 z^k$.

Taking the answer from (a)

$$\sum_{k=1}^{\infty} k z^k = \frac{z}{(z-1)^2}$$

$$\frac{d}{dz}\left[\sum_{k=1}^{\infty} k z^k\right] = \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$\sum_{k=1}^{\infty} k^2 z^{k-1} = \cancel{\frac{-(z+1)}{(z-1)^3}} \quad \frac{1}{(z-1)^2} - \frac{2z}{(z-1)^3}$$

multiplying both ~~sideds~~ sides by z, we get

$$\sum_{k=1}^{\infty} k^2 z^k = \cancel{\frac{-z(z+1)}{(z-1)^3}} \quad \frac{z}{(z-1)^2} - \frac{2z}{(z-1)^3}$$

V.4 EXERCISES.

2. $f(z) = \frac{z^2-1}{z^3-1}$ about $z=2$.

singularities: $z^3-1=0$

$(z-1)(z^2+z+1)=0$

$z_0=1$, $w_0 = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ $w_1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Calculating the distance from $z=2$ for each singularity,

z_0 : $|2-1| = 1$

w_0 : $\left|2 - \left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right)\right| = \left|2.5 - \frac{i\sqrt{3}}{2}\right| = \sqrt{(2.5)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{7}$

w_1 : $\left|2 - \left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right)\right| = \left|2.5 + \frac{i\sqrt{3}}{2}\right| = \sqrt{(2.5)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{7}$

$\therefore$ Radius of convergence $= \sqrt{7}$.

5. We know,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots \qquad |z|$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\cancel{e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

$$\cancel{e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{iz} = 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

$$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \cdots$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \cos z + i \sin z.$$

V.5 EXERCISES.

1 (b) $f(z) = \frac{z^2}{z^3 - 1}$ find the power series expansion about $z = \infty$

consider $g(w) = f(1/w) = \frac{(1/w)^2}{(1/w)^3 - 1} = \frac{w}{1 - w^3}$ is analytic

at $w = 0$.

Expanding $g(w)$ about $w = 0$,

We know, $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$, $|t| < 1$. Taking $t = w^3$,

$$\sum_{k=0}^{\infty} (w^3)^k = \frac{1}{1-w^3} \qquad |w^3| < 1$$

$$\frac{w}{1-w^3} = \sum_{k=0}^{\infty} w^{3k+1} \qquad |w| < 1$$

Replacing w by $1/z$ as $f(z) = g(1/z)$

$$\frac{w}{1-w^3} = \sum_{k=1}^{\infty} w^{3k+1} \qquad |w| < 1$$

$$f(z) = \frac{1/z}{1-(1/z)^3} = \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^{3k+1}, \quad |1/z| < 1$$

$$= \sum_{k=0}^{\infty} z^{-3k-1} \qquad |z| > 1$$

2. Given $f(z)$ is analytic at ∞, such that

$$f(z) = \sum_{k=0}^{\infty} \frac{b_k}{z^k} = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} \ldots \qquad |z| > 1/\rho \quad \text{--- (1)}$$

such that $f(\infty) = b_0$, $f'(\infty) = b_1$

As, f is analytic at ∞, f is differentiable at infinity, f is continuous at ∞.

$$\lim_{z\to\infty} f(z) = f(\infty) = b_0$$

from (1), $\lim_{z\to\infty} f'(z) = 0$

Since f is analytic at ∞, $f'(\infty) = \lim_{z\to\infty} f'(z) = b_1 = 0$

Now, $\lim_{z\to\infty} z(f(z) - f(\infty))$

Since $\lim_{z\to\infty} f(z) = f(\infty) = b_0$.

Hence, $\lim_{z\to\infty} z(f(z) - f(\infty)) = 0$

So, $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$

V.6 EXERCISES.

3. $f(z) = \frac{e^z}{1+z}$

Expansion of e^z : $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

Expansion of $\frac{1}{1+z}$: $\sum_{n=0}^{\infty} (-1)^n z^n \quad |z| < 1$

Combining the Two series using Cauchy's product formula,

$$\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty}(-1)^m z^m\right) = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \frac{(-1)^{k-n}}{n!}\right) z^k$$

First few terms:

$a_0 = 1$

$a_1 = 1 - \frac{1}{1!} = 0$

$a_2 = \frac{1}{2!} - \frac{1}{1!} = -\frac{1}{2}$

$a_3 = -\frac{1}{3!} + \frac{1}{2!} = \frac{1}{3}$

General term: $a_n = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right]$

Singularity : $z = -1$

When expanding about $z=0$, the distance to the closest singularity is 1. Thus radius of convergence, $R = 1$.