Solution Assignment # 7

(1) jt. m.g.f.

$$M_{X_1,X_2}(t_1,t_2) = E\left(e^{t_1X_1+t_2X_2}\right)$$

$$= \int_0^\infty\int_0^\infty e^{t_1x_1+t_2x_2}\, e^{-(x_1+x_2)}\, dx_2\, dx_1$$

$$= \int_0^\infty e^{-x_1(1-t_1)}\, dx_1 \int_0^\infty e^{-x_2(1-t_2)}\, dt_2$$

$$= (1-t_1)^{-1}(1-t_2)^{-1} \quad \text{If } t_1, t_2 < 1$$

Note : Since X_1 & X_2 are indep, we can write

$$M_{X_1,X_2}(t_1,t_2) = M_{X_1}(t_1)\, M_{X_2}(t_2)$$

m.g.f. of $Z = X_1 + X_2$

$$M_Z(t) = E\left(e^{t(X_1+X_2)}\right) = (1-t)^{-2}, \quad t<1$$

$$E(Z) = \left.\frac{\partial M_Z(t)}{dt}\right|_{t=0} = \left.2(1-t)^{-3}\right|_{t=0} = 2$$

$$E(Z^2) = \left.\frac{\partial^2 M_Z(t)}{dt^2}\right|_{t=0} = \left.6(1-t)^{-4}\right|_{t=0} = 6 \quad \Rightarrow V(Z) = 2$$

(2)

$$M_{X_1,X_2}(t_1,t_2) = E\left(e^{t_1X_1+t_2X_2}\right)$$

$$= EE\left(e^{t_1X_1+t_2X_2}\,\middle|\,X_1\right)$$

$$= E\left(e^{t_1X_1}\, E\left(e^{t_2X_2}\,\middle|\,X_1\right)\right)$$

Since $X_2 \mid X_1 \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$

$E\left(e^{t_2X_2}\mid X_1\right) \rightarrow$ cond'al m.g.f of X_2 given X_1

$$M_{X_1,X_2}(t_1,t_2) = E\left(e^{t_1 X_1}\left(e^{t_2\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(X_1-\mu_1)\right)+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)}\right)\right)$$

$$= e^{t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)}\, E\left(e^{t_1X_1+t_2\rho\frac{\sigma_2}{\sigma_1}X_1}\right) e^{-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}$$

$$= e^{t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1} \times E\left(e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)X_1}\right)$$

$$= e^{t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}\, e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)\mu_1+\frac{\sigma_1^2}{2}\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)^2}$$

$$= \exp\left(t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\cancel{\rho^2})-\cancel{t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}+t_1\mu_1+\cancel{t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}\right.$$
$$\left.+\frac{\sigma_1^2}{2}\left(t_1^2+\cancel{t_2^2\rho^2\frac{\sigma_2^2}{\sigma_1^2}}+2t_1t_2\rho\frac{\sigma_2}{\sigma_1}\right)\right)$$

$$= \exp\left(t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2+t_1\mu_1+\frac{t_1^2}{2}\sigma_1^2+t_1t_2\rho\sigma_1\sigma_2\right)$$

$$= \exp\left(t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2t_1t_2\sigma_1\sigma_2\rho\right)\right)$$

$$\left.\frac{\partial M_{X_1,X_2}(t_1,t_2)}{\partial t_1}\right|_{t_1=0,t_2=0} = \mu_1 \quad \text{sly} \quad \left.\frac{\partial M_{X_1,X_2}}{\partial t_2}\right|_{t_1=0,t_2=0} = \mu_2$$

$\&\ V(X_1)=\sigma_1^2,\ V(X_2)=\sigma_2^2$

$$E(X_1X_2) = \left.\frac{\partial^2 M_{X_1,X_2}(t_1,t_2)}{\partial t_1\,\partial t_2}\right|_{t_1=0,t_2=0} = \rho\sigma_1\sigma_2+\mu_1\mu_2$$

$$\Rightarrow \mathrm{Cov}(X_1,X_2) = (\rho\sigma_1\sigma_2+\mu_1\mu_2)-\mu_1\mu_2 = \rho\sigma_1\sigma_2$$

$$\Rightarrow \mathrm{Corr}(X_1,X_2) = \rho.$$

(3)

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & o/w \end{cases}$$

$$f_X(x) = \int_x^1 2\,dy = 2(1-x), \quad 0 < x < 1$$

$$= 0 \quad o/w$$

$$f_Y(y) = \int_0^y 2\,dx = 2y, \quad 0 < y < 1$$

$$= 0 \quad o/w$$

$$f_{Y|X=x} = \begin{cases} \frac{2}{2(1-x)}, & x < y < 1 \\ 0 & o/w \end{cases}$$

$$f_{X|Y=y} = \begin{cases} \frac{2}{2y}, & 0 < x < y \\ 0 & o/w. \end{cases}$$

$$E(Y|X) = \int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$$

$$E(Y^2|X) = \int_x^1 y^2 \frac{1}{1-x}\,dy = \frac{1-x^3}{3(1-x)}.$$

$$\Rightarrow V(Y|X) = E(Y^2|X) - E^2(Y|X) = \frac{1-x^3}{3(1-x)} - \frac{1+x}{2}$$

Sly $E(X|Y)$, $E(X^2|Y)$ and hence $V(X|Y)$.

4) (a) $\text{Cov}(X, b) = E(X - E(X))(b - E(b)) = 0 = \text{Cov}(Y, b) = \text{Cov}(Z, b)$

(b) $\text{Cov}(X, aY+b) = E(X - E(X))(aY + b - E(aY+b))$

$= E(X - E(X))(aY + \cancel{b} - aE(Y) - \cancel{b})$

$= a\,\text{Cov}(X, Y)$

(c) $\text{Cov}(X, Y+Z) = E(X - E(X))(Y + Z - E(Y) - E(Z))$

$= E(X - E(X))[(Y - E(Y)) + (Z - E(Z))]$

$= \text{Cov}(X, Y) + \text{Cov}(X, Z)$

(d) $\text{Cov}(X, aY+b) = a\,\text{Cov}(X, Y)$

$$\text{Corr}^n(X, aY+b) = \frac{\text{Cov}(X, aY+b)}{[V(X)\;V(aY+b)]^{1/2}} = \frac{a\,\text{Cov}(X,Y)}{[V(X)\;a^2\,V(Y)]^{1/2}}$$

$= \text{Corr}(X, Y)$

(5) $\text{Cov}(W_1, W_2) = \text{Cov}\left(X_1, \frac{\sqrt{3}-1}{2} X_1 + \frac{3-\sqrt{3}}{2} X_2\right)$

$= \frac{\sqrt{3}-1}{2} V(X_1) + \frac{3-\sqrt{3}}{2}\text{Cov}(X_1, X_2) = \frac{\sqrt{3}-1}{2}\sigma^2$

$V(W_1) = \sigma^2$ & $V(W_2) = \left(\frac{\sqrt{3}-1}{2}\right)^2\sigma^2 + \left(\frac{3-\sqrt{3}}{2}\right)^2\sigma^2 = (\sqrt{3}-1)^2\sigma^2$

$\Rightarrow \rho_{W_1, W_2} = \frac{1}{2}$

Sly ρ_{W_1, W_3} & ρ_{W_2, W_3}

$\text{Cov}(W_1, W_3) = \text{Cov}(X_1, (\sqrt{2}-1)X_2 + (2-\sqrt{2})X_3) = 0$

$\Rightarrow \rho_{W_1, W_3} = 0$

(6) (a) $P(3 < Y < 8) \qquad Y \sim N(1, 25)$

$$= P\left(\frac{3-1}{5} < \frac{Y-1}{5} < \frac{8-1}{5}\right) = \Phi\left(\frac{7}{5}\right) - \Phi\left(\frac{2}{5}\right)$$

$= \ldots$ (from table)

(b) $P(3 < Y < 8 \mid X = 7) \qquad \left[Y|X \sim N\left(1 + \rho \frac{5}{4}(x-3), 25(1-\rho^2)\right) \right.$

$\left. \text{i.e. } Y|X=7 \sim N(4, 16) \right]$

$$= P\left(\frac{3-4}{4} < \frac{Y-4}{4} < \frac{8-4}{4} \,\middle|\, X=7\right)$$

$$= \Phi(1) - \Phi(-0.25) = \ldots$$

(c) $P(-3 < X < 3) \qquad X \sim N(3, 16)$

$$= P\left(\frac{-3-3}{4} < \frac{X-3}{4} < \frac{3-3}{4}\right) = \Phi(0) - \Phi\left(-\frac{6}{4}\right) = \ldots$$

(d) $P(-3 < X < 3 \mid Y = 4) \qquad \left[X|Y \sim N\left(3 + \rho \frac{4}{5}(y-1), 16(1-\rho^2)\right) \right]$

$\text{i.e } X|Y=4 \sim N(4.44, (3.2)^2)$

$$= P\left(\frac{-3-4.44}{3.2} < \frac{X-4.44}{3.2} < \frac{3-4.44}{3.2} \,\middle|\, Y=4\right)$$

$$= \Phi\left(-\frac{1.44}{3.2}\right) - \Phi\left(-\frac{7.44}{3.2}\right) = \ldots$$

(7) $(X, Y) \sim N_2(5, 10, 1, 25, \rho) \quad ; \rho > 0$

$Y|X=5 \sim N_2(10, 25(1-\rho^2))$

$$P(4 < Y < 16 \mid X=5) = P\left(\frac{4-10}{5\sqrt{1-\rho^2}} < \frac{Y-10}{5\sqrt{1-\rho^2}} < \frac{16-10}{5\sqrt{1-\rho^2}} \,\middle|\, X=5\right)$$

$$= \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \Phi\left(-\frac{6}{5\sqrt{1-\rho^2}}\right)$$

$$= 2\,\Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - 1 = 0.954 \text{ (given condition)}$$

$$\Rightarrow \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = 0.977 = \Phi(2)$$

$$\Rightarrow \frac{6}{5\sqrt{1-\rho^2}} = 2 \Rightarrow 1-\rho^2 = 0.36 \Rightarrow \rho = 0.8 \text{ (as } \rho > 0).$$

(8) $E(Y) = \sum_1^{15} E(X_i) = 30$

$V(Y) = 15\, V(X_i) = 45$; $V(Z) = 10 \times 3 = 30$

$\mathrm{Cov}(Y, Z) = \mathrm{Cov}\left(\sum_1^{15} X_i, \sum_{11}^{20} X_i\right) = 5\, V(X_i) = 15$

$$\rho_{Y,Z} = \frac{15}{[45 \times 30]^{1/2}}$$

(9) $U = X - Y$; $V = 2X - 3Y$

$E(U) = -5$ $\qquad$ $E(V) = 2 \times 15 - 3 \times 20 = -30$

$V(U) = V(X) + V(Y) - 2\mathrm{Cov}(X,Y)$ $\qquad$ $V(V) = 4V(X) + 9V(Y) - 12\,\mathrm{Cov}(X,Y)$

Now,

$$\rho_{X,Y} = -0.6 = \frac{\mathrm{Cov}(X,Y)}{\sqrt{25 \times 100}} = \frac{\mathrm{Cov}(X,Y)}{50} \Rightarrow \mathrm{Cov}(X,Y) = -30$$

$\Rightarrow V(U) = 185$ & $V(V) = 100 + 900 + 360 = 1360$

$\Rightarrow \mathrm{Cov}(U,V) = \mathrm{Cov}(X-Y, 2X-3Y) = \longrightarrow$

$= 2V(X) - 3\,Cov(X,Y) - 2\,Cov(Y,X) + 3V(Y)$

$= 2 \times 25 - 5(-30) + 3 \times 100 = 500$

$\Rightarrow \rho_{U,V} = \dfrac{500}{\sqrt{185 \times 1360}}$

10) X : r.v. denoting life time

$X \sim Exp(50)$ p.d.f. $f(x) = \begin{cases} \frac{1}{50} e^{-x/50}, & x > 0 \\ 0 & \text{o/w.} \end{cases}$

$F_X(x) = 1 - e^{-x/50}, \quad x > 0$

Y_1 : # of bulbs out of 8 to have lifetime < 40

Y_2 : . . . . ≥ 40 & < 60

Y_3 : . . . . ≥ 60 & ≤ 80

Y_4 : . . . . > 80

$P(X < 40) = F_X(40) = 1 - e^{-40/50} = p_1$, say

$P(40 \leq X < 60) = F_X(60) - F_X(40) = e^{-\frac{40}{50}} - e^{-60/50} = p_2$, say

$P(60 \leq X \leq 80) = F_X(80) - F_X(60) = e^{-60/50} - e^{-80/50} = p_3$, say.

$P(X > 80) = 1 - p_1 - p_2 - p_3 = e^{-80/50}$.

jt. p.m.f. of Y_1, Y_2, Y_3 is multinomial. $(8, p_1, p_2, p_3)$.

$\Rightarrow P(Y_1 = 2, Y_2 = 3, Y_3 = 2) = \dfrac{8!}{2!\,3!\,2!\,1!} p_1^2\, p_2^3\, p_3^2\, (1 - p_1 - p_2 - p_3)$.

Marginal dist^n $Y_3 \sim Bin\left(8,\ p_3 = e^{-60/50} - e^{-80/50}\right)$

$$E(Y_3) = 8\left(e^{-60/50} - e^{-80/50}\right)$$

Cond^n al $Y_3 \mid Y_2 = y_2 \sim Bin\left(8 - y_2,\ \frac{p_3}{1-p_2}\right)$

$$\Rightarrow E(Y_3 \mid Y_2 = 1) = (8-1)\left(\frac{e^{-60/50} - e^{-80/50}}{1 - e^{-40/50} + e^{-60/50}}\right).$$

(11) (a)

X \ Y	0	1	2	
0	1/3	0	0	1/3
1	0	1/3	0	1/3
2	0	0	1/3	1/3
	1/3	1/3	1/3	

$E(X) = 1 = E(Y)$

$V(X) = E(X^2) - 1$

$= \frac{5}{3} - 1 = \frac{2}{3} = V(Y)$

$E(XY) = (0\times0)\frac{1}{3} + (1\times1)\frac{1}{3} + (2\times2)\times\frac{1}{3}$

$= 5/3$

$Cov(X,Y) = \frac{5}{3} - 1 = \frac{2}{3}$

$\rho_{X,Y} = 1$

(b)

X \ Y	0	1	2
0	0	0	1/3
1	0	1/3	0
2	1/3	0	0

Sly

$\Rightarrow \rho_{X,Y} = -1$

(c)

X \ Y	0	1	2
0	1/3	0	0
1	0	1/3	0
2	1/3	0	0

$\rho_{X,Y} = 0.$

(12)

$X \backslash Y$	1	2	
1	1/10	0	1/10
2	2/10	4/10	6/10
3	3/10	0	3/10
	6/10	4/10	

$$E(X) = \frac{1}{10} + 2\,\frac{6}{10} + 3\,\frac{3}{10} = \frac{22}{10}$$

$$E(Y) = \frac{6}{10} + 2\,\frac{4}{10} = \frac{14}{10}$$

$$E(X^2) = \frac{1}{10} + 4\,\frac{6}{10} + 9\,\frac{3}{10} = \frac{52}{10}$$

$$E(Y^2) = \frac{6}{10} + 4\,\frac{4}{10} = \frac{22}{10}$$

$$V(X) = \frac{52}{10} - \left(\frac{22}{10}\right)^2 = \cdots$$

$$V(Y) = \left(\frac{22}{10}\right)^2 - \left(\frac{14}{10}\right)^2 = \cdots$$

$$E(XY) = (1\times1)\,\frac{1}{10} + (2\times1)\,\frac{2}{10} + (2\times2)\,\frac{4}{10} + (3\times1)\,\frac{3}{10}$$

$$= \frac{30}{10} = 3$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\,E(Y) = 3 - \frac{22}{10}\cdot\frac{14}{10} = \cdots$$

$$\mathrm{Corr}^{n}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{[V(X)\,V(Y)]^{1/2}} = \cdots$$

jt. m.g.f. of (X,Y)

$$M_{X,Y}(t_1,t_2) = \sum_{x,y} e^{t_1x+t_2y}\,P(X=x, Y=y)$$

$$= e^{t_1+t_2}\times\frac{1}{10} + e^{2t_1+t_2}\,\frac{2}{10} + e^{2(t_1+t_2)}\,\frac{4}{10}$$

$$+ e^{3t_1+t_2}\,\frac{3}{10}$$

(13) $M_{X,Y}(u,v) = E\left(e^{uX+vY}\right)$

$$\Psi(u,v) = \log M_{X,Y}(u,v)$$

$$\frac{\partial\Psi(u,v)}{\partial u} = \frac{1}{M_{X,Y}(u,v)}\cdot\frac{\partial M_{X,Y}(u,v)}{\partial u}$$

$$\left.\frac{\partial\Psi(u,v)}{\partial u}\right|_{u=0,\,v=0} = \frac{1}{M(0,0)}\cdot E(X) = E(X)$$

Sly $\dfrac{\partial \psi(0,0)}{\partial v} = \dfrac{\partial \psi(u,v)}{\partial v}\Big|_{u=0,v=0} = E(Y)$

$$\frac{\partial^2 \psi(u,v)}{\partial u^2} = \frac{1}{M(u,v)}\frac{\partial^2 M(u,v)}{\partial u^2} + \left[\frac{-1}{(M(u,v))^2}\frac{\partial M(u,v)}{\partial u}\right]\left(\frac{\partial M(u,v)}{\partial u}\right)$$

$$= \frac{1}{M(u,v)}\cdot\frac{\partial^2 M(u,v)}{\partial u^2} - \left(\frac{\partial M(u,v)}{\partial u}\cdot\frac{1}{M(u,v)}\right)^2$$

$$\frac{\partial^2 \psi(u,v)}{\partial u^2}\Big|_{u=0,v=0} = E(X^2) - E^2(X) = V(X) = \frac{\partial^2 \psi(0,0)}{\partial u^2}$$

Sly $\dfrac{\partial^2 \psi(u,v)}{\partial v^2}\Big|_{u=0,v=0} = V(Y)$

$$\frac{\partial^2 \psi(u,v)}{\partial v\,\partial u} = \frac{1}{M(u,v)}\cdot\frac{\partial^2 M(u,v)}{\partial v\,\partial u} - \frac{1}{(M(u,v))^2}\cdot\frac{\partial M(u,v)}{\partial v}\cdot\frac{\partial M(u,v)}{\partial u}$$

$$\frac{\partial^2 \psi(u,v)}{\partial v\,\partial u}\Big|_{u=0,v=0} = E(XY) - E(X)E(Y)$$

i.e. $\dfrac{\partial^2 \psi(u,v)}{\partial v\,\partial u}\Big|_{u=0,v=0} = Cov(X,Y)$

(14) Marginal p.d.f. of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f_\rho(x,y)\,dy + \frac{1}{2}\int_{-\infty}^{\infty} f_{-\rho}(x,y)\,dy$$

$$= \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x) \quad [\phi(x) \text{ p.d.f. of } N(0,1)]$$

$$= \phi(x) \Rightarrow X \sim N(0,1)$$

Sly $f_Y(y) = \phi(y) \Rightarrow Y \sim N(0,1)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{X,Y}(x,y)\,dx\,dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{\rho}(x,y)\,dx\,dy + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{-\rho}(x,y)\,dx\,dy$$

$$= \frac{1}{2}(\rho) + \frac{1}{2}(-\rho) = 0$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 0 - 0.0 = 0$$

$\rho(X,Y) = 0 \Rightarrow X$ & Y are uncorrelated

Since, $f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$.

X & Y are not independent

(15) $\int_0^1 \int_{-x}^{x} k\,dy\,dx = 1 \Rightarrow k\int_0^1 2x\,dx = 1 \Rightarrow k = 1$

Marginal of X, $f_X(x) = \int_{-x}^{x} k\,dy = \begin{cases} 2x, & 0<x<1 \\ 0 & \text{o/w} \end{cases}$

Marginal of Y, $f_Y(y) = \int_{|y|}^{1} dx = \begin{cases} 1-|y|, & -1<y<1 \\ 0 & \text{o/w} \end{cases}$

Conditional distn of $Y|X=x$; $f_{Y|X=x} = \begin{cases} \frac{1}{2x}, & -x<y<x \\ 0, & \text{o/w} \end{cases}$

$$E(Y|X=x) = \int_{-x}^{x} y\,\frac{1}{2x}\,dy = 0$$

Sly $E(X|Y=y) = \int_{|y|}^{1} x\,\frac{1}{1-|y|}\,dx = \frac{1-y^2}{2(1-|y|)}$.

$\left(f_{X|Y=y} = \begin{cases} (1-|y|)^{-1}, & |y|<x<1 \\ 0 & \text{o/w} \end{cases} \right)$

$$f_{X,Y}(x,y) = 1 \neq f_X(x)\, f_Y(y)$$

$\Rightarrow$ X & Y are not indep.

$$E(XY) = \int_0^1 \int_{-x}^{x} xy \, dy \, dx = 0$$

$$E(Y) = E\,E(Y|X) = 0.$$

$$\Rightarrow \operatorname{Cov}(X,Y) = \rho_{X,Y} = 0$$

$\Rightarrow$ X & Y are uncorrelated

(16) $H_{X,Y}(s,t) = \{a(e^{s+t}+1) + b(e^s+e^t)\}$, $\left(a,b>0,\ a+b=\frac{1}{2}\right)$

$$E(X) = \frac{\partial}{\partial s}\left(a(e^{s+t}+1)+b(e^s+e^t)\right)\Big|_{s=t=0}$$

$$= a e^t e^s + b e^s\Big|_{t=s=0} = a+b = \frac{1}{2} = E(Y)$$

$$E(X^2) = \frac{\partial^2}{\partial s^2}\left(a(e^{s+t}+1)+b(e^s+e^t)\right)\Big|_{s=t=0}$$

$$= a e^t e^s + b e^s\Big|_{s=t=0} = a+b = \frac{1}{2} = E(Y^2)$$

$$V(X) = V(Y) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(XY) = \frac{\partial^2}{\partial t\,\partial s}\left(a(e^{s+t}+1)+b(e^s+e^t)\right)\Big|_{s=t=0}$$

$$= a e^t e^s\Big|_{s=t=0} = a$$

$$\therefore \operatorname{Cov}(X,Y) = a - \frac{1}{4} \Rightarrow \rho_{X,Y} = \frac{a-\frac{1}{4}}{\frac{1}{4}} = \underline{4a-1}$$

(17)

$$Var\left(\frac{X}{3}+\frac{2Y}{3}\right)\left(= Var\left(\frac{2X}{3}+\frac{Y}{3}\right)\right) \quad \because V(X)=V(Y)$$

$$= \frac{1}{9}V(X) + \frac{4}{9}V(Y) + 2\,Cov\left(\frac{X}{3}, \frac{2Y}{3}\right)$$

$$= \frac{2}{9} + \frac{8}{9} + \frac{4}{9}\times\frac{2}{3} = \frac{2}{9}+\frac{8}{9}+\frac{8}{27} = \frac{38}{27}$$

$$Cov\left(\frac{X}{3}+\frac{2Y}{3}, \frac{2X}{3}+\frac{Y}{3}\right)$$

$$= \frac{2}{9}V(X) + \frac{1}{9}Cov(X,Y) + \frac{4}{9}Cov(X,Y) + \frac{2}{9}V(Y)$$

$$= \frac{4}{9} + \frac{2}{27} + \frac{8}{27} + \frac{4}{9} = \frac{34}{27}$$

$$Corr|^2\left(\frac{X}{3}+\frac{2Y}{3}, \frac{2X}{3}+\frac{Y}{3}\right) = \frac{34/27}{38/27} = \frac{34}{38}$$