Problem set # 5

(1)

Belt 1

$X \sim$ Exp with mean $\alpha \sim \frac{1}{\alpha} e^{-x/\alpha}$; $x > 0$

Belt 2 $Y \sim$ Exp with mean $2\alpha \sim \frac{1}{2\alpha} e^{-x/2\alpha}$; $x > 0$

P(system works beyond α)

$$P(X > \alpha \cap Y > \alpha) = P(X>\alpha)\, P(Y > \alpha)$$

$$= \left(\int_\alpha^\infty \frac{1}{\alpha} e^{-x/\alpha} dx\right) \times \left(\int_\alpha^\infty \frac{1}{2\alpha} e^{-y/2\alpha} dx\right)$$

$$= e^{-1} \times e^{-1/2} = \underline{e^{-3/2}}$$

(2) (a) $P(X>5) = P\left(\frac{X-10}{6} > \frac{5-10}{6}\right) = P\left(Z > -\frac{5}{6}\right)$; $Z \sim N(0,1)$

$$= 1 - \Phi(-5/6)$$

$$= 1 - (1 - \Phi(5/6))$$

$$= \Phi(5/6) = .7967$$

(b) $P(4 < X < 16) = P\left(\frac{4-10}{6} < Z < \frac{16-10}{6}\right) = P(-1 < Z < 1)$

$$= \Phi(1) - \Phi(-1) = 2\,\Phi(1) - 1$$

$$= \ldots$$

(c) $P(X < 8) = P\left(Z < \frac{8-10}{6}\right) = \Phi(-\frac{1}{3}) = 1 - \Phi(1/3) = \ldots$

(3) $P(X \le 0) = \frac{1}{2} = P(X \ge 0) \Rightarrow \mu = 0$

$$P(-1.96 \le X \le 1.96) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \le \frac{X}{\sigma} \le \frac{1.96}{\sigma}\right) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \le Z \le \frac{1.96}{\sigma}\right) = 0.95 \text{ ; } Z \sim N(0,1)$$

$$2\,\Phi\left(\frac{1.96}{\sigma}\right) - 1 = 0.95$$

$$\Phi\left(\frac{1.96}{\sigma}\right) = 0.975$$

$$\Rightarrow \frac{1.96}{\sigma} = \Phi^{-1}(0.975) = 1.96$$

$$\Rightarrow \sigma = 1$$

(4) X : lifetime r.v.

$X \sim N(\mu, \sigma^2)$

$\mu = 1.4 \times 10^6$ hrs

$\sigma = 3 \times 10^5$ hrs

$P(X < 1.8 \times 10^6)$

$= P\left(\frac{X - 1.4 \times 10^6}{3 \times 10^5} < \frac{0.4 \times 10^6}{3 \times 10^5} \right)$

$= \cancel{\Phi(Z < \frac{4}{3})} = P(Z < \frac{4}{3})$ $[Z \sim N(0,1)]$

$= \Phi(\frac{4}{3}) = 0.918$

Y : r.v. denoting # of chips that have lifetime $< 1.8 \times 10^6$ hr

$Y \sim Bin(10, 0.918)$

$\Rightarrow P(Y \geq 2) = 1 - P(Y < 2)$

$= 1 - P(Y=0) - P(Y=1)$

$= 1 - \binom{10}{0} (.918)^0 (1-.918)^{10} - \binom{10}{1} (0.918)^1 (1-.918)^9$

$= \ldots$

(5) $X \sim N(0,1)$

$$\forall\, t > 0 \quad P(|X| \geq t) = 1 - P(|X| < t)$$
$$= 1 - P(-t < X < t)$$
$$= 1 - [\Phi(t) - \Phi(-t)]$$
$$= 1 - [2\Phi(t) - 1]$$
or
$$= 1 - [2(1 - P(X > t)) - 1]$$
$$= 2 - 2 + 2P(X > t) = 2P(X > t)$$

$$P(X > t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2}\, dx$$
$$\leq \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{x}{t} e^{-x^2/2}\, dx \quad [t < x < \infty]$$
$$y = x^2/2$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{t} \int_{t^2/2}^{\infty} e^{-y}\, dy = \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}$$
$$\Rightarrow P(|X| \geq t) \leq 2 \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t} = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$$

(6)

$X = x$	0	1	2	......
$P(X = x)$	p_0	p_1	p_2	......

$$\sum_{k=0}^{\infty} (1 - F(k)) = \sum_{k=0}^{\infty} P(X > k) = P(X > 0) + P(X > 1) + P(X > 2) + \cdots$$
$$= (p_1 + p_2 + p_3 + \cdots)$$
$$+ (p_2 + p_3 + \cdots)$$
$$+ (p_3 + p_4 + \cdots)$$
$$= p_1 + 2p_2 + 3p_3 + \cdots$$

$$= \sum_{i=1}^{\infty} i\, p_i = \sum_{i=0}^{\infty} i\, P(X=i) = E(X)$$

(7) d.f. $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\beta x^2}, & x \geq 0. \end{cases}$

$\beta > 0$

p.d.f. $f(x) = \begin{cases} 2\beta\, x\, e^{-\beta x^2}, & x \geq 0 \\ 0 & \text{o/w}. \end{cases}$

$$E(X) = 2\beta \int_0^{\infty} x^2 e^{-\beta x^2} dx$$

$y = x^2$

$$= \beta \int_0^{\infty} y^{1/2} e^{-\beta y} = \beta \cdot \frac{\Gamma 3/2}{\beta^{3/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} = \mu.$$

$$E\, X^2 = 2\beta \int_0^{\infty} x^3 e^{-\beta x^2} dx \overset{y = x^2}{=} \beta \int_0^{\infty} y\, e^{-\beta y} dy = \beta \frac{\Gamma 2}{\beta^2} = \frac{1}{\beta}$$

$$V(X) = E(X^2) - (E\,X)^2 = \frac{1}{\beta} - \mu^2 = \frac{1}{\beta} - \frac{\pi}{4\beta}$$

median : m_0

$m_0 \ni F(m_0) = \frac{1}{2} = 1 - F(m_0)$

i.e. $2\beta \int_0^{m_0} x e^{-\beta x^2} dx = 2\beta \int_{m_0}^{\infty} x e^{-\beta x^2} dx = \frac{1}{2}$

i.e. $1 - e^{-\beta m_0^2} = \frac{1}{2}$

$\Rightarrow m_0 = \ldots$

(8) $$1-\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_x^{\infty} \frac{1}{y}\left(y e^{-y^2/2}\right) dy\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{y}\cdot\left(-e^{-y^2/2}\right)\Big|_x^{\infty} - \int_x^{\infty}\left(-\frac{1}{y^2}\right)\left(-e^{-y^2/2}\right) dy\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{x} e^{-x^2/2} - \underbrace{\int_x^{\infty}\frac{1}{y^2} e^{-y^2/2}\, dy}_{\geq 0.}\right]$$

$$\Rightarrow \quad 1-\Phi(x) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{\phi(x)}{x}.$$

(9) p.d.f. of (7)
Mode - pt at which $f(x)$ is max^m.

$$f'(x) = 2\beta\left(x e^{-\beta x^2}(-2\beta x) + e^{-\beta x^2}\right)$$

$$f'(x) = 0 \Rightarrow 2\beta x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2\beta}}.$$

$$f''(x) = 2\beta \frac{d}{dx}\left(e^{-\beta x^2}(1 - 2\beta x^2)\right)$$

$$= 2\beta\left(e^{-\beta x^2}(-4\beta x) + (1 - 2\beta x^2) e^{-\beta x^2}(-2\beta x)\right).$$

$$f''(x)\Big|_{x = \frac{1}{\sqrt{2\beta}}} = 2\beta\left(e^{-\frac{1}{2}}\left(-4\sqrt{\beta/2}\right)\right) < 0.$$

$\beta > 0$

$\Rightarrow$ m^*, the mode of the dist^n is at $\frac{1}{\sqrt{2\beta}}$.

$$m^* = \frac{1}{\sqrt{2\beta}}$$

$$\mu = E(X) = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\beta}} = \left(\frac{\sqrt{\pi}}{2}\right)\sqrt{2}\ m^*$$

i.e. $\mu = \sqrt{\frac{\pi}{2}}\ m^*$.

& $2m^{*2} - \mu^2 = 2\left(\frac{2}{\pi}\mu^2\right) - \mu^2$

$$= \frac{4}{\pi}\mu^2 - \mu^2 = \frac{\cancel{4}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{4}} \cdot \frac{1}{\beta} - \mu^2$$

i.e.

$2m^{*2} - \mu^2 = \sigma^2$

$$\cancel{= \left(\frac{4}{\pi} - 1\right)\mu^2} = \frac{1}{\beta} - \mu^2$$

$$= E X^2 - \mu^2$$

$$= V(X).$$

(10) $X \sim P(\lambda)$

p.m.f. $P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}, & x = 0, 1, 2, \ldots \\ 0, & o/w. \end{cases}$

$Y = X^2 - 5 \Rightarrow$ Range space of $Y = \{-5, -4, -1, 4, 11, \ldots\}$.
$= \mathcal{Y}$

$P(Y=y) = P(X^2 - 5 = y) = P(X^2 = y+5)$

p.m.f. of Y : $P(Y=y) = P(X = \sqrt{y+5}) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{\sqrt{y+5}}}{(\sqrt{y+5})!}, & y \in \mathcal{Y} \\ 0, & o/w. \end{cases}$

(11)

$X \sim B(n, p)$

p.m.f. $P(X=x) = \begin{cases} \binom{n}{x} p^{x} (1-p)^{n-x}, & x = 0, 1, \ldots n \\ 0, & o/w. \end{cases}$

$Y = n - X \Rightarrow \mathcal{Y} = \{0, 1, \ldots n\}$.

$P(Y=y) = P(n - X = y) = P(X = n-y)$

$\Rightarrow$ p.m.f. of Y

$$P(Y=y) = \begin{cases} \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}; & y = 0, 1, \ldots n \\ 0, & o/w \end{cases}$$

$$= \begin{cases} \binom{n}{y} (1-p)^{y} p^{n-y}, & y = 0, 1, \ldots n \\ 0, & o/w. \end{cases}$$

$\Rightarrow Y \sim B(n, 1-p)$.

$\mathcal{X} = \{-2, -1, 0, 1, 2, 3\}$

(12) $Y = X^2 \rightarrow$ range sp $= \{0, 1, 4, 9\}$

p.m.f. $$P(Y=y) = \begin{cases} P(X=0) & y=0 \\ P(X=-1)+P(X=1) & y=1 \\ P(X=-2)+P(X=2) & y=4 \\ P(X=3) & y=9 \end{cases}$$

$$= \begin{cases} \frac{1}{5}, & y=0 \\ \frac{1}{6}+\frac{1}{15}, & y=1 \\ \frac{1}{5}+\frac{1}{3}, & y=4 \\ \frac{1}{30} & y=9 \end{cases}$$

(13) $$P(X=x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x, & x=0,1,2,\ldots \\ 0, & o/w \end{cases}$$

$$Y = \frac{X}{X+1} \Rightarrow X = \frac{Y}{1-Y}$$

range space of $Y = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

$$P(Y=y) = P\left(\frac{X}{X+1} = y\right) = P\left(X = \frac{y}{1-y}\right)$$

$$= \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, & y = 0, \frac{1}{2}, \frac{2}{3}, \ldots \\ 0 & o/w. \end{cases}$$

(14) p.m.f. of X

$$P(X=x) = \begin{cases} e^{-1}, & x=0 \\ \dfrac{e^{-1}}{2(|x|)!}, & x \in \{\pm 1, \pm 2, \ldots\} \\ 0, & o/w. \end{cases}$$

$Y = |X| \qquad \mathcal{Y} = \{0, 1, 2, \ldots\}.$

~~P(y=y~~ $P(Y=0) = P(X=0) = e^{-1}$

$$P(Y=1) = P(X=-1) + P(X=1)$$
$$= \frac{e^{-1}}{2} + \frac{e^{-1}}{2} = e^{-1}$$

$$P(Y=2) = P(X=-2) + P(X=2)$$
$$= \frac{e^{-1}}{2.2!} + \frac{e^{-1}}{2.2!} = \frac{e^{-1}}{2!}$$

Sly for $K=1,2,\ldots$

$$P(Y=K) = P(X=-K) + P(X=K)$$
$$= \frac{e^{-1}}{2.K!} + \frac{e^{-1}}{2.K!} = \frac{e^{-1}}{K!}$$

p.m.f. of Y

$$P(Y=y) = \begin{cases} \dfrac{e^{-1}}{y!}, & y = 0, 1, 2, \ldots \\ 0, & o/w \end{cases}$$

$Y \sim P(1)$