MSO 201 A
Homework 2

[1] Let $\Omega = \{0,1,2,\ldots\}$. If for an event A,

(a) $P(A) = \sum\limits_{x \in A} \frac{e^{-\lambda}\lambda^{x}}{x!}, \lambda > 0.$

(b) $P(A) = \sum\limits_{x \in A} p\,(1-p)^{x}, 0 < p < 1.$

(c) $P(A) = \begin{cases} 1 & \text{if the number of elements in A is finite} \\ 0 & \text{otherwise.} \end{cases}$

Determine in each of the above cases whether P is a probability measure.
In cases where your answer is in the affirmative, determine $P(E), P(F), P(G)$, $P(E \cap F)$, $P(E \cup F)$, $P(F \cup G), P(E \cap G)$ and $P(F \cap G)$, where $E = \{x \in \Omega : x > 2\}$, $F = \{x \in \Omega : 0 < x < 3\}$ and $G = \{x \in \Omega : 3 < x < 6\}$.

[2] Let $\Omega = \Re$. In each of the following cases determine whether $P(.)$ is a probability measure. For an interval I,

(a) $P(I) = \int\limits_{I} \frac{1}{2} e^{-|x|}\, dx$

(b) $P(I) = \begin{cases} 0, & \text{if } I \subset (-\infty, 0), \\ \int\limits_{I} 2x\, e^{-x^{2}}\, dx, & \text{if } I \subset [0, \infty). \end{cases}$

(c) $P(I) = \begin{cases} 1 & \text{if length of } I \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$

[3] Show that the probability of exactly one of the events A or B occurring is $P(A) + P(A) - 2P(A \cap B)$.

[4] Prove that

$$\begin{aligned} P(A \cap B) - P(A)P(B) &= P(A)P(B^{C}) - P(A \cap B^{C}) \\ &= P(A^{C})P(B) - P(A^{C} \cap B) \\ &= P\left[(A \cup B)^{C}\right] - P(A^{C})P(B^{C}) \end{aligned}$$

[5] For events $A_1, A_2, \ldots, A_n$ show that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P\left(A_{i_1} \cap A_{i_2}\right) + \sum_{1 \le i_1 < i_2 < i_3 \le n} P\left(A_{i_1} \cap A_{i_2} \cap A_{i_3}\right) - \ldots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

[6] Consider the sample space $\Omega = \{0,1,2,\ldots\}$ and $\mathcal{F}$ the $\sigma-$field of subsets of Ω. To the elementary event $\{j\}$ assign the probability

$$P(\{j\}) = c\,\frac{2^j}{j!}, \quad j = 0,1,2,....$$

(a) Determine the constant c.

(b) Define the events A, B and C by $A = \{j : 2 \le j \le 4\}$, $B = \{j : j \ge 3\}$ and

$C = \{j : j \text{ is an odd integer}\}$.

Evaluate $P(A), P(B), P(C), P(A \cap B), P(A \cap C), P(C \cap B), P(A \cap B \cap C)$ and verify the formula for $P(A \cup B \cup C)$.

[7] Each packet of a certain cereal contains a small plastic model of one of the five different dinosaurs; a given packet is equally likely to contain any one of the five dinosaurs. Find the probability that someone buying six packets of the cereal will acquire models of three favorite dinosaurs.

[8] Suppose n cards numbered *1, 2, …,* n are laid out at random in a row. Let A_i denote the event that 'card i appears in the i^{th} position of the row', which is termed as a match. What is the probability of obtaining at least one match?

[9] A man addresses n envelopes and writes n cheques for payment of n bills.

(a) If the n bills are placed at random in the n envelopes, what would be the probability that each bill would be placed in the wrong envelope?

(b) If the n bills and n cheques are placed at random in the n envelopes, one bill and one cheque in each envelope, what would be the probability that in no instance would the enclosures be completely correct?

[10] For events A, B and C such that $P(C) > 0$, prove that

(a) $P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(AB \mid C)$

(b) $P(A^C \mid C) = 1 - P(A \mid C)$.

[11] Let A and B be two events such that $0 < P(A) < 1$. Which of the following statements are true?

(a) $P(A \mid B) + P(A^C \mid B) = 1$; (b) $P(A \mid B) + P(A \mid B^C) = 1$; (c) $P(A \mid B) + P(A^C \mid B^C) = 1$

[12] Consider the two events A and B such that $P(A) = 1/4$, $P(B \mid A) = 1/2$ and $P(A \mid B) = 1/4$. Which of the following statements are true?

(a) A and B are mutually exclusive events,

(b) $A \subset B$,

(c) $P(A^C \mid B^C) = 3/4$,

(d) $P(A \mid B) + P(A \mid B^C) = 1$

[13] Consider an urn in which 4 balls have been placed by the following scheme. A fair coin is tossed, if the coin comes up heads, a white ball is placed in the urn otherwise a red ball is placed in the urn.

(a) What is the probability that the urn will contain exactly 3 white balls?

(b) What is the probability that the urn will contain exactly 3 white balls, given that the first ball placed in the urn was white?

[14] A random experiment has three possible outcomes, ***A***, ***B*** and ***C***, with probabilities p_A, p_B and p_C. What is the probability that, in independent performances of the experiment, ***A*** will occur before ***B***?

[15] A system composed of ***n*** separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component ***i***, independent of other components, functions with probability p_i, $i=1(1)n$, what is the probability that the system functions?

[16] A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, the student needs to answer correctly all the three questions. What is the probability that the student will pass the examination if he remembers correctly answers to 90 questions on the list?

[17] A person has three coins in his pocket, two fair coins (heads and tails are equally likely) but the third one is biased with probability of heads $2/3$. One coin selected at random drops on the floor, landing heads up. How likely is it that it is one of the fair coins?

[18] A slip of paper is given to A, who marks it with either a $+$ or a $-$ sign, with a probability $1/3$ of writing a $+$ sign. A passes the slip to B, who may either leave it unchanged or change the sign before passing it to C. C in turn passes the slip to D after perhaps changing the sign; finally D passes it to a referee after perhaps changing the sign. It is further known that B, C and D each change the sign with probability $2/3$. Find the probability that A originally wrote a $+$ given that the referee sees a $+$ sign on the slip.

[19] Each of the three boxes A, B and C, identical in appearance, has two drawers. Box A contains a gold coin in each drawer, box B contains a silver coin in each drawer and box C contains a gold coin in one drawer and silver coin in the other. A box is chosen at random and one of its drawers is then chosen at random and opened, and a gold coin is found. What is the probability that the other drawer of this box contains a silver coin?

[20] Each of four persons fires one shot at a target. Let C_k denote the event that the target is hit by person ***k***, ***k = 1, 2, 3, 4***. If the events C_1, C_2, C_3, C_4 are independent and if $P(C_1)=P(C_2)=0.7$, $P(C_3)=0.9$ and $P(C_4)=0.4$, compute the probability that: (a) all of them hit the target; (b) no one hits the target; (c) exactly one hits the target; (d) at least one hits the target.

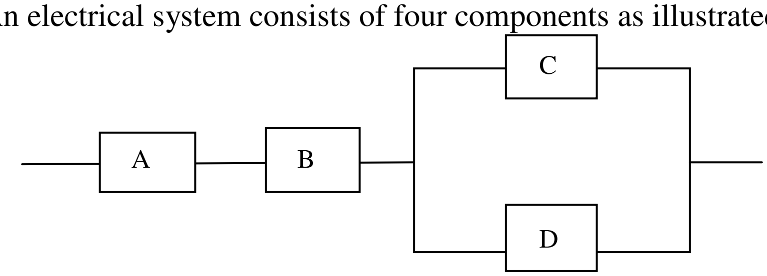
[21] Let $A_1, A_2, \ldots A_n$ be n independent events. Show that $P\left(\bigcap_{i=1}^{n} A_i^C\right) \le \exp\left(-\sum_{i=1}^{n} P(A_i)\right)$.

[22] Give a counter example to show that pairwise independence of a set of events $A_1, A_2, \ldots A_n$ does not imply mutual independence.

[23] We say that B carries negative information about event A if $P(A \mid B) < P(A)$. Let A, B and C be three events such that B carries negative information about A and C carries negative information about B. Is it true that C carries negative information about A? Prove your assertion.

[24] Suppose in a class there are 5 boys and 3 girl students. A list of 4 students, to be interviewed, is made by choosing 4 students at random from this class. If the first student selected at random from the list, for interview, is a girl, then find the conditional probability of selecting a boy next from among the remaining 3 students in the list.

[25] An electrical system consists of four components as illustrated in the figure below.



The system works if components A and B work and either of the components C or D work. It is known that the components work independently and that

$$P(A \text{ works}) = P(\text{B works}) = 0.9 \text{ and } P(\text{C works}) = P(\text{D works}) = 0.8.$$

Find the probability that (a) the entire system works, and (b) the component C does not work, given that the entire system works.

[26] During the course of an experiment with a particular brand of a disinfectant on flies, it is found that 80% are killed in the first application. Those which survive develop a resistance, so that the percentage of survivors killed in any later application is half of that in the preceding application. Find the probability that (a) a fly will survive 4 applications; (b) it will survive 4 applications, given that it has survived the 1^{st} one.

[27] An art dealer receives a group of 5 old paintings and on the basis of past experience, he thinks that the probabilities are, 0.76, 0.09, 0.02, 0.01,0.02 and 0.10 that 0, 1, 2, 3, 4 or all 5 of them, respectively, are forgeries. The art dealer sends one randomly chosen (out of 5) paintings for authentication. If this painting turns out to be a forgery, then what probability should he now assign to the possibility that the other 4 are also forgeries.