① Let X denote the # of throws reqd to get a 6

$$\mathcal{X} = \{1, 2, 3, \ldots\}$$

$$P(X = x) = \left(\frac{5}{6}\right)^{x-1} \frac{1}{6} \qquad x \in \mathcal{X}$$

$$= 0 \qquad \text{o/w}$$

$$E(X) = \sum_{1}^{\infty} x \left(\frac{5}{6}\right)^{x-1} \frac{1}{6} = \frac{1}{6} \sum_{1}^{\infty} x \left(\frac{5}{6}\right)^{x-1} = \left(1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + \cdots\right)\frac{1}{6}$$

$$= \frac{1}{6}\Bigg[\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right)$$
$$+ \frac{5}{6}\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right)$$
$$+ \left(\frac{5}{6}\right)^2\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right)$$
$$+ \cdots\Bigg]$$

$$= \frac{1}{6}\left[\frac{1}{1 - \frac{5}{6}} + \frac{5}{6}\,\frac{1}{1 - \frac{5}{6}} + \left(\frac{5}{6}\right)^2 \frac{1}{1 - 5/6} + \cdots\right]$$

$$= \frac{1}{6}\left[6\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right)\right] = 6$$

② X : length of run of of heads or tails staring with trial 1

$$\mathcal{X} = \{1, 2, \ldots\}$$

$$P(X = x) = \underset{\uparrow\ \text{run of } x\ \text{Ts}}{(1-p)^x p} + \underset{\uparrow\ \text{run of } x\ \text{H}}{p^x (1-p)}$$

$$E(X) = \sum_{1}^{\infty} x \left((1-p)^x p + p^x (1-p)\right)$$

$$= p(1-p)\left(\sum_{1}^{\infty} x (1-p)^{x-1} + \sum_{1}^{\infty} x\, p^{x-1}\right)$$

$$= p(1-p)\left(\frac{1}{p^2} + \frac{1}{(1-p)^2}\right) \quad \leftarrow \text{as in } 1$$

$$= \frac{1-2p+2p^2}{p(1-p)}.$$

3 (a) $E(|X|) = \sum_{1}^{\infty} |x| \frac{1}{x(x+1)} = \sum_{1}^{\infty} \frac{1}{x+1}$ not convergent

$\Rightarrow E(X)$ does not exist

(b) $E|X| = \int_{|x|>1} |x| \frac{1}{2x^2}\, dx = \infty$

$\Rightarrow E(X)$ does not exist

(c) $E|X| = \int_{-\infty}^{\infty} \frac{|x|}{\pi} \frac{1}{1+x^2}\, dx = \frac{2}{\pi}\int_0^{\infty} \frac{x}{1+x^2}\, dx$

$= \frac{1}{\pi}\left(\log(1+x^2)\right)\Big|_0^{\infty} = \infty$

$\Rightarrow E(X)$ does not exist

4 Trivial calculations

e.g. (a) $E(X) = a\int_0^1 x^a\, dx = \frac{a}{a+1}$; $E(X^2) = \frac{a}{a+2}$

$V(X) = E(X^2) - (E(X))^2 = \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2$

$= \ldots$

5 $E(X) = \frac{c}{a}\int_{\mu}^{\infty} x\left(\frac{x-\mu}{a}\right)^{c-1} e^{-\left(\frac{x-\mu}{a}\right)^c}\, dx$

$y = \left(\frac{x-\mu}{a}\right)^c \qquad dy = \frac{c}{a}\left(\frac{x-\mu}{a}\right)^{c-1}$

$\Rightarrow E(X) = \int_0^{\infty} (a y^{1/c} + \mu) e^{-y}\, dy = a\,\Gamma\left(\frac{1}{c}+1\right) + \mu$

$$E(X^2) = \frac{c}{a}\int_{\mu}^{\infty} x^2 \left(\frac{x-\mu}{a}\right)^{c-1} e^{-\left(\frac{x-\mu}{a}\right)^c} dx$$

as in $E(X)$; $$= \int_0^{\infty} (a y^{1/c} + \mu)^2 e^{-y} dy$$

$$= a^2 \Gamma\left(\tfrac{2}{c}+1\right) + 2a\mu\,\Gamma\left(\tfrac{1}{c}+1\right) + \mu^2$$

$$V(X) = E X^2 - (E X)^2$$

$$= \left(a^2 \Gamma\left(\tfrac{2}{c}+1\right) + 2a\mu\,\Gamma\left(\tfrac{1}{c}+1\right) + \mu^2\right) - \left(a\,\Gamma\left(\tfrac{1}{c}+1\right) + \mu\right)^2$$

$$= \ldots$$

(6) $$\int_0^m 3x^2\,dx = \int_m^1 3x^2\,dx = \frac{1}{2}$$

$$\Rightarrow \quad m = \ldots$$

(7) $$\int_0^{\infty} (1-F(x))\,dx = \int_0^{\infty}\int_x^{\infty} f_X(y)\,dy\,dx$$

$$\overbrace{}^{0<x<y<\infty}$$

$$= \int_0^{\infty} f_X(y)\int_0^{y} dx\;dy = \int_0^{\infty} y\, f_X(y)\,dy = E(X)$$

(8) Let Z denote the r. v. $\ni$

Z : score in a shot $\quad \mathcal{X}_Z = \{0, 2, 3, 4\}$.

$$P(Z=0) = P(X > \sqrt{3}) = \frac{2}{\pi}\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2}\,dx = \frac{2}{\pi} \tan^{-1}x \Big|_{\sqrt{3}}^{\infty} = \frac{1}{3}$$

$$P(Z=2) = P(1 < X < \sqrt{3}) = \frac{2}{\pi}\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{6}$$

$$P(Z=3) = P\left(\frac{1}{\sqrt{3}} < X < 1\right) = \frac{2}{\pi}\int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1+x^2}\,dx = \frac{1}{6}$$

$$P(Z=4) = P\left(0 < X < \frac{1}{\sqrt{3}}\right) = \frac{2}{\pi}\int_{0}^{1/\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{3}$$

Expected score,

$$E(Z) = 0 \times \frac{1}{3} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{2} + \frac{4}{3} = \ldots$$

9 (a) $$M_X(t) = E(e^{tX}) = \sum_0^n e^{tx}\binom{n}{x}p^x(1-p)^{n-x}$$

$$= \sum_0^n \binom{n}{x}(pe^t)^x(1-p)^{n-x}$$

$$= (\overline{1-p} + pe^t)^n$$

$\leftarrow q = 1-p$

$$\left.\frac{d}{dt}M_X(t)\right|_{t=0} = \left. n(q+pe^t)^{n-1}pe^t\right|_{t=0}$$

$$= np = E(X)$$

$$\frac{d^2}{dt^2}M_X(t) = n(n-1)(q+pe^t)^{n-2}(pe^t)^2 + n(q+pe^t)^{n-1}pe^t$$

$$\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = n(n-1)p^2 + np = \mu_2' = E(X^2)$$

$$V(X) = EX^2 - (EX)^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p) = npq$$

Sly (b)

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(c) $X \sim G(\alpha, \beta)$

$$M_X(t) = E(e^{tX}) = \frac{1}{\Gamma\alpha\ \beta^\alpha} \int_0^\infty e^{tx}\ x^{\alpha-1}\ e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma\alpha\ \beta^\alpha} \int_0^\infty x^{\alpha-1}\ e^{-x\left(\frac{1}{\beta}-t\right)}\, dx$$

(region of existence $t < \frac{1}{\beta}$)

$$= \frac{\Gamma\alpha}{\Gamma\alpha\ \beta^\alpha} \cdot \frac{1}{\left(\frac{1}{\beta}-t\right)^\alpha} = \left(\frac{1}{1-\beta t}\right)^\alpha$$

$$E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = -\alpha\,(1-\beta t)^{-\alpha-1}(-\beta)\Big|_{t=0}$$

$$= \alpha\beta$$

$$E(X^2) = \frac{d^2 M_X(t)}{dt^2}\Big|_{t=0}$$

$$= \alpha\beta\left(-(\alpha+1)\,(1-\beta t)^{-\alpha-2}(-\beta)\right)\Big|_{t=0}$$

$$= \cancel{\alpha\beta}\ \alpha(\alpha+1)\,\beta^2$$

$$V(X) = E(X^2) - (E(X))^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

Sly (e).

10 $M_X(t) = e^{-5t}\ \frac{1}{2} + e^{4t}\ \frac{1}{6} + e^{5t}\ \frac{1}{8} + \frac{5}{24}\ e^{25t}$

a 4 pt distn

p.m.f.

$X = x$	-5	4	5	25
$P(X = x)$	$1/2$	$1/6$	$1/8$	$5/24$

d.f. $$F_X(x) = \begin{cases} 0 & x < -5 \\ \frac{1}{2} & -5 \le x < 4 \\ \frac{1}{2}+\frac{1}{6} & 4 \le x < 5 \\ \frac{1}{2}+\frac{1}{6}+\frac{1}{8} & 5 \le x < 25 \\ \frac{1}{2}+\frac{1}{6}+\frac{1}{8}+\frac{5}{24} = 1 & x \geq 25 \end{cases}$$

(11) X is (+)ve values r.v., by Markov's inequality

$$P(X \geq 2\mu) = P(|X| \geq 2\mu) \leq \frac{E(|X|)}{2\mu} = \frac{1}{2}$$

(12) $$P(-2 < X < 8) = P\left(\frac{-2-3}{2} < \frac{X - E(X)}{\sqrt{V(X)}} < \frac{8-3}{2}\right)$$

$E(X) = 3$
$V(X) = 4$

$$= P\left(-5/2 < \frac{X - E(X)}{\sqrt{V(X)}} < 5/2\right)$$

$$= P\left(|X-\mu| \leq \frac{5}{2}\sqrt{V(X)}\right)$$

$$= 1 - P\left(|X-\mu| \geq 5/2\sqrt{V(X)}\right)$$

$$\geq 1 - \frac{V(X)}{\frac{25}{4}\cdot V(X)} \quad \text{(chebyshev's ineq)}$$

$$= 1 - \frac{4}{25} = \frac{21}{25}$$

(13) $E(X) = -\frac{1}{8} + \frac{1}{8} = 0 = \mu$; $V(X) = EX^2 = \frac{1}{8}+\frac{1}{8} = \frac{1}{4} = \sigma^2$

By chebyshev's inequality

$\forall \epsilon > 0$. $$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

i.e $$P(|X| \geq \epsilon) \leq \frac{1}{4\epsilon^2}$$

Also, $P(|X| \geq \epsilon) = \begin{cases} \frac{1}{8}+\frac{1}{8}=\frac{1}{4} & 0<\epsilon \leq 1 \\ 0 & \epsilon > 1 \end{cases}$

i.e. $P(|X| \geq \epsilon) = \frac{1}{4} \quad \forall\ \epsilon \ni 0<\epsilon \leq 1$

$\Rightarrow$ for $\epsilon = 1$, bound from Chebyshev's ineq is attained exactly and hence cannot be improved

(14) Let X be the r.v. denoting the # of components functioning

$X \sim B(n, p)$

$P(\text{system works effectively}) = P(X \geq [n/2]+1)$

(a) $P(5 \text{ comp. system works}) = p_5$

i.e. $p_5 = P_5(X \geq 3) = \binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + p^5$

& $p_3 = \binom{3}{2} p^2(1-p) + p^3 = P(3 \text{ comp system works})$

$p_5 > p_3$

Iff $\binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p) + p^5 > \binom{3}{2} p^2(1-p) + p^3$

simplify to get the condition as $\underline{p > \frac{1}{2}}$

(b) $P_{2k+1}(X \geq k+1) = p_{2k+1} = P_{2k-1}(X \geq k+1)$

$+ P_{2k-1}(X = k)\ P_2(X \geq 1)$

$+ P_{2k-1}(X = k-1)\ P_2(X = 2)$

i.e. $p_{2K+1} = P_{2K-1}(X \geq K+1) + P_{2K-1}(X=K)\left(1-(1-p)^2\right)$
$+ P_{2K-1}(X=K-1)\, p^2$

Further $p_{2K-1} = P_{2K-1}(X \geq K)$
$= P_{2K-1}(X=K) + P_{2K-1}(X \geq K+1)$

Since $p_{2K+1} = \underleftrightarrow{P_{2K-1}(X \geq K+1) + P_{2K-1}(X=K)}$
$- P_{2K-1}(X=K)\,(1-p)^2$
$+ P_{2K-1}(X=K-1)\, p^2$

$\Rightarrow p_{2K+1} = p_{2K-1} - P_{2K-1}(X=K)\,(1-p)^2$
$+ P_{2K-1}(X=K-1)\, p^2$

$\Rightarrow p_{2K+1} > p_{2K-1}$ if

$P_{2K-1}(X=K)\left(-(1-p)^2\right) + P_{2K-1}(X=K-1)\, p^2 > 0$

i.e. $\cancel{\binom{2K-1}{K}}\, p^K (1-p)^{K-1}\left(-1-p^2+2p\right) + \cancel{\binom{2K-1}{K-1}}\, p^{K-1}(1-p)^K p^2 > 0$

i.e. $p^K (1-p)^{K-1}\left(-1-p^2+2p+p-p^2\right) > 0$

i.e. $-1-2p^2+3p > 0$

i.e. $(2p-1)(1-p) > 0$

i.e. $p > \frac{1}{2}$ reqd condition.

(15) X : # of interview attempts to get 5 interviews.

$$P(X=x) = \binom{x-1}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{x-5} \times \frac{2}{3} \quad ; x = 5, 6, \ldots$$

reqd prob $P(X \le 8) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= \binom{4}{4}\left(\frac{2}{3}\right)^5 + \binom{5}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1 + \ldots + \ldots$$

$$= \ldots$$

(16) P(selecting Box 1) = P(selecting Box 2) $= \frac{1}{2}$

Suppose Box 2 is found empty, then Box 2 has been chosen $(N+1)^{th}$ times, at this time Box 1 contains K matches if it has been chosen N−K times.

⇒ Choosing Box 2 ≡ success. Choosing Box 1 ≡ failure } Bernoulli trial. $p = \frac{1}{2}$

⇒ Box 2 found empty with K matches left in Box 1

≡ N − K failures preceeding $(N+1)^{th}$ success

$$\text{prob} = \binom{N+(N-K)}{N}\left(\frac{1}{2}\right)^N\left(\frac{1}{2}\right)^{N-K} \cdot \frac{1}{2}$$

$$= \binom{2N-K}{N}\left(\frac{1}{2}\right)^{2N-K+1}$$

Sly Box 1 found empty with K matchs in Box 2

$$\text{prob} = \binom{2N-K}{N}\left(\frac{1}{2}\right)^{2N-K+1}$$

$$\Rightarrow \text{reqd prob} = \binom{2N-K}{N}\left(\frac{1}{2}\right)^{2N-K}.$$