[1] The joint p.d.f. of (X,Y)

$$f(x,y)=\begin{cases} e^{-(x+y)} & 0<x,y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the joint m.g.f. of (X,Y).

(b) Find the m.g.f. of $Z=X+Y$ and hence $V(Z)$.

[2] Derive the joint m.g.f. of $(X_1,X_2)\sim N_2\left(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho\right)$ and using the joint m.g.g.f find $\rho(X_1,X_2)$.

[3] Let the joint p.d.f. of (X,Y) be

$$f(x,y)=\begin{cases} 2 & 0<x<y<1 \\ 0 & \text{otherwise,} \end{cases}$$

Find the conditional mean and conditional variance of X given $Y=y$ and that of Y given $X=x$. Compute further $\rho(X,Y)$.

[4] Let X,Y and Z be three random variables and a and b be two scalar constants. Prove that

(a) $Cov(X,b)=Cov(Y,b)=Cov(Z,b)=0$

(b) $Cov(X,aY+b)=a\,Cov(X,Y)$

(c) $Cov(X,Y+Z)=Cov(X,Y)+Cov(X,Z)$

(d) $\rho(X,aY+b)=\rho(X,Y)$ for $a>0$.

[5] Let X_1,X_2 and X_3 be three independent random variables each with a variance σ^2. Define the new random variables

$W_1=X_1,\ W_2=\dfrac{\sqrt{3}-1}{2}X_1+\dfrac{3-\sqrt{3}}{2}X_2$ and $W_3=\left(\sqrt{2}-1\right)X_2+\left(2-\sqrt{2}\right)X_3$.

Find $\rho(W_1,W_2)$, $\rho(W_1,W_3)$ and $\rho(W_2,W_3)$.

[6] Let $(X,Y)\sim N_2(3,1,16,25,0.6)$. Find

(a) $P(3<Y<8)$

(b) $P(3<Y<8 \mid X=7)$

(c) $P(-3<X<3)$

(d) $P(-3<X<3 \mid Y=4)$

[7] Let $(X,Y) \sim N_2(5,10,1,25,\rho)$ with $\rho>0$. If it is given that

$$P(4<Y<16 \mid X=5)=0.954$$

and $\Phi(2)=0.977$, find the value of ρ.

[8] Let $X_1, X_2, \ldots, X_{20}$ be independent random variables with identical distributions, each with a mean 2 and variance 3. Define $Y=\sum_{i=1}^{15} X_i$ and $Z=\sum_{i=11}^{20} X_i$. Find $E(Y)$, $E(Z)$, $V(Y)$, $V(Z)$ and $\rho(Y,Z)$.

[9] Let X and Y be a jointly distributed random variables with $E(X)=15, E(Y)=20$, $V(X)=25, V(Y)=100$ and $\rho(X,Y)=-0.6$. Find $\rho(X-Y, 2X-3Y)$.

[10] Suppose that the lifetime of light bulbs of a certain kind follows exponential distribution with p.d.f.

$$f_X(x)=\begin{cases} \dfrac{1}{50} e^{-x/50} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that among 8 such bulbs, 2 will last less that 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours. Find the expected number of bulbs in a lot of 8 bulbs with lifetime between 60 and 80 hours and also the expected number of bulbs in a lot of 8 with lifetime between 60 and 80 hours, given that the number of bulbs with lifetime anywhere between 40 and 60 hours is 2.

[11] Let the random variables X and Y have the following joint p.m.f.s

(a) $P(X=x, Y=y)=1/3$, if $(x,y)\in\{(0,0),(1,1),(2,2)\}$ and 0 otherwise.

(b) $P(X=x, Y=y)=1/3$, if $(x,y)\in\{(0,2),(1,1),(2,0)\}$ and 0 otherwise.

(c) $P(X=x, Y=y)=1/3$, if $(x,y)\in\{(0,0),(1,1),(2,0)\}$ and 0 otherwise.

In each of the above cases find the coefficient of correlation between X and Y.

[12] The joint p.m.f. of (X,Y) is

$P(X=x, Y=y) = xy/10$, if $(x,y) \in \{(1,1),(2,1),(2,2),(3,1)\}$ and 0 otherwise. Find the joint m.g.f. of X and Y and the coefficient of correlation between X and Y. Using the joint m.g.f., find the p.m.f. $Z = X + Y$.

[13] Let $M_{X,Y}(u,v)$ denote the joint m.g.f. (X,Y) and $\psi(u,v) = \log(M_{X,Y}(u,v))$. Show that

$$\left.\frac{\partial \psi(u,v)}{\partial u}\right|_{u=v=0}, \left.\frac{\partial \psi(u,v)}{\partial v}\right|_{u=v=0}, \left.\frac{\partial^2 \psi(u,v)}{\partial u^2}\right|_{u=v=0}, \left.\frac{\partial^2 \psi(u,v)}{\partial v^2}\right|_{u=v=0} \text{ and } \left.\frac{\partial^2 \psi(u,v)}{\partial u \, \partial v}\right|_{u=v=0}$$

yields the means, the variances and the covariance of the two random variables.

[14] The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{2}\left(f_\rho(x,y) + f_{-\rho}(x,y)\right); \qquad -\infty < x, y < \infty$$

where, $f_\rho(x,y)$ is the probability density function of $N_2(0,0,1,1,\rho)$ and $f_{-\rho}(x,y)$ is the probability density function of $N_2(0,0,1,1,-\rho)$. Find the marginal p.d.f.s of X and Y, the correlation coefficient between X and Y. Are the 2 variables independent?

[15] Let the joint p.d.f. of X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} k, & \text{if } -x < y < x;\ 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant k and obtain the conditional expectations $E(X \mid Y = y)$ and $E(Y \mid X = x)$. Verify whether the 2 random variables are independent and/or uncorrelated.

[16] The joint moment generating function of X and Y is given by

$$M_{X,Y}(s,t) = \left\{a\left(e^{s+t}+1\right) + b\left(e^s + e^t\right)\right\}, \ a, b > 0; \quad a + b = 1/2.$$

Find the correlation coefficient between X and Y.

[17] Let X and Y be jointly distributed random variables with

$$E(X) = E(Y) = 0,\ E\left(X^2\right) = E\left(Y^2\right) = 2 \text{ and } \rho(X,Y) = 1/3$$

Find $\rho(X/3 + 2Y/3,\ 2X/3 + Y/3)$.